RECENT WORK ON BEAUVILLE SURFACES, STRUCTURES AND GROUPS

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Abstract

Beauville surfaces are a class of complex surfaces defined by letting a finite group G act on a product of Riemann surfaces. These surfaces possess many attractive geometric properties several of which are dictated by properties of the group G. In this survey we discuss the groups that may be used in this way. *En route* we discuss several open problems, questions and conjectures.

1 Introduction

Roughly speaking (precise definitions will be given in the next section), a Beauville surface is a complex surface $\mathcal{S}$ defined by taking a pair of complex curves, i.e. Riemann surfaces, $\mathcal{C}_1$ and $\mathcal{C}_2$ and letting a finite group G act freely on their product to define $\mathcal{S}$ as a quotient $(\mathcal{C}_1 \times \mathcal{C}_2)/G$. These surfaces have a wide variety of attractive geometric properties: they are surfaces of general type; their automorphism groups [50] and fundamental groups [20] are relatively easy to compute (being closely related to G - see Section 7.2 and 7.3); these surfaces are rigid surfaces in the sense of admitting no nontrivial deformations [10] and thus correspond to isolated points in the moduli space of surfaces of general type [37].

Much of this good behaviour stems from the fact that the surface $(\mathcal{C}_1 \times \mathcal{C}_2)/G$ is uniquely determined by a particular pair of generating sets of G known as a 'Beauville structure'. This converts the study of Beauville surfaces to the study of groups with Beauville structures, i.e. Beauville groups.

Beauville surfaces were first defined by Catanese in [20] as a generalisation of an earlier example of Beauville [14, Exercise X.13(4)] (native English speakers may find the English translation [15] somewhat easier to read and get hold of) in which $\mathcal{C} = \mathcal{C}'$ and the curves are both the Fermat curve defined by the equation $X^5 + Y^5 + Z^5 = 0$ being acted on by the group $(\mathbb{Z}/5\mathbb{Z}) \times (\mathbb{Z}/5\mathbb{Z})$ (this choice of group may seem somewhat odd at first, but the reason will become clear later). Bauer, Catanese and Grunewald went on to use these surfaces to construct examples of smooth regular surfaces with vanishing geometric genus [11]. Early motivation came from the consideration of the 'Friedman-Morgan speculation' – a technical conjecture concerning when two algebraic surfaces are diffeomorphic which Beauville surfaces provide counterexamples to. More recently, they have been used to construct interesting orbits of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (connections with Gothendeick's theory of dessins d'enfant make it possible for this group to act

on the set of all Beauville surfaces). We will discuss this in slightly more detail in Section 7.6. Furthermore, Beauville's original example has also recently been used by Galkin and Shinder in [34] to construct examples of exceptional collections of line bundles.

Like any survey article, the topics discussed here reflect the research interests of the author. Slightly older surveys discussing related geometric and topological matters are given by Bauer, Catanese and Pignatelli in [12, 13]. Other notable works in the area include [7, 51, 58, 63].

We remark that throughout we shall use the standard 'Atlas' natation for finite groups and related concepts as described in [24], excepting that we will occasionally deviate to minimise confusion with similar notation for geometric concepts.

In Section 2 we will introduce the preliminary definitions before proceeding in Section 3 to discuss the case of the finite simple groups. We then go on in Section 4 to discuss the abelian and nilpotent groups. Next, we focus our attention on special types of Beauville structures when we discuss strongly real Beauville structures in Section 5 and mixed Beauville structures in Section 6. Finally, we discuss a miscellany of related but less well studied topics in Section 7.

2 Preliminaries

Definition 2.1 A surface $\mathcal{S}$ is a **Beauville surface of unmixed type** if

- the surface $\mathcal{S}$ is isogenous to a higher product, that is, $\mathcal{S} \cong (\mathcal{C}_1 \times \mathcal{C}_2)/G$ where $\mathcal{C}_1$ and $\mathcal{C}_2$ are algebraic curves of genus at least 2 and G is a finite group acting faithfully on $\mathcal{C}_1$ and $\mathcal{C}_2$ by holomorphic transformations in such a way that it acts freely on the product $\mathcal{C}_1 \times \mathcal{C}_2$, and
- each $\mathcal{C}_i/G$ is isomorphic to the projective line $\mathbb{P}_1(\mathbb{C})$ and the covering map $\mathcal{C}_i \to \mathcal{C}_i/G$ is ramified over three points.

There also exists a concept of Beauville surfaces of mixed type but we shall postpone our discussion of these until Section 6. In the first of the above conditions the genus of the curves in question needs to be at least 2. It was later proved by Fuertes, González-Diez and Jaikin-Zapirain in [32] that in fact we can take the genus as being at least 6. The second of the above conditions implies that each $\mathcal{C}_i$ carries a regular dessin in the sense of Grothendieck's theory of *dessins d'enfants* (childeren's drawings) [45]. Furthermore, by Belyĭ's Theorem [16] this ensures that $\mathcal{S}$ is defined over an algebraic number field in the sense that when we view each Riemann surface as being the zeros of some polynomial we find that the coeffcients of that polynomial belong to some number field. Equivalently they admit an orientably regular hypermap [52], with G acting as the orientation-preserving automorphism group. A modern account of dessins d'enfants and proofs of Belyĭ's theorem may be found in the recent book of Girondo and González-Diez [38].

This can also be described instead in terms of uniformisation and the language of Fuchsian groups [40, 61].

What makes this class of surfaces so good to work with is the fact that all of the above definition can be 'internalised' into the group. It turns out that a group

G can be used to define a Beauville surface if and only if it has a certain pair of generating sets known as a Beauville structure.

Definition 2.2 Let G be a finite group. Let $x, y \in G$ and let

$$\Sigma(x,y) := \bigcup_{i=1}^{|G|} \bigcup_{g \in G} \{(x^i)^g, (y^i)^g, ((xy)^i)^g\}.$$

An **unmixed Beauville structure** for the group G is a set of pairs of elements $\{\{x_1, y_1\}, \{x_2, y_2\}\} \subset G \times G$ with the property that $\langle x_1, y_1 \rangle = \langle x_2, y_2 \rangle = G$ such that

$$\Sigma(x_1, y_1) \cap \Sigma(x_2, y_2) = \{e\}.$$

If G has a Beauville structure we say that G is a **Beauville group**. Furthermore we say that the structure has **type**

$$((o(x_1), o(y_1), o(x_1y_1)), (o(x_2), o(y_2), o(x_2y_2))).$$

Traditionally, authors have defined the above structure in terms of so-called 'spherical systems of generators of length 3', meaning $\{x, y, z\} \subset G$ with $xyz = e$, but we omit $z = (xy)^{-1}$ from our notation in this survey. (The reader is warned that this terminology is a little misleading since the underlying geometry of Beauville surfaces is hyperbolic thanks to the below constraint on the orders of the elements.) Furthermore, many earlier papers on Beauville structures add the condition that for $i = 1, 2$ we have that

$$\frac{1}{o(x_i)} + \frac{1}{o(y_i)} + \frac{1}{o(x_iy_i)} < 1,$$

but this condition was subsequently found to be unnecessary following Bauer, Catanese and Grunewald's investigation of the wall-paper groups in [9]. A triple of elements and their orders satisfying this condition are said to be hyperbolic. Geometrically, the type gives us considerable amounts of geometric information about the surface: the Riemann-Hurwitz formula

$$g(\mathcal{C}_i) = 1 + \frac{|G|}{2}\left(1 - \frac{1}{o(x_i)} - \frac{1}{o(y_i)} - \frac{1}{o(x_iy_i)}\right)$$

tells us the genus of each of the curves used to define the surface $\mathcal{S}$ and by a theorem of Zeuthen-Segre this in turn gives us the Euler number of the surface $\mathcal{S}$ since

$$e(\mathcal{S}) = 4\frac{(g(\mathcal{C}_1) - 1)(g(\mathcal{C}_2) - 1)}{|G|}$$

which in turn gives us the holomorphic Euler-Poincaré characteristic of $\mathcal{S}$, namely $4\chi(\mathcal{S}) = e(\mathcal{S})$ (see [20, Theorem 3.4]).

Furthermore, if a group can be generated by a pair of elements of orders a and b whose product has order c then G is a homomorphic image of the triangle group

$$T_{a,b,c} = \langle x, y, z | x^a = y^b = z^c = xyz = 1 \rangle.$$

Homomorphic images of the triangle group $T_{2,3,7}$ are known as Hurwitz groups. In several places in the literature, knowing that a particular group is a Hurwitz group has proved useful for deciding its status as a Beauville group. For a discussion of known results on Hurwitz groups see the excellent surveys of Conder [22, 23].

3 Finite Simple Groups

A necessary condition for a group to be a Beauville group is that it is 2-generated. In [1, 59] it is proved that all non-abelian finite simple groups are 2-generated. For a long time it was conjectured that every non-abelian finite simple group, aside from the alternating group A_5, is a Beauville group [10, Conjecture 7.17], providing a rich source of examples. Various authors proved special cases of this [10, 31, 33]. The full result comes from the following Theorem which is proved by the author, Magaard and Parker in [27, 28].

Theorem 3.1 *With the exceptions of $SL_2(5)$ and $PSL_2(5)(\cong A_5 \cong SL_2(4))$, every finite quasisimple group is a Beauville group.*

Similar results were proved at around the same time by Garion, Larsen and Lubotzky in [36] (using probabilistic results concerning triangle groups from the PhD thesis of Marion [55]) and by Guralnick and Malle in [46] using the theory of linear algebraic groups. Since the overriding ideas behind the proofs given in [27, 36, 46] are in many ways quite general we sketch these ideas in the hope that they may be useful in proving other conjectures that appear later in this survey.

First note that the alternating groups can be dealt with using classical permutation group theory. Furthermore, the low rank groups of Lie type may be dealt with using explicit matrix calculations (see for instance the work of Fuertes and Jones in [33] concerning the groups $PSL_2(q)$, ${}^2B_2(2^{2n+1})$ and ${}^2G_2(3^{2n+1})$.) The sporadic simple groups are easily dealt with on a case by case basis with structure constant calculations being useful for the larger groups. The real difficulty lies with the groups of Lie type of unbounded rank.

Let G be a finite simple group of Lie type of characteristic p. To ensure that we can choose elements of the group G whose product behaves as we require we use a theorem of Gow [44] (a slight generalisation of this result to quasisimple groups is given in [27, Theorem 2.6]). An element of G is said to be 'semisimple' if its order is coprime to p and is said to be 'regular semisimple' if its centralizer in G has order coprime to p.

Theorem 3.2 *Let G be a finite simple group of Lie type of characteristic p and let $s \in G$ be a semisimple element. Let $R_1, R_2 \subset G$ be conjugacy classes of regular semisimple elements of G. Then there exist elements $x \in R_1$ and $y \in R_2$ such that $s = xy$.*

To ensure that the conjugacy part of the definition of a Beauville structure is satisfied we aim to choose $x_1, x_2, y_1, y_2 \in G$ such that $o(x_1)o(y_1)o(x_1y_1)$ is coprime to $o(x_2)o(y_2)o(x_2y_2)$. This is made possible by a classical theorem of Zsigmondy

[64] (or rather Bang [2] in the case $p = 2$.) Whilst [2] and [64] are over a century old and therefore difficult to read and get hold of, a more recent account of a proof is given by Lünburg in [54].

Theorem 3.3 *For any positive integers a and n there exists a prime that divides $a^n - 1$ but not $a^k - 1$ for any $k < n$ with the following exceptions:*

- *$a = 2$ and $n = 6$; and*
- *$a + 1$ is a power of 2 and $n = 2$.*

The real significance of the above results stems from the fact that most groups of Lie type have an order that is a product of numbers of the form $p^k - 1$ and so the above result guarantees the existence of a rich supply of distinct primes that can be taken as being the orders of the elements of our Beauville structure.

It remains to decide if a given triple will generate the group. Since our elements have orders given by Theorem 3.3 we can use a theorem of Guralnick, Pentilla, Praeger and Saxl [47] concerning subgroups of the general linear group $GL_n(p^a)$ containing elements of these orders and closely related results of Niemeyer and Praeger [56] for the other classical groups to show that no proper subgroups contain our elements. It follows that our chosen elements will generate the group.

4 Abelian and Nilpotent Groups

The abelian Beauville groups were essentially classified by Catanese in [20, page 24.] and the full argument is given explicitly in [9, Theorem 3.4] where the following is proved.

Theorem 4.1 *Let G be an abelian group. Then G is a Beauville group if, and only if, $G = (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})$ where $n > 1$ is coprime to 6.*

This explains why Beauville's original example used the group $(\mathbb{Z}/5\mathbb{Z}) \times (\mathbb{Z}/5\mathbb{Z})$ - it is the smallest abelian Beauville group.

Theorem 4.1 has been put to great use by González-Diez, Jones and Torres-Teigell in [42] where several structural results concerning the surfaces defined by abelian Beauville groups are proved. For each abelian Beauville group they describe all the surfaces arising from that group, enumerate them up to isomorphism and impose constraints on their automorphism groups. As a consequence they show that all such surfaces are defined over $\mathbb{Q}$.

After the abelian groups, the next most natural class of finite groups to consider are the nilpotent groups. In [3, Lemma 1.3] Barker, Boston and the author note the following easy Lemma.

Lemma 4.2 *Let G and G' be Beauville groups and let $\{\{x_1, y_1, \}, \{x_2, y_2\}\}$ and $\{\{x'_1, y'_1, \}, \{x'_2, y'_2\}\}$ be their respective Beauville structures. Suppose that*

$$gcd(o(x_i), o(x'_i)) = gcd(o(y_i), o(y'_i)) = 1$$

for $i = 1, 2$. Then $\{\{(x_1, x'_1,), (y_1, y'_1)\}, \{(x_2, x'_2), (y_2, y'_2)\}\}$ is a Beauville structure for the group $G \times G'$.

Recall that a finite group is nilpotent if, and only if, it isomorphic to the direct product of its Sylow subgroups. It thus follows that this lemma, and its easy to prove converse, reduces the study of nilpotent Beauville groups to that of Beauville p-groups. Note that Theorem 4.1 gives us infinitely many examples of Beauville p-groups for every prime $p > 3$ - simply let n be any power of p. Early examples of Beauville 2-groups and 3-groups were constructed by Fuertes, González-Diez and Jaikin-Zapirain in [32] where a Beauville group of order 2^{12} and another of order 3^{12} were constructed. Even earlier than this, two Beauville 2-groups of order 2^8 arose as part of a classification due to Bauer, Catanese and Grunewald in [11] of certain classes of surfaces of general type.

More recently, in [3] Barker, Boston and the author classified the Beauville p-groups of order at most p^4 and made substantial progress on the cases of groups of order p^5 and p^6. In particular, the number of Beauville p-groups of order p^4 is two for every $p > 3$ and zero otherwise, but for p^5 we have the following.

Conjecture 4.3 For all $p \geq 5$, the number of Beauville p-groups of order p^5 is given by $p + 10$.

In [3, Theorem 1.4] we prove that there are at least $p + 8$ Beauville groups of order p^5. Furthermore, the above conjecture has been verified computationally for all primes p such that $5 \leq p \leq 19$. Perhaps more interestingly, other results proved in [3] verify that the proportion of 2-generated p-groups of order p^5 that are Beauville tends to 1 as p tends to infinity, however this fails to to be true for p-groups of order p^6.

Question 4.4 If $n > 6$ what is the behaviour, as p tends to infinity, of the proportion of 2-generated p-groups that are Beauville?

Another consequence of this work was determining the smallest Beauville p-group for all primes. In the below presentations, if no relationship between two generators is specified by a relation or relator then it should be assumed that the two generators commute.

Theorem 4.5 *The smallest Beauville p-groups are as follows.*

- *For $p = 2$ the group*

$$\langle x, y \mid x^4, y^4, [x, y^2]^2 2, [x, y^3]^2, [x^2, y^3] \rangle$$

  of order 2^7.

- *For $p = 3$ the group*

$$\langle x, y, z, w, t \mid x^3, y^3, z^3, w^3, t^3, y^x = yz, z^x = zw, z^y = zt \rangle$$

  of order 3^5.

- *For $p \geq 5$ the group*

$$\langle x, y, z \mid x^5, y^5, z^5, [x, y] = z \rangle$$

  of order p^3.

Further examples are given by the following unpublished constructions due to Jones and Wolfart.

Theorem 4.6 *Let G be a finite group of exponent $n = p^e > 1$ for some prime $p \geq 5$, such that the abelianisation G/G' of G is isomorphic to $\mathbb{Z}_n \times \mathbb{Z}_n$. Then G has a Beauville structure.*

Corollary 4.7 *Let G be a 2-generated finite group of exponent p for some prime $p \geq 5$. Then G has a Beauville structure.*

As noted earlier Beauville p-groups for $p > 3$ are in bountiful supply. Several examples of Beauville 2-groups and 3-groups are constructed by Barker, Boston, Peyerimhoff and Vdovina in [5, 6] by considering sections of groups defined using projective planes. More recently, in [4] Barker, Boston, Peyerimhoff and Vdovina using similar ideas constructed the first infinite family of Beauville 2-groups. At the time of writing, as far as the author is aware, only finitely many Beauville 3-groups are known leading to the following natural problem.

Problem 4.8 Construct infinitely many Beauville 3-groups.

More recently in [60] Stix and Vdovina give a construction of Beauville p-groups that provides infinitely many examples for every $p \geq 5$. More specifically they prove the following.

Theorem 4.9 *Let p be a prime, $n, m \in \mathbb{N}$ and $\lambda \in (\mathbb{Z}/p^m\mathbb{Z})^\times$ with $\lambda^{p^n} \cong 1$ mod p^m. The semidirect product $\mathbb{Z}/p^m\mathbb{Z} : \mathbb{Z}/p^n\mathbb{Z}$ with action $\mathbb{Z}/p^n\mathbb{Z} \to Aut(\mathbb{Z}/p^m\mathbb{Z})$ sending $1 \mapsto \lambda$ admits an unmixed Beauville structure if and only if $p \geq 5$ and $n = m$.*

They go on to prove related results using the theory of pro-p groups.

We conclude this section with the following remarks. Nigel Boston has recently undertaken some substantial and as yet unpublished computations regarding the relationship between p-groups' status as Beauville groups and their position on the so-called 'O'Brien Trees' [57]. Whilst little global pattern appears to exist in general, there does appear to be some mysterious relationship with an invariant known as the 'nuclear rank' of the group - see [17]. Since defining this concept is somewhat involved we shall say no more about this here.

5 Strongly Real Beauville Groups

Given any complex surface $\mathcal{S}$ it is natural to consider the complex conjugate surface $\overline{\mathcal{S}}$. In particular it is natural to ask if the surfaces are biholomorphic.

Definition 5.1 Let $\mathcal{S}$ be a complex surface. We say that $\mathcal{S}$ is **real** if there exists a biholomorphism $\sigma : \mathcal{S} \to \overline{\mathcal{S}}$ such that σ^2 is the identity map.

As noted earlier this geometric condition can be translated into algebraic terms.

Definition 5.2 Let G be a Beauville group and let $X = \{\{x_1, y_1\}, \{x_2, y_2\}\}$ be a Beauville structure for G. We say that G and X are **strongly real** if there exists an automorphism $\phi \in \mathrm{Aut}(G)$ and elements $g_i \in G$ for $i = 1, 2$ such that

$$g_1\phi(x_i)g_1^{-1} = x_i^{-1} \text{ and } g_2\phi(y_i)g_2^{-1} = y_i^{-1}$$

for $i = 1, 2$.

It is often, but not always, convenient to take $g_1 = g_2$.

Our first examples come immediately from Theorem 4.1 since for any abelian group the function $x \mapsto -x$ is an automorphism.

Corollary 5.3 *Every Beauville structure of an abelian Beauville group is strongly real.*

A little more generally, when it comes to strongly real Beauville p-groups the examples given by Theorem 4.1 are, as far as the author is aware, the only known examples. Furthermore, the Beauville 2-groups constructed by Barker, Boston, Peyerimhoff and Vdovina in [4] are explicitly shown to not be strongly real. However, a combination of Corollary 5.3 and the fact that p-groups in general tend to have large automorphism groups [18, 19] it seems likely that most Beauville p-groups are in fact strongly real Beauville groups. This makes the following problem particularly pressing.

Problem 5.4 Construct examples of strongly real Beauville p-groups.

In [25] the following conjecture, a refinement of an earlier conjecture of Bauer, Catanese and Grunewald [9, Section 5.4], is made.

Conjecture 5.5 All non-abelian finite simple groups apart from A_5, M_{11} and M_{23} are strongly real Beauville groups.

Only a few cases of this conjecture are known.

- In [31] Fuertes and González-Diez showed that the alternating groups A_n ($n \geq 7$) and the symmetric groups S_n ($n \geq 5$) are strongly real Beauville groups by explicitly writing down permutations for their generators and the automorphisms used and applying some of the classical theory of permutation groups to show that their elements had the properties they claimed. It was subsequently found that the group A_6 is also strongly real.
- In [33] Fuertes and Jones proved that the simple groups $PSL_2(q)$ for prime powers $q > 5$ and the quasisimple groups $SL_2(q)$ for prime powers $q > 5$ are strongly real Beauville groups. As with the alternating and symmetric groups, these results are proved by writing down explicit generators, this time combined with a celebrated theorem usually (but historically inaccurately) attributed to Dickson for the maximal subgroups of $PSL_2(q)$. (For a full statement of this result and related theorems as well a detailed historical account of the maximal subgroups of low dimensional classical groups see the excellent survey of King in [53].) General lemmas for lifting structures from a group to its covering groups are also used.

- In [26] the author determined which of the sporadic simple groups are strongly real Beauville groups, including the '27th sporadic simple group', the Tits group ${}^2F_4(2)'$. Only the Mathieu groups M_{11} and M_{23} are not strongly real. For all of the other sporadic groups smaller than the Baby Monster group $\mathbb{B}$ explicit words in the 'standard generators' [62] for a strongly real Beauville structure were given. For the Baby Monster group $\mathbb{B}$ and Monster group $\mathbb{M}$ character theoretic methods were used.
- In [25] the author also verified this conjecture for the Suzuki groups ${}^2B_2(2^{2n+1})$. Again, this was achieved by writing down explicit elements of the group which using the list of maximal subgroups of the Suzuki group are shown to generate.
- In [25] the author extended earlier computations of Bauer, Catanese and Grunewald, verifying this conjecture for all non-abelian finite simple groups of order at most 100 000 000.

We remark that several of the groups mentioned in the above bullet points are not simple. More generally we ask the following.

Question 5.6 Which groups are strongly real Beauville groups?

Finally, we remark that in [25] the author constructs many further examples of strongly real Beauville groups. This includes the characteristically simple groups A_n^k for moderate values of k and sufficiently large values of n, the groups $S_n \times S_n$ for $n \geq 5$ and the almost simple sporadic groups. This last calculation combined with the earlier remarks on the symmetric group lead to the following conjecture.

Corollary 5.7 *A split extension of a simple group is a Beauville group if, and only if, it is a strongly real Beauville group.*

6 The Mixed Case

When we defined Beauville surfaces and groups we considered the action of a group G on the product of two curves $\mathcal{C}_1 \times \mathcal{C}_2$. In an unmixed structure this action comes solely from the action of G on each curve individually, however there is nothing to stop us considering an action on the product that interchanges the two curves and it is precisely this situation that we discuss in this section. Recall from Definition 2.2 that given $x, y \in G$ we write

$$\Sigma(x,y) := \bigcup_{i=1}^{|G|} \bigcup_{g \in G} \{(x^i)^g, (y^i)^g, ((xy)^i)^g\}.$$

Definition 6.1 Let G be a finite group. A **mixed Beauville structure** for G is a quadruple (G^0, g, h, k) where G^0 is an index 2 subgroup and $g, h, k \in G$ are such that

- $\langle g, h \rangle = G^0$;
- $k \notin G^0$;

- for every $\gamma \in G^0$ we have that $(k\gamma)^2 \notin \Sigma(g,h)$ and
- $\Sigma(g,h) \cap \Sigma(g^k,h^k) = \{e\}$

A Beauville surface defined by a mixed Beauville structure is called a **mixed Beauville surface** and group possessing a mixed Beauville structure is called a **mixed Beauville group**.

In terms of the curves defining the surface, the group G^0 is the stabiliser of the curves with the elements of $G \setminus G^0$ interchanging the two terms of $\mathcal{C}_1 \times \mathcal{C}_2$. Moreover it is only possible for a Beauville surface $(\mathcal{C}_1 \times \mathcal{C}_2)/G$ to come from a mixed Beauville structure if $\mathcal{C}_1 \cong \mathcal{C}_2$. The above conditions also ensure that $\{\{g,h\},\{g^k,h^k\}\} \subset G^0 \times G^0$ is a Beauville structure for G^0.

In general, mixed Beauville structures are much harder to construct than their unmixed counterparts. The following lemma of Fuertes and González-Diez imposes a strong condition on a group with a mixed Beauville structure [31, Lemma 5].

Lemma 6.2 *Let $(\mathcal{C}_1 \times \mathcal{C}_2)/G$ be a mixed Beauville surface and let G^0 be the subgroup of G consisting of the elements which do not interchange the two curves. Then the order of any element in $G \setminus G^0$ is divisible by 4.*

Clearly no simple group can have a mixed Beauville structure since it is necessary to have a subgroup of index 2 and the cyclic group of order 2 is not a Beauville group, however that does not preclude the possibility of almost simple groups having mixed Beauville structures. The above lemma was originally used to show that no symmetric group has a mixed Beauville structure. In [26] the author used the above to show that no almost simple sporadic group has a mixed Beauville structure (though the almost simple Tits group ${}^2F_4(2)$ is not excluded by the above lemma) and in general most almost simple groups are ruled out by it (though as the groups $P\Sigma L_2(p^2)$ show there are infinitely many exceptions to this). A further restriction comes from [9, Theorem 4.3] where Bauer, Catanese and Grunewald prove that G^0 must be non-abelian. Various geometric constraints are proved by Torres-Teigell in his PhD thesis [61]. Most notably the genus of a mixed Beauville surface is odd and at least 17. Furthermore, this bound is sharp. This naturally leads to the following problem.

Problem 6.3 Find mixed Beauville structures.

The earliest examples of groups that do possess mixed Beauville structures were given by Bauer, Catanese and Grunewald in [9]. Their general construction is of the form $(H \times H) : (\mathbb{Z}/4\mathbb{Z})$, the generator of the group $\mathbb{Z}/4\mathbb{Z}$ acting on the direct product by interchanging its two factors and $G^0 = H \times H \times \mathbb{Z}/2\mathbb{Z}$.

Lemma 6.4 *Let H be a finite group and let $x_1, y_1, x_2, y_2 \in H$. Suppose that*

(1) $o(x_1)$ and $o(y_1)$ are even;

(2) $\langle x_1^2, y_1^2, x_1y_1\rangle = H$;

(3) $\langle x_2, y_2\rangle = H$ and

(4) $o(x_1)o(y_1)o(x_1y_1)$ is coprime to $o(x_2)o(y_2)o(x_2y_2)$.

If the above conditions are satisfied then (G^0, x, y, g) *is a mixed Beauville structure for some* $g \in (H \times H) : (\mathbb{Z}/4\mathbb{Z})$ *where* $x = (x_1, x_2, 2), y = (y_1, y_2, 2) \in H \times H \times \mathbb{Z}/2\mathbb{Z}$ *(note that* $2 \in \mathbb{Z}/4\mathbb{Z}$ *generates the subgroup isomorphic to* $\mathbb{Z}/2\mathbb{Z}$*). Furthermore, if* H *is a perfect group then we cane replace condition (2) with the condition*

(2') $\langle x_1, y_1 \rangle = H$.

Note that in [9] this last hypothesis was incorrectly stated in terms of the perfectness of G rather than H. Bauer, Catanese and Grunewald go on to use the above lemma to construct examples in the cases with the property that if H is taken to be a sufficiently large alternating group or a special linear groups $SL_2(p)$ with $p \neq 2, 3, 5, 17$ (though their argument also does not apply in the case $p = 7$), then $(H \times H) : (\mathbb{Z}/4\mathbb{Z})$ has a mixed Beauville structure. Given the extent to which mixed Beauville groups are in short supply it would be interesting to see if the above construction can be used in other cases.

Problem 6.5 Find other groups H that the above lemma can be applied to.

In [29] the author and Pierro prove a slight generalisation of Lemma 6.4 that replaces the cyclic group of order 4 with the dicyclic group of order $4k$ defined by the presentation

$$\langle x, y \mid x^{2k} = y^4 = 1, x^y = x^{-1}, x^k = y^2 \rangle$$

for some positive integer k. In particular, when finding examples of groups that satisfy the hypotheses of this generalisation (which is sufficient to show that such groups satisfy the hypotheses of Lemma 6.4) we obtain new examples of mixed Beuville groups from the groups H and $H \times H$ where H is any of the alternating groups A_n $(n \geq 6)$, the linear groups $PSL_2(q)$ $(q \geq 7$ odd), the unitary groups $PSU_3(q)$ $(q \geq 3)$, the Suzuki groups ${}^2B_2(2^{2n+1})$ $(n \geq 1)$, the small Ree groups ${}^2G_2(3^{2n+1})$ $(n \geq 1)$, the large Ree groups ${}^2F_4(q)$ $(q \geq 8)$, the Steinberg triality groups ${}^3D_4(q)$ $(q \geq 2)$ and the sporadic simple groups (including the Tits group ${}^2\mathrm{F}_4(2)'$) as well as the groups $PSL_2(2^n) \times PSL_2(2^n)$ $(n \geq 3)$.

What about p-groups? If p is odd then again, the absence of index 2 subgroups ensures that there exist no mixed Beauville p-groups. In the construction described above the technical constraints on H ensure that it cannot be a 2-group, stopping this providing a source of examples. Early examples of mixed Beauville 2-groups were given by Bauer, Cataneses and Grunewald constructed in [11] where they constructed two mixed Beauville groups of order 2^8. Even so, the lack of known Beauville 2-groups makes the following a natural problem.

Problem 6.6 Construct infinitely many mixed Beauville 2-groups.

7 Miscellanea

7.1 $PSL_2(q)$ and $PGL_2(q)$

In [10, Question 7.7] Bauer, Catanese and Grunewald asked the following question,

Existence and classification of Beauville surfaces, i.e.,
a) which finite groups G can occur?
b) classify all possible Beauville surfaces for a given finite group G.

In [35] Garion answered the above in the case of the groups $PSL_2(q)$ and $PGL_2(q)$. For $PSL_2(q)$ we have the following.

Theorem 7.1 *Let $G = PSL_2(q)$ where $5 < q = p^e$ for some prime number p and some positive integer e. Let $\tau_1 = (r_1, s_1, t_1)$, $\tau_2 = (r_2, s_2, t_2)$ be two hyperbolic triples of integers. Then G admits an unmixed Beauville structure of type (τ_1, τ_2) if, and only if, the following hold:*

(i) the group G is a quotient of the triangle groups T_{r_1,s_1,t_1} and T_{r_2,s_2,t_2} with torsion-free kernel;

(ii) if $p = 2$ or e is odd or $q = 9$, then $r_1s_1t_1$ is coprime to $r_2s_2t_2$. If p is odd, e is even and $q > 9$, then $g = gcd(r_1s_1t_1, r_2s_2t_2) \in \{1, p, p^2\}$. Moreover, if p divides g and τ_1 (respectively τ_2) is up to a permutation (p, p, n) then $n \neq p$ and n is a 'good G-order'.

Here by 'good G-order' we mean the following. Let q be an odd prime power and let $n > 1$ be an integer. Then n is a good G-order if either

- n divides $(q-1)/2$ and a primitive root of unity a of order $2n$ in $\mathbb{F}_q$ has the property that $-a = c^2$ for some $c \in \mathbb{F}_q$ or
- n divides $(q+1)/2$ and a primitive root of unity a of order $2n$ in $\mathbb{F}_q^2$ has the property that $-a = c^2$ for some $c \in \mathbb{F}_{q^2}$ such that $c^{q+1} = 1$.

A similar theorem is given for the groups $PGL_2(q)$.

Given that generic lists of maximal subgroups of other low rank groups of Lie type are well known in numerous other cases, it seems likely that analogous results for these groups can also be obtained. We thus reiterate Bauer, Catanese and Grunewald's earlier question in this case.

Problem 7.2 Obtain results analogous to the above for other classes of finite simple groups.

7.2 Fundamental Groups of Beauville Surfaces

We mentioned in the introduction that Beauville surfaces have fundamental groups that are easy to work with. To make this vague remark a little more specific we note the following. Suppose that if G is a Beauville group with a Beauville structure of type $((a_1, b_1, c_1), (a_2, b_2, c_2))$, then for $i = 1, 2$ there exist surjective homomorphisms $\rho_i : T_{a_i,b_i,c_i} \to G$. The direct product $\ker(\rho_1) \times \ker(\rho_2)$ is the fundamental group of the product $\mathcal{C}_1 \times \mathcal{C}_2$. The fundamental group of the surface $(\mathcal{C}_1 \times \mathcal{C}_2)/G$ is now an extension of a normal subgroup $\ker(\rho_1) \times \ker(\rho_2)$ by G, or more precisely the inverse image in $T_{a_1,b_1,c_1} \times T_{a_2,b_2,c_2}$ of the diagonal subgroup of $G \times G$ under the epimorphism $\rho_1 \times \rho_2$. It turns out that this simple description of the fundamental group is responsible for the rigidity of Beauville surfaces and this in turn ensures

that the topological and geometric features of the surfaces are closely intertwined - see [51, Section 9] for details.

Unsurprisingly, since a Beauville group dictates so many features of its corresponding Beauville surface which in turn determines its fundamental group we also have the reverse relationship whereby the fundamental group determines the original Beauville group. The following is proved by González-Diez and Torres-Teigell [41, 61]. (It also worth noting related results given by Bauer, Catanese and Grunewald in [10] and by Catanese in [20]).

Theorem 7.3 *Two Beauville surfaces are isometric if and only if their fundamental groups are isomorphic.*

The fundamental group is one of the most basic tools in algebraic topology. It is, however, somewhat limited in its usefulness and topologists have found several important higher dimensional analogues of the fundamental group and so it is natural to pose the following question.

Question 7.4 Do the higher homotopy/homology/cohomology groups of a Beauville surface have similar descriptions in terms of triangle groups and the corresponding Beauville group and to what extent do they uniquely determine the surface?

By way of partial progress on this question in [8] Bauer, Catanese and Frapporti recently showed that for any Beauville surface $\mathcal{S}$ the homology group $H_1(\mathcal{S}, \mathbb{Z})$ is finite. They also give a much more detailed discussion of geometric aspects of the study of fundamental groups of Beauville surfaces and related objects as well as computer calculations of these objects in some cases.

7.3 Automorphism Groups of Beauville Surfaces

In [50] Jones investigated the automorphism groups of unmixed Beauville surfaces. Some of these results were obtained independently by Fuertes and González-Diez in [30] and were later extended to mixed Beauville surfaces by González-Diez and Torres-Teigell in [40, Section 5.3].

Theorem 7.5 *The automorphism group Aut($\mathcal{S}$) of a Beauville surface $\mathcal{S} = (\mathcal{C}_1 \times \mathcal{C}_2)/G$ has a normal subgroup Inn($\mathcal{S}$) $\triangleleft Z(G)$ with Aut($\mathcal{S}$)/Inn($\mathcal{S}$) isomorphic to a subgroup of the wreath product $S_3 \wr S_2$. In particular Aut($\mathcal{S}$) is a finite soluble group of order dividing $72|Z(G)|$ and of derived length at most 4.*

Here the subgroup $\mathrm{Inn}(\mathcal{S})$ consists of automorphisms preserving the two curves (or more precisely, induced by automorphisms of $\mathcal{C} \times \mathcal{C}'$ preserving them) though it does not necessarily contain all of them: they form a subgroup of index at most 2 in $\mathrm{Aut}(\mathcal{S})$, whereas $\mathrm{Inn}(\mathcal{S})$ can have index up to 72. The results in the mixed case are similar.

7.4 Beauville Genus Spectra

In [10, Question 7.7(b)] Bauer, Catanese and Grunewald ask us to classify all possible Beauville surfaces for a given finite group G.

As a partial answer to this, in [32, Section 4] Fuertes, González-Diez and Jaikin-Zapirain introduce the concept of Beauville genus spectrum which we define as follows.

Definition 7.6 Let G be a finite group. The **Beauville genus spectrum** of G is the set $Spec(G)$ of pairs of integers (g_1, g_2) such that $g_1 \leq g_2$ and there are curves $\mathcal{C}_1$ and $\mathcal{C}_2$ of genera g_1 and g_2 with an action of G on $\mathcal{C}_1 \times \mathcal{C}_2$ such that $(\mathcal{C}_1 \times \mathcal{C}_2)/G$ is a Beauville surface.

By the Riemann-Hurwitz formula each g_i is bounded above by $1 + \frac{|G|}{2}$ and so this set is always finite. Fuertes, González-Diez and Jaikin-Zapirain determine the Beauville spectra of several small groups.

Proposition 7.7 *1.* $Spec(S_5) = \{(19, 21)\}$

2. $Spec(PSL_2(7)) = \{(8, 49), (15, 49), (17, 22), (22, 33), (22, 49)\}$

3. $Spec(S_6) = \{(49, 91), (91, 121), (91, 169), (121, 169), (151, 169)\}$

4. If $gcd(n, 6) = 1$ *and* $n > 1$ *then*

$$Spec((\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})) = \{(\frac{(n-1)(n-2)}{2}, \frac{(n-1)(n-2)}{2})\}.$$

Unpublished calculations of the author's PhD student, Emilio Pierro, has added a few more finite simple and almost simple groups to the above list, the largest being the Mathieu group M_{23}. Furthermore, the Beauville genus structures of $PSL_2(q)$ and $PGL_2(q)$ may be deduced from the results discussed in Subsection 7.1. This naturally leads us to ask the following.

Problem 7.8 Determine the Beauville genus spectrum of more groups.

7.5 Characteristically Simple Groups

Characteristically simple groups are usually defined in terms of characteristic subgroups, but for finite groups this turns out to be equivalent to the following.

Definition 7.9 A finite group G is said to be **characteristically simple** if G is isomorphic to the direct product H^k where H is a finite simple group for some positive integer k.

If we fix H then for large values of k the group H^k will not be 2-generated and therefore will not be Beauville. For more modest values of k there is, however, still hope. These groups have recently been investigated by Jones in [48, 49] where the following conjecture is investigated.

Conjecture 7.10 Let G be a finite characteristically simple group. Then G is Beauville if and only if it is 2-generated and not isomorphic to the alternating group A_5.

Theorem 3.1 shows that this conjecture is true for the characteristically simple group H^k in the case $k = 1$ for every non-abelian finite simple group H. If G is abelian then this conjecture holds by Theorem 4.1 following the convention that a cyclic group is not considered to be 2-generated. In [39] the above conjecture is verified for the alternating groups and in [49] it is verified for the linear groups $PSL_2(q)$ and $PSL_3(q)$, the unitary groups $PSU_3(q)$, the Suzuki groups ${}^2B_2(2^{2n+1})$, the small Ree groups ${}^2G_2(3^{2n+1})$ and the sporadic simple groups. In addition to the above the author has performed computations that verify the above conjecture for all characteristically simple groups of order at most 10^{30}. As an amusing aside we note that this shows that whilst A_5 is not a Beauville group, the direct product of nineteen copies of A_5 is!

In [25] the author considers which of the characteristically simple groups are strongly real Beauville groups. The main conjecture is the following.

Conjecture 7.11 If G is a finite simple group of order greater than 3, then $G \times G$ is a strongly real Beauville group.

It is likely that many larger direct products are also strongly real, however the precise statement of a conjecture along these lines is likely to be much more complicated. For example, a straightforward computation verifies that neither of the groups $\mathrm{M}_{11}\times\mathrm{M}_{11}\times\mathrm{M}_{11}$ and $\mathrm{M}_{23}\times\mathrm{M}_{23}\times\mathrm{M}_{23}$ are strongly real despite the fact that both of the groups $\mathrm{M}_{11}\times\mathrm{M}_{11}\times\mathrm{M}_{11}\times\mathrm{M}_{11}$ and $\mathrm{M}_{23}\times\mathrm{M}_{23}\times\mathrm{M}_{23}\times\mathrm{M}_{23}$ are.

The above conjecture has been verified for the alternating groups (though slightly stronger results are true in this case), the sporadic simple groups, the linear groups $PSL_2(q)$ $(q > 5)$, the Suzuki groups ${}^2B_2(2^{2n+1})$, the sporadic simple groups (including the Mathieu groups M_{11} and M_{23}, despite the statement of Conjecture 5.5) and all of the finite simple groups of order at most 100 000 000.

7.6 Orbits of the Absolute Galois Group

The task of understanding the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is of central importance in algebraic number theory and is related to the Inverse Galois Problem (it is equivalent to asking if every finite group is a quotient of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ under a topologically closed normal subgroup) and this is arguably the hardest open problem in algebra today. As things stand $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ remains very poorly understood. A natural approach to understanding any group is to study some action(s) of the group. An immeidate consequence of Belyĭ's Theorem is that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the set of all Beauville surfaces. Recently there has been much interest in constructing orbits consisting of mutually non-homeomorphic pairs of Beauville surfaces. In [41, 43] González-Diez, Jones and Torres-Teigell have constructed arbitrarily large orbits of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ consisting of mutually non-homeomorphic pairs of Beauville surfaces defined by the Beauville groups $PSL_2(q)$ and $PGL_2(q)$.

Problem 7.12 Construct arbitrarily large orbits of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ consisting of mutually non-homeomorphic pairs of Beauville surfaces using other groups.

A slightly different motivation for addressing the above problem comes from the following. Knowing whether or not $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on the set of Beauville surfaces is equivalent to the longstanding question of whether or not $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on the set of regular dessins. This was recently resolved by González-Diez and Jaikin-Zapirain in [39] by showing that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on the set of Beauville surfaces.

References

[1] M. Aschbacher and R. Guralnick, Some applications of the first cohomology group, J. Algebra 90 (1984), no. 2, 446–460

[2] A. S. Bang, Talteoretiske undersølgelser, Tidskrifft, Math. (5)4 (1886), 130–137

[3] N. W. Barker, N. Bosten and B. T. Fairbairn, A note on Beauville p-groups, Exp. Math., 21(3): 298–306 (2012) `arXiv:0907.2028v3` doi:10.1080/10586458.2012.669267

[4] N. W. Barker, N. Boston, N. Peyerimhoff and A. Vdovina, An infinite family of 2-groups with mixed Beauville structures, preprint 2013, to appear in Int. Math. Res. Not. `arXiv:1304.4480v1`

[5] N. W. Barker, N. Boston, N. Peyerimhoff and A. Vdovina, Regular algebraic surfaces isogenous to a higher product constructed from group representations using projective planes, preprint (2011), `arxiv.org:1109.6053`

[6] N. W. Barker, N. Boston, N. Peyerimhoff and A. Vdovina, New examples of Beauville surfaces, Monatsh. Math., 166 (2012), no. 3–4, pp. 319–327 DOI: 10.1007/s00605-011-0284-6

[7] I. Bauer, Product-Quotient Suraces: Result and Problems, preprint 2012 `arxiv:1204.3409`

[8] I. Bauer, F. Catanese and D. Frapporti, The fundamental group and torsion group of Beauville surfaces, preprint 2014, `arXiv:1402.2109`

[9] I. Bauer, F. Catanese and F. Grunewald, Beauville surfaces without real structures, in Geometric methods in algebra and number theory pp. 1–42, Progr. Math., 235, Birkhuser Boston, Boston, MA, 2005.

[10] I. Bauer, F. Catanese and F. Grunewald, Chebycheff and Belyi Polynomials, Dessins d'Enfants, Beauville Surfaces and Group Theory, Mediterr. J. math. 3 (2006), 121–146

[11] I. Bauer, F. Catanese and F. Grunewald, The classiffication of surfaces with $p_g = q = 0$ isogenous to a product of curves, Pre Apple. Math. Q. 4 (2008), no. 2, Special Issue: In honor of Fedor Bogomolov. Part 1, 547–586

[12] I. Bauer, F. Catanese and R. Pignatelli, Surfaces of General Type with Geometric Genus Zero: A Survey, in Complex and differential geometry 1–48, Springer Proc. Math., 8, Springer-Verlag, Heidelberg, 2011

[13] I. C. Bauer, F. Catanese and R. Pignatelli, Complex surfaces of general type: some recent progress, in Global aspects of complex geometry, 1–58, Springer, Berlin, 2006

[14] A. Beauville, Surfaces algébriques complexes (Astérisque 54 1978)

[15] A. Beauville, Complex Algebraic Surfaces, (London Mathematical Socity Student Texts 34, Cambridge University Press, Cambridge, 1996)

[16] G. V. Belyĭ, On Galois extensions of a maximal cyclotomic field, Math. USSR Izvestija 14 (1980), 247–256

[17] N. Boston, A Survey of Beauville p-groups, to appear in the Proceedings of conference on Beauville Surfaces and Group, Newcastle 2012 (eds I. Bauer, S. Garion and

A. Vdovina)

[18] N. Boston, Embedding 2-groups in groups generated by involutions, J. Algebra 300 (2006), no. 1, 73-76

[19] N. Boston, M. R. Bush and F. Hajir, Heuristics for p-class Towers of Imaginary Quadratic Fields, preprint 2011, `arXiv:1111.4679v1`

[20] F. Catanese, Fibered surfaces, varieties isogenous to a product and related moduli spaces, Amer. J. Math. 122 (2000), no. 1, 1–44

[21] F. Catanese, Moduli spaces of surfaces and real structures, Ann. of Math.(2) 158 (2003), no. 2, 577–592

[22] M. D. E. Conder, Hurwitz groups: a brief survey, Bull. Amer. Math. Soc. 23 (1990), 359–370

[23] M. D. E. Conder, An update on Hurwitz groups, Groups Complexity Cryptology, Volume 2, Issue 1 (2010) 35–49

[24] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups* (Clarendon Press, Oxford 1985)

[25] B. T. Fairbairn, Strongly Real Beauville Groups, to appear in the Proceedings of conference on Beauville Surfaces and Groups, Newcastle 2012 (eds I. Bauer, S. Garion and A. Vdovina), Springer-Verlag London, 2014

[26] B. T. Fairbairn, Some Exceptional Beauville Structures, J. Group Theory, 15(5), pp. 631–639 (2012) `arXiv:1007.5050` DOI: 10.1515/jgt-2012-0018

[27] B. T. Fairbairn, K. Magaard and C. W. Parker, Generation of finite simple groups with an application to groups acting on Beauville surfaces, Proc. London Math. Soc. (2013) doi: 10.1112/plms/pds097

[28] B. T. Fairbairn, K. Magaard and C. W. Parker, Corrigendum to Generation of finite simple groups with an application to groups acting on Beauville surfaces, to appear

[29] B. T. Fairbairn and E. Pierro, New examples of mixed Beauville groups, preprint 2014 `arXiv:1404.7303`

[30] Y. Fuertes and G. González-Diez, On the number of automorphisms of unmixed Beauville surfaces, preprint

[31] Y. Fuertes and G. González-Diez, On Beauville structures on the groups S_n and A_n, Math. Z. 264 (2010), no. 4, 959–968

[32] Y. Fuertes, G. González-Diez and A. Jaikin-Zapirain, On Beauville surfaces, Groups Geom. Dyn. 5 (2011), no. 1, 107–119

[33] Y. Fuertes and G. Jones, Beauville surfaces and finite groups, J. Algebra 340 (2011) 13–27

[34] S. Galkin and E. Shnider, Exceptional collections of line bundles on the Beuville surface, preprint 2012, `arxiv:1210.3339`

[35] S. Garion, On Beauville Structures for $PSL(2, q)$, preprint (2010) `arxiv:1003.2792v2`

[36] S. Garion, M. Larsen and A. Lubotzky, Beauville surfaces and finite simple groups, J. Reine Angew. Math. 666 (2012), 225–243

[37] S. Garion and M. Penegini, New Beauville surfaces, moduli spaces and finite groups, Comm. Algebra. 42, Issue 5 (2014), 2126–2155 `arxiv:0910.5402`

[38] E. Girondo and G. González-Diez, Introduction to Compact Riemann Surfaces and Dessins d'Enfants (London Mathematical Society Student texts 79) Cambridge University Press, Cambridge 2011

[39] G. González-Diez and A. Jaikin-Zapirain, The absolute Galois group acts faithfully on regular dessins and on Beauville surfaces, preprint 2012 http://www.uam.es/personal_pdi/ciencias/gabino/Jule03.pdf

[40] G. González-Diez and D. Torres-Teigell, An introduction to Beauville surfaces via uniformization, in Quasiconformal mappings, Riemann surfaces, and Teichmüller spaces, 123–151,Contemp. Math., 575, Amer. Math. Soc., Providence, RI, 2012

[41] G. González-Diez and D. Torres-Teigell, Non-homeomorphic Galois conjugate Beauville structures on $PSL(2,p)$, Advances in Mathematics 229 (2012) pp. 3096–3122

[42] G. González-Diez, G. A. Jones and D. Torres-Teigell, Beauville surfaces with abelian Beauville group, `arxiv:1102.4552`

[43] G. González-Diez, G. A. Jones and D. Torres-Teigell, Arbitrarily large Galois orbits of non-homeomorphic surfaces, `arXiv:1110.4930`

[44] R. Gow, Commutators in finite simple groups of Lie type, Bull. London Math. Soc. 32 (2000), no. 3, 311–315

[45] A. Grothendieck, Esquisse d'un Programme, in Geometric Galois Actions 1. Around Grothendieck's Esquisse d'un Programme, eds P. Lochak and L. Schneps, London Math. Soc. Lecture Note Ser. 242, Cambridge University Press, 1997, pp. 5–84

[46] R. Guralnick and G. Malle, Simple groups admit Beauville structures, J. Lond. Math. Soc. (2) 85 (2012), no. 3, 694–721

[47] R. Guralnick, T. Pentilla, C. Praeger and J. Saxl, Linear groups with orders having certain large prime divisors, Proc. London Math. Soc. (3) 78 (1999), no. 1, 167–214

[48] G. A. Jones, Characteristically simple Beauville groups, I: cartesian powers of alternating groups, preprint 2013 `arXiv:1304.5444v1`

[49] G. A. Jones, Characteristically simple Beauville groups, II: low rank and sporadic groups, preprint 2013, `arXiv:1304.5450v1`

[50] G. A. Jones, Automorphism groups of Beauville surfaces, J. Group Theory. Volume 16, Issue 3, Pages 353–381, DOI: 10.1515/jgt-2012-0049, 2013

[51] G. A. Jones, Beauville surfaces and groups: a survey, Fields Inst. Comm., to appear

[52] G. A. Jones and D. Singerman, Belyi functions, hypermaps and Galois groups, Bull. Lond. Math. Soc. 28 (1996) 561–590

[53] O. King, The subgroup stucture of finite classical groups in terms of geometric configurations, in *Surveys in combinatorics, 2005* (ed B. S. Webb), London Math. Soc. Lecture Nore Ser., 327. Cambridge University Press, Cambridge 2005, 29–56

[54] H. Lünburg, Ein einfacher Beweis für den Satz von Zsigmondy über primitive Primteiler von $a^N - 1$, in *Geometries and Groups*, Lecture Notes in Mathematics 893, Springer, New York, 1981, pp.219–222

[55] C. Marion, Triangle groups and finite simple groups, PhD thesis, Imperial College London, 2009

[56] A. C. Niemeyer and C. E. Praeger, A recognition algorithm for classical groups over finite fields, Proc. London Math. Soc. (3) 77 (1998), no.1, 117–169

[57] E. A. O'Brien, The p-group generation algorithm, J. Symbolic Comput. 9 (1990), no. 5-6, 677–698

[58] J. Širáň, How symmetric can maps on surfaces be?, in *Surveys in Combinatorics 2013* (Simon R. Blackburn, Stefanie Gerke and Mark Wildon eds.), London Math. Soc. Lecture Note Ser. 409 (Cambridge University Press, Cambridge, 2013), 161–238

[59] R. Steinberg, Generators for simple groups, Canad. J. Math., 14 (1962), pp. 277–283

[60] J. Stix and A. Vdovina, Series of p-groups with Beauville structure, preprint 2014 `arXiv:1405.3872`

[61] D. Torres-Teigell, Triangle groups, dessins d'enfants and Beauville surfaces, PhD thesis, Universidad Autonoma de Madrid, 2012

[62] R. A. Wilson, Standard generators for sporadic simple groups, J. Algebra 184 (1996), no. 2, 505–515

[63] J. Wolfart, ABC for polynomials, dessins d'enfants and uniformization - a survey. Elementare und analytische Zahlentheorie, Schr. Wiss. Ges. Johann Wolfgang Goethe Univ. Frankfurt am Main, 20, Franz Steiner Verlag Stuttgart, Stuttgart, 313–345 (2006) (http://www.math.uni-frankfurt.de/∼wolfart/)

[64] K. Zsigmondy, Zur Theorie der Potenzreste, Monatsh. Math. Phys. 3 (1892), no. 1,

265–284