The size of a graph is reconstructible from any $n-2$ cards

Paul Brown* and Trevor Fenner†
Department of Computer Science and Information Systems
Birkbeck College, University of London, United Kingdom

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Abstract

Let G and H be graphs of order n. The *number of common cards* of G and H is the maximum number of disjoint pairs (v, w), where v and w are vertices of G and H, respectively, such that $G - v \cong H - w$. We prove that if the number of common cards of G and H is at least $n - 2$ then G and H must have the same number of edges when $n \geq 29$. This is the first improvement on the 25-year-old result of Myrvold that if G and H have at least $n - 1$ common cards then they have the same number of edges. It also improves on the result of Woodall and others that the numbers of edges of G and H differ by at most 1 when they have $n - 2$ common cards.

Keywords: *Graph reconstruction, vertex-deleted subgraphs, common cards, size reconstruction*

1 Introduction

In this paper all graphs are finite, undirected and contain no loops or multiple edges. Any graph-theoretic terminology and notation not explicitly explained can be found in Bondy and Murty's text [1].

Let G be a graph of order n, and let $v \in V(G)$. The *vertex-deleted subgraph* or *card* $G - v$ of G is obtained from G by deleting v together with all edges of G incident to v. The multi-set of all n unlabelled cards of G is called the *deck* of G, which we denote by $\mathcal{D}(G)$. The *Reconstruction Conjecture*, first proposed by Kelly and Ulam in 1941 [5, 6, 13], asserts that, when $n > 2$, two graphs G and H are isomorphic if and only if $\mathcal{D}(G) = \mathcal{D}(H)$. However, despite the efforts of many graph theorists, the status of the sufficiency of the condition remains unresolved.

One approach to tackling this problem has been to consider the *number of common cards* between pairs of graphs. A *common card of*, or *between*, two graphs G and H is any card in the multi-set intersection $\mathcal{D}(G) \cap \mathcal{D}(H)$. The number of common cards of G and H, denoted by $b(G, H)$, is the cardinality of this multi-set intersection. It follows that there exist labellings $v_1, v_2, \ldots, v_n$ of $V(G)$ and $w_1, w_2, \ldots, w_n$ of $V(H)$ such that $G - v_j$ is isomorphic to $H - w_j$ for all j, $1 \leq j \leq b(G, H)$. The Reconstruction Conjecture can then be reformulated as follows:

*paulb@dcs.bbk.ac.uk
†trevor@dcs.bbk.ac.uk

$b(G, H) < n$ unless G and H are isomorphic when $n > 2$. Examples of families of pairs of non-isomorphic graphs that have a large number of common cards relative to n can be found in [2] and [4].

It is also of interest to obtain bounds on $b(G, H)$ when G and H differ on certain graph parameters. A graph parameter θ is *reconstructible* from a subset S of the deck $\mathcal{D}(G)$ if $\theta(H) = \theta(G)$ for every graph H for which $\mathcal{D}(H)$ contains S. For example, it was shown by Bowler et al [3] that $b(G, H) \leq \frac{n}{2} + 1$ when G is connected and H is disconnected, i.e., connectedness is reconstructible from any $\lfloor \frac{n}{2} \rfloor + 2$ cards. Other authors have considered pairs for which the *sizes*, i.e., the numbers of edges, of G and H differ. However, despite the fact that it is easy to show that $b(G, H) < n$ in this case [8], obtaining stronger bounds has proved difficult. Myrvold [9] showed that $b(G, H) \leq n - 2$ for such pairs when $n \geq 7$. More recently, Woodall [14] generalised results of Kocay, Ramachandran, Monikandan and Balakumar [7] [11] [10] to obtain a number of relationships between $b(G, H)$ and $\big||E(G)| - |E(H)|\big|$.

A consequence of one of Woodall's results is that if $\big||E(G)| - |E(H)|\big| \geq 2$ then $b(G, H) \leq n-3$ when n is sufficiently large. However, he stated that it was currently unknown whether, for n sufficiently large, $|E(G)| = |E(H)|$ when $b(G, H) = n - 2$. In this paper we prove that this is indeed the case when $n \geq 29$, i.e., the size of a graph is reconstructible from any $n-2$ cards in its deck for $n \geq 29$. We note that this lower bound on n is almost certainly too high. However, for the pair of graphs of order 8 in Figure 1, $b(G, H) = 6$ and $|E(G)| = |E(H)| + 1$. Indeed, this pair, together with their complements, are the only two pairs of non-isomorphic graphs of order 8 that have 6 common cards [12]. It is easy to check that $G - v_j \cong H - w_j$ for $1 \leq j \leq 6$.

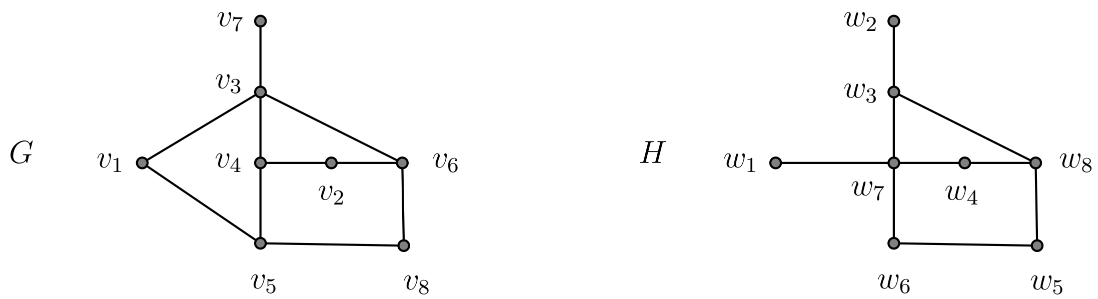

Figure 1: A pair of graphs of order 8 with different sizes having 6 common cards.

Using similar methods to those employed in this paper, we are hopeful that it can be shown that the size of a graph is reconstructible from any set of β cards in its deck for β less than $n - 2$. Indeed, it likely that this result holds for significantly smaller values of β, since, as was stated in [2], there is currently no known pair G and H for which $|E(G)| \neq |E(H)|$ and $b(G, H) > \frac{2n}{3}$, when $n \geq 22$. Indeed, a stronger version of the Reconstruction Conjecture was proposed in that paper, namely that $\frac{2n}{3}$ is, in fact, an upper bound on $b(G, H)$ for large n when G and H are not isomorphic.

2 Notation and preliminary results

Let G be a graph and let v be a vertex of G. The *neighbourhood* of v in G is the subset $N_G(v)$ of $V(G)$ consisting of all vertices of G adjacent to v, and its *degree* in G, denoted by $d_G(v)$, is the cardinality of this set, i.e., $d_G(v) = |N_G(v)|$. We denote the minimum and maximum degrees of the vertices of G by $\delta(G)$ and $\Delta(G)$, respectively.

We define $D_\alpha(G) = \{v \in V(G) \mid d_G(v) = \alpha\}$ and $d_\alpha(G) = |D_\alpha(G)|$, i.e., $d_\alpha(G)$ is the number of vertices of G of degree α. For any $S \subseteq V(G)$, we further define $D_\alpha(S) = S \cap D_\alpha(G)$ and $d_\alpha(S) = |D_\alpha(S)|$. So $d_\alpha(N_G(v))$ is the number of neighbours of v of degree α in G. We note that $d_\alpha(G) = 0$ when $\alpha < \delta(G)$ or $\alpha > \Delta(G)$, and that $\sum_i d_i(S) = |S|$, so $\sum_i d_i(G) = n$.

We denote the *complement* of G by $\overline{G}$, and note that $d_{\overline{G}}(v) = n - 1 - d_G(v)$ for all v; thus $D_i(\overline{G}) = D_{n-1-i}(G)$ for all i. Since $\delta(\overline{G}) = n - 1 - \Delta(G)$, it follows that $D_{\delta(\overline{G})}(\overline{G}) = D_{n-1-\Delta(G)}(\overline{G}) = D_{\Delta(G)}(G)$. Clearly, for any $v \in V(G)$ and $w \in V(H)$, we have $\overline{G} - v \cong \overline{H} - w$ if and only if $G - v \cong H - w$. This implies that $b(\overline{G}, \overline{H}) = b(G, H)$.

Finally, we define

$$E_{\alpha\beta}(G) = \{e \in E(G) \mid e = uv, \text{ where } d_G(u) = \alpha \text{ and } d_G(v) = \beta, \text{ or vice versa}\}, \quad (1)$$

and $e_{\alpha\beta}(G) = |E_{\alpha\beta}(G)|$. We note that $e_{\alpha\beta}(G) \le \min\{\alpha d_\alpha(G), \beta d_\beta(G)\}$.

Lemma 2.1. Let G be a graph and let $S \subseteq V(G)$. Then, for all α,

$$\alpha d_\alpha(G) \ge \sum_{u \in S} d_\alpha(N_G(u)), \quad (2)$$

$$\sum_{u \in S} d_\alpha(N_G(u)) \ge (\alpha + 1 + |S| - n) d_\alpha(G) - d_\alpha(S). \quad (3)$$

Proof Let $T = \{(u, v) \in S \times D_\alpha(G) \mid uv \in E(G)\}$. For each $v \in D_\alpha(G)$, there are precisely $|N_G(v) \cap S|$ elements in T. On the other hand, for each $u \in S$, there are precisely $d_\alpha(N_G(u))$ elements in T. Thus $\sum_{v \in D_\alpha(G)} |N_G(v) \cap S| = |T| = \sum_{u \in S} d_\alpha(N_G(u))$.

Since $|N_G(v) \cap S| \le |N_G(v)| = \alpha$ for every vertex $v \in D_\alpha(G)$, it is clear that $\sum_{v \in D_\alpha(G)} |N_G(v) \cap S| \le \alpha d_\alpha(G)$, yielding (2). Now, each vertex in $D_\alpha(G) \setminus D_\alpha(S)$ is adjacent to at least $\alpha - (n - 1 - |S|)$ vertices of S and each vertex in $D_\alpha(S)$ is adjacent to at least $\alpha - (n - |S|)$ vertices of S, so it follows that

$$\begin{aligned}\sum_{v \in D_\alpha(G)} |N_G(v) \cap S| &\ge (\alpha + 1 + |S| - n)(d_\alpha(G) - d_\alpha(S)) + (\alpha + |S| - n) d_\alpha(S) \\ &= (\alpha + 1 + |S| - n) d_\alpha(G) - d_\alpha(S),\end{aligned}$$

yielding (3). □

Lemma 2.2. Let G be a graph and let $v \in V(G)$. If $d_G(v) = \alpha$, then

$$d_\alpha(G - v) = d_\alpha(G) + d_{\alpha+1}(N_G(v)) - d_\alpha(N_G(v)) - 1 \quad (4)$$

and, for $i \neq \alpha$,

$$d_i(G - v) = d_i(G) + d_{i+1}(N_G(v)) - d_i(N_G(v)). \quad (5)$$

Proof These follow immediately since $d_{G-v}(u) = d_G(u) - 1$ for $u \in N_G(v)$, and $d_{G-v}(u) = d_G(u)$ for $u \in V(G - v) \setminus N_G(v)$. □

Lemma 2.3. (Handshaking) Let G be a graph. Then $\sum_{v \in V(G)} d_G(v) = 2|E(G)|$. □

Lemma 2.4. Let G be a graph and let $\Lambda_\alpha(G) = E_{\alpha\alpha}(G) \cup E_{\alpha\alpha+1}(G) \cup E_{\alpha+1\alpha+1}(G)$. Then $\Lambda_{\delta(G)-1}(G-v) \subseteq \Lambda_{\delta(G)}(G)$ for all $v \in V(G)$.

Proof Let $\delta = \delta(G)$. Clearly, $E_{\delta-1\delta-1}(G) = E_{\delta-1\delta}(G) = E_{\delta-1\delta+1}(G) = \emptyset$. So, if $xy \in E_{\delta-1\delta-1}(G-v) \cup E_{\delta-1\delta}(G-v) \cup E_{\delta\delta}(G-v)$, then $xy \in E_{\delta\delta}(G) \cup E_{\delta\delta+1}(G) \cup E_{\delta+1\delta+1}(G)$. □

Lemma 2.5. Let G be a graph. Suppose that $e_{\alpha\alpha+1}(G) = 0$. Then

$$\left|\{v \in V(G) \mid e_{\alpha\alpha+1}(G-v) \geq 1\}\right| \leq (\alpha+1)d_{\alpha+1}(G) + (\alpha+2)e_{\alpha\alpha+2}(G). \tag{6}$$

Proof Let $v \in V(G)$. Suppose there exists $xy \in E_{\alpha\alpha+1}(G-v)$, where $d_{G-v}(x) = \alpha$ and $d_{G-v}(y) = \alpha+1$. Then, either $v \in N_G(x)$ and $d_G(x) = \alpha+1$, or $v \in N_G(y) \setminus N_G(x)$, $d_G(x) = \alpha$ and $d_G(y) = \alpha+2$. The result now follows since there are at most $(\alpha+1)d_{\alpha+1}(G)$ vertices of G that have a neighbour x of degree $\alpha+1$, and at most $(\alpha+2)e_{\alpha\alpha+2}(G)$ vertices of G that have a neighbour y of degree $\alpha+2$ that is adjacent to a vertex x of degree α. □

3 General results

From now on, we assume that G and H are non-isomorphic graphs of order n. We define $\delta = \min\{\delta(G), \delta(H)\}$ and $\Delta = \max\{\Delta(G), \Delta(H)\}$.

We label the vertices of G and H so that $G - v_j \cong H - w_j$ for $j = 1, 2, \ldots, b(G, H)$, where $d_G(v_1) \leq d_G(v_2) \ldots \leq d_G(v_{b(G,H)})$. We define $\mathcal{A}(G) = \{v_j \mid 1 \leq j \leq b(G, H)\}$ and $\mathcal{A}(H) = \{w_j \mid 1 \leq j \leq b(G, H)\}$. In addition, we define $\overline{\mathcal{A}(G)} = V(G) \setminus \mathcal{A}(G)$, and $\overline{\mathcal{A}(H)} = V(H) \setminus \mathcal{A}(H)$. Clearly, $|\mathcal{A}(G)| = |\mathcal{A}(H)| = b(G, H)$, and $|\overline{\mathcal{A}(G)}| = |\overline{\mathcal{A}(H)}| = n - b(G, H)$.

For all of the results in this section, we shall assume that $|E(G)| = |E(H)| + 1$, *without stating this explicitly.* The justification for this assumption will become apparent in Section 4.

Lemma 3.1. Let $v_j \in \mathcal{A}(G)$. Then $d_G(v_j) = d_H(w_j) + 1$.

Proof This follows since $|E(G)| - d_G(v_j) = |E(G - v_j)| = |E(H - w_j)| = |E(H)| - d_H(w_j)$. □

Corollary 3.2. $d_H(w_1) \leq d_H(w_2) \ldots \leq d_H(w_{b(G,H)})$. □

Corollary 3.3. The following relationships between $\mathcal{A}(G)$ and $\mathcal{A}(H)$ follow immediately.

(a) $d_{i+1}(\mathcal{A}(G)) = d_i(\mathcal{A}(H))$ for all i.

(b) $d_{i+1}(G) = d_i(\mathcal{A}(H)) + d_{i+1}(\overline{\mathcal{A}(G)})$ for all i.

(c) $d_i(H) = d_{i+1}(\mathcal{A}(G)) + d_i(\overline{\mathcal{A}(H)})$ for all i.

(d) $d_\delta(\mathcal{A}(G)) = d_\Delta(\mathcal{A}(H)) = 0$, $d_\delta(G) = d_\delta(\overline{\mathcal{A}(G)})$ and $d_\Delta(H) = d_\Delta(\overline{\mathcal{A}(H)})$.

(e) $d_\delta(G) \leq n - b(G, H)$.

□

Lemma 3.4.

$$\sum_{y \in \overline{\mathcal{A}(H)}} d_H(y) - \sum_{x \in \overline{\mathcal{A}(G)}} d_G(x) = b(G, H) - 2. \tag{7}$$

Proof Since $|\mathcal{A}(H)| = b(G, H)$, it follows from Lemmas 2.3 and 3.1 that

$$2|E(G)| = \sum_{v_j \in \mathcal{A}(G)} d_G(v_j) + \sum_{x \in \overline{\mathcal{A}(G)}} d_G(x) = b(G, H) + \sum_{w_j \in \mathcal{A}(H)} d_H(w_j) + \sum_{x \in \overline{\mathcal{A}(G)}} d_G(x).$$

Similarly, since $|E(G)| = |E(H)| + 1$, it follows from Lemma 2.3 that

$$2|E(G)| = 2|E(H)| + 2 = \sum_{w_j \in \mathcal{A}(H)} d_H(w_j) + \sum_{y \in \overline{\mathcal{A}(H)}} d_H(y) + 2.$$

The above two equations immediately yield (7). □

Lemma 3.5. Let $\psi = d_\delta(H) - d_\delta(G)$. Then, for all $v_j \in \mathcal{A}(G)$ and $w_j \in \mathcal{A}(H)$:

(a) $d_\delta(N_G(v_j)) = d_\delta(N_H(w_j))$;

(b) if $d_G(v_j) = \delta+1$, or equivalently $d_H(w_j) = \delta$, then $d_{\delta+1}(N_G(v_j)) - d_{\delta+1}(N_H(w_j)) = \psi - 1$;

(c) if $d_G(v_j) \geq \delta+2$, or equivalently $d_H(w_j) \geq \delta+1$, then $d_{\delta+1}(N_G(v_j)) - d_{\delta+1}(N_H(w_j)) = \psi$.

Proof (a) Since $d_{\delta-1}(G) = d_{\delta-1}(H) = 0$, it follows from (5) that $d_\delta(N_G(v_j)) = d_{\delta-1}(G - v_j) = d_{\delta-1}(H - w_j) = d_\delta(N_H(w_j))$.

(b) By putting $i = \delta$ in (5) for $v_j \in D_{\delta+1}(\mathcal{A}(G))$, and $\alpha = \delta$ in (4) for $w_j \in D_\delta(\mathcal{A}(H))$, we obtain

$$d_\delta(G - v_j) = d_\delta(G) + d_{\delta+1}(N_G(v_j)) - d_\delta(N_G(v_j)), \tag{8}$$

$$d_\delta(H - w_j) = d_\delta(H) + d_{\delta+1}(N_H(w_j)) - d_\delta(N_H(w_j)) - 1. \tag{9}$$

Since $G - v_j \cong H - w_j$, on using part (a) and the definition of ψ, the result follows from (8) and (9).

(c) Since $d_\delta(\mathcal{A}(G)) = 0$ by Corollary 3.3(d), this result follows similarly by putting $i = \delta$ in (5) for $v_j \in \mathcal{A}(G) \setminus D_{\delta+1}(\mathcal{A}(G))$ and $w_j \in \mathcal{A}(H) \setminus D_\delta(\mathcal{A}(H))$. □

Corollary 3.6. Let $b = b(G, H)$. Then

$$\delta d_\delta(G) \geq (\delta + b - n) d_\delta(H), \tag{10}$$

$$\delta d_\delta(H) \geq (\delta + 1 + b - n) d_\delta(G). \tag{11}$$

Proof Using inequality (2) with $\alpha = \delta$ and $S = \mathcal{A}(G)$, and then inequality (3) with $\alpha = \delta$ and $S = \mathcal{A}(H)$, it follows from Lemma 3.5(a) that

$$\delta d_\delta(G) \geq \sum_{v_j \in \mathcal{A}(G)} d_\delta(N_G(v_j)) = \sum_{w_j \in \mathcal{A}(H)} d_\delta(N_H(w_j)) \geq (\delta + 1 + b - n) d_\delta(H) - d_\delta(\mathcal{A}(H)).$$

This implies inequality (10). Inequality (11) follows similarly, noting that $d_\delta(\mathcal{A}(G)) = 0$ by Corollary 3.3(d). □

Corollary 3.7. Let $\psi = d_\delta(H) - d_\delta(G)$ and let $b = b(G, H)$. Then

$$(\delta + 1) d_{\delta+1}(G) \geq \sum_{w_j \in \mathcal{A}(H)} d_{\delta+1}(N_H(w_j)) + \psi b - d_{\delta+1}(\mathcal{A}(G)), \tag{12}$$

$$(\delta + 1) d_{\delta+1}(H) \geq \sum_{v_j \in \mathcal{A}(G)} d_{\delta+1}(N_G(v_j)) - \psi b + d_{\delta+1}(\mathcal{A}(G)). \tag{13}$$

Proof By Corollary 3.3(d), $d_G(v_j) \geq \delta+1$ for all $v_j \in \mathcal{A}(G)$. So, since $|\mathcal{A}(G)| = b$, on summing $d_{\delta+1}(N_G(v_j))$ for all $v_j \in \mathcal{A}(G)$ using Lemma 3.5(b) and (c), it follows that

$$\sum_{v_j \in \mathcal{A}(G)} d_{\delta+1}(N_G(v_j)) = \sum_{w_j \in \mathcal{A}(H)} d_{\delta+1}(N_H(w_j)) + \psi b - d_{\delta+1}(\mathcal{A}(G)).$$

Inequality (12) follows from this equation on using inequality (2) with $\alpha = \delta+1$ and $S = \mathcal{A}(G)$. Inequality (13) follows similarly on using inequality (2) with $\alpha = \delta + 1$ and $S = \mathcal{A}(H)$. □

Lemma 3.8. Let $\psi = d_\delta(H) - d_\delta(G)$. Suppose that $\psi \geq 1$. Then

$$b(G,\,H) \leq \left\lfloor \tfrac{(\delta+2)(n+\psi)}{\psi+\delta+2} \right\rfloor \leq \left\lfloor \tfrac{(\delta+2)(n+1)}{\delta+3} \right\rfloor.$$

Proof Let $b = b(G,\,H)$. By Corollary 3.3(a) and (d),

$$d_{\delta+1}(\mathcal{A}(G)) = d_\delta(\mathcal{A}(H)) \leq d_\delta(H) = d_\delta(G) + \psi = d_\delta(\overline{\mathcal{A}(G)}) + \psi.$$

Hence

$$d_{\delta+1}(G) \leq d_{\delta+1}(\overline{\mathcal{A}(G)}) + d_\delta(\overline{\mathcal{A}(G)}) + \psi \leq |\overline{\mathcal{A}(G)}| + \psi = (n-b) + \psi.$$

So, since $\psi b \leq (\delta+2)d_{\delta+1}(G)$ by (12), it follows that $\psi b \leq (\delta+2)(n-b+\psi)$. Thus $(\psi+\delta+2)b \leq (\delta+2)(n+\psi)$, yielding the first inequality. The second inequality follows since $n \geq \delta+2$, as otherwise $G \cong H \cong K_n$. □

Corollary 3.9. Suppose that $d_\delta(H) > d_\delta(G)$.

(a) If $\delta = 0$, then $b(G,\,H) \leq \left\lfloor \frac{2}{3}(n+1) \right\rfloor$, and $b(G,\,H) \leq n-3$ when $n \geq 9$.

(b) If $\delta = 1$ then $b(G,\,H) \leq \left\lfloor \frac{3}{4}(n+1) \right\rfloor$, and $b(G,\,H) \leq n-3$ when $n \geq 12$.

(c) If $\delta \leq 6$ then $b(G,\,H) \leq \left\lfloor \frac{8}{9}(n+1) \right\rfloor$, and $b(G,\,H) \leq n-3$ when $n \geq 27$.

□

Lemma 3.10. Let $\psi = d_\delta(H) - d_\delta(G)$. Suppose that $\psi \leq -1$ and $d_\delta(\mathcal{A}(H)) = 0$. Then

$$b(G,\,H) \leq \left\lfloor \tfrac{(\delta+1)(n+\delta+3-\psi)}{\delta+1-\psi} \right\rfloor \leq \left\lfloor \tfrac{(\delta+1)(n+\delta+4)}{\delta+2} \right\rfloor.$$

Proof Let $b = b(G,\,H)$. If $d_{\delta+1}(\mathcal{A}(H)) = 0$, then $-\psi b \leq (\delta+1)d_{\delta+1}(\overline{\mathcal{A}(H)}) \leq (\delta+1)(n-b)$ by (13); thus $b \leq \left\lfloor \frac{(\delta+1)n}{\delta+1-\psi} \right\rfloor \leq \left\lfloor \frac{(\delta+1)(n+\delta+3-\psi)}{\delta+1-\psi} \right\rfloor$. So suppose that $d_{\delta+1}(\mathcal{A}(H)) \geq 1$. Hence $d_H(w_1) = \delta+1$, and thus $d_G(v_1) = \delta+2$ by Lemma 3.1. Therefore, by Lemma 3.5(c), $d_{\delta+1}(N_G(v_1)) - d_{\delta+1}(N_H(w_1)) = \psi$. On putting $\alpha = \delta+1 = d_H(w_1)$ in (4) and $i = \delta+1 \neq d_G(v_1)$ in (5), it follows that

$$d_{\delta+1}(G) = d_{\delta+1}(H) + d_{\delta+2}(N_H(w_1)) - d_{\delta+2}(N_G(v_1)) + \psi - 1 \geq d_{\delta+1}(H) - (\delta+3-\psi), \quad (14)$$

as $d_{\delta+2}(N_G(v_1)) \leq d_G(v_1) = \delta+2$. Since $d_{\delta+1}(G) = d_{\delta+1}(\overline{\mathcal{A}(G)})$ by Corollary 3.3(b), it follows from (14) that

$$n - b = |\overline{\mathcal{A}(G)}| \geq d_{\delta+1}(\overline{\mathcal{A}(G)}) = d_{\delta+1}(G) \geq d_{\delta+1}(H) - (\delta+3-\psi). \quad (15)$$

From (13), we see that $(\delta+1)d_{\delta+1}(H) \geq -\psi b$. Combining this with (15) then gives $n - b \geq -\frac{\psi b}{\delta+1} - (\delta+3-\psi)$. Thus $(\delta+1-\psi)b \leq (\delta+1)(n+\delta+3-\psi)$, yielding the first inequality. The second inequality follows immediately as $n+\delta+3 > \delta+1$. □

Corollary 3.11. Suppose that $d_\delta(G) > d_\delta(H)$ and $d_\delta(\mathcal{A}(H)) = 0$.

(a) If $\delta = 0$ then $b(G,\,H) \leq \left\lfloor \frac{1}{2}(n+4) \right\rfloor$, and $b(G,\,H) \leq n-3$ when $n \geq 9$.

(b) If $\delta = 1$ then $b(G,\,H) \leq \left\lfloor \frac{2}{3}(n+5) \right\rfloor$, and $b(G,\,H) \leq n-3$ when $n \geq 17$.

□

We note that, although these bounds could be improved, they are sufficient for our purposes.

4 The case when $b(G, H) = n - 2$

We now prove the main result of this paper: if $n \geq 29$ and $b(G, H) \geq n - 2$, then $|E(G)| = |E(H)|$. We first state the results of Myrvold and Woodall. The first theorem is a restatement of Theorem 2.3 in [9]. The second follows from Theorem 1.3(b) in [14] (this result was also proved for $n \geq 9$ by Kocay, Ramachandran, Monikandan and Balakumar [7] [11] [10]).

Theorem 4.1. Let G and H be graphs of order n, where $n \geq 7$ and $b(G, H) \geq n - 1$. Then $|E(G)| = |E(H)|$. □

Theorem 4.2. Let G and H be graphs of order n, where $n \geq 10$ and $b(G, H) = n - 2$. Then $\big||E(G)| - |E(H)|\big| \leq 1$. □

We assume from now on that $|E(G)| = |E(H)| + 1$ *and* $b(G, H) = n - 2$. We shall show that $n \leq 28$ under these assumptions. Our main result will then follow from Theorems 4.1 and 4.2. We let $\overline{\mathcal{A}(G)} = \{x_1,\ x_2\}$ and $\overline{\mathcal{A}(H)} = \{y_1,\ y_2\}$, where we assume that $d_G(x_1) \leq d_G(x_2)$ and $d_H(y_1) \leq d_H(y_2)$, without loss of generality. We recall from Lemma 3.1 that $d_G(v_j) = d_H(w_j)+1$ for all j.

Since $\overline{H} - w_j \cong \overline{G} - v_j$ for all j, we may assume that $\mathcal{A}(\overline{H}) = \mathcal{A}(H)$ and $\mathcal{A}(\overline{G}) = \mathcal{A}(G)$. So $\overline{\mathcal{A}(\overline{H})} = \{y_1, y_2\}$ and $\overline{\mathcal{A}(\overline{G})} = \{x_1, x_2\}$, where we note that $d_{\overline{H}}(y_2) \leq d_{\overline{H}}(y_1)$ and $d_{\overline{G}}(x_2) \leq d_{\overline{G}}(x_1)$. Furthermore, since $|E(\overline{H})| = |E(\overline{G})| + 1$, we make frequent use of the following **complementarity principle**:

For any result for G and H, we may deduce a corresponding result for $\overline{H}$ and $\overline{G}$ by replacing all occurrences of G, H, δ, Δ, x_1, x_2, y_1 and y_2 by $\overline{H}$, $\overline{G}$, $n-1-\Delta$, $n-1-\delta$, y_2, y_1, x_2 and x_1, respectively.

Lemma 4.3. $d_H(y_1) + d_H(y_2) - d_G(x_1) - d_G(x_2) = n - 4$.

Proof This follows immediately from Lemma 3.4. □

We first show that $\delta \geq 2$ under our assumptions.

Lemma 4.4. Suppose that $n \geq 9$. Then $1 \leq \delta \leq \Delta \leq n - 2$.

Proof Suppose that $\delta = 0$. If $d_0(H) > d_0(G)$ then $b(G, H) \leq n - 3$ by Corollary 3.9(a), which is impossible. Hence $d_0(H) \leq d_0(G)$. Now, if $d_H(w_1) = \delta = 0$, then Lemma 3.5(b) implies that $d_1(N_G(v_1)) = d_0(H) - d_0(G) - 1 < 0$, which is impossible. Thus $d_0(\mathcal{A}(H)) = 0$. It now follows from Corollary 3.11(a) that $b(G, H) \leq n - 3$ when $d_0(G) > d_0(H)$. Therefore $d_0(G) = d_0(H)$. Hence $G \cong G_1 \cup K_1$ and $H \cong H_1 \cup K_1$, where G_1 and H_1 are non-isomorphic graphs of order $n-1$. Now $d_0(\mathcal{A}(G)) = 0$ by Corollary 3.3(d). Since we also have $d_0(\mathcal{A}(H)) = 0$, clearly $b(G_1, H_1) = b(G, H) = n - 2$. So $|E(G_1)| = |E(H_1)|$ by Theorem 4.1. However, since $|E(G_1)| = |E(G)| = |E(H)| + 1 = |E(H_1)| + 1$, this is impossible. Therefore $\delta \geq 1$. By the complementarity principle, it immediately follows that $n - 1 - \Delta \geq 1$, and thus $\Delta \leq n - 2$. □

We recall the definition of $E_{\alpha\beta}(G)$ from (1), and that $e_{\alpha\beta}(G) = |E_{\alpha\beta}(G)|$. These will be used frequently in several of the remaining proofs.

Lemma 4.5. Suppose that $n \geq 16$ and $\delta = 1$. Then $d_1(H) \leq d_1(G)$ and $d_1(\mathcal{A}(H)) = 0$.

Proof If $d_1(H) > d_1(G)$, then $b(G, H) \leq n-3$ by Corollary 3.9(b), which is impossible. Hence $d_1(H) \leq d_1(G)$. We now assume that $d_1(\mathcal{A}(H)) \geq 1$ and derive a contradiction.

Since $d_1(G) = d_1(\overline{\mathcal{A}(G)})$ by Corollary 3.3(d), we have

$$1 \leq d_1(\mathcal{A}(H)) \leq d_1(H) \leq d_1(G) = d_1(\overline{\mathcal{A}(G)}) \leq 2. \tag{16}$$

So $d_G(x_1) = 1$ and $d_H(w_1) = 1$, and thus $d_G(v_1) = 2$ by Lemma 3.1. Hence, by Lemma 3.5(b),

$$d_2(N_G(v_1)) + (d_1(G) - d_1(H)) + 1 = d_2(N_H(w_1)) \leq d_H(w_1) = 1.$$

This implies that $d_2(N_G(v_1)) = 0$, $d_1(G) = d_1(H)$ and $d_2(N_H(w_1)) = 1$. Therefore, let $N_H(w_1) = \{t\}$ and $N_H(t) = \{w_1, t'\}$, and thus $e_{11}(H) = 0$ as $d_1(H) \leq 2$ by (16). In addition, it follows from Corollary 3.3(b) and (16) that

$$d_2(G) = d_1(\mathcal{A}(H)) + d_2(\overline{\mathcal{A}(G)}) \leq d_1(\overline{\mathcal{A}(G)}) + d_2(\overline{\mathcal{A}(G)}) \leq 2. \tag{17}$$

Hence $e_{22}(G) = 0$ as $d_2(N_G(v_1)) = 0$.

Case (a): $e_{12}(G) = 0$.

Let $S = \{v_j \in \mathcal{A}(G) \mid e_{12}(G - v_j) \geq 1\}$. From (6) with $\alpha = 1$, we have $|S| \leq 2d_2(G) + 3e_{13}(G)$. Hence $|S| \leq 10$, since $d_2(G) \leq 2$ by (17) and $e_{13}(G) \leq d_1(G) \leq 2$ by (16). Now let $T = \{w_j \in \mathcal{A}(H) \mid e_{12}(H - w_j) \geq 1\}$. Since $w_1 t \in E_{12}(H)$, clearly $\mathcal{A}(H) \setminus \{w_1, t, t'\} \subseteq T$, so $|T| \geq n - 5$. Since $|T| = |S|$, this implies that $n \leq 15$, which is impossible.

Case (b): $e_{12}(G) \geq 1$.

Without loss of generality, we assume that $d_2(N_G(x_1)) = 1$. Now, since $e_{11}(H) = 0$, it follows that $d_1(N_H(w_j)) = 0$ if $d_H(w_j) = 1$. So, by Lemmas 3.1 and 3.5(a), $d_1(N_G(v_j)) = 0$ if $d_G(v_j) = 2$. Since $d_2(N_G(x_1)) = 1$, this implies that $E_{12}(G) = \{x_1x_2\}$, and thus $D_1(G) = \{x_1\}$. Therefore, $D_1(H) = \{w_1\}$ by (16) and $D_2(G) = \{v_1, x_2\}$ by (17). Moreover, $d_1(N_G(v_j)) = 0$ for all j, so it follows from Lemma 3.5(a) that t is either y_1 or y_2. By Lemma 4.3, $d_H(y_1) + d_H(y_2) = n - 1$. Now $d_H(t) = 2$ and $d_H(y_1) \leq d_H(y_2)$, so t must be y_1, and therefore $d_H(y_1) = 2$ and $d_H(y_2) = n - 3$. Clearly, y_1 is the unique leaf of $H - w_1$; so $G - v_1$ also has a unique leaf, namely x_1. Now $v_1x_2 \notin E(G)$ since $e_{22}(G) = 0$. So $D_2(N_{G-v_1}(x_1)) = \{x_2\}$, and hence $d_2(N_{H-w_1}(y_1)) = 1$. Thus $d_H(t') = 2$, so t' is not y_2. It therefore follows that, without loss of generality, we may assume that t' is w_2, i.e., $N_H(y_1) = \{w_1, w_2\}$. Furthermore, since $d_H(y_2) = n - 3$, we have $N_H(y_2) = V(H) \setminus \{w_1, y_1, y_2\}$ and $N_H(w_2) = \{y_1, y_2\}$. This implies that y_2 is the unique vertex at distance 2 from the unique leaf y_1 of $H - w_1$. Now $w_1y_1 \in E_{11}(H - w_2)$, and thus $e_{11}(G - v_2) \geq 1$. So, since $D_1(G) = \{x_1\}$, $E_{12}(G) = \{x_1x_2\}$ and $e_{22}(G) = 0$, clearly $v_2x_2 \in E(G)$. This implies that v_2 is the unique vertex at distance 2 from the unique leaf x_1 of $G - v_1$. However, $d_G(v_2) = 3$ by Lemma 3.1, so $d_{G-v_1}(v_2) \leq 3$. This is impossible as $d_{H-w_1}(y_2) = n - 3 \geq 13$. Therefore $d_1(\mathcal{A}(H)) = 0$. □

Corollary 4.6. Suppose that $n \geq 16$ and $\Delta = n - 2$. Then $d_1(\overline{G}) \leq d_1(\overline{H})$ and $d_1(\mathcal{A}(\overline{G})) = 0$.

Proof This follows immediately by the complementarity principle. □

Lemma 4.7. Suppose that $n \geq 17$. Then $2 \leq \delta \leq \Delta \leq n - 3$.

Proof By Lemma 4.4, it follows that $1 \leq \delta \leq \Delta \leq n - 2$. We now assume that $\delta = 1$ and show that this leads to a contradiction.

By Lemma 4.5, $d_1(H) \leq d_1(G)$ and $d_1(\mathcal{A}(H)) = 0$, so $d_1(H) = d_1(\overline{\mathcal{A}(H)})$. Now, if $d_1(G) > d_1(H)$ then $b(G, H) \leq n - 3$ by Corollary 3.11(b), which is impossible; hence $d_1(G) = d_1(H)$. So, by Corollary 3.3(d),

$$d_1(\overline{\mathcal{A}(G)}) = d_1(G) = d_1(H) = d_1(\overline{\mathcal{A}(H)}).$$

Therefore $d_G(x_1) = d_H(y_1) = 1$ as $\delta = 1$; moreover, $d_G(x_2) = 1$ if and only if $d_H(y_2) = 1$. Hence $d_H(y_2) = n - 4 + d_G(x_2)$ by Lemma 4.3, and it follows that $d_G(x_2) = 2$ and $d_H(y_2) = n - 2$, as $\Delta \le n-2$. Thus $\Delta = n-2$, and hence $\min(\delta(\overline{H}), \delta(\overline{G})) = 1$. By Corollary 4.6, $d_1(\mathcal{A}(\overline{G})) = 0$. So $d_1(\overline{G}) = 0$ as $d_{\overline{G}}(x_1) = n-2$ and $d_{\overline{G}}(x_2) = n-3$. However, since $d_{\overline{H}}(y_2) = 1$, we have $d_1(\overline{H}) > d_1(\overline{G})$. By applying Corollary 3.11(b) to the pair $\overline{H}$ and $\overline{G}$, we see that $b(\overline{H}, \overline{G}) \le n-3$, which is impossible since $b(\overline{H}, \overline{G}) = b(G, H)$. Therefore $\delta \ge 2$. By the complementarity principle, it follows that $n - 1 - \Delta \ge 2$, and hence $\Delta \le n-3$. □

Lemma 4.8. Suppose that $n \ge 17$. Then $d_H(y_1) \ge \delta + 1$ and $D_\delta(H) = D_\delta(\mathcal{A}(H))$.

Proof By Lemmas 4.3 and 4.7, $d_H(y_1) \ge (n-4) - (n-3) + 2\delta = 2\delta - 1 \ge \delta + 1$. Since $d_H(y_2) \ge d_H(y_1)$, it immediately follows that $D_\delta(H) = D_\delta(\mathcal{A}(H))$. □

We note that Lemma 4.8 implies that if $\delta(H) = \delta$ then $d_H(w_1) = \delta$.

Theorem 4.9. Let G and H be graphs of order n, where $n \ge 27$. Suppose that $|E(G)| = |E(H)| + 1$ and $b(G, H) = n-2$. Then $d_\delta(G) = d_\delta(H)$ and $d_\Delta(G) = d_\Delta(H)$.

Proof We first note that $2 \le \delta \le \Delta \le n-3$ by Lemma 4.7, and $d_\delta(G) \le 2$ by Corollary 3.3(e). We show that both of the cases $d_\delta(H) > d_\delta(G)$ and $d_\delta(G) > d_\delta(H)$ lead to contradictions, from which it will follow that $d_\delta(G) = d_\delta(H)$. By applying the complementarity principle, it will then follow that $d_\Delta(G) = d_{n-1-\Delta}(\overline{G}) = d_{n-1-\Delta}(\overline{H}) = d_\Delta(H)$.

Suppose first that $d_\delta(H) > d_\delta(G)$. Then, by (10), $\delta d_\delta(G) \ge (\delta-2)d_\delta(H) \ge (\delta-2)(d_\delta(G)+1)$, so $\delta \le 2 + 2d_\delta(G) \le 6$. This contradicts Corollary 3.9(c).

Suppose instead that $d_\delta(G) > d_\delta(H)$, so $d_\delta(H) \le 1$. Then, by (11), $\delta d_\delta(H) \ge (\delta-1)d_\delta(G) \ge (\delta-1)(d_\delta(H)+1)$, so $\delta \le 1 + d_\delta(H) \le 2$. It immediately follows that $\delta = 2$, $d_2(H) = 1$ and $d_2(G) = 2$. Therefore $D_2(G) = \{x_1, x_2\}$ by Corollary 3.3(d), and $D_2(H) = \{w_1\}$ by Lemma 4.8. So $d_G(v_1) = 3$ by Lemma 3.1, and thus $D_3(G) = \{v_1\}$ by Corollary 3.3(b).

By Lemma 3.5(a), $d_2(N_G(v_j)) = d_2(N_H(w_j))$ for all j. Since $D_2(H) = \{w_1\}$, it follows that $d_2(N_G(v_1)) = 0$, and thus $e_{23}(G) = 0$; moreover, $d_2(N_G(v_j)) \le 1$ for all j. Now, if $x_1x_2 \notin E(G)$, there would be four distinct vertices $v_j \in \mathcal{A}(G)$ such that $d_2(N_G(v_j)) = 1$ and, therefore, four corresponding vertices $w_j \in \mathcal{A}(H)$ such that $d_2(N_H(w_j)) = 1$. This is clearly impossible as $d_2(H) = 1$; hence x_1x_2 must be in $E(G)$. In addition, since $d_2(N_G(v_1)) = 0$, it follows that $x_1x_2 \in E_{22}(G - v_1)$ and therefore $e_{22}(H - w_1) \ge 1$. As $D_2(H) = \{w_1\}$, this implies that $N_H(w_1) = \{t, t'\}$, where $d_H(t) = d_H(t') = 3$ and $tt' \in E(H)$.

Let $S = \{v_j \in \mathcal{A}(G) \mid e_{23}(G - v_j) \ge 1\}$. Since $e_{23}(G) = 0$, we have $|S| \le 3d_3(G) + 4e_{24}(G)$ by (6). Now $e_{24}(G) \le 2$, as $D_2(G) = \{x_1, x_2\}$ and $x_1x_2 \in E(G)$. Hence $|S| \le 11$, as $d_3(G) = 1$. Now let $T = \{w_j \in \mathcal{A}(H) \mid e_{23}(H - w_j) \ge 1\}$. As both w_1t and w_1t' are in $E_{23}(H)$, clearly $\mathcal{A}(H) \setminus (N_H(t) \cup N_H(t')) \subseteq T$; so $|T| \ge n-7$. Since $|S| = |T|$, this implies that $n \le 18$, contradicting the hypothesis. □

Corollary 4.10. Suppose that $n \ge 27$. Then $\delta(G) = \delta(H) = \delta$ and $\Delta(G) = \Delta(H) = \Delta$. □

Lemma 4.11. Suppose that $n \ge 27$. Then

$$2 \le \delta = d_G(x_1) \le d_G(x_2) < d_H(y_1) \le d_H(y_2) = \Delta \le n-3. \tag{18}$$

Proof Since $d_\delta(G) = d_\delta(\overline{\mathcal{A}(G)})$ and $d_\Delta(H) = d_\Delta(\overline{\mathcal{A}(H)})$ by Corollary 3.3(d), it follows from Corollary 4.10 that $d_G(x_1) = \delta$ and $d_H(y_2) = \Delta$. So $d_H(y_1) - d_G(x_2) = n - 4 - \Delta + \delta \ge 1$ by Lemmas 4.3 and 4.7. The other inequalities in (18) follow from Lemma 4.7. □

Corollary 4.12. Suppose that $n \geq 27$. Then $d_H(w_1) = \delta$, $d_G(v_1) = \delta + 1$, $d_G(v_{n-2}) = \Delta$ and $d_H(w_{n-2}) = \Delta - 1$. □

For vertices $s \in V(G)$ and $t \in V(H)$, we define $N_G^{\mathcal{A}}(s) = N_G(s) \cap \mathcal{A}(G)$ and $N_H^{\mathcal{A}}(t) = N_H(t) \cap \mathcal{A}(H)$, respectively.

Lemma 4.13. Suppose that $n \geq 27$ and $d_\delta(G) = d_\delta(H) = 1$. Then $D_\delta(G) = \{x_1\}$ and $D_\delta(H) = \{w_1\}$. In addition, either $e_{\delta\delta+1}(G) \geq 1$ or $e_{\delta+1\delta+1}(G) \geq 1$.

Proof Since $d_\delta(G) = d_\delta(H) = 1$, it follows that $D_\delta(G) = D_\delta(\overline{\mathcal{A}(G)}) = \{x_1\}$ by Lemma 4.11, and $D_\delta(H) = D_\delta(\mathcal{A}(H)) = \{w_1\}$ by Corollary 4.12. This implies that $d_{\delta+1}(G) \leq 2$ by Corollary 3.3(b).

Suppose now that $e_{\delta\delta+1}(G) = e_{\delta+1\delta+1}(G) = 0$. Then, since $e_{\delta\delta}(G) = 0$, it follows from Lemma 2.4 that $e_{\delta-1\delta}(G - v_j) = e_{\delta\delta}(G - v_j) = 0$ for all v_j. Now $d_{\delta+1}(N_H(w_1)) \geq 1$ by Lemma 3.5(b), so $e_{\delta\delta+1}(H) \geq 1$, and thus there exists $t \in D_{\delta+1}(N_H(w_1))$. Furthermore, since $e_{\delta-1\delta}(H - w_j) = e_{\delta\delta}(H - w_j) = 0$ for all w_j, it is easy to see that $N_H^{\mathcal{A}}(t) = \{w_1\}$. Since $\delta \geq 2$ by (18), this implies that $\delta = 2$ and $N_H(t) = \{w_1, y_1, y_2\}$.

Let $S = \{v_j \mid e_{23}(G - v_j) \geq 1\}$. Since $e_{23}(G) = 0$, we have $|S| \leq 3d_3(G) + 4e_{24}(G)$ by (6). Hence $|S| \leq 14$, as $d_3(G) \leq 2$ and $d_2(G) = 1$. Now let $T = \{w_j \mid e_{23}(H - w_j) \geq 1\}$. As $w_1t \in E_{23}(H)$, clearly $\mathcal{A}(H) \setminus (N_H^{\mathcal{A}}(w_1) \cup N_H^{\mathcal{A}}(t)) \subseteq T$, so $|T| \geq n - 5$. As $|S| = |T|$, this implies that $n \leq 19$, which is impossible. Therefore, either $e_{\delta\delta+1}(G) \geq 1$ or $e_{\delta+1\delta+1}(G) \geq 1$. □

Corollary 4.14. Suppose that $n \geq 27$ and $d_\delta(G) = d_\delta(H) = 1$. Then $D_{\delta+1}(G) = \{v_1, x_2\}$.

Proof By Lemma 4.13 and Corollary 3.3(a), $D_\delta(G) = \{x_1\}$ and $d_{\delta+1}(\mathcal{A}(G)) = d_\delta(\mathcal{A}(H)) = 1$. So $D_{\delta+1}(\mathcal{A}(G)) = \{v_1\}$ and thus $D_{\delta+1}(G) \subseteq \{v_1, x_2\}$. Suppose that $d_G(x_2) \geq \delta + 2$. Then $e_{\delta+1\delta+1}(G) = 0$, so $e_{\delta\delta+1}(G) \geq 1$ by Lemma 4.13. This implies that $D_\delta(N_G(v_1)) = \{x_1\}$, and thus $d_\delta(N_H(w_1)) = 1$ by Lemma 3.5(a). This is impossible since $D_\delta(H) = \{w_1\}$ by Lemma 4.13. Hence $d_G(x_2) = \delta + 1$. □

Lemma 4.15. Suppose that $n \geq 27$. Then $n - 4 \leq 2(\Delta - \delta) \leq n - 2$.

Proof By Lemma 4.11, $d_G(x_1) = \delta$ and $d_H(y_2) = \Delta$. Now, if $d_G(x_2) > d_G(x_1)$ then $d_\delta(H) = d_\delta(G) = 1$ by Theorem 4.9 and Corollary 3.3(d); so $d_G(x_2) = \delta + 1$ by Corollary 4.14. Hence $d_G(x_2) \in \{\delta, \delta + 1\}$. It immediately follows from the complementarity principle that $d_{\overline{H}}(y_1) \in \{n - 1 - \Delta, n - \Delta\}$, and thus $d_H(y_1) \in \{\Delta, \Delta - 1\}$. Since $(\Delta - \delta) + d_H(y_1) - d_G(x_2) = n - 4$ by Lemma 4.3, the result now follows easily. □

Lemma 4.16. Suppose that $n \geq 27$, $d_{\delta+1}(G) = 2$ and $\delta \geq d_\delta(H) + 6$. Then $d_{\delta+1}(H) = d_{\delta+1}(\mathcal{A}(H)) = 2$.

Proof If $d_H(y_1) \leq \delta + 1$ then $d_H(y_1) = \delta + 1$ and $d_G(x_1) = d_G(x_2) = \delta$ by (18). However, this would imply that $d_H(y_2) = n - 5 + \delta$ by Lemma 4.3, which is impossible since $\delta \geq 6$. So $d_H(y_1) \geq \delta + 2$, and therefore $d_{\delta+1}(H) = d_{\delta+1}(\mathcal{A}(H))$. We now show that $d_{\delta+1}(H) = 2$.

Suppose first that $d_{\delta+1}(H) \leq 1$. Then, by (13) and Theorem 4.9,

$$\delta + 1 \geq \sum_{v_j \in \mathcal{A}(G)} d_{\delta+1}(N_G(v_j)) + d_{\delta+1}(\mathcal{A}(G)). \tag{19}$$

Putting $\alpha = \delta + 1$ and $S = \mathcal{A}(G)$ in inequality (3), we see that the right hand side of (3) reduces to $\delta d_{\delta+1}(G) - d_{\delta+1}(\mathcal{A}(G))$. Combining inequalities (3) and (19) then implies that $\delta + 1 \geq \delta d_{\delta+1}(G) = 2\delta$, contradicting the bound on δ.

So suppose instead that $d_{\delta+1}(H) \geq 3$. By (12), Theorem 4.9 and Corollary 3.3(a),

$$2(\delta+1) \geq \sum_{w_j \in \mathcal{A}(H)} d_{\delta+1}(N_H(w_j)) - d_\delta(\mathcal{A}(H)). \tag{20}$$

Using inequality (3) for H with $\alpha = \delta + 1$ and $S = \mathcal{A}(H)$, together with inequality (20), it follows that

$$2(\delta+1) \geq \delta d_{\delta+1}(H) - d_{\delta+1}(\mathcal{A}(H)) - d_\delta(\mathcal{A}(H)) \geq 3(\delta-1) - d_\delta(H).$$

Again, this contradicts the assumption that $\delta \geq d_\delta(H) + 6$. Therefore $d_{\delta+1}(H) = 2$. □

We note that $1 \leq d_\delta(G) \leq 2$ by Theorem 4.9 and Corollary 3.3(e). To complete the proof of the upper bound on n, we find upper bounds on δ when $d_\delta(G) = 1$ and when $d_\delta(G) = 2$. The complementarity principle then gives corresponding lower bounds on Δ. These bounds, together with Lemma 4.15, will then yield the required upper bound on n.

We define $\Lambda_\alpha(G) = E_{\alpha\alpha}(G) \cup E_{\alpha\alpha+1}(G) \cup E_{\alpha+1\alpha+1}(G)$, as in Lemma 2.4, and let $\lambda_\alpha(G) = |\Lambda_\alpha(G)|$.

Lemma 4.17. Suppose that $n \geq 27$ and $d_\delta(G) = d_\delta(H) = 1$. Then $\delta \leq 6$.

Proof We assume that $\delta \geq 7$ and obtain a contradiction.

Clearly, $e_{\delta\delta}(G) = e_{\delta\delta}(H) = 0$. Also, by Lemma 4.13 and Corollary 4.14, $D_\delta(G) = \{x_1\}$, $D_\delta(H) = \{w_1\}$ and $D_{\delta+1}(G) = \{v_1, x_2\}$. Therefore, since $\delta \geq 7$, it follows from Lemma 4.16 that $D_{\delta+1}(H) = D_{\delta+1}(\mathcal{A}(H)) = \{w_2, w_3\}$, and hence $D_{\delta+2}(G) = D_{\delta+2}(\mathcal{A}(G)) = \{v_2, v_3\}$ by Corollary 3.3(a). We summarise the degrees of the vertices of G and H in Table 1.

v_1	v_2	v_3	$v_4, \ldots, v_{n-2}$	x_1	x_2
$\delta+1$	$\delta+2$	$\delta+2$	$\geq \delta+3$	δ	$\delta+1$
w_1	w_2	w_3,	$w_4, \ldots, w_{n-2}$	y_1	y_2
δ	$\delta+1$	$\delta+1$	$\geq \delta+2$	$\geq \delta+2$	$\geq \delta+2$

Table 1: The degrees of the vertices of G and H in Lemma 4.17.

We note that $D_{\delta+1}(N_G(v_1)) \subseteq \{x_2\}$ and $D_\delta(N_H(w_1)) = \emptyset$. So, by Lemma 3.5(a), $d_\delta(N_G(v_1)) = 0$, which implies that $v_1x_1 \notin E(G)$.

Case (a): $v_1x_2 \in E(G)$.

In this case, $D_{\delta+1}(N_G(v_1)) = \{x_2\}$. So it follows from Lemma 3.5(b) that $d_{\delta+1}(N_H(w_1)) = d_{\delta+1}(N_G(v_1)) + 1 = 2$. Therefore $D_{\delta+1}(N_H(w_1)) = \{w_2, w_3\}$, so $E_{\delta\delta+1}(H) = \{w_1w_2, w_1w_3\}$. We recall from Lemma 2.4 that, $\Lambda_{\delta-1}(G-v_j) \subseteq \Lambda_\delta(G)$ for all v_j. So $\Lambda_{\delta-1}(G-v_j) \subseteq \{v_1x_2, x_1x_2\}$ for all v_j, since $v_1x_1 \notin E(G)$.

Now let $S = \{v_j \in \mathcal{A}(G) \mid \lambda_{\delta-1}(G - v_j) \geq 1\}$. Clearly, $S \subseteq N_G(x_2)$, so $|S| \leq \delta + 1$. Now let $T = \{w_j \in \mathcal{A}(H) \mid \lambda_{\delta-1}(H - w_j) \geq 1\}$. Since $E_{\delta\delta+1}(H) = \{w_1w_2, w_1w_3\}$, it is easy to see that $(N_H^{\mathcal{A}}(w_2) \cup N_H^{\mathcal{A}}(w_3)) \setminus \{w_1\} \subseteq T$, so $|T| \geq 2(\delta-1) - |N_H^{\mathcal{A}}(w_2) \cap N_H^{\mathcal{A}}(w_3)| - 1$. We shall show that $N_H^{\mathcal{A}}(w_2) \cap N_H^{\mathcal{A}}(w_3) = \{w_1\}$, hence $|T| \geq 2\delta - 4$. As $|S| = |T|$, it will immediately follow that $\delta \leq 5$, contradicting our assumption that $\delta \geq 7$.

Suppose first that $x_1x_2 \notin E(G)$. Then $\Lambda_{\delta-1}(G - v_j) \subseteq \{v_1x_2\}$ for all v_j, so $\lambda_{\delta-1}(H - w_j) \leq 1$ for all w_j. Clearly, if there exists $w_k \in (N_H^{\mathcal{A}}(w_2) \cap N_H^{\mathcal{A}}(w_3)) \setminus \{w_1\}$, then $\{w_1w_2, w_1w_3\} \subseteq \Lambda_{\delta-1}(H - w_k)$. This is impossible, so it immediately follows that $N_H^{\mathcal{A}}(w_2) \cap N_H^{\mathcal{A}}(w_3) = \{w_1\}$.

Suppose, on the other hand, that $x_1x_2 \in E(G)$. Then $E_{\delta\delta}(G-v_1) = \{x_1x_2\}$ since $v_1x_2 \in E(G)$, by hypothesis, and $v_1x_1 \notin E(G)$. So $e_{\delta\delta}(H-w_1) = 1$, which implies that $w_2w_3 \in E_{\delta+1\delta+1}(H)$. Thus, if there exists $w_k \in (N_H^{\mathcal{A}}(w_2) \cap N_H^{\mathcal{A}}(w_3)) \setminus \{w_1\}$, then $\{w_1w_2,\, w_1w_3,\, w_2w_3\} \subseteq \Lambda_{\delta-1}(H-w_k)$. However, this is impossible since $\Lambda_{\delta-1}(G-v_j) \subseteq \{v_1x_2,\, x_1x_2\}$ for all v_j, so it again follows that $N_H^{\mathcal{A}}(w_2) \cap N_H^{\mathcal{A}}(w_3) = \{w_1\}$.

Case (b): $v_1x_2 \notin E(G)$.

In this case, $D_{\delta+1}(N_G(v_1)) = \emptyset$. Thus $d_{\delta+1}(N_H(w_1)) = 1$ by Lemma 3.5(b). Therefore, since $D_{\delta+1}(H) = \{w_2,\, w_3\}$, without loss of generality we may assume that $D_{\delta+1}(N_H(w_1)) = \{w_2\}$, so $E_{\delta\delta+1}(H) = \{w_1w_2\}$. This implies that $d_\delta(N_H(w_3)) = 0$, so $d_\delta(N_G(v_3)) = 0$ by Lemma 3.5(a). Hence $v_3x_1 \notin E(G)$.

Now $e_{\delta+1\delta+1}(G) = 0$ as $v_1x_2 \notin E(G)$, so it follows from Lemma 4.13 that $e_{\delta\delta+1}(G) \geq 1$. Hence $x_1x_2 \in E(G)$ as $v_1x_1 \notin E(G)$. In addition, by Lemma 3.5(a), $d_\delta(N_G(v_2)) = d_\delta(N_H(w_2)) = 1$, so $v_2x_1 \in E(G)$. Now $e_{\delta-1\delta}(H-w_2) = 0$, so $e_{\delta-1\delta}(G-v_2) = 0$, and therefore $v_2x_2 \notin E(G)$. It follows that $x_1x_2 \in E_{\delta-1\delta+1}(G-v_2)$, so $e_{\delta-1\delta+1}(H-w_2) \geq 1$. This implies that there exists $t \in D_{\delta+2}(N_H(w_1) \cap N_H(w_2))$, so $w_1w_2tw_1$ is a cycle in H. Since $\delta \geq 3$, there must also exist $w_k \in N_H^{\mathcal{A}}(t) \setminus \{w_1,\, w_2\}$. It follows that $w_1w_2tw_1$ is a cycle in $H-w_k$, each vertex of which has degree at most $\delta+1$. Since $G-v_k \cong H-w_k$, there must exist such a cycle in $G-v_k$. Clearly, the vertices of this latter cycle are contained in $\{v_1,\, v_2,\, v_3,\, x_1,\, x_2\}$. Since G contains none of the edges v_1x_1, v_1x_2, v_2x_2 or v_3x_1, it is straightforward to check that the only possible such cycle is $v_1v_2v_3v_1$. Hence $d_{\delta+1}(N_G(v_3)) \geq 1$. Necessarily, $d_{\delta+1}(N_H(w_3)) \leq 1$, so it follows from Lemma 3.5(c) that $d_{\delta+1}(N_G(v_3)) = d_{\delta+1}(N_H(w_3)) = 1$, and thus $v_3x_2 \notin E(G)$ and $w_2w_3 \in E(H)$. However, this implies that $E_{\delta\delta}(G-v_3) = \emptyset$, which is impossible since $w_1w_2 \in E_{\delta\delta}(H-w_3)$. □

Lemma 4.18. Suppose that $n \geq 27$ and $d_\delta(G) = d_\delta(H) = 2$. Then $\delta \leq 7$.

Proof We assume that $\delta \geq 8$ and obtain a contradiction.

Since $d_\delta(G) = d_\delta(H) = 2$, it easy to see, using Corollary 3.3(d) and (18) that $D_\delta(G) = D_\delta(\overline{\mathcal{A}(G)}) = \{x_1,\, x_2\}$ and $D_\delta(H) = D_\delta(\mathcal{A}(H)) = \{w_1,\, w_2\}$. So, by Corollary 3.3(a), $D_{\delta+1}(G) = \{v_1,\, v_2\}$. Therefore, since $\delta \geq 8$, it follows from Lemma 4.16 that $D_{\delta+1}(H) = D_{\delta+1}(\mathcal{A}(H)) = \{w_3,\, w_4\}$, and hence $D_{\delta+2}(G) = D_{\delta+2}(\mathcal{A}(G)) = \{v_3,\, v_4\}$ by Corollary 3.3(a). We summarise the degrees of the vertices of G and H in Table 2.

v_1	v_2	v_3	v_4	$v_5, \ldots, v_{n-2}$	x_1	x_2
$\delta+1$	$\delta+1$	$\delta+2$	$\delta+2$	$\geq \delta+3$	δ	δ
w_1	w_2	w_3	w_4	$w_5, \ldots, w_{n-2}$	y_1	y_2
δ	δ	$\delta+1$	$\delta+1$	$\geq \delta+2$	$\geq \delta+2$	$\geq \delta+2$

Table 2: The degrees of the vertices of G and H in Lemma 4.18.

Now $d_\delta(N_G(v_j)) = d_\delta(N_H(w_j))$ for all j, by Lemma 3.5(a). In addition, by Lemma 3.5(b),

$$d_{\delta+1}(N_H(w_1)) = d_{\delta+1}(N_G(v_1)) + 1 \qquad \text{and} \qquad d_{\delta+1}(N_H(w_2)) = d_{\delta+1}(N_G(v_2)) + 1. \tag{21}$$

We may therefore assume, without loss of generality, that $w_1w_3 \in E(H)$, so $d_\delta(N_H(w_3)) \geq 1$. If $d_\delta(N_H(w_3)) = 1$ then $w_2w_3 \notin E(H)$, so (21) implies that $D_{\delta+1}(N_H(w_2)) = \{w_4\}$ and $d_{\delta+1}(N_G(v_2)) = 0$. Thus

$$d_\delta(N_H(w_3)) = 1 \Rightarrow D_{\delta+1}(N_H(w_2)) = \{w_4\} \text{ and } e_{\delta+1\delta+1}(G) = 0. \tag{22}$$

Case (a): $w_1w_2 \notin E(H)$, so $e_{\delta\delta}(H) = 0$.

In this case, $d_\delta(N_G(v_1)) = d_\delta(N_H(w_1)) = 0$ and $d_\delta(N_G(v_2)) = d_\delta(N_H(w_2)) = 0$. Therefore $e_{\delta\delta+1}(G) = 0$.

(a)(i): $x_1x_2 \notin E(G)$, so $e_{\delta\delta}(G) = 0$.
Now $e_{\delta\delta}(G - v_1) = 0$ as $e_{\delta\delta+1}(G) = 0$, so $e_{\delta\delta}(H - w_1) = 0$. It immediately follows that $w_2w_3 \notin E(H)$, and thus $d_\delta(N_H(w_3)) = 1$. Therefore $e_{\delta+1\delta+1}(G) = 0$ by (22), and thus $\lambda_\delta(G) = 0$. Hence $\lambda_{\delta-1}(G - v_j) = 0$ for all v_j by Lemma 2.4. Now, since $\delta \geq 3$, there exists some $w_k \in N_H^{\mathcal{A}}(w_3) \setminus \{w_1\}$. However, it is easy to see that $w_1w_3 \in \Lambda_{\delta-1}(H - w_k)$ for any such w_k, which is impossible since $\lambda_{\delta-1}(G - v_k) = 0$.

(a)(ii): $x_1x_2 \in E(G)$, so $e_{\delta\delta}(G) = 1$.
By the hypothesis of case (a), $e_{\delta-1\delta-1}(H - w_3) = 0$, so $e_{\delta-1\delta-1}(G - v_3) = 0$, and hence $d_\delta(N_G(v_3)) \leq 1$. Therefore $d_\delta(N_H(w_3)) = 1$, as $w_1w_3 \in E(H)$ and $d_\delta(N_H(w_3)) = d_\delta(N_G(v_3))$. So $D_{\delta+1}(N_H(w_2)) = \{w_4\}$ and $e_{\delta+1\delta+1}(G) = 0$, by (22). It immediately follows that $d_{\delta+1}(N_G(v_1)) = 0$, so $d_{\delta+1}(N_H(w_1)) = 1$ by (21). Hence $w_1w_4 \notin E(H)$, and thus $e_{\delta\delta}(H - w_1) = 0$. However, this is impossible since $d_\delta(N_G(v_1)) = 0$, which implies that $x_1x_2 \in E_{\delta\delta}(G - v_1)$.

Case (b): $w_1w_2 \in E(H)$, so $e_{\delta\delta}(H) = 1$.

In this case, $d_\delta(N_G(v_1)) = d_\delta(N_H(w_1)) = 1$ and similarly $d_\delta(N_G(v_2)) = d_\delta(N_H(w_2)) = 1$. Therefore $e_{\delta\delta+1}(G) = 2$.

(b)(i): $x_1x_2 \notin E(G)$, so $e_{\delta\delta}(G) = 0$.
Now $e_{\delta-1\delta-1}(G - v_j) = 0$ for all v_j, so $e_{\delta-1\delta-1}(H - w_j) = 0$ and thus $d_\delta(N_H(w_j)) \leq 1$ for all w_j. Hence $d_\delta(N_H(w_3)) = 1$, so $D_{\delta+1}(N_H(w_2)) = \{w_4\}$ by (22). Therefore $D_\delta(N_H(w_4)) = \{w_2\}$, which implies that $D_{\delta+1}(N_H(w_1)) = \{w_3\}$.

Let $S = \{v_j \in \mathcal{A}(G) \mid \lambda_{\delta-1}(G - v_j) \geq 2\}$. Since $e_{\delta\delta}(G) = 0$ and $d_\delta(N_G(v_1)) = d_\delta(N_G(v_2)) = 1$, it is easy to see that $S = N_G^{\mathcal{A}}(v_1) \cap N_G^{\mathcal{A}}(v_2)$, so $|S| \leq \delta$. Now let $T = \{w_j \in \mathcal{A}(H) \mid \lambda_{\delta-1}(H - w_j) \geq 2\}$. Since $\{w_1w_2,\, w_1w_3,\, w_2w_4\} \subseteq E(H)$, it is clear that $(N_H^{\mathcal{A}}(w_3) \cup N_H^{\mathcal{A}}(w_4)) \setminus \{w_1,\, w_2\} \subseteq T$, so $|T| \geq 2(\delta - 1) - |N_H^{\mathcal{A}}(w_3) \cap N_H^{\mathcal{A}}(w_4)| - 2$.

Suppose there exists $w_k \in N_H^{\mathcal{A}}(w_3) \cap N_H^{\mathcal{A}}(w_4)$. Now w_k is not w_1 or w_2, since $d_{\delta+1}(N_H(w_1)) = d_{\delta+1}(N_H(w_2)) = 1$. So $\{w_1w_2,\, w_1w_3,\, w_2w_4\} \subseteq \Lambda_{\delta-1}(H - w_k)$. However, since $e_{\delta\delta}(G) = 0$ by hypothesis (b)(i) and $e_{\delta+1\delta+1}(G) = 0$ by (22), it follows that $\lambda_\delta(G) = e_{\delta\delta+1}(G) = 2$. Since $\lambda_{\delta-1}(G - v_j) \leq \lambda_\delta(G)$ for all v_j by Lemma 2.4, this contradicts the fact that $\lambda_{\delta-1}(H - w_k) \geq 3$. Hence $N_H^{\mathcal{A}}(w_3) \cap N_H^{\mathcal{A}}(w_4) = \emptyset$, and therefore $|T| \geq 2\delta - 4$. As $|S| = |T|$, this implies that $\delta \leq 4$, contradicting our assumption that $\delta \geq 8$.

(b)(ii): $x_1x_2 \in E(G)$, so $e_{\delta\delta}(G) = 1$.
Since $d_\delta(N_G(v_1)) = 1$, we have $e_{\delta-1\delta}(G - v_1) \geq 1$, so $e_{\delta-1\delta}(H - w_1) \geq 1$, which implies that w_3 or w_4 is in $N_H(w_1) \cap N_H(w_2)$. It follows that $w_1w_2w_3w_1$ or $w_1w_2w_4w_1$ is a cycle in H. Without loss of generality, we assume that $w_1w_2w_4w_1$ is a cycle in H.

For any graph F, let $C(F)$ be the set of cycles of length three in F in which each vertex on the cycle has degree at most δ. Now $\delta \geq 4$, so $d_H(w_4) \geq 5$, and thus there must exist some $w_k \in N_H^{\mathcal{A}}(w_4) \setminus \{w_1,\, w_2\}$. Hence $w_1w_2w_4w_1 \in C(H - w_k)$, and therefore $C(G - v_k) \neq \emptyset$. Clearly, the vertices of any cycle in $C(G - v_k)$ must be contained in $\{v_1,\, v_2,\, x_1,\, x_2\}$. However, since $d_\delta(N_G(v_1)) = d_\delta(N_G(v_2)) = 1$, no cycle in $C(G - v_k)$ can contain the edge x_1x_2. It therefore follows that $v_k \in N_G(v_1) \cap N_G(v_2)$, and that $C(G - v_k)$ contains precisely one cycle, namely $v_1v_2x_1v_1$ or $v_1v_2x_2v_1$.

It follows that $v_1v_2 \in E(G)$, hence $d_{\delta+1}(N_G(v_1)) = d_{\delta+1}(N_G(v_2)) = 1$. So $d_{\delta+1}(N_H(w_1)) = d_{\delta+1}(N_H(w_2)) = 2$ by (21), and thus $\{w_3, w_4\} \subseteq N_H(w_1) \cap N_H(w_2)$. Now $d_{\delta+1}(N_G(v_k)) = 2$, so $d_{\delta+1}(N_H(w_k)) = 2$ by Lemma 3.5(c), and hence $w_k \in N_H(w_3) \cap N_H(w_4)$. However, this implies that both $w_1w_2w_3w_1$ and $w_1w_2w_4w_1$ are cycles in $C(H - w_k)$, contradicting the fact that $|C(G - v_k)| = 1$. □

Combining the above results leads us to our main theorem.

Theorem 4.19. Let G and H be graphs of order n, where $n \geq 29$ and $b(G, H) \geq n-2$. Then $|E(G)| = |E(H)|$.

Proof Suppose that $|E(G)| \neq |E(H)|$. By Theorems 4.1 and 4.2, it is sufficient to consider the case when $b(G, H) = n - 2$ and $|E(G)| = |E(H)| + 1$, and show that this leads to a contradiction. In this case, $1 \leq d_\delta(H) = d_\delta(G) \leq 2$ by Theorem 4.9 and Corollary 3.3(e). Therefore $\delta \leq 7$, by Lemmas 4.17 and 4.18. It now follows from the complementarity principle that $n - 1 - \Delta \leq 7$, and thus $n - 15 \leq \Delta - \delta$. So, since $2(\Delta - \delta) \leq n - 2$ by Lemma 4.15, we have $n \leq 28$. This contradicts the premise that $n \geq 29$, so $|E(G)| = |E(H)|$. □

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