Some Examples Related to Conway Groupoids and their Generalisations

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Abstract. We discuss the recently introduced notion of a Conway Groupoid. In particular we consider various generalisations of the concept including infinite analogues.

1. Introduction

In [**5**] Conway introduced a construction, akin to Loyd's celebrated 15-puzzle, of a pseudogroup that gave a new construction of the Mathieu group M_{12} which he called M_{13}. (Note that some parts of the literature erroneously claim that the first appearance of these ideas was ten years later in [**6**].) This was originally motivated by the well-known similarities in the 3-local structure (i.e. the relationship between the Sylow 3-subgroups and the group as a whole) of the groups $\mathrm{L}_3(3)$ and M_{12}. We briefly describe Conway's original construction.

Recall that the projective plane of order three is a set Ω of thirteen points and a collection of thirteen subsets of size four that we call lines with the property that any pair of points is contained in precisely one line. In other words it is a (2,4,13) Steiner system or a 2-(13,4,1) design. Given any two $a, b \in \Omega$ if the line these two points belong to is $\{a, b, c, d\}$ then we can define the permutation $[a, b] := (ab)(cd)$. If we further fix a point $\infty \in \Omega$ we can think of a 'hole' as sitting at this point and being moved by $[\infty, a]$ to a. We call this a 'move' of the game. Repeating this procedure, a sequence of moves will transport the hole around the plane. If we consider sequences of moves that eventually return the hole to the original point, ∞, then, as Conway proved, the resulting permutations form a group that acts on the remaining twelve point and this group is the sporadic simple group M_{12}. In categorical terms the set of all permutations, M_{13} is a groupoid (i.e. a category in which all arrows are invertible).

There have been recent attempts to generalise the above construction along the following lines. Let Ω be a finite set. A function $[\cdot,\cdot]\colon \Omega^2 \to \mathrm{Sym}(\Omega)$ is said to be *pliable* if for each $a, b \in \Omega$ the permutation $[a, b]$ sends a to b and $[a, b]^{-1} = [a, b]$.

1991 *Mathematics Subject Classification.* Primary 20B15; Secondary 20B25, 05B05.
Key words and phrases. Mathieu Groupoid, Conway Groupoid, M_{13} Groupoids.

This naturally extends by defining for each $a_1, \dots, a_k \in \Omega$,

$$[a_1, a_2, \dots, a_k] := \prod_{i=1}^{k-1} [a_i, a_{i+1}].$$

For each $x \in \Omega$ we define

- $\mathcal{L}_x([\cdot,\cdot]) := \{[x, a_1, a_2, \cdots, a_k] \mid k \in \mathbb{Z}^+,\ a_i \in \Omega\}$ (we call this the **Conway groupoid** of $[\cdot,\cdot]$);
- $\pi_x([\cdot,\cdot]) := \{[x, a_1, a_2, \cdots, a_k, x] \mid k \in \mathbb{Z}^+,\ a_i \in \Omega\}$ (we call this the **hole stabilizer** of $[\cdot,\cdot]$).

Several examples have been constructed and Conway groupoids with carious properties have been classified — for details see [**12, 13, 14**]. Other notable works in the area include [**3, 6, 10, 23, 25**]. Here we aim to try and generalise the above concepts showing that several interesting new examples emerge if we relax the various defining conditions in the above.

Throughout we shall use the standard Atlas notation for finite groups and related concepts as described in the introductory chapters of [**7**].

This paper is organised as follows. In the next section we will discuss a number of variants of Conway groupoids that hint at generalisations of the concept. In the third section our attention turns to the specific variant of Conway groupoids in which the underlying set is infinite. Finally, in the fourth section we discuss a number of open questions and problems.

Acknowledgement The author wishes to express his deepest gratitude to the organisers of the meeting 'Finite Simple Groups: Thirty Years of the Atlas and Beyond Celebrating the Atlases and Honoring John Conway' for their work pulling together the meeting. Furthermore the author wishes to thank Jason Semeraro for introducing the author to the subject in the first place and for helpful conversations and correspondence on the matter.

2. Generalisations

2.1. $[a, b]$ **Not Always Being Defined.** Since we are working in the setting of groupoids where composition of $[a, b]$ and $[c, d]$ is not always possible it seems natural to consider the possibility of $[a, b]$ not necessarily even existing.

Definition 2.1. We define a **partial groupoid** $\mathcal{L}'_x([\cdot,\cdot]) := \{[x, a_1, a_2, \cdots, a_k] \mid k \in \mathbb{Z}^+,\ a_i \in \Omega$ and all of $[x, a_1], \dots, [a_{k-1}, a_k]$ exist$\}$.

For instance, it may be the case that a and b correspond to the vertices of a graph with $[a, b]$ only being defined when a and b are a certain distance apart from one another. We could then define $[a, b]$ to be an involution in the automorphism group of the graph (for example) that interchanges a and b. This naturally condemns the hole stabilizer π_x to being a subgroup of the graph's automorphism group, however several interesting examples still arise.

Example 2.2. In 1965 Zvonimir Janko discovered the first of the modern sporadic groups, which became known as the Janko group J_1, just a couple of years of before Conway discovered his eponymous groups. As part of the race to discover as much as possible about this strange new object, in [**22**] Livingstone introduced a beautiful 11-regular graph Γ on 266 vertices on which J_1 acts vertex transitively and is its full automorphism group (see also [**7**, p. 36]). The stabilizer of a vertex is

Γ	n	$\mathrm{Aut}(\Gamma)$	vertex	edge
$\mathrm{U}_3(3)$	36	$\mathrm{U}_3(3):2$	$\mathrm{PGL}_2(7)$	$[2^4 \cdot 3]$
Hall-Janko	100	$\mathrm{J}_2:2$	$\mathrm{U}_3(3):2$	$2{\times}\mathrm{PGL}_2(7)$
$\mathrm{G}_2(4)$	416	$\mathrm{G}_2(4):2$	$\mathrm{J}_2:2$	$2{\times}\mathrm{U}_3(3):2$
Suzuki	1 782	Suz: 2	$\mathrm{G}_2(4):2$	$2{\times}\mathrm{J}_2:2$

TABLE 1. The Suzuki Tower. For each graph Γ we give n, the number of its vertices; its automorphism group, $\mathrm{Aut}(\Gamma)$; the stabilizer of a vertex in the column denoted 'vertex' and the stabilizer of an edge in the column denoted 'edge'. (Some authors include the Heawood graph in the tower, but the central involution of its edge stabiliser does not interchange the two endpoints.)

isomorphic to $\mathrm{L}_2(11)$ and acts transitively on the 11 neighbours hence J_1 also acts edge transitively on Γ. Moreover, the stabilizer of an edge is isomorphic to $2{\times}\mathrm{A}_5$ with the central involution interchanging the two vertices adjoined to the edge. If a and b are two vertices adjoined by an edge we can then define $[a,b]$ to be the central involution of the stabilizer of this edge. It turns out that Γ has girth five and if $a_0, a_1, \ldots, a_4$ is one such 5-cycle, then $[a_0, a_1, \ldots, a_4, a_0]$ is also an involution (and is in fact is equal to $[a_2, a_3]$). It turns out that in $\mathcal{L}'_x$ if we loosen the definition of π_x for Conway groupoids to allow the possibility of $[a,b]$ not always existing, then in this case π_x in fact generates the full stabilizer of a_0 in this case.

Thompson's celebrated 'Suzuki chain' of groups gives another sequence of graphs, sometimes referred to as the 'Suzuki Tower', that may be used in this manner. In each case the full automorphism group acts vertex transitively on the graph and the stabilizer of an edge has a center of order two, the central involution interchanging the two ends of the edge (and is typically of the form $2 \times H$ where H is an almost simple group). We give some details in Table 1 and slightly more explicit descriptions of these graphs are given in [**7**, pp. 42, 97 & 131].

We remark that several results from elsewhere in the literature have analogues for these games as long as the hypotheses are amended to 'if the composition exists' conditions. As a typical example we have the following analogue of [**13**, Lemma 2.7].

Lemma 2.3. *The following are equivalent.*

(1) *For all $a,b,c,d \in \Omega$ such that each of $[a,b]$, $[c,d]$, $[a^{[c,d]}, b^{[c,d]}]$ and $[a,b] \cdot [c,d]$ exist, we have that $[a,b]^{[c,d]} = [a^{[c,d]}, b^{[c,d]}]$;*
(2) *For all $a,b,c \in \Omega$ such that $[a,b]$ and $[b,c]$ exist we have that $[a,b]^{[b,c]} = [a^{[b,c]}, c]$;*
(3) *For all $b,c \in \Omega$ such that $[a,b]$, $[b,c]$ and $[c, a^{[b,c]}]$ exist we have that $[b,c] = [a,b,c,a^{[b,c]}]$.*

PROOF. All the calculations performed in the proof of [**13**, Lemma 2.7] apply whenever the relevant compositions and elements exist. □

As an example of a game in which $[a,b]$ does no always exist but the conditions of the above lemma hold we have the following.

Example 2.4. Consider the isometry group of the icosahedron. The stabilizer of an edge is a Klein fours group, however of the three involutions one stands out as

Γ	n	Y	N
K_4	4	2	trivial
$\mathrm{K}_{3,3}$	6	-	trivial
Cube	8	-	3
Petersen	10	$\mathrm{PGL}_2(7)$	$\mathrm{L}_2(7)$
Heawood	14	-	A_6
Dodecahedron	20	S_{18}	A_{18}

TABLE 2. Some famous small cubic graphs and the groups obtained by considering the 'double hole' construction. In the Γ column we give the name of the graph, n is the number of vertices and the remaining two columns list the groups obtained depending on whether or not we are allowed to interchange the positions of the two holes (which is impossible in the bipartite cases.)

special being the only involution that is a rotation. If a and b are vertices that are adjoined by an edge then we can define $[a, b]$ to be this involution. If c is a common neighbour of a and b and $d \neq b$ is the common neighbour of a and c, then a simple direct calculation verifies that $[a, b]^{[a,c]} = [a^{[a,c]}, b^{[a,c]}] = [c, d]$.

Several related constructions are described in [**24, 26, 27**].

2.2. Multiple Holes. One of Conway's early attempts to generalise his M_{13} game is briefly described in [**6**, Section 8] by considering a variant in which we play with multiple holes that he described as follows.

> On the Petersen graph, we can play the game using two holes, which must always be adjacent. A move consists of permuting cyclically the three neighbours of a hole (the other hole and two further vertices). We obtain the group $\mathrm{L}_2(7)$ (if the holes return to their original positions) or $\mathrm{L}_2(7).2$=PGL(2,7) (if they are allowed to be interchanged by the move sequence.)

It is clear that we can play the same 'multihole game' on any cubic graph. There is a long history of studying and classifying cubic graphs that are vertex and edge transitive (without some sort of symmetry assumption different starting points for the holes may give different games and groups.) In particular, in [**4**] all symmetric cubic graphs with certain nice transitivity properties and small girth are classified, there being only finitely many exceptional cases for every girth up to 9. We list some examples in Table 2. Clearly if a graph has no odd cycles (i.e. is bipartite) then the resulting group cannot act transitively on the vertices, since the permutations obtained when moving the holes around necessarily keep the vertices of each half of the graph separate.

A related discussion is given by Ekenta, Jang and Siehler in [**9**].

Even more generally, if we drop the edge transitivity condition we may consider cubic graphs in which different orbits of edges in the action of the automorphism group of the graph may define different groups.

Alternatively, if we would like to use permutations with fewer fixed points, we may instead consider an embedding of a k-regular graph on an orientable surface. One may now insist that hole b can only be moved to a neighbour of a that lies on the same face as b and the orientation of the surface now defines a cyclic permutation of all the neighbours of a in the direction dictated by the choice of c. Conway's Petersen graph construction described in the above quote may be viewed as an example of this using the natural embedding of the Petersen graph on the torus. This avenue of research opens up the intriguing possibility of connections bridging the world of Conway groupoids with that of Grothendieck's theory of *dessins d'enfant*, the dessin corresponding to a certain permutation group defined on the edges of the graph and the groupoid corresponding to a certain permutation group acting on the vertices. (More specifically, the non-hole vertices, if we insist on the holes returning to their original positions — for natural reasons we tend to restrict our attention to bipartite graphs when considering dessins d'enfant). Several good introductions *dessins d'enfant* have appeared in recent years — see for instance any of [**15, 17, 18**].

2.3. Formally moving the hole. There are plenty of instances in which every pair $a, b \in \Omega$ may be associated with a unique permutation that, alas, does not interchange a and b. Under such circumstances we may still set $[a, b]$ to be the permutation determined by a and b and consider the hole to have formally moved from a to b by some other mysterious means (it is tempting to talk of the hole being moved 'as if by the invisible hand of Conway') without actually having been moved there by $[a, b]$. The following gives a natural infinite family of such examples.

Example 2.5. Galois himself described in his fateful letter to Chevalier the action of the groups $L_2(q)$ on $q + 1$ points. This is beautifully described in some detail by Conway in the first of his famous 'three lectures on exceptional groups' reproduced in Conway and Sloane [**8**, Chapter 10]. This action is 2-transitive and the stabilizer of two points is a dihedral group of order $q - 1$. In particular, if $q \equiv 1 \pmod 4$ and $q > 5$, then this stabilizer will have a unique non-trivial central element and for $a, b \in \Omega$ we can define $[a, b]$ to be this involution. Note that this involution dose not interchange a and b. For example, in terms of the natural action of $L_2(q)$ on the projective line, an element $x \mapsto -1/x$ will fix any y such that $y^2 + 1 = 0$. Since $q \equiv 1 \pmod 4$ two such values exist and if we call these a and b, then $[a, b]$ is precisely this element.

3. Infinite Conway Groupoids

In every previously investigated example of a Conway groupoid and its variants the underlying set has been taken to be finite. In many ways this is odd: finite t-designs are much less well behaved than their infinite counterparts, particularly when it comes to the question of existence. For example, showing that the obvious necessary conditions for a finite combinatorial design to exist are in fact sufficient was a problem that was finally settled by Keevash in [**19**] as recently as 2014 — a century and a half after the problem was first posed by Steiner in 1853! Their infinite counterparts, at least with finite blocks, however, are much better understood — see for instance [**1, 2, 11, 16, 20**]. In particular, much is known about infinite t-designs and there are numerous examples in the literature. Our first example uses one such design to give a direct generalisation of a finite Conway groupoid.

Example 3.1. The following are direct analogues of the 'Boolean quadruple systems' of [**13**, Example 1.1]. Let V be an infinite vector space defined over the

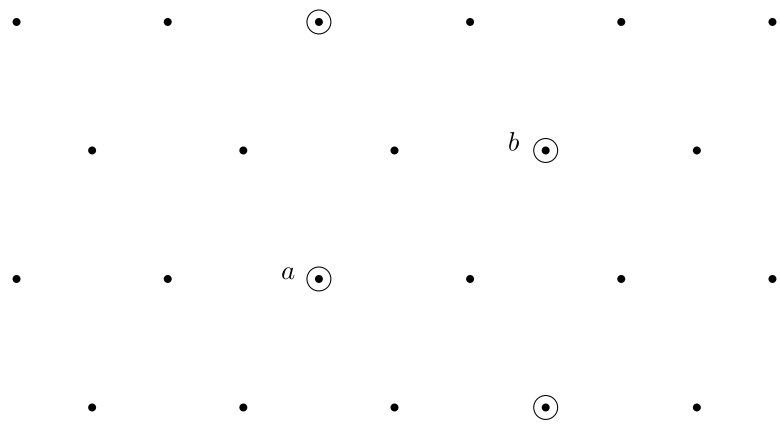

FIGURE 1. A triangular lattice defining a $2-(\aleph_0, 4, 1)$ design: the points labelled a and b define the block whose points are circled in the diagram.

field of two elements $\mathbb{F}_2$ (for example taking the algebraic closure $\overline{\mathbb{F}_2}$ will do). We consider a 3–($\aleph_0$,4,1) design $(V, \mathcal{B})$ whose blocks are the members of the set

$$\mathcal{B} := \{(v_1, v_2, v_3, v_4)\,|v_i \in V\,, \sum_{i=1}^{4} v_i = 0\,\}.$$

Equivalently, we can write

$$\mathcal{B} := \{v + W \,|\, v \in V\,, W < V\,, \dim W = 2\}$$

and so $\mathcal{B}$ is the set of all affine subspaces of V. We note that this groupoid also has the property that

($\triangle$) if $B_1, B_2 \in \mathcal{B}$ are such that $|B_1 \cap B_2| = 2$, then $B_1 \triangle B_2 \in \mathcal{B}$

since being in characteristic 2 means that adding together the two expressions $v_1 + v_2 + v_3 + v_4 = v_1 + v_2 + v_5 + v_6 = 0$ gives us $v_3 + v_4 + v_5 + v_6 = 0$. We can also see that $(V, \mathcal{B})$ has the property of being 'supersimple': any two distinct blocks intersect in at most two points. Further note that any set of three vectors is contained in a unique line and given any two blocks in $\mathcal{B}$ they intersect in at most two points. Finally, we note that we can also obtain a similar 3–($\aleph_1$,4,1) example by instead using an $\mathbb{F}_2$ vector space like the 2-adic numbers.

There are plenty of infinite $2-(\aleph_0, 4, \lambda)$ designs that do not have natural finite analogues.

Example 3.2. Consider the familiar two-dimensional hexagonal lattice that we depict in Figure 3.2. Given any two points of the lattice a and b there exist precisely two equilateral triangles whose corners are points of the lattice two of which are a and b. We thus have a $2-(\aleph_0, 4, 1)$ design (and a $2-(\aleph_0, 3, 2)$ design). Unlike the design in the last example, this is not supersimple since it is possible to find pairs of blocks that intersect in three points. If a and b are points of this lattice and $\{a, b, c, d\}$ is the corresponding block then we can define $[a, b] := (a, b)(c, d)$ and play the usual game.

Many of the general results proved for Conway groupoids whose underlying set is finite do not critically depend on the underlying set being finite (though some caution is required since some results are inherently finitary in nature, for instance the heavy use of Fischer's classification of the finite 3-transposition groups in [**13**]). As an example, the proof of the following is entirely analogous to that of its finite counterpart.

Lemma 3.3. *Suppose that $\mathcal{D}$ is a pliable hypergraph with blocks of size 4 such that any pair of points are colinear. Fix an element*

$$f := [a_0, a_1, \ldots, a_n] \in \mathcal{L}_{\mathcal{D}}.$$

The following statements hold:

(a) $f = [a_0, a_1, \ldots, a_i] \cdot [a_i, a_{i+1}, \ldots, a_n]$ *for all* $1 \leq i \leq n-1$*;*
(b) $f = [a_0, a_1, \ldots, a_i, x, a_i, a_{i+1}, \ldots, a_n]$ *for each* $0 \leq i \leq n$ *and* $x \in \Omega$*;*
(c) *for each* $x \in \Omega$*,*
  (i) $\mathcal{L}_{\mathcal{D}} = \cup_{a,b\in\Omega}[a, x] \cdot \pi_x(\mathcal{D}) \cdot [x, b]$ *and*
  (ii) *if* $a, b \in \Omega$ *are distinct, then* $[a, x] \cdot \pi_x(\mathcal{D}) \cap [b, x] \cdot \pi_x(\mathcal{D}) = \emptyset$*;*
(d) $\pi_x(\mathcal{D}) = \langle [x, a, b, x] \mid a, b \in \Omega \setminus \{x\}$

Proof. The proof is entirely analogous to the parts of [**12**, Lemma 3.1] that make no reference to any object being finite. □

4. Concluding Remarks and Open Problems

In this final section we discuss a number of open questions and problems relating to Conway groupoids.

4.1. The Large Mathieu Groups.

Perhaps the most obvious and natural question is the following.

Question 4.1. Is there a natural analogue to M_{13} for the large Mathieu groups?

It would be natural to call such an object an M_{25}, however since 25 is not of the form n^2+n+1 there is no projective plane that may be used in an entirely analogous way to the M_{13}, though there are several 2-designs on 25 points providing candidates for an alternative. There has also been some suggestion of using the structure of the vector space $\mathbb{F}_2^5$ to provide a sort of M_{31} [**21**]. Moreover, Conway himself has suggested that it would be more natural to consider a 'doubled up' version of M_{13} itself, an object that may be more naturally called M_{26} or even M_{13+13}. A similar idea for a $2\mathrm{M}_{13}$ (an analogue of the covering group $2\mathrm{M}_{12}$) is discussed in [**6**, Section 5].

Recall from the introduction that the initial motivation for M_{13} came from the 3-local structure of M_{12}. Since the small Mathieu groups are naturally associated with ternary Golay code and the large Mathieu groups with the binary Golay code, then it seems sensible to consider groups with a similar 2-local structure to that of M_{24}. It is well known that the Sylow 2-subgroup of M_{24} is isomorphic to that of $\mathrm{L}_5(2)$, but an even more striking candidate is isomorphic to the sporadic simple Held group. Not only does it have the same Sylow 2-subgroup but they lie inside the larger group in the same manner: both have odd index subgroups with structure $2^6 : 3 \cdot \mathrm{S}_6$. Indeed the Held group naturally acts on a certain regular digraph with 2058 vertices [**7**, p. 104] — might there be an M_{2058}?

4.2. Exotic Groupoids. We reiterate [**13**, Question 1.5]. We call a Conway groupoid $\mathcal{L}_x(\mathcal{D})$ associated to a supersimple design $\mathcal{D}$ *exotic* if $\mathcal{L}_x(\mathcal{D})$ is not a group and $\pi_x(\mathcal{D})$ is primitive. The original M_{13} is exotic and [**14**, Theorem D] gives strong bounds on the possible parameters defining $\mathcal{D}$ for which $\mathcal{L}_x(\mathcal{D})$ is exotic.

Question 4.2. Is M_{13} the only exotic Conway groupoid?

4.3. The condition ($\triangle$) in supersimple designs. We briefly discuss a variant of [**13**, Question 1.4]. We noted in Example 3.1 that the groupoids defined there satisfy the condition ($\triangle$). In [**13**, Theorems B and C] finite groupoids corresponding to supersimple designs satisfying condition ($\triangle$) along with certain other constraints are classified however the authors of [**13**] express the view that groupoids not satisfying condition ($\triangle$) may be possible.

Question 4.3. Is it possible to classify (finite or infinite) groupoids not satisfying condition ($\triangle$)?

4.4. Other Structures. We briefly discuss a variant of [**13**, Question 1.6]. Whilst Conway groupoids and the related structures described here are often associated with combinatorial designs, there are plenty of other structures out there that may be considered: codes, vertex transitive graphs, the 27 lines of a general cubic surface [**7**, p. 26], the 28 bitangents of a general quartic curve [**7**, p. 46] to name a few.

Question 4.4. Are there alternative combinatorial structures which can be used to define interesting groupoids and related structures?

We remark that in [**6**, Section 8] Conway considers similar constructions coming from the projective plane of order 2 (leading to the groups $3\mathrm{A}_6$ and $\mathrm{A}_6.2$) and, as mentioned earlier, on using the Petersen graph (leading to the groups $(\mathrm{PG})\mathrm{L}_2(7)$)

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