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Radial basis functions for the sphere

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Abstract

In this paper we compute the ultraspherical series expansions for the more commonly used radial basis functions. In several special cases we provide asymptotic estimates for the decay rate of the coefficients involved. Knowledge of the decay of these coefficients is useful because they enable error estimates for spherical interpolation.

1 Introduction

The multivariate interpolation problem is as follows. Given values $\{f_i\}_{i=1}^N$ of a function $f : \mathbb{R}^d \to \mathbb{R}$ at distinct locations (nodes) $\{x_i\}_{i=1}^N$ in $\mathbb{R}^d$, find an *interpolant* $s : \mathbb{R}^d \to \mathbb{R}$, in a suitable linear space of functions T (the interpolation space), satisfying

$$s(x_i) = f_i, \quad 1 \leq i \leq N. \tag{1.1}$$

1.1 Radial basis function method

One of the most promising ways of solving this problem is to employ the *Radial Basis Function* (RBF) method. This method specifies the interpolation space

$$T_\phi = \text{span}\ \{\phi(d(\cdot, x_1)), \ldots, \phi(d(\cdot, x_N))\}, \tag{1.2}$$

where $d(x, y) = \|x - y\|$, $\|\cdot\|$ usually being the Euclidean norm (other norms have been considered; see, for example, [2] and [3]), and $\phi : [0, \infty) \to \mathbb{R}$ is the *radial basis function*.

Now posing the interpolation problem in T_ϕ amounts to finding a function of the form

$$s(x) = \sum_{j=1}^N \lambda_j \phi(d(x, x_j)), \qquad \text{for } \lambda_j \in \mathbb{R}, \quad 1 \leq j \leq N,$$

satisfying conditions (1.1). This is equivalent to solving the following linear system:

$$A\lambda = f, \tag{1.3}$$

where $A \in \mathrm{IR}^{N\times N}$ is defined by

$$A_{i,j} = \phi(d(x_i, x_j)), \qquad 1 \leq i, j \leq N. \tag{1.4}$$

Thus a unique interpolant $s \in T_\phi$ exists for any f if and only if the interpolation matrix A is non-singular.

Definition 1.1. *A function $\phi : [0,\infty) \to \mathrm{IR}$ is said to be:*

(i) **Strictly positive definite** *(SPD) on IR^d whenever its associated interpolation matrix (1.4) is positive definite on IR^N, for all distinct $\{x_i\}_{i=1}^N$ in IR^d.*

(ii) **Conditionally strictly positive definite** *of order m (CSPD(m)) on IR^d whenever its associated interpolation matrix (1.4) is positive definite on the* **subspace** *of IR^N defined by*

$$V_{m-1} = \{\lambda = (\lambda_1, \dots \lambda_N)^T \in \mathrm{IR}^N : \sum_{i=1}^N \lambda_i p(x_i) = 0 \ \text{for all} \ p \in \Pi_{m-1}(\mathrm{IR}^d)\},$$

for all distinct $\{x_i\}_{i=1}^N$ in IR^d. Here $\Pi_{m-1}(\mathrm{IR}^d)$ denotes the space of all d-variate polynomials of degree at most $m-1$.

If ϕ is SPD on IR^d, then there exists a unique interpolant $s \in T_\phi$ since the interpolation matrix is, by definition, positive definite and hence non-singular. If ϕ is $CSPD(m)$ on IR^d however, it can be shown that if the interpolation nodes are $\Pi_{m-1}(\mathrm{IR}^d)$-unisolvent — the only element of $\Pi_{m-1}(\mathrm{IR}^d)$ that vanishes at every node is the zero polynomial — then there exists a unique interpolant $s \in T_\phi \oplus \Pi_{m-1}(\mathrm{IR}^d)$, that is

$$s(x) = \sum_{j=1}^N \lambda_j \phi(d(x, x_j)) + p(x),$$

where $\lambda = (\lambda_1, \dots \lambda_N)^T \in V_{m-1}$ and $p \in \Pi_{m-1}(\mathrm{IR}^d)$; see [13] for details.

The functions used in the RBF method are usually either SPD or $CSPD(m)$ on IR^d. The following is a list of the more common examples.

$(Gaussian) : \phi(r) = e^{-\alpha r^2}, \ \alpha > 0;$

$(Potential\, Spline) : \phi(r) = (-1)^{[\beta]+1} r^{2\beta},\ \beta > 0$ and $\beta \notin \mathbf{Z}_+ = \{1, 2, ...\}$;

$(Thin\, Plate\, Spline) : \phi(r) = (-1)^{k+1} r^{2k} \log(r),\ k \in \mathbf{Z}_+$;

$(Multiquadric) : \phi(r) = (-1)^{[\beta]+1} (r^2 + c^2)^{\beta},\ \beta > 0,\ \beta \notin \mathbf{Z}_+$, and $c > 0$;

$(Inverse\, Multiquadric) : \phi(r) = (r^2 + c^2)^{\beta},\ -d/2 < \beta < 0,\ \beta \notin \mathbf{Z}$, and $c > 0$.

1.2 Zonal basis function method

The RBF method can be specialised if attention is turned to the case where the distinct locations $x_1, \dots x_N$ are known to lie on the unit sphere S^{d-1} in $\mathbb{R}^d$, $d \geq 2$. To transfer the method we consider the interpolation space

$$T_\psi = \text{span } \{\psi(g(\cdot, x_1)), \dots, \psi(g(\cdot, x_N))\}, \tag{1.5}$$

where $g(x, y) = \arccos(x^T y)$ denotes the geodesic metric, and $\psi : [0, \pi] \to \mathbb{R}$ is called a *zonal basis function* (ZBF).

Following the development of the RBF method, it is clear that interpolation is unique in T_ψ if and only if the associated interpolation matrix $B \in \mathbb{R}^{N \times N}$ defined by

$$B_{i,j} = \psi(g(x_i, x_j)), \qquad 1 \leq i, j \leq N, \tag{1.6}$$

is non-singular.

Definition 1.2. *A function $\psi : [0, \pi] \to \mathbb{R}$ is said to be:*

(i) **Strictly positive definite** *(SPD) on S^{d-1} whenever its associated interpolation matrix (1.6) is positive definite on $\mathbb{R}^N$, for all distinct $\{x_i\}_{i=1}^N$ on S^{d-1}.*

(ii) **Conditionally strictly positive definite** *of order m (CSPD(m)) on S^{d-1} whenever its associated interpolation matrix (1.6) is positive definite on the* **subspace** *of $\mathbb{R}^N$ given by*

$$W_{m-1} = \{\lambda = (\lambda_1, \dots \lambda_N)^T \in \mathbb{R}^N : \sum_{i=1}^N \lambda_i Y(x_i) = 0 \text{ for all } Y \in H_{m-1}(S^{d-1})\},$$

for all distinct $\{x_i\}_{i=1}^N$ on S^{d-1}. Here $H_{m-1}(S^{d-1})$ denotes the space of all spherical harmonics on S^{d-1} of order at most $m-1$.

With Definition 1.2, the specialisation of the RBF method to the sphere (*the ZBF method*) is complete. In particular, interpolation is unique in T_ψ if ψ

is SPD on S^{d-1}. If ψ is $CSPD(m)$ on S^{d-1} however, it can be shown that if the interpolation nodes are $H_{m-1}(S^{d-1})$-unisolvent — the only element of $H_{m-1}(S^{d-1})$ that vanishes at every node is the zero spherical harmonic — then there exists a unique interpolant $s \in T_\psi \oplus H_{m-1}(S^{d-1})$, that is

$$s(x) = \sum_{j=1}^{N} \lambda_j \psi(g(x, x_j)) + Y(x),$$

where $\lambda = (\lambda_1, \ldots \lambda_N)^T \in W_{m-1}$ and $Y \in H_{m-1}(S^{d-1})$; see [5] for details. We remark that the role of the spherical harmonic space $H_{m-1}(S^{d-1})$ within the ZBF method is equivalent to the role of the polynomial space $\Pi_{m-1}(\mathbb{R}^d)$ within the RBF method, indeed $H_{m-1}(S^{d-1}) = \Pi_{m-1}(\mathbb{R}^d)|_{S^{d-1}}$; (see [11] or [14]).

Using the work of Schoenberg [16], and extensions thereof [5], we can formulate the following theorem.

Theorem 1.3. *If ψ is $CSPD(m)$ on S^{d-1}, then ψ has the following form*

$$\psi(\theta) = \sum_{k=0}^{\infty} a_k P_k^\lambda(\cos(\theta)), \tag{1.7}$$

where

$$a_k \geq 0 \quad for \quad k \geq m \quad and \quad \sum_{k=0}^{\infty} a_k P_k^\lambda(1) < \infty. \tag{1.8}$$

Here $\{P_k^\lambda\}$ denote the ultraspherical polynomials ([1], 22.2.3) and $\lambda = (d-2)/2$.

Remarks 1.4. *(i) The case $\psi \in SPD(m)$ is covered by setting $m = 0$ in Theorem 1.3.*

(ii) In [12], a framework is established for solving the interpolation problem on a compact Riemannian manifold $\mathbf{M}$ *using SPD kernels $\kappa : \mathbf{M} \times \mathbf{M} \to \mathbb{R}$. The ZBF method with ψ SPD is a specific instance of this more general approach for* $\mathbf{M} = S^{d-1}$.

(iii) In view of Theorem 1.3 we choose to consider each zonal function ψ as a function of the inner product, $x^T y$, since $\cos(g(x, y)) = x^T y$.

The complete characterization of the class of functions of the form (1.7) satisfying (1.8) that are $CSPD(m)$ on S^{d-1} remains an open problem. Several researchers have investigated this in recent papers; in particular, in [18], it is shown that a sufficient condition is $a_k > 0$, for $k \geq m$. (See [15] for an extension of this work). One can use this condition to generate candidate zonal functions to be used within the ZBF method. The following is a list of functions ψ SPD on S^2 for example:

$\psi(t) = (1 + h^2 - 2ht)^{-1/2}$, where $a_k = h^k$, for $0 < h < 1$;

$\psi(t) = (1 - h^2)(1 + h^2 - 2ht)^{-3/2}$, where $a_k = (2k + 1)h^k$, for $0 < h < 1$;

$\psi(t) = 1 - \sqrt{\frac{1-t}{2}}$, where $a_0 = 1/3$ and $a_k = \frac{2}{(2k-1)(2k+3)}$, $k \geq 1$.

2 Radial Functions For Spheres

Most of the recent research regarding the ZBF method is of a theoretical nature, and very little has been reported of its performance in practice (see, however [6]). Much more is known about the RBF method and so a potential user may wish to take a common radial function and use it as a zonal basis function; indeed a radial function ϕ that is $CSPD(m)$ on $\mathbb{R}^d$ is also $CSPD(m)$ on S^{d-1}. Furthermore, the RBFs remain well defined if the interpolation problem is set on a perturbed sphere, which is likely to be the case for several practical applications.

In order to take advantage of the extant ZBF theory (especially convergence results [7] and [10]), it is desirable to have the ultraspherical series expansions (1.7) for all the common radial functions. The remainder of this section addresses precisely this issue. In order to use radial functions on the sphere one usually employs

$$d(x, y) = \|x - y\| = \sqrt{2 - 2x^T y}, \qquad x, y \in S^{d-1}. \tag{2.1}$$

In particular, if ϕ is $SCPD(m)$ on $\mathbb{R}^d$ then the zonal function $\psi(t) = \phi(\sqrt{2 - 2t})$ is $SCPD(m)$ on S^{d-1}, and so, by Theorem (1.1), has an expansion

$$\psi(t) = \sum_{n=0}^{\infty} a_n P_n^{\lambda}(t), \qquad -1 \leq t \leq 1,$$

where the coefficients (a_n) satisfy (1.8). The ultraspherical polynomials P_n^{λ} are given by Rodrigues' formula ([1], 22.11.2)

$$P_n^{\lambda}(t) = c_n(\lambda)(1 - t^2)^{1/2-\lambda} \frac{d^n}{dt^n}(1 - t^2)^{n+\lambda-1/2}, \quad n \in \mathbb{N} = \{0, 1, \ldots\}, \tag{2.2}$$

where

$$c_n(\lambda) = \frac{(-1)^n \pi^{1/2} 2^{(1-n-2\lambda)} \Gamma(n+2\lambda)}{\Gamma(n+\lambda+\frac{1}{2})\Gamma(n+1)\Gamma(\lambda)}. \tag{2.3}$$

We note that these are simply the Legendre polynomials when $\lambda = 1/2$. They satisfy the orthogonality relation ([1], 22.2.3)

$$\int_{-1}^{1} P_m^\lambda(t) P_n^\lambda(t)(1-t^2)^{\lambda-1/2} dt = \begin{cases} 0, & m \neq n, \\ d_n, & m = n, \end{cases} \tag{2.4}$$

where

$$d_n = \frac{\pi \Gamma(n+2\lambda)}{2^{2\lambda-1}(n+\lambda)\Gamma(n+1)\Gamma(\lambda)^2}, \tag{2.5}$$

and thus the series coefficients are given by

$$a_n = \frac{1}{d_n} \int_{-1}^{1} \psi(t)(1-t^2)^{\lambda-1/2} P_n^\lambda(t) dt, \quad n \in \mathbb{N}. \tag{2.6}$$

Employing (2.2) and integrating by parts n times gives

$$a_n = \frac{(-1)^n c_n(\lambda)}{d_n} \int_{-1}^{1} \psi^{(n)}(t)(1-t^2)^{n+\lambda-1/2} dt. \tag{2.7}$$

2.1 Multiquadrics

Here we consider the function $\phi(r) = (r^2 + c^2)^\beta$, $c > 0$, where $\beta \in \mathbb{R} \backslash \mathbf{Z}$. It is known that ϕ is SPD for $-d/2 < \beta < 0$, and $(-1)^{[\beta]+1}\phi$ is $CSPD([\beta]+1)$ for $\beta > 0$ (see [13]). To use the multiquadric on the sphere we consider $\psi(t) = (2 + c^2 - 2t)^\beta$. Applying (2.7) for $n \in \mathbb{N}$ gives

$$a_n(\beta, \lambda) = \frac{2^{\beta-n}\Gamma(\beta+1)}{\Gamma(\beta-n+1)} \frac{c_n(\lambda)}{d_n} \int_{-1}^{1} (1 + \frac{c^2}{2} - t)^{\beta-n}(1-t^2)^{n+\lambda-1/2} dt.$$

Setting $A = \frac{c^2}{2}$ and $u = \frac{t+1}{2}$,we find

$a_n(\beta, \lambda) =$

$$\frac{2^{2n+2\lambda+\beta}(2+A)^{\beta-n}\Gamma(\beta+1)}{\Gamma(\beta-n+1)}\frac{c_n(\lambda)}{d_n}\int_0^1(1-\frac{2u}{2+A})^{\beta-n}(1-u)^{n+\lambda-1/2}u^{n+\lambda-1/2}du.$$

Using the identity (see [1], 15.3.1)

$$\int_0^1(1-zu)^{-a}(1-u)^{c-b-1}u^{b-1}du=\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}F(a,b;c;z),$$

with $a=n-\beta$, $b=n+\lambda+1/2$, $c=2b$ and $z=\frac{2}{2+A}$, we see that

$$a_n(\beta,\lambda)=\alpha_n(\beta,\lambda)F(n-\beta,n+\lambda+1/2;2(n+\lambda+1/2);\frac{2}{2+A}), \qquad (2.8)$$

where

$$\alpha_n(\beta,\lambda)=\frac{2^{2n+2\lambda+\beta}(2+A)^{\beta-n}\Gamma(\beta+1)\Gamma(n+\lambda+\frac{1}{2})^2}{\Gamma(\beta-n+1)\Gamma(2(n+\lambda+\frac{1}{2}))}\frac{c_n(\lambda)}{d_n} \qquad (2.9)$$

which on substituting (2.3) and (2.5)

$$=\frac{(-1)^n2^{n+2\lambda+\beta}(2+A)^{\beta-n}(n+\lambda)\Gamma(\lambda)\Gamma(\beta+1)}{\Gamma(\beta-n+1)\pi^{1/2}}\frac{\Gamma(n+\lambda+\frac{1}{2})}{\Gamma(2(n+\lambda+\frac{1}{2}))} \qquad (2.10)$$

and $F(a,b;c;z)$ is the Gauss Hypergeometric series (see [1], 15.1.1) defined by

$$F(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)}\frac{z^k}{k!}. \qquad (2.11)$$

This series is absolutely convergent for $|z|\le 1$ provided $\Re(c-a-b)>0$, that is, $-d/2<\beta$. Thus (2.8) holds for all multiquadrics.

2.2 Potential splines

Here we consider the function $\phi(r)=r^{2\beta}$, for $\beta>0$ and $\beta\notin \mathbf{Z}_+$. It is known that $(-1)^{[\beta]+1}\phi$ is $CSPD([\beta]+1)$ (see [13]). To use the potential splines on the sphere we consider, $\psi(t) = (2-2t)^\beta$. This can be derived from the multiquadric case above by simply setting $A = \frac{c^2}{2} = 0$ i.e. $\frac{2}{2+A} = 1$. Using the results from Section 2.1 and the following identity ([1], 15.1.20)

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

we can deduce the ultraspherical coefficients of the potential splines

$$a_n(\beta,\lambda) = (-1)^n \pi^{-1/2} 2^{2\lambda}(n+\lambda)\Gamma(\lambda)\frac{2^{2\beta}\Gamma(\beta+1)\Gamma(\beta+\lambda+\frac{1}{2})}{\Gamma(\beta+1-n)\Gamma(\beta+n+1+2\lambda)}. \quad (2.12)$$

2.3 Thin plate splines

Here we consider the function $\phi(r) = r^{2k}\log r$, for $k \in \mathbf{Z}_+$. It is known that $(-1)^{k+1}\phi$ is $CSPD(k+1)$ (see [13]). To use the thin plate splines on the sphere we consider, $\psi(t) = \frac{1}{2}(2-2t)^k \log(2-2t)$. This function can be derived from the potential spline using the observation

$$\psi(t) = \frac{1}{2}\frac{\partial}{\partial\beta}(2-2t)^\beta\Big|_{\beta=k}. \quad (2.13)$$

Thus the ultraspherical coefficients of the thin plate splines $b_n(k,\lambda)$ are given by

$$b_n(k,\lambda) = \frac{1}{2}\frac{\partial}{\partial\beta}a_n(\beta,\lambda)\Big|_{\beta=k}, \quad (2.14)$$

where $a_n(\beta,\lambda)$ are as in (2.12). In particular, we rewrite (2.12) as

$$a_n(\beta,\lambda) = (-1)^n \pi^{-1/2} 2^{2\lambda}(n+\lambda)\Gamma(\lambda)h(\beta), \quad (2.15)$$

where

$$h(\beta) = \frac{2^{2\beta}\Gamma(\beta+1)\Gamma(\beta+\lambda+\frac{1}{2})}{\Gamma(\beta+1-n)\Gamma(\beta+n+1+2\lambda)}. \quad (2.16)$$

In order to differentiate $h(\beta)$, we consider the so called digamma function ([1], 6.3.1), which is defined by $\Psi(z) = \Gamma'(z)/\Gamma(z)$ for $z \neq 0,-1,-2,\ldots$. Then for $\beta = k \geq n$

$$h'(k) = h(k)\{\Psi(k+1) + \Psi(k+\lambda+\frac{1}{2}) + 2\log 2 \quad (2.17)$$

$$-\Psi(k+1-n)-\Psi(k+n+2\lambda+1)\}$$

We can also write $\Gamma(\beta+1)=\beta(\beta-1)\cdots(\beta-n+1)\Gamma(\beta-n+1)$ and so consider $h(\beta)$ as

$$h(\beta)=\frac{2^{2\beta}\Gamma(\beta+\lambda+\frac{1}{2})\beta(\beta-1)\cdots(\beta-n+1)}{\Gamma(\beta+n+2\lambda+1)}=\frac{u(\beta)v(\beta)}{w(\beta)},$$

where $u(\beta)=2^{2\beta}\Gamma(\beta+\lambda+1/2)$, $v(\beta)=\beta(\beta-1)\cdots(\beta-n+1)$ and $w(\beta)=\Gamma(\beta+n+2\lambda+1)$. Thus

$$h'(k)=\frac{w(k)\{u'(k)v(k)+u(k)v'(k)\}-u(k)v(k)w'(k)}{w(k)^2}$$

and, since $v(k)=0$, for all $k<n$, this is simply

$$h'(k)=\frac{u(k)v'(k)}{w(k)}$$

Furthermore $v'(k)=(-1)^{n-(k+1)}\Gamma(k+1)\Gamma(n-k)$, from which we can see

$$h'(k)=\frac{(-1)^{n-(k+1)}2^{2k}\Gamma(k+\lambda+\frac{1}{2})\Gamma(k+1)\Gamma(n-k)}{\Gamma(k+n+2\lambda+1)}. \qquad (2.18)$$

We can now use equations (2.17) and (2.18) to deduce the ultraspherical coefficients for the thin plate spline; for $k\geq n$

$$b_n(k,\lambda)=a_n(k,\lambda)\{\Psi(k+1)+\Psi(k+\lambda+\frac{1}{2})+2\log 2-\Psi(k+1-n) \qquad (2.19)$$

$$-\Psi(k+n+2\lambda+1)\};$$

whilst for $k<n$

$$b_n(k,\lambda)=\frac{(-1)^{k+1}2^{2(k+\lambda)}(n+\lambda)\Gamma(\lambda)\Gamma(k+\lambda+\frac{1}{2})\Gamma(k+1)\Gamma(n-k)}{2\pi^{1/2}\Gamma(k+n+1+2\lambda)}. \qquad (2.20)$$

2.4 Gaussians

Here we consider the function $\phi(r) = e^{-\alpha r^2}$, for $\alpha > 0$. It is well known that ϕ is SPD (see [13]). To use the Gaussian on the sphere we consider, $\psi(t) = e^{-2\alpha} e^{2\alpha t}$. Again we apply formula (2.7) to obtain

$$a_n(\alpha, \lambda) = \frac{(-1)^n e^{-2\alpha}(2\alpha)^n c_n(\lambda)}{d_n} \int_{-1}^{1} e^{2\alpha t}(1-t^2)^{n+\lambda-1/2} dt.$$

The integral in the above formula represents the modified Bessel function $I_{n+\lambda}$; specifically we have ([17], 3.71)

$$I_{n+\lambda}(2\alpha) = \frac{\alpha^{n+\lambda}}{\Gamma(n+\lambda+\frac{1}{2})\pi^{1/2}} \int_{-1}^{1} e^{2\alpha t}(1-t^2)^{n+\lambda-1/2} dt. \qquad (2.21)$$

Therefore we deduce the ultraspherical coefficients

$$a_n(\alpha, \lambda) = \frac{(-1)^n \pi^{1/2} 2^n e^{-2\alpha}\Gamma(n+\lambda+\frac{1}{2})}{\alpha^\lambda} \frac{c_n(\lambda)}{d_n} I_{n+\lambda}(2\alpha), \quad n \in \mathbb{N},$$

and, on substituting (2.3) and (2.5),

$$a_n(\alpha, \lambda) = \frac{(n+\lambda)\Gamma(\lambda)e^{-2\alpha}}{\alpha^\lambda} I_{n+\lambda}(2\alpha). \qquad (2.22)$$

3 Common RBF's for the 2-Sphere

In this concluding section we specialise the results of Section 2 to the sphere S^2, in which case $\lambda = 1/2$. Furthermore we apply the results to the radial functions in their more familiar form.

3.1 The Inverse Multiquadric: $\phi(r) = (r^2 + c^2)^{-1/2}$.

Here we apply (2.8) with $\lambda = 1/2$ and $\beta = -1/2$ giving, for $n \in \mathbb{N}$,

$$a_n = a_n(-1/2, 1/2) = \alpha_n(-1/2, 1/2) F(n+1/2, n+1; 2(n+1); \frac{4}{4+c^2}).$$

Considering the following identity ([1], 15.1.13):

$$F(a,\frac{1}{2}+a;1+2a;z) = 2^{2a}(1+\sqrt{1-z})^{-2a}$$

setting $a = n+\frac{1}{2}$ and $z = \frac{4}{4+c^2}$ allows us to deduce:

$$a_n = \frac{(n+\frac{1}{2})(-1)^n\pi}{\Gamma(\frac{1}{2}-n)\Gamma(n+\frac{3}{2})}\Big(\frac{2}{c+\sqrt{4+c^2}}\Big)^{2n+1}, \tag{3.1}$$

this can be simplified further, using the identity ([1], 6.1.17)

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}. \tag{3.2}$$

In particular setting $z = 1/2 - n$ yields

$$\Gamma(1/2-n)\Gamma(n+3/2) = (n+\frac{1}{2})\Gamma(1/2-n)\Gamma(n+1/2) = (n+\frac{1}{2})(-1)^n\pi$$

giving

$$a_n = h^{2n+1} \tag{3.3}$$

where $h = \frac{2}{c+\sqrt{4+c^2}} < 1$, thus the coefficients decay at an exponential rate.

3.2 The Multiquadric: $\phi(r) = (r^2+c^2)^{1/2}$.

Here we apply (2.8) again with $\lambda = 1/2$, $\beta = 1/2$, giving for $n \in \mathbb{N}$,

$$a_n = a_n(1/2,1/2) = \alpha_n(1/2,1/2)F(n-1/2,n+1;2(n+1);\frac{4}{4+c^2}).$$

A closed form representation for $F(n-1/2,n+1;2(n+1);\frac{4}{4+c^2})$ can be derived quite easily ([1] Section 15). In particular, we have:

$$F(n-1/2,n+1;2(n+1);z) = \frac{(n+\frac{1}{2})\sqrt{1-z}+(1-\frac{z}{2})}{(n+\frac{3}{2})}2^{2n+1}(1+\sqrt{1-z})^{-(2n+1)}.$$

Setting $z = \frac{4}{4+c^2}$ and multiplying by $\alpha_n(1/2,1/2)$ gives

$$a_n = \frac{(-1)^n\pi(n+\frac{1}{2})(2+c^2+(n+\frac{1}{2})c\sqrt{4+c^2})}{2\Gamma(\frac{3}{2}-n)\Gamma(n+\frac{5}{2})}\Big(\frac{2}{c+\sqrt{4+c^2}}\Big)^{2n+1}. \tag{3.4}$$

Further simplification is possible, setting $z = 3/2 - n$ in (3.2) gives

$$\Gamma(3/2-n)\Gamma(n+5/2) = (n+\frac{3}{2})(n+\frac{1}{2})(n-\frac{1}{2})\Gamma(3/2-n)\Gamma(n-1/2)$$
$$= (n+\frac{3}{2})(n+\frac{1}{2})(n-\frac{1}{2})(-1)^{n-1}\pi,$$

thus

$$a_n = -\frac{(2+c^2+(n+\frac{1}{2})c\sqrt{4+c^2})}{2(n+\frac{3}{2})(n-\frac{1}{2})}h^{2n+1} = O\Big(\frac{h^{2n+1}}{n}\Big). \tag{3.5}$$

3.3 The Pseudo Cubic: $\phi(r) = r^3$

Here we simply set $\beta = 3/2$ and $\lambda = 1/2$ in (2.12) giving, for $n \in \mathbb{N}$,

$$a_n = a_n(3/2, 1/2) = (-1)^n 2^4 \frac{(n+\frac{1}{2})\Gamma(\frac{5}{2})^2}{\Gamma(\frac{5}{2}-n)\Gamma(n+\frac{7}{2})} \tag{3.6}$$

Simplification is again possible, setting $z = 5/2 - n$ in (3.2) gives

$$\Gamma(5/2-n)\Gamma(n+7/2) = (n+\frac{5}{2})(n+\frac{3}{2})(n+\frac{1}{2})(n-\frac{1}{2})(n-\frac{3}{2})(-1)^n\pi,$$

thus

$$a_n = \frac{9}{(n+\frac{5}{2})(n+\frac{3}{2})(n-\frac{1}{2})(n-\frac{3}{2})} = O\Big(\frac{1}{n^4}\Big). \tag{3.7}$$

3.4 The Thin Plate Spline: $\phi(r) = r^2 \log r$

Here we simply set $k = 1$ and $\lambda = 1/2$ in (2.20), for $n > 1$ this provides

$$a_n = b_n(1, 1/2) = 4(n+1/2)\frac{\Gamma(2)^2\Gamma(n-1)}{\Gamma(n+3)}$$
$$= \frac{4(n+\frac{1}{2})}{(n+2)(n+1)n(n-1)} = O\Big(\frac{1}{n^3}\Big). \tag{3.8}$$

3.5 The Gaussian: $\phi(r) = e^{-\alpha r^2}$

Setting $\lambda = 1/2$ in (2.22) yields, for $n \in \mathbb{N}$,

$$a_n = a_n(\alpha, 1/2) = \sqrt{\frac{\pi}{\alpha}}(n + 1/2)e^{-2\alpha} I_{n+\frac{1}{2}}(2\alpha),$$

employing 2.21 gives

$$a_n = \frac{(n+\frac{1}{2})e^{-2\alpha}\alpha^n}{\Gamma(n+1)} \int_{-1}^{1} e^{2\alpha t}(1-t^2)^n dt. \tag{3.9}$$

We can derive the asymptotic behaviour using the well-known method of Laplace. However, we prefer a direct approach, which we present for the convenience of the reader. Consider the integral appearing in (3.9), that is

$$G_n = \int_{-1}^{1} e^{2\alpha t}(1-t^2)^n dt. \tag{3.10}$$

Setting $\tau = \sqrt{n}t$, we obtain

$$\sqrt{n} G_n = \int_{-\infty}^{\infty} f_n(\tau) d\tau$$

where

$$f_n(\tau) = \begin{cases} e^{\frac{2\alpha\tau}{\sqrt{n}}}(1-\tau^2/n)^n, & |\tau| \le \sqrt{n} \\ 0, & |\tau| > \sqrt{n}. \end{cases} \tag{3.11}$$

Observing that $0 \le f_n(\tau) \le e^{2\alpha}e^{-\tau^2}$ and $\lim_{n\to\infty} f_n(\tau) = e^{-\tau^2}$ allows us to employ the dominated convergence theorem,

$$\sqrt{n} G_n = \int_{-\infty}^{\infty} f_n(\tau) d\tau \ \to \int_{-\infty}^{\infty} e^{-\tau^2} d\tau = \sqrt{\pi}, \quad as \quad n \to \infty. \tag{3.12}$$

Also we have Stirling's formula ([1], 6.1.38)

$$\Gamma(n+1) = n! \sim \sqrt{2\pi} n^{n+1/2} e^{-n}, \quad as \quad n \to \infty. \tag{3.13}$$

Employing (3.12) and (3.13) together in (3.9) gives

$$a_n \sim \frac{e^{-2\alpha}}{2^{1/2}} \Big(\frac{e\alpha}{n}\Big)^n, \quad as \quad n \to \infty,$$

i.e. the Gaussian coefficients decay at an exponential rate.

3.6 Concluding Remarks

The motivation for this work stems from recent results on error estimates for spherical interpolation ([4], [7] and [10]). These topics have been investigated by several mathematicians, in particular the Leicester group ([8] and [9]). Specifically the report [8] calculates the ultraspherical coefficients for the Duchon splines which are also contained in this paper, the approach taken (private communication) however is quite distinct from the one given here.

For practical purposes a potential user would prefer to work with a basis function ψ with a closed form representation and with provably good approximation properties. The results of this paper allow us to provide convergence results for the common RBF's restricted to the sphere.

The process of restricting radial functions to the sphere clearly provides suitable ZBF's. However the class of **all** suitable ZBF's is much larger, containing, in addition, the *truly zonal* functions i.e. those that are SPD or $CSPD(m)$ on S^{d-1} but not on $\mathbb{R}^d$. It is not clear whether choosing a truly zonal function provides any advantages over a restricted RBF, and this is an obvious topic for further research.

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