Knowledge Base Exchange: The Case of OWL 2 QL

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Abstract

In this article, we define and study the problem of exchanging knowledge between a source and a target knowledge base (KB), connected through mappings. Differently from the traditional database exchange setting, which considers only the exchange of data, we are interested in exchanging implicit knowledge. As representation formalism we use Description Logics (DLs), thus assuming that the source and target KBs are given as a DL TBox+ABox, while the mappings have the form of DL TBox assertions. We define a general framework of KB exchange, and study the problem of translating the knowledge in the source KB according to the mappings expressed in OWL 2 QL, the profile of the standard Web Ontology Language OWL 2 based on the description logic *DL-Lite*$_{\mathcal{R}}$. We develop novel game- and automata-theoretic techniques, and we provide complexity results that range from NLogSpace to ExpTime.

Keywords: Description Logic, Knowledge Exchange, DL-Lite, Data Exchange, Query Inseparability

1. Introduction

Ontologies are at the heart of various Computer Science disciplines, among which the most prominent ones are Semantic Web, Biomedical informatics, and of course, Artificial Intelligence and Knowledge Representation. Here, for simplicity, by *ontology* we mean a formal representation of the knowledge about a domain in terms of concepts (unary predicates) and roles (binary predicates). In the biomedical domain, e.g., *Pneumonia* and *Lung* could be concepts, and *finding_site* could be a role, and the knowledge about the domain could be asserted in an axiom of the form *"The finding site of pneumonia is lungs"* [1, 2]. The advantages of using ontologies are that, on the one hand, they provide a framework for organizing and structuring information, and on the other hand, they are equipped with capabilities to reason about concepts and roles.

When representing the knowledge about a domain of interest in terms of an ontology, on the one hand the designer is free to choose the formalism in which to express the ontology, among a variety of different alternatives (e.g., a relational database possibly with constraints, Datalog, or Description Logics). On the other hand, she can select the specific terminology she considers more appropriate to convey the domain semantics. For instance, when creating a biomedical ontology about deseases, the lungs can be modeled as *Pair_of_lungs* or *Both_lungs*. This leads to having complex forms of information, maintained in different formats and organized according to different structures. Often, this information needs to be shared between agents: to reuse the existing ontologies, to integrate knowledge from different agents, and so on. Therefore in recent years, both in the data management and in the knowledge representation communities, several settings have been investigated that address this problem from various perspectives: *(i)* in *information integration*, uniform access is provided to a collection of data sources by means of an ontology (or global schema) to which the sources are mapped [3]; *(ii)* in *peer-to-peer systems*, a set of peers declaratively linked to each other collectively provide access to the information assets they maintain [4, 5, 6]; *(iii)* in *ontology matching*, the aim

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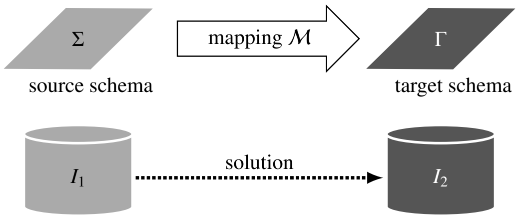


Figure 1: Data Exchange Framework.

is to understand and derive the correspondences between elements in two ontologies [7, 8, 9]; *(iv)* in *ontology modularity*, the aim is to extract independent, possibly small, subsets of an ontology, so called modules [10, 11, 12]; *(v)* in *knowledge translation*, axioms are being translated from one representation (i.e., logical language and vocabulary) into another [13, 14, 15]; and, finally, *(vi)* in *data exchange*, the information stored according to a source schema needs to be restructured and translated so as to conform to a target schema [16, 17]. The work we present in this article is inspired by this latter setting investigated in databases.

Data exchange is a field of database theory, motivated by several applications from industry [18, 19], that deals with transferring data between differently structured databases. In the seminal article [16], the data exchange problem was defined as the problem of transforming data structured under a source schema into data structured under a target schema, given a mapping specifying how to translate data from the source to the target schema. This problem is depicted in Figure 1, where the obtained target data instance is referred to as a *solution*. The data exchange problem has been studied for different combinations of languages used to specify the source schema, the target schema and the mapping [17, 20, 21]. Most of the results in the literature consider source-to-target tuple generating dependencies (tgds) as the language to specify mappings. The dependencies in this class allow one to express containment of conjunctive queries: if a conjunction of several predicates holds, then a conjunction of some other predicates must hold as well. For example, the tgd

$$\forall a, b \, . \, \mathit{AuthorOf}(a, b) \rightarrow \exists y, g \, . \, \mathit{BookInfo}(b, a, y) \wedge \mathit{BookGenre}(b, g) \tag{1}$$

says that if a is the author of a book b, then there exist y and g such that b is a book with author a that was published in year y, and b has genre g. Many database integrity constraints can be expressed by tgds, so these dependencies have been widely used in databases. Source-to-target tgds (st-tgds) are tgds of a special shape: the conjunction on the left-hand side uses only symbols from a source schema, while the conjunction on the right-hand side uses only symbols from a target schema.

A fundamental assumption in the (traditional) data exchange framework is that the source is a complete database: every fact is either true or false. On the other hand, a target instance can be incomplete and a source instance can have many different solutions, as incomplete information can be introduced by the mapping layer (see also [22]).

Example 1.1. If we consider the mapping consisting of the constraint (1), and a source instance consisting of one entry $\mathit{AuthorOf}(\mathsf{tolkien}, \mathsf{lotr})$, encoding that Tolkien is the author of 'The Lord of the Rings', then the following two target instances, I_2 and I'_2, are solutions:

$$\begin{aligned} I_2 &= \{\mathit{BookInfo}(\mathsf{lotr}, \mathsf{tolkien}, 1937),\ \mathit{BookGenre}(\mathsf{lotr}, \mathsf{fantasy})\}, \\ I'_2 &= \{\mathit{BookInfo}(\mathsf{lotr}, \mathsf{tolkien}, \textsc{null}_1),\ \mathit{BookGenre}(\mathsf{lotr}, \textsc{null}_2)\}. \end{aligned}$$

Note that here incompleteness is caused by the existential restriction $\exists y, g \ldots$, which can be satisfied by introducing new objects: either named individuals (or constants), like $\mathsf{fantasy}$, or anonymous objects, like $\textsc{null}_1$. Note also that $\textsc{null}_1$ and $\textsc{null}_2$ are *labeled nulls*, which are widely used in databases to represent anonymous objects. ■

To characterize *good* transformations, several criteria have been considered [23]. We emphasize two types of good translations, *universal solutions* and *query solutions*. Universal solutions are the most general solutions: any other solution is more specific (I'_2 in Example 1.1 is a universal solution), while query solutions are good solutions from the point of view of answering target queries, i.e., queries formulated over the target schema.

Data exchange with incomplete information. As mentioned before, in the (traditional) data exchange framework, source instances are assumed to contain complete information. However, there are natural scenarios where source instances may contain incomplete information, as shown in [24, 25, 21]. In particular, the problem of data exchange with incomplete source data was extensively studied in [25], where an incomplete specification is understood as an object with (possibly infinitely) many interpretations. A simple example of such an object is a database with nulls: assume that we have a table storing information about book genres, and that 'The Lord of The Rings' is a book whose genre is unkown. In this case, the table would consist of an entry of the form $BookGenre(\mathsf{lotr}, \textsc{null})$, which represents all different instances containing a concrete value for the genre of 'The Lord of The Rings': $BookGenre(\mathsf{lotr}, \mathsf{fantasy})$, $BookGenre(\mathsf{lotr}, \mathsf{history})$, $BookGenre(\mathsf{lotr}, \mathsf{scifi})$, etc.

A knowledge base is another example of an object with multiple interpretations. A *knowledge base (KB)* is a description of a domain of interest that includes two kinds of information: *(i) ground facts*, i.e., extensional information of the form *"John is a student"*, *"Databases is a course"*, *"John attends the Databases course"*, etc., which assert properties of individual objects that are part of the domain; and *(ii) logical axioms*, i.e., intensional information of the form *"Every course must be taught by somebody"*, *"A student cannot be a professor"*, etc., which structure the knowledge about the domain. We also call the second type of information an *ontology*. It is implicit in the standard semantics of a KB that the knowledge it describes is only a partial description of a domain of interest, which means that the KB represents many actual states of the world. For instance, if we consider the KB consisting of the five axioms mentioned above, then it could represent one possible state of the world, where John also attends the Statistics course, David teaches Databases and Peter teaches Statistics. The framework proposed in [25] is general enough to accommodate KBs. In fact, a general *knowledge exchange framework* is proposed in [25], for the case where the source is a KB as opposed to a relational database. Moreover, it is shown in that work that some natural problems (such as query answering over the target schema) become undecidable if KBs are specified by tgds and mappings are specified by source-to-target tgds. Thus, some decidability results are obtained in [25] by considering some restricted fragments of the class of tgds when specifying KBs.

An alternative to the approach proposed in [25] to achieve decidability is to consider less expressive ontological languages when specifying both KBs and mappings. A good candidate for that role is the formalism of Description Logics, which come in variants that provide fair expressive power, and at the same time possess good computational properties.

Description Logics as ontology language. Description Logics (DLs) [26] are a family of formal languages, more precisely, fragments of first-order logic, that are specifically designed to serve as ontology languages. They exhibit a reasonable tradeoff between their expressive power and the computational complexity of logical inference tasks. Nice computational properties in DLs are achieved by restricting attention to unary and binary predicates, called *concepts* and *roles*, respectively, and to restricted forms of axioms. Ground facts in DLs are encoded in the form of an *ABox*, which is a set of membership assertions, and logical axioms are encoded in the form of a *TBox*, which is a set of concept and role inclusions. For instance, the DL KB containing the five axioms describing the university domain mentioned before looks as follows:

$Student(\mathsf{john})$ $Course(\mathsf{databases})$ $attends(\mathsf{john}, \mathsf{databases})$	$Course \sqsubseteq \exists teaches^-$ $Student \sqsubseteq \neg Professor$

Notice that both inclusions above are between concepts.

Thus, the starting point for our work is the knowledge exchange framework defined in [25], and the main motivation is to find ontology and mapping specification languages where the fundamental problem of *knowledge base exchange* can be solved, and which are both natural and useful in practice. For this purpose, we focus on the Description Logic underlying OWL 2 QL, which is the profile [27] of the standard Web Ontology Language OWL 2 [28] that has been specifically designed for efficient query answering. Next we describe our contributions in this respect.

Our contributions. First, we propose and develop a framework for KB exchange based on DLs; both source and target are KBs constituted by a DL TBox, representing intensional information, and an ABox, representing extensional information, and mappings are sets of DL concept and role inclusions. We then specialize this framework to the case of lightweight DLs of the *DL-Lite* family [29]. In particular, we consider $\textit{DL-Lite}_{\mathcal{R}}$, which is the logic underlying the OWL 2 QL profile of OWL 2. In this framework, we are interested in three types of solutions: universal solutions,

Universal solutions	Membership	Non-emptiness
simple ABoxes	PTime-complete (Th. 6.10)	PTime-complete (Th. 6.11)
extended ABoxes	NP-complete (Th. 6.14)	PSpace-hard (Lem. 6.15), in ExpTime (Th. 6.19)

Universal UCQ-solutions	Membership	Non-emptiness
simple ABoxes	ExpTime-complete ([30, Th. 45])	in ExpTime ([30])
extended ABoxes	ExpTime-complete ([30, Th. 46])	PSpace-hard (Lem. 6.16)

UCQ-representations	Membership	Non-emptiness
	NLogSpace-complete (Th. 7.6)	NLogSpace-complete (Th. 7.17)

Table 1: Complexity results for the membership and non-emptiness problems.

universal UCQ-solutions, and UCQ-representations. *Universal solutions* are the most precise solutions: a target KB $\mathcal{K}_t$ is a universal solution for a source KB $\mathcal{K}_s$ under a mapping $\mathcal{M}$ if it preserves all the interpretations of $\mathcal{K}_s$ with respect to $\mathcal{M}$. *Universal* UCQ-*solutions* is a relaxation of the notion of universal solutions: a target KB $\mathcal{K}_t$ is a universal UCQ-solution for a source KB $\mathcal{K}_s$ under a mapping $\mathcal{M}$ if it preserves all answers to unions of conjunctive queries (UCQs) formulated over the target signature. UCQ-*representations* are similar to universal UCQ-solutions, but they do not depend on the source ABox, only on the source TBox and the mapping: a target TBox $\mathcal{T}$ is a UCQ-representation of a source TBox $\mathcal{S}$ under a mapping $\mathcal{M}$ if for each possible source ABox $\mathcal{A}_s$, it holds that $\mathcal{T}$, $\mathcal{M}$, and $\mathcal{A}_s$ give the same answers to UCQs as $\mathcal{S}$, $\mathcal{M}$, and $\mathcal{A}_s$. The rationale behind the notion of UCQ-representation is to maximize the implicit knowledge translated to the target. Thus, a UCQ-representation of a source TBox captures at best the intensional information that can be extracted from this source TBox according to a mapping and using UCQs.

Second, we study each one of the three notions of solution just described, and their relationship to each other for the case of KBs and mappings defined using *DL-Lite*$_\mathcal{R}$. We provide examples that justify the need for target ABoxes with labeled nulls in order for universal solutions and universal UCQ-solutions to exist, as the language of *DL-Lite*$_\mathcal{R}$ is capable of implying the existence of new objects. Such ABoxes mentioning anonymous objects are called *extended ABoxes*, as opposed to *simple ABoxes*, which mention only named individuals (or constants).

Finally, in order to obtain a good understanding of the knowledge base exchange problem, we study the computational complexity of the *membership* and *non-emptiness* problems for universal solutions, universal UCQ-solutions and UCQ-representations. For universal solutions (resp., universal UCQ-solutions), the membership problem verifies, given a source KB $\mathcal{K}_s$, a mapping $\mathcal{M}$, and a target KB $\mathcal{K}_t$, whether $\mathcal{K}_t$ is a universal solution (resp., universal UCQ-solution) for $\mathcal{K}_s$ under $\mathcal{M}$; instead, the non-emptiness problem addresses the question whether there exists a universal solution (resp., universal UCQ-solution) for a given source KB $\mathcal{K}_s$ and a given mapping $\mathcal{M}$. For UCQ-representations, the membership problem verifies, given a source TBox $\mathcal{S}$, a mapping $\mathcal{M}$, and a target TBox $\mathcal{T}$, whether $\mathcal{T}$ is a UCQ-representation for $\mathcal{S}$ under $\mathcal{M}$; instead, the non-emptiness problem addresses the question whether there exists a UCQ-representation for a given source TBox $\mathcal{S}$ and a given mapping $\mathcal{M}$, that is, whether $\mathcal{S}$ is UCQ-representable under $\mathcal{M}$. Notice that the non-emptiness problem is directly related to the task of materializing a translation; moreover, determining UCQ-*representability* is a crucial task, since it allows one to use the obtained target TBox to infer new knowledge in the target, thus reducing the amount of extensional information to be transferred from the source.

The complexity results obtained in this article (for *DL-Lite*$_\mathcal{R}$) are summarized in Table 1, where we also mentioned the theorems and lemmas where the results are proved. For universal solutions with simple ABoxes, we show that both the membership and the non-emptiness problems are PTime-complete, where the upper bound is obtained by considering infinite games on graphs with the reachability acceptance condition, for which it is known that the problem of finding a winning strategy is in PTime. Then, for universal solutions with extended ABoxes, we prove that the membership problem is NP-complete, while the non-emptiness problem is PSpace-hard, and provide for the latter an ExpTime upper bound based on a novel approach exploiting two-way alternating tree automata. For UCQ-representations, we show that both the membership and non-emptiness problems are NLogSpace-complete, the key condition for this low complexity being the fact that UCQ-representations do not depend on the shape of ABoxes. As for universal UCQ-solutions, the main results have been established in [30], where it has been shown that the membership problem (both for simple and for extended ABoxes) is ExpTime-complete. The upper bound immediately provides also an ExpTime algorithm for solving the non-emptiness problem with simple ABoxes [30]. For extended ABoxes, we prove instead a PSpace lower bound, which does not carry over to simple ABoxes.

It should be noticed that in the non-emptiness problem mentioned before, the target signature is assumed to be part of the input. Thus, the constructed solutions (i.e., universal solutions, universal UCQ-solutions and UCQ-representations) are not allowed to use any new concept or role symbols not included in the given target signature. The problem of allowing additional symbols in these constructions is certainly interesting and worth investigating in the future. However it is a different problem from the one we are studying here. In fact, the problem we are investigating is a natural one, fully in line with the work done in data exchange [16, 17, 20, 25, 21]. Moreover, there are several reasons why it may be undesirable or even impossible to allow for additional concepts or roles in the target. First, the target signature might be given and not under control of the user, therefore it might not be extensible. Second, there might be privacy issues that prevent the use of all the information in a source KB, so only the information about some concepts and roles have to be displayed. This problem can be viewed as a knowledge exchange problem where the target signature stores the symbols to be displayed, and which cannot include some new concepts or roles. Third, a source signature might be very large, hence the user would like to switch to a smaller target signature. In this case, it is not desirable to add new symbols that can make the target signature to grow. Finally, an instance of data exchange could be part of the more general problem of schema evolution [31, 32], where one needs to consider a sequence of several instances of data exchange. In this context, allowing for keeping existing symbols or adding new symbols at each step, might result in an unacceptable (and undesired) growth of the signature.

Organization of the article. The rest of the article is structured as follows. We start with related work in Section 2, and then we provide in Section 3 the preliminary notions and terminology needed in the rest of the article. In Section 4, we introduce our knowledge base exchange framework: we formally define the three notions of solution, and we set up the space of computational complexity-related problems that we consider. Section 5 gives some intuition and basic results about each kind of solution, and provides several examples about these notions. Then, the complexity results and the technical development are presented in Section 6 for universal solutions, and in Section 7 for UCQ-representations. Finally, we provide in Section 8 some concluding remarks. Detailed proofs of many of the results are provided in an appendix, so as to ease the presentation in the main body of the article.

2. Related Work

Data exchange, including the case with incomplete information, which is the most important area related to our work, has already been discussed in the introduction. Below we discuss other related areas.

Knowledge Translation. The problem of *knowledge translation* was addressed in [13] with the goal of formalizing the task of reusing/sharing existing encoded knowledge in the process of the development of new intelligent systems. This problem had emerged already in the early nineties, and in [33] an interlingua-based methodology for this problem was proposed, where logical theories encoded in one representation (source) are translated to another representation (target). *Interlingua* is a mediating first-order logic based language designed for communicating knowledge between the source and the target representations, where a *representation* is formed using a declarative language, a vocabulary, and a base theory (associated with the language). In [34, 35, 36], the authors devised a formalism for producing translations based on a theory of contexts; a translation is specified as a set of first-order logic sentences, each of which describes a rule for deriving a formula in a target output context that is a translation of a formula in a source input context. Such an approach, first, provides a formal semantics for translation, and second, enables translations to be computed by standard theorem provers.

A decade later there has been a revival of interest in knowledge translation in the context of the Semantic Web, where the problem of communicating knowledge between heterogeneous agents is especially relevant [14, 37, 15]. The focus of these works is to translate axioms represented in a rule-based formalism, where the mapping axioms, that is, the axioms defining how the source and target vocabularies are related, are represented in a simple fragment of first-order logic. In this context, algorithms for translating axioms have been developed and implemented.

While the first work [13] gives a rather abstract and high-level view on the problem of knowledge translation, the more recent contributions [14, 37, 15] are more on the practical side and lack solid theoretical foundations. Thus, none of these results provides a precise understanding of the complexity of the problems related to translating knowledge.

Data and Information Integration. A problem closely related to data exchange is that of *data integration*, which is concerned with the task of combining data coming from a variety of heterogeneous sources [3, 38, 39, 40, 41]. The main aim is to provide a uniform view of these data so that users can query and access them in an integrated way. This problem is relevant in many real-world applications, both in commercial and scientific domains [42]. The problem of data integration is addressed by defining a *global schema* (i.e., a schema available to the user) and mappings between the schemas of the data sources and the global schema. While the combination of the schemas of the sources to be integrated naturally corresponds to the source schema in data exchange, the global schema plays the role of the target schema.

Information integration has also been studied under the assumption that the global schema is expressed by means of an ontology, which provides a layer that captures the semantics of the domain of interest and that helps to overcome the semantic heterogeneity of the data sources [43, 44]. In fact, the problem of integration has also been considered when applied to ontologies themselves, i.e., when the sources to be integrated are incompletely specified, in terms of logical constraints encoded in an ontology [45].

Although the data and ontology integration settings bear similarity to the one we are studying here, the techniques developed there are not applicable towards our goals, due to the difference in focus between information integration and exchange: while in information integration the aim is to query the source through the target via the mappings, possibly without materializing any data at the target, the aim of exchange is precisely to understand which data to materialize and how to do this efficiently.

Ontology and Knowledge Base Maintenance. There are various scenarios where one ontology or KB needs to be compared against another or against its own part. On one hand, this occurs when an ontology was updated and the update needs to be verified. On the other hand, *modularization* (or module extraction) aims at splitting a given ontology into smaller sub-ontologies, each of which can be used autonomously, when only a subset of the ontology signature is of interest [10, 12, 46, 47]. Such sub-ontologies are called *modules*, and since they are typically of a small size (compared to the entire ontology, which can be very large), it is easier to understand them and perform reasoning with them. Another mechanism to extract information relevant to a subset of the ontology signature, is *uniform interpolation*, also known as *forgetting* [48, 49, 50]. As opposed to modules, uniform interpolates are not restricted to subsets of the original ontology, but can be arbitrary sets of axioms over the restricted signature that at best capture the semantics. It is important to observe that, in general, the restriction to a smaller signature can lead to a much larger ontology [49].

In the Description Logics domain, ontology modularity and uniform interpolation rely on the notion of inseparability for a signature Σ, or Σ-*inseparability*, as a main technical tool. This notion has been studied for expressive DLs [51, 47, 52] and for Horn variants of DLs [53, 54, 55, 30]. Two major forms of inseparability have been considered in the literature. First, two KBs are said to be Σ-model inseparable, if every model of one of these KBs can be extended to a model of the other one in such a way that they agree on the symbols from Σ, and vice-versa. In other words, these KBs cannot be logically distinguished in the signature Σ. The second notion is query-based: two KBs are Σ-query inseparable if they give the same answers to all queries formulated over Σ. So intuitively, such KBs cannot be distinguished as far as answering queries formulated over Σ is concerned. This work is relevant for our investigation, as the notions of Σ-model and Σ-query inseparability are tightly related to some of the concepts studied in this paper. We formally define these notions in Section 3, and make these connections precise in Section 4.

3. Preliminaries

3.1. The Description Logic *DL-Lite*$_\mathcal{R}$

In this work we are concerned with OWL 2 QL, which is grounded on the lightweight DLs of the *DL-Lite* family [29]. Such DLs are characterized by the fact that conjunctive query answering is first-order rewritable and that standard reasoning can be done in polynomial time. Specifically, the formal counterpart of OWL 2 QL is *DL-Lite*$_\mathcal{R}$, for which we present now syntax and semantics.

Let N_C, N_R, N_a, N_ℓ be pairwise disjoint countably infinite sets of *concept names*, *role names*, *constants*, and *labeled nulls*, respectively. Assume in the following that $A \in N_C$ and $P \in N_R$; in *DL-Lite*$_\mathcal{R}$, B and C are used to denote basic and arbitrary (or complex) concepts, respectively, and R and Q are used to denote basic and arbitrary (or complex) roles, respectively, which are defined as follows:

$$R ::= P \mid P^- \qquad B ::= A \mid \exists R$$
$$Q ::= R \mid \neg R \qquad C ::= B \mid \neg B$$

From now on, for a basic role R, we use R^- to denote P^- when $R = P$, and P when $R = P^-$.

A TBox, usually denoted $\mathcal{O}$, is a finite set of *concept inclusions* $B \sqsubseteq C$ and *role inclusions* $R \sqsubseteq Q$ encoding relevant domain knowledge. We call an inclusion of the form $B_1 \sqsubseteq \neg B_2$ or $R_1 \sqsubseteq \neg R_2$ a *disjointness axiom*. An ABox $\mathcal{A}$ is a finite set of *membership assertions* $B(a)$, $R(a, b)$, where $a, b \in N_a$, indicating which individuals belong to the concepts and how they are related by the roles in the ontology. We use $\mathsf{ind}(\mathcal{A})$ to denote the set of constants appearing in $\mathcal{A}$.

Example 3.1. We define now an ontology PhotoCamera about digital photo cameras, underlying the structure of an electronics selling website. Specifically, we want to capture the fact that DSLR (digital single lens reflex) cameras have exchangeable lenses, and that there are different types of connectors between the camera and the lens, which are called *mounts*. For instance, some camera manufacturers have proprietary mounts, which allow one to connect to a camera only lenses of that manufacturer. Instead other manufacturers adopt standard mounts, e.g., the Micro Four Thirds system, that work across camera and lens models of different manufacturers. We define first a TBox $\mathcal{O}_{\mathsf{cam}}$ introducing some concepts and roles that are relevant for this domain. For clarity, we use strings beginning with capital letters to denote concepts, and strings beginning with lowercase letters to denote roles. The concept *DigitalCamera* denotes digital cameras, while *DSLRCamera* denotes digital reflex cameras. *ExchangeLens* denotes exchange lenses that can be mounted onto DSLR cameras through lens mounts, which in turn are grouped together in the concept *Mount*. The role *cameraMounts* relates DSLR cameras to their mounts, and hence has *Mount* as its range. The role *lensMounts* relates exchange lenses to their mounts, and its domain is *ExchangeLens*. Moreover, we require that every *Mount* is the mount of some *ExchangeLens* to which it is connected via the inverse of the role *lensMounts*. This knowledge is captured by the following *DL-Lite*$_\mathcal{R}$ TBox $\mathcal{O}_{\mathsf{cam}}$ of the ontology PhotoCamera:

$$\begin{array}{r}
\mathit{DSLRCamera} \sqsubseteq \mathit{DigitalCamera}, \quad \mathit{DSLRCamera} \sqsubseteq \exists \mathit{cameraMounts}, \quad \exists \mathit{cameraMounts}^- \sqsubseteq \mathit{Mount} \\
\mathit{Mount} \sqsubseteq \exists \mathit{lensMounts}^-, \quad \exists \mathit{lensMounts} \sqsubseteq \mathit{ExchangeLens}
\end{array}$$

The ABox $\mathcal{A}_{\mathsf{cam}} = \{\mathit{DSLRCamera}(\mathsf{canon5d})\}$ of PhotoCamera simply introduces an instance of a DSLR camera. ■

In this paper, we also consider extended ABoxes, which are obtained by allowing labeled nulls in membership assertions. Formally, an *extended ABox* is a finite set of membership assertions $B(u)$ and $R(u, v)$, where $u, v \in (N_a \cup N_\ell)$. Moreover, a(n *extended*) *KB* $\mathcal{K}$ is a pair $\langle \mathcal{O}, \mathcal{A} \rangle$, where $\mathcal{O}$ is a TBox and $\mathcal{A}$ is an (extended) ABox. When we need to emphasize the distinction between ABoxes and extended ABoxes, we might also use the term *simple ABox* to refer to an ABox that is not extended; likewise for *simple KBs*. Note that labeled nulls are quite natural in the Semantic Web, since RDF (and hence OWL) in fact supports "extended ABoxes" by allowing blank nodes to occur in membership assertions. Similarly to labeled nulls, blank nodes are used to refer to unnamed objects.

A *signature* Σ is a finite set of concept and role names. A KB $\mathcal{K}$ is said to be *defined over* (or simply, *over*) Σ if all the concept and role names occurring in $\mathcal{K}$ belong to Σ (and likewise for TBoxes, ABoxes, concept inclusions, role inclusions and membership assertions). Moreover, an *interpretation* $\mathcal{I}$ *of* Σ is a pair $\langle \Delta^\mathcal{I}, \cdot^\mathcal{I} \rangle$, where $\Delta^\mathcal{I}$ is a non-empty domain and $\cdot^\mathcal{I}$ is a *partial* interpretation function over $N_C \cup N_R \cup N_a$, such that: (1) $A^\mathcal{I}$ is defined and $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$, for every concept name $A \in \Sigma$; (2) $P^\mathcal{I}$ is defined and $P^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$, for every role name $P \in \Sigma$; and (3) $a^\mathcal{I} \in \Delta^\mathcal{I}$, for every constant $a \in N_a$, such that $a^\mathcal{I}$ is defined (such constants are called *interpreted*). Function $\cdot^\mathcal{I}$ is also extended to interpret concept and role constructs:

$$\begin{array}{rclcrcl}
(\exists R)^\mathcal{I} & = & \{o \in \Delta^\mathcal{I} \mid \exists o' \in \Delta^\mathcal{I} \text{ such that } (o, o') \in R^\mathcal{I}\}; & \quad & (\neg B)^\mathcal{I} & = & \Delta^\mathcal{I} \setminus B^\mathcal{I}; \\
(P^-)^\mathcal{I} & = & \{(o, o') \in \Delta^\mathcal{I} \times \Delta^\mathcal{I} \mid (o', o) \in P^\mathcal{I}\}; & & (\neg R)^\mathcal{I} & = & (\Delta^\mathcal{I} \times \Delta^\mathcal{I}) \setminus R^\mathcal{I}.
\end{array}$$

Note that, consistently with the semantics of OWL 2 QL, we do *not* make the unique name assumption (UNA), i.e., we allow distinct constants $a, b \in N_a$ to be interpreted as the same object, that is, $a^\mathcal{I} = b^\mathcal{I}$. Observe also that labeled nulls are *not* interpreted by $\mathcal{I}$. Finally, note that interpretations do not have to interpret all constants in N_a. This is required first of all to avoid that both the canonical model and the generating structure (as defined in Section 3.2) are forced to be infinite. Moreover, this allows for finite interpretation domains without the need for interpreting an infinite number of constants as the same object.

Let $\mathcal{I} = \langle \Delta^\mathcal{I}, \cdot^\mathcal{I} \rangle$ be an interpretation of a signature Σ. Then $\mathcal{I}$ is said to *satisfy* a concept inclusion $B \sqsubseteq C$ over Σ, denoted by $\mathcal{I} \models B \sqsubseteq C$, if $B^\mathcal{I} \subseteq C^\mathcal{I}$; $\mathcal{I}$ is said to *satisfy* a role inclusion $R \sqsubseteq Q$ over Σ, denoted by $\mathcal{I} \models R \sqsubseteq Q$, if $R^\mathcal{I} \subseteq Q^\mathcal{I}$; and $\mathcal{I}$ is said to *satisfy* a TBox $\mathcal{O}$ over Σ, denoted by $\mathcal{I} \models \mathcal{O}$, if $\mathcal{I} \models \alpha$ for every $\alpha \in \mathcal{O}$. Moreover, satisfaction of membership assertions over Σ is defined as follows. A *substitution* over $\mathcal{I}$ is a partial function $h_\mathcal{I} : (N_a \cup N_\ell) \rightarrow \Delta^\mathcal{I}$

such that for every $a \in N_a$, (1) $h_{\mathcal{I}}(a)$ is defined if and only if $a^{\mathcal{I}}$ is defined; and (2) if $h_{\mathcal{I}}(a)$ is defined, then $h_{\mathcal{I}}(a) = a^{\mathcal{I}}$. Then, $\mathcal{I}$ is said to *satisfy* an (extended) ABox $\mathcal{A}$, denoted by $\mathcal{I} \models \mathcal{A}$, if there exists a substitution $h_{\mathcal{I}}$ over $\mathcal{I}$ such that:

- for every $B(u) \in \mathcal{A}$, it holds that $h_{\mathcal{I}}(u)$ is defined and $h_{\mathcal{I}}(u) \in B^{\mathcal{I}}$; and
- for every $R(u, v) \in \mathcal{A}$, it holds that $h_{\mathcal{I}}(u)$ and $h_{\mathcal{I}}(v)$ are defined and $(h_{\mathcal{I}}(u), h_{\mathcal{I}}(v)) \in R^{\mathcal{I}}$.

Finally, $\mathcal{I}$ is said to *satisfy* a(n extended) KB $\mathcal{K} = \langle \mathcal{O}, \mathcal{A}\rangle$, denoted by $\mathcal{I} \models \mathcal{K}$, if $\mathcal{I} \models \mathcal{O}$ and $\mathcal{I} \models \mathcal{A}$. Such $\mathcal{I}$ is called a *model* of $\mathcal{K}$, and we use $\textsc{Mod}(\mathcal{K})$ to denote the set of all models of $\mathcal{K}$. We say that $\mathcal{K}$ is *consistent* if $\textsc{Mod}(\mathcal{K}) \neq \emptyset$.

As is customary, given a(n extended) KB $\mathcal{K}$ over a signature Σ and a membership assertion or an inclusion α over Σ, we use notation $\mathcal{K} \models \alpha$ to indicate that for every interpretation $\mathcal{I}$ of Σ, if $\mathcal{I} \models \mathcal{K}$, then $\mathcal{I} \models \alpha$. Similarly, we use $\mathcal{O} \models \alpha$ for a TBox $\mathcal{O}$, and $\mathcal{A} \models \alpha$, for an ABox $\mathcal{A}$.

3.2. The Canonical and Generating Models

Throughout this section we consider only simple KBs. Horn logics in general, and *DL-Lite*$_\mathcal{R}$ in particular, enjoy the *canonical model property*. It means that, given a KB $\mathcal{K}$, if $\mathcal{K}$ is consistent, then it is possible to construct a model of $\mathcal{K}$ that is more general than any of the other models of this KB. We now introduce this notion formally, and show how the canonical model can be constructed for a *DL-Lite*$_\mathcal{R}$ KB.

The canonical model. Let $\mathcal{K} = \langle \mathcal{O}, \mathcal{A}\rangle$ be a consistent simple *DL-Lite*$_\mathcal{R}$ KB. To define the canonical model of $\mathcal{K}$, we need to introduce some terminology. For every basic role R in $\mathcal{K}$, we define the equivalence class $[R]$ as

$$[R] = \{S \mid S \text{ is a basic role}, \mathcal{O} \models R \sqsubseteq S, \text{ and } \mathcal{O} \models S \sqsubseteq R\}.$$

We introduce a witness $w_{[R]}$ for each $[R]$, and write $[R] \sqsubseteq_\mathcal{O} [S]$ if $\mathcal{O} \models R \sqsubseteq S$. Then the *generating relation* $\rightsquigarrow_\mathcal{K}$ between the set $N_a \cup \{w_{[R]} \mid R \text{ is a basic role}\}$ and the set $\{w_{[R]} \mid R \text{ is a basic role}\}$ is defined as follows:

- $a \rightsquigarrow_\mathcal{K} w_{[R]}$, if (1) $\mathcal{K} \models \exists R(a)$; (2) $\mathcal{K} \not\models R(a, b)$, for every $b \in N_a$; and (3) $[R'] = [R]$, for every $[R']$ such that $[R'] \sqsubseteq_\mathcal{O} [R]$ and $\mathcal{K} \models \exists R'(a)$.
- $w_{[S]} \rightsquigarrow_\mathcal{K} w_{[R]}$, if (1) $\mathcal{O} \models \exists S^- \sqsubseteq \exists R$; (2) $[S^-] \neq [R]$; and (3) $[R'] = [R]$ for every $[R']$ such that $[R'] \sqsubseteq_\mathcal{O} [R]$ and $\mathcal{O} \models \exists S^- \sqsubseteq \exists R'$.

Intuitively, the generating relation defines when an existing object can be reused to satisfy an axiom of the form $B \sqsubseteq \exists R$, or a new object has to be generated.

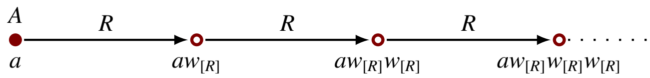
A sequence $aw_{[R_1]} \dots w_{[R_n]}$, where $a \in \mathsf{ind}(\mathcal{A})$, $n \geq 0$, $a \rightsquigarrow_\mathcal{K} w_{[R_1]}$ and $w_{[R_i]} \rightsquigarrow_\mathcal{K} w_{[R_{i+1}]}$ for $i \in \{1, \dots, n-1\}$, is called a *$\mathcal{K}$-path*. We denote by $\mathsf{path}(\mathcal{K})$ the set of all $\mathcal{K}$-paths, and by $\mathsf{wit}(\mathcal{K})$ the set of all $w_{[R]}$ such that $aw_{[R_1]} \dots w_{[R_n]}$ is a $\mathcal{K}$-path, $n \geq 1$ and $w_{[R]} = w_{[R_n]}$. A $\mathcal{K}$-path $aw_{[R_1]} \dots w_{[R_n]}$ with $n \geq 1$ encodes an object that has to be generated to satisfy all axioms in $\mathcal{K}$, and which is called an *anonymous individual* as it is distinct from any named individual (i.e., constant). Finally, for every $\sigma \in \mathsf{path}(\mathcal{K})$, denote by $\mathsf{tail}(\sigma)$ the last element in σ. With this we have the necessary ingredients to define the *canonical* (or, *universal*) *model* of $\mathcal{K}$, which is denoted by $\mathsf{uni}(\mathcal{K})$. Formally, $\mathsf{uni}(\mathcal{K})$ is defined as an interpretation such that:

$$\begin{aligned}
\Delta^{\mathsf{uni}(\mathcal{K})} &= \mathsf{path}(\mathcal{K}),\\
a^{\mathsf{uni}(\mathcal{K})} &= a,\\
A^{\mathsf{uni}(\mathcal{K})} &= \{a \in \mathsf{ind}(\mathcal{A}) \mid \mathcal{K} \models A(a)\} \cup \{\sigma \cdot w_{[R]} \in \mathsf{path}(\mathcal{K}) \mid \mathcal{O} \models \exists R^- \sqsubseteq A\},\\
P^{\mathsf{uni}(\mathcal{K})} &= \{(a, b) \in \mathsf{ind}(\mathcal{A}) \times \mathsf{ind}(\mathcal{A}) \mid \mathcal{K} \models P(a, b)\} \cup \{(\sigma, \sigma \cdot w_{[R]}) \mid \mathsf{tail}(\sigma) \rightsquigarrow_\mathcal{K} w_{[R]} \text{ and } [R] \sqsubseteq_\mathcal{O} [P]\} \cup\\
&\qquad \{(\sigma \cdot w_{[R]}, \sigma) \mid \mathsf{tail}(\sigma) \rightsquigarrow_\mathcal{K} w_{[R]} \text{ and } [R^-] \sqsubseteq_\mathcal{O} [P]\},
\end{aligned}$$

where $a \in \mathsf{ind}(\mathcal{A})$, A is a concept name, and P is a role name occurring in $\mathcal{K}$.

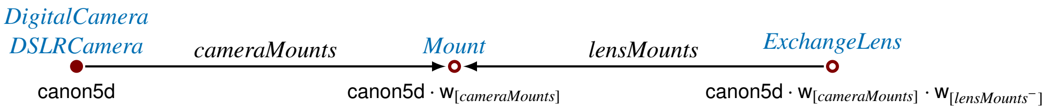
Notice that the part of $\mathsf{uni}(\mathcal{K})$ formed by the anonymous individuals is tree shaped. On the other hand, individuals in $\mathsf{ind}(\mathcal{A})$ can be connected in an arbitrary way in $\mathsf{uni}(\mathcal{K})$, and they are the only individuals that are interpreted by $\mathsf{uni}(\mathcal{K})$.

Example 3.2. Let $\mathcal{K} = \langle \mathcal{O}, \mathcal{A}\rangle$, where $\mathcal{O} = \{A \sqsubseteq \exists R, \exists R^- \sqsubseteq \exists R\}$ and $\mathcal{A} = \{A(a)\}$. Then the canonical model $\mathsf{uni}(\mathcal{K})$ can be seen as an infinite R-path starting in a, which can depicted as follows:



In this figure, dots represent domain elements, a label A on a node x represents the fact $x \in A^{\mathsf{uni}(\mathcal{K})}$, and a label R on an arrow between x and y represents the fact $(x, y) \in R^{\mathsf{uni}(\mathcal{K})}$. ■

Example 3.3. Assume that $\mathcal{K}_{\mathsf{cam}} = \langle \mathcal{O}_{\mathsf{cam}}, \mathcal{A}_{\mathsf{cam}} \rangle$, where $\mathcal{O}_{\mathsf{cam}}$ and $\mathcal{A}_{\mathsf{cam}}$ are as in Example 3.1. Then the canonical model $\mathsf{uni}(\mathcal{K})$ can be depicted as follows:



■

The interpretation $\mathsf{uni}(\mathcal{K})$ is called the canonical model because every other model of $\mathcal{K}$ is less general than $\mathsf{uni}(\mathcal{K})$. We formalize generality in terms of homomorphisms. For an interpretation $\mathcal{I}$ and a signature Σ, the *Σ-types* $\mathbf{t}^{\mathcal{I}}_{\Sigma}(o)$ and $\mathbf{r}^{\mathcal{I}}_{\Sigma}(o, o')$ for $o, o' \in \Delta^{\mathcal{I}}$ are given by the set of concepts B (respectively, roles R) over Σ, such that $o \in B^{\mathcal{I}}$ (respectively, $(o, o') \in R^{\mathcal{I}}$). We also use $\mathbf{t}^{\mathcal{I}}(o)$ and $\mathbf{r}^{\mathcal{I}}(o, o')$ to refer to the types over the signature of all concepts and roles names. Then, a *Σ-homomorphism* from an interpretation $\mathcal{I}$ to $\mathcal{J}$ is a function $h : \Delta^{\mathcal{I}} \mapsto \Delta^{\mathcal{J}}$ such that: (1) for every $a \in N_a$ such that $a^{\mathcal{I}}$ is defined, it holds that $a^{\mathcal{J}}$ is defined and $h(a^{\mathcal{I}}) = a^{\mathcal{J}}$; (2) $\mathbf{t}^{\mathcal{I}}_{\Sigma}(o) \subseteq \mathbf{t}^{\mathcal{J}}_{\Sigma}(h(o))$ and $\mathbf{r}^{\mathcal{I}}_{\Sigma}(o, o') \subseteq \mathbf{r}^{\mathcal{J}}_{\Sigma}(h(o), h(o'))$ for all $o, o' \in \Delta^{\mathcal{I}}$. We say that $\mathcal{I}$ is *Σ-homomorphically embeddable* into $\mathcal{J}$ if there exists a Σ-homomorphism from $\mathcal{I}$ to $\mathcal{J}$, and $\mathcal{I}$ is *Σ-homomorphically equivalent* to $\mathcal{J}$, if they are Σ-homomorphically embeddable into each other. If Σ is the set of all concepts and roles names, we call a Σ-homomorphism simply *homomorphism*.

The theorem below establishes the relationship between the canonical model $\mathsf{uni}(\mathcal{K})$ and an arbitrary model of $\mathcal{K}$.

Theorem 3.4 ([55]). *If $\mathcal{K}$ is consistent, then $\mathsf{uni}(\mathcal{K})$ is a model of $\mathcal{K}$. Moreover, for every model $\mathcal{I}$ of $\mathcal{K}$, there exists a homomorphism from $\mathsf{uni}(\mathcal{K})$ to $\mathcal{I}$.*

The generating structure. In general, the canonical model of a *DL-Lite*$_{\mathcal{R}}$ KB $\mathcal{K}$ can be infinite, which makes it impossible to deal with it in practice. Thus, we define here an alternative notion that is called the *generating structure* of $\mathcal{K}$. This structure is always finite and can be used for deciding various reasoning tasks efficiently. Formally, given a simple KB $\mathcal{K} = \langle \mathcal{O}, \mathcal{A} \rangle$, the *generating structure* $\mathsf{gen}(\mathcal{K}) = \langle \Delta^{\mathsf{gen}(\mathcal{K})}, \cdot^{\mathsf{gen}(\mathcal{K})} \rangle$ of $\mathcal{K}$, is defined as:

$$
\begin{aligned}
\Delta^{\mathsf{gen}(\mathcal{K})} &= \mathsf{ind}(\mathcal{A}) \cup \mathsf{wit}(\mathcal{K}), \\
a^{\mathsf{gen}(\mathcal{K})} &= a, \\
A^{\mathsf{gen}(\mathcal{K})} &= \{a \in \mathsf{ind}(\mathcal{A}) \mid \mathcal{K} \models A(a)\} \cup \{w_{[R]} \in \mathsf{wit}(\mathcal{K}) \mid \mathcal{O} \models \exists R^- \sqsubseteq A\}, \\
P^{\mathsf{gen}(\mathcal{K})} &= \{(a, b) \in \mathsf{ind}(\mathcal{A}) \times \mathsf{ind}(\mathcal{A}) \mid \mathcal{K} \models P(a, b)\} \cup \{(x, w_{[R]}) \mid x \rightsquigarrow_{\mathcal{K}} w_{[R]} \text{ and } [R] \sqsubseteq_{\mathcal{O}} [P]\} \cup \\
&\qquad \{(w_{[R]}, x) \mid x \rightsquigarrow_{\mathcal{K}} w_{[R]} \text{ and } [R^-] \sqsubseteq_{\mathcal{O}} [P]\},
\end{aligned}
$$

where $a \in \mathsf{ind}(\mathcal{A})$, A is a concept name, and P is a role name occurring in $\mathcal{K}$. It is easy to see that $\mathsf{gen}(\mathcal{K})$ is of polynomial size in the size of $\mathcal{K}$. Note that the canonical model $\mathsf{uni}(\mathcal{K})$ can be obtained by unraveling [56, Ch.2] the generating structure $\mathsf{gen}(\mathcal{K})$, i.e., by introducing a new domain element for every path starting from (the interpretation of) a constant.

3.3. Queries and Certain Answers

In this paper, we deal with conjunctive queries and their unions. A *conjunctive query* (CQ) (of *arity* $k \geq 0$) over a signature Σ is a formula of the form $q(\vec{x}) = \exists \vec{y}.\, \varphi(\vec{x}, \vec{y})$, where $\vec{x}, \vec{y}$ are tuples of variables, $\vec{x} = (x_1, \ldots, x_k)$ is the tuple of free variables of $q(\vec{x})$, and $\varphi(\vec{x}, \vec{y})$ is a conjunction of atoms of the form $A(z)$ and $P(z, z')$, where A is a concept name in Σ, P is a role name in Σ, and each of z, z' is a variable from $\vec{x} \cup \vec{y}$. Given an interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ of Σ and a k-tuple $\vec{o}$ of elements of $\Delta^{\mathcal{I}}$, we write $\mathcal{I} \models q[\vec{o}]$, if there exist a tuple $\vec{o}_1$ of elements of $\Delta^{\mathcal{I}}$ such that $\mathcal{I}, \xi \models \varphi(\vec{x}, \vec{y})$, where ξ is the substitution that assigns $\vec{x}$ to $\vec{o}$ and $\vec{y}$ to $\vec{o}_1$, and we write $\mathcal{I} \not\models q[\vec{o}]$ otherwise. A *union of conjunctive queries* (UCQ) over a signature Σ is a first-order formula of the form $q(\vec{x}) = \bigvee_{i=1}^{n} q_i(\vec{x})$, where each q_i, for $i \in \{1, \ldots, n\}$, is a

CQ over Σ. Then, $\mathcal{I} \models q[\vec{o}]$ if $\mathcal{I} \models q_i[\vec{o}]$ for some $i \in \{1, \dots, n\}$, and $\mathcal{I} \not\models q[\vec{o}]$ otherwise. If $k = 0$, then q is said to be a *Boolean query*, and we simply write $\mathcal{I} \models q$ if $\mathcal{I} \models q[()]$, and $\mathcal{I} \not\models q$ otherwise.

Given a query q of arity k and a KB $\mathcal{K}$ defined over a signature Σ, the *certain answers* to q over $\mathcal{K}$ are defined as:

$$cert(q, \mathcal{K}) = \{(a_1, \dots, a_k) \mid \{a_1, \dots, a_k\} \subseteq N_a \text{ and } \mathcal{I} \models q[a_1^{\mathcal{I}}, \dots, a_k^{\mathcal{I}}], \text{ for every } \mathcal{I} \in \textsc{Mod}(\mathcal{K})\}.$$

Each tuple $\vec{a} = (a_1, \dots, a_k)$ in $cert(q, \mathcal{K})$ is called a *certain answer* for q over $\mathcal{K}$, and we write $\mathcal{K} \models q[\vec{a}]$. Notice that, by definition, the certain answers to a query do not contain labeled nulls. If q is a Boolean query, then $cert(q, \mathcal{K}) = \{()\}$ (representing the value true) if $\mathcal{I} \models q$ for every $\mathcal{I} \in \textsc{Mod}(\mathcal{K})$, and $cert(q, \mathcal{K}) = \emptyset$ (representing the value false) otherwise. Observe also that, if $\mathcal{K}$ is unsatisfiable, then $cert(q, \mathcal{K})$ is trivially the set of all possible tuples $(a_1, \dots, a_k)$ of constants in N_a, which we denote by *AllTup*(q).

It is important to notice that the notion of certain answers can be characterized through the notion of canonical model. The following lemma establishes that the canonical model can be used for checking certain answers to UCQs.

Lemma 3.5. *Let $\mathcal{K}$ be a consistent KB, $q(\vec{x})$ a* UCQ*, and $\vec{a}$ a tuple of constants. Then $\mathcal{K} \models q[\vec{a}]$ iff* $\mathsf{uni}(\mathcal{K}) \models q[\vec{a}]$.

Proof. Let $\vec{a} = (a_1, \dots, a_k)$ for $a_i \in N_a$, and $q(\vec{x}) = \exists y_1 \cdots \exists y_m . \varphi(x_1, \dots, x_k, y_1, \dots, y_m)$.

($\Rightarrow$) Assume $\mathcal{K} \models q[\vec{a}]$. Then for each model $\mathcal{I}$ of $\mathcal{K}$, we have that $\mathcal{I} \models q[a_1^{\mathcal{I}}, \dots, a_k^{\mathcal{I}}]$. Since $\mathsf{uni}(\mathcal{K})$ is a model of $\mathcal{K}$, and $a_i^{\mathsf{uni}(\mathcal{K})} = a_i$, for each constant a_i, it follows that $\mathsf{uni}(\mathcal{K}) \models q[\vec{a}]$.

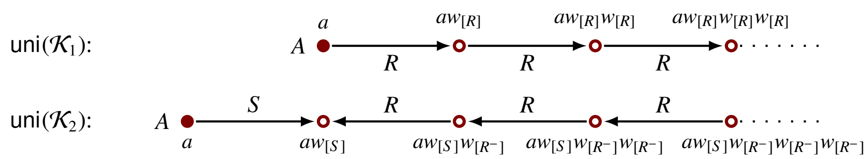
($\Leftarrow$) Assume $\mathsf{uni}(\mathcal{K}) \models q[\vec{a}]$. Then there exist $\sigma_1, \dots, \sigma_m \in \Delta^{\mathsf{uni}(\mathcal{K})}$ such that $\mathsf{uni}(\mathcal{K}) \models \varphi[a_1, \dots, a_k, \sigma_1, \dots, \sigma_m]$. Let $\mathcal{I}$ be a model of $\mathcal{K}$. By Theorem 3.4, there exists a homomorphism h from $\mathsf{uni}(\mathcal{K})$ to $\mathcal{I}$. Then it is easy to see that $\mathcal{I} \models \varphi[a_1^{\mathcal{I}}, \dots, a_k^{\mathcal{I}}, h(\sigma_1), \dots, h(\sigma_m)]$, hence $\mathcal{I} \models q[a_1^{\mathcal{I}}, \dots, a_k^{\mathcal{I}}]$. Thus, as $\mathcal{I}$ is an arbitrary model of $\mathcal{K}$, we conclude that $\mathcal{K} \models q[\vec{a}]$. □

3.4. Σ-Query Entailment

We refine the notion of Σ-query entailment studied in [55]. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be KBs, and Σ a signature. Then, $\mathcal{K}_1$ *Σ-query entails* $\mathcal{K}_2$ if for each UCQ q over Σ, $cert(q, \mathcal{K}_2) \subseteq cert(q, \mathcal{K}_1)$. Moreover, $\mathcal{K}_1$ and $\mathcal{K}_2$ are *Σ-query inseparable*, if they Σ-query entail each other. Note that we define Σ-query entailment and inseparability with respect to UCQs, whereas in [55] these notions are defined with respect to CQs. Since *DL-Lite*$_{\mathcal{R}}$ enjoys the canonical model property, it is easy to see that our definitions and the previous ones coincide.

It is well known that homomorphisms preserve answers to UCQs [57], in particular, if $\mathsf{uni}(\mathcal{K}_2)$ is Σ-homomorphically embeddable into $\mathsf{uni}(\mathcal{K}_1)$, then $\mathcal{K}_1$ Σ-entails $\mathcal{K}_2$. However, for a characterization of Σ-query entailment one has to consider finite Σ-homomorphisms, as illustrated in the following example.

Example 3.6 ([55])**.** Let $\mathcal{K}_1 = \langle \mathcal{O}_1, \mathcal{A} \rangle$ and $\mathcal{K}_2 = \langle \mathcal{O}_2, \mathcal{A} \rangle$, where $\mathcal{A} = \{A(a)\}$, $\mathcal{O}_1 = \{A \sqsubseteq \exists R, \exists R^- \sqsubseteq \exists R\}$ and $\mathcal{O}_2 = \{A \sqsubseteq \exists S, \exists S^- \sqsubseteq \exists R^-, \exists R \sqsubseteq \exists R^-\}$, and $\Sigma = \{A, R\}$. The canonical models of $\mathcal{K}_1$ and $\mathcal{K}_2$ are as follows:



In this case there is no Σ-homomorphism from $\mathsf{uni}(\mathcal{K}_2)$ to $\mathsf{uni}(\mathcal{K}_1)$, although $\mathcal{K}_1$ Σ-query entails $\mathcal{K}_2$. ■

Given an interpretation $\mathcal{I}$ over a signature Σ, we say that $\mathcal{I}'$ is a finite *sub-interpretation of* $\mathcal{I}$ (induced by a finite set D) if: *(1)* $\Delta^{\mathcal{I}'} = \Delta^{\mathcal{I}} \cap D$; *(2)* $A^{\mathcal{I}'} = A^{\mathcal{I}} \cap D$ for every concept name $A \in \Sigma$; *(3)* $P^{\mathcal{I}'} = P^{\mathcal{I}} \cap (D \times D)$ for every role name $P \in \Sigma$; and *(4)* $a^{\mathcal{I}'} = a^{\mathcal{I}}$ for every $a \in N_a$ such that $a^{\mathcal{I}}$ is defined and $a^{\mathcal{I}} \in \Delta^{\mathcal{I}'}$. We say that $\mathcal{I}$ is *finitely Σ-homomorphically embeddable* into an interpretation $\mathcal{J}$ if there exists a Σ-homomorphism from every finite sub-interpretation $\mathcal{I}'$ of $\mathcal{I}$ to $\mathcal{J}$.

Lemma 3.7 ([55])**.** *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be consistent KBs, and Σ a signature. Then $\mathcal{K}_1$ Σ-query entails $\mathcal{K}_2$ iff* $\mathsf{uni}(\mathcal{K}_2)$ *is finitely Σ-homomorphically embeddable into* $\mathsf{uni}(\mathcal{K}_1)$.

By using this lemma, we can confirm that KB $\mathcal{K}_1$ Σ-query entails KB $\mathcal{K}_2$ in Example 3.6, as $\mathsf{uni}(\mathcal{K}_2)$ is finitely Σ-homomorphically embeddable into $\mathsf{uni}(\mathcal{K}_1)$.

4. Knowledge Base Exchange

The goal of this section is to generalize the setting proposed in [25] to consider *DL-Lite*$_{\mathcal{R}}$, and to formalize the problems studied in this paper. The former is done in Section 4.1, while the latter is done in Section 4.2.

4.1. A knowledge base exchange framework

We start by defining the fundamental notion of mapping, which plays a key role in both data and knowledge exchange. Assume that Σ, Γ are signatures with no concepts or roles in common. An inclusion $E_s \sqsubseteq E_t$ is said to be *from* Σ *to* Γ, if E_s is a concept or a role over Σ and E_t is a concept or a role over Γ. Then we have that

Definition 4.1 ([16, 58])**.** *A* mapping *is a tuple* $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$*, where* $\mathcal{B}$ *is a TBox consisting of inclusions from* Σ *to* Γ.

Example 4.2. Consider the ontology PhotoCamera defined in Example 3.1, and a second ontology DigitalPhoto talking about digital photo camera. This new ontology uses the following vocabulary Γ_{cam}:

$$\mathit{DigitalPhotoCamera}(\cdot),\ \mathit{ReflexCamera}(\cdot),\ \mathit{InterchangeableLens}(\cdot),\ \mathit{MountType}(\cdot),\ \mathit{hasMountType}(\cdot,\cdot),\ \mathit{mountsOn}(\cdot,\cdot)$$

Then we can specify the relation between the terms in the different ontologies by means of a mapping. Formally, let $\mathcal{M}_{\mathsf{cam}} = (\Sigma_{\mathsf{cam}}, \Gamma_{\mathsf{cam}}, \mathcal{B}_{\mathsf{cam}})$, where Σ_{cam} is the vocabulary from Example 3.1, and $\mathcal{B}_{\mathsf{cam}}$ consists of the following inclusions:

$$\begin{array}{rlrl} \mathit{DigitalCamera} \sqsubseteq & \mathit{DigitalPhotoCamera} & \mathit{Mount} \sqsubseteq & \mathit{MountType} \\ \mathit{DSLRCamera} \sqsubseteq & \mathit{ReflexCamera} & \mathit{cameraMounts} \sqsubseteq & \mathit{hasMountType} \\ \mathit{ExchangeLens} \sqsubseteq & \mathit{InterchangeableLens} & \mathit{lensMounts} \sqsubseteq & \mathit{mountsOn} \end{array}$$

Thus, $\mathcal{M}_{\mathsf{cam}}$ relates the concepts and roles of the PhotoCamera ontology with the concepts and roles of the DigitalPhoto ontology. ■

The semantics of such a mapping was initially defined in [58]. Here we adapt it to the setting without the unique name assumption (and, more generally, without the standard name assumption). More specifically, given interpretations $\mathcal{I}, \mathcal{J}$ of Σ and Γ, respectively, the pair $(\mathcal{I}, \mathcal{J})$ *satisfies* TBox $\mathcal{B}$, denoted by $(\mathcal{I}, \mathcal{J}) \models \mathcal{B}$, if

- for every $a \in N_a$ such that $a^{\mathcal{I}}$ or $a^{\mathcal{J}}$ is defined, it holds that both $a^{\mathcal{I}}$ and $a^{\mathcal{J}}$ are defined and $a^{\mathcal{I}} = a^{\mathcal{J}}$,
- for every concept inclusion $B \sqsubseteq C \in \mathcal{B}$, it holds that $B^{\mathcal{I}} \subseteq C^{\mathcal{J}}$, and
- for every role inclusion $R \sqsubseteq Q \in \mathcal{B}$, it holds that $R^{\mathcal{I}} \subseteq Q^{\mathcal{J}}$.

Notice that the connection between the information in $\mathcal{I}$ and $\mathcal{J}$ is established through the constants that move from source to target according to the mapping. For this reason, we require constants to be interpreted in the same way in $\mathcal{I}$ and $\mathcal{J}$, i.e., they preserve their meaning when they are transferred. Besides, notice that this is the only restriction imposed on the domains of $\mathcal{I}$ and $\mathcal{J}$ (in particular, we require neither that $\Delta^{\mathcal{I}} = \Delta^{\mathcal{J}}$ nor that $\Delta^{\mathcal{I}} \subseteq \Delta^{\mathcal{J}}$). Finally, $\textsc{Sat}_{\mathcal{M}}(\mathcal{I})$ is defined as the set of interpretations $\mathcal{J}$ of Γ such that $(\mathcal{I}, \mathcal{J}) \models \mathcal{B}$, and given a set $\mathcal{X}$ of interpretations of Σ, $\textsc{Sat}_{\mathcal{M}}(\mathcal{X})$ is defined as $\bigcup_{\mathcal{I} \in \mathcal{X}} \textsc{Sat}_{\mathcal{M}}(\mathcal{I})$.

The main problem studied in the knowledge exchange framework is the problem of translating a KB according to a mapping. We formalize this problem through three different notions of translation introduced below (see Section 5 for a comparison of these different notions of solution). We start by introducing the concepts of solution and universal solution. More precisely,

Definition 4.3. *Given a mapping* $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$ *and KBs* $\mathcal{K}_{\mathsf{s}}$, $\mathcal{K}_{\mathsf{t}}$ *over* Σ *and* Γ*, respectively,* $\mathcal{K}_{\mathsf{t}}$ *is a* solution *(resp.,* universal solution*) for* $\mathcal{K}_{\mathsf{s}}$ *under* $\mathcal{M}$ *if* $\textsc{Mod}(\mathcal{K}_{\mathsf{t}}) \subseteq \textsc{Sat}_{\mathcal{M}}(\textsc{Mod}(\mathcal{K}_{\mathsf{s}}))$ *(resp.,* $\textsc{Mod}(\mathcal{K}_{\mathsf{t}}) = \textsc{Sat}_{\mathcal{M}}(\textsc{Mod}(\mathcal{K}_{\mathsf{s}}))$*).*

Thus, $\mathcal{K}_{\mathsf{t}}$ is a solution for $\mathcal{K}_{\mathsf{s}}$ under $\mathcal{M}$ if every interpretation of $\mathcal{K}_{\mathsf{t}}$ is a valid translation of an interpretation of $\mathcal{K}_{\mathsf{s}}$ according to $\mathcal{M}$. Although natural, this is a mild restriction, which gives rise to the stronger notion of universal solution. More precisely, if $\mathcal{K}_{\mathsf{t}}$ is a universal solution for $\mathcal{K}_{\mathsf{s}}$ under $\mathcal{M}$, then $\mathcal{K}_{\mathsf{t}}$ is designed to exactly represent the space of interpretations obtained by translating the interpretations of $\mathcal{K}_{\mathsf{s}}$ under $\mathcal{M}$. It should be noticed that this definition of universal solution can be restated in terms of the notion of model inseparability presented in Section 2. More precisely, we have that $\mathcal{K}_{\mathsf{t}}$ is a universal solution for $\mathcal{K}_{\mathsf{s}} = \langle \mathcal{S}, \mathcal{A}_{\mathsf{s}} \rangle$ under $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$ if and only if $\mathcal{K}_{\mathsf{t}}$ is Γ-model inseparable with $\langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_{\mathsf{s}} \rangle$.

Example 4.4. Let $\mathcal{K}_{\mathsf{cam}} = \langle O_{\mathsf{cam}}, \mathcal{A}_{\mathsf{cam}} \rangle$ where O_{cam} and $\mathcal{A}_{\mathsf{cam}}$ are respectively the TBox and the ABox of the PhotoCamera KB from Example 3.1, and $\mathcal{M}_{\mathsf{cam}}$ the mapping from Example 4.2. Then $\mathcal{K}'_{\mathsf{cam}} = \langle O'_{\mathsf{cam}}, \mathcal{A}'_{\mathsf{cam}} \rangle$ is a universal solution for $\mathcal{K}_{\mathsf{cam}}$ under $\mathcal{M}_{\mathsf{cam}}$, where $O'_{\mathsf{cam}} = \emptyset$ and $\mathcal{A}'_{\mathsf{cam}}$ contains the following assertions:

$$\begin{aligned}&\mathit{ReflexCamera}(\mathsf{canon5d}),\ \mathit{DigitalPhotoCamera}(\mathsf{canon5d}),\ \mathit{hasMountType}(\mathsf{canon5d}, m),\\ &\qquad\qquad\mathit{MountType}(m),\ \mathit{mountsOn}(l, m),\ \mathit{InterchangeableLens}(l).\end{aligned}$$

Here m and l are distinct labeled nulls. For more examples of universal solutions see Section 5.1. ■

A second class of translations is obtained by observing that solutions and universal solutions are too restrictive for some applications, in particular when one only needs a translation storing enough information to properly answer some queries. For the particular case of UCQ, this gives rise to the notions of UCQ-solution and universal UCQ-solution.

Definition 4.5. *Given a mapping $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$, a KB $\mathcal{K}_{\mathsf{s}} = \langle \mathcal{S}, \mathcal{A}_{\mathsf{s}} \rangle$ over Σ and a KB $\mathcal{K}_{\mathsf{t}}$ over Γ, $\mathcal{K}_{\mathsf{t}}$ is a* UCQ-solution *for $\mathcal{K}_{\mathsf{s}}$ under $\mathcal{M}$ if $\mathcal{K}_{\mathsf{t}}$ Γ-query entails $\langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_{\mathsf{s}} \rangle$. Moreover, $\mathcal{K}_{\mathsf{t}}$ is a* universal UCQ-solution *for $\mathcal{K}_{\mathsf{s}}$ under $\mathcal{M}$ if $\mathcal{K}_{\mathsf{t}}$ and $\langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_{\mathsf{s}} \rangle$ are Γ-query inseparable.*

Example 4.6. Consider $\mathcal{K}_{\mathsf{cam}}$ and $\mathcal{M}_{\mathsf{cam}}$ from Example 4.4. Then $\mathcal{K}''_{\mathsf{cam}} = \langle O''_{\mathsf{cam}}, \mathcal{A}''_{\mathsf{cam}} \rangle$ is a universal UCQ-solution for $\mathcal{K}_{\mathsf{cam}}$ under $\mathcal{M}_{\mathsf{cam}}$, where $\mathcal{A}''_{\mathsf{cam}} = \{\mathit{ReflexCamera}(\mathsf{canon5d})\}$ and O''_{cam} is the following TBox:

$$\begin{aligned}&\mathit{ReflexCamera} \sqsubseteq \mathit{DigitalPhotoCamera},\quad \mathit{ReflexCamera} \sqsubseteq \exists \mathit{hasMountType},\\ &\qquad \exists \mathit{hasMountType}^- \sqsubseteq \mathit{MountType},\quad \mathit{MountType} \sqsubseteq \exists \mathit{mountsOn}^-,\quad \exists \mathit{mountsOn} \sqsubseteq \mathit{InterchangeableLens}.\end{aligned}$$

This can be straightforwardly verified using Lemma 3.7 and the fact that the canonical models of $\langle O_{\mathsf{cam}} \cup \mathcal{B}_{\mathsf{cam}}, \mathcal{A}_{\mathsf{cam}} \rangle$ and $\mathcal{K}''_{\mathsf{cam}}$ are finite. Note that, the universal solution $\mathcal{K}'_{\mathsf{cam}}$ of Example 4.4 is also a universal UCQ-solution. This holds in general, as shown in Section 5.2. ■

Finally, a last class of solutions is obtained by considering that users want to translate as much of the knowledge in a TBox as possible, as a lot of effort is put in practice when constructing a TBox. This observation gives rise to the notion of UCQ-representation, which formalizes the idea of translating a source TBox according to a mapping. We present an alternative to the formalization of this notion given in [58], which is appropriate for our setting where disjointness axioms are considered.[1]

Definition 4.7. *Assume that $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$ and $\mathcal{S}$, $\mathcal{T}$ are TBoxes over Σ and Γ, respectively. Then $\mathcal{T}$ is a* UCQ-representation *of $\mathcal{S}$ under $\mathcal{M}$ if for every ABox $\mathcal{A}_{\mathsf{s}}$ over Σ that is consistent with $\mathcal{S}$, it holds that $\langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_{\mathsf{s}} \rangle$ and $\langle \mathcal{T} \cup \mathcal{B}, \mathcal{A}_{\mathsf{s}} \rangle$ are Γ-query inseparable.*

Notice that $\mathcal{A}_{\mathsf{s}}$ is required to be consistent with $\mathcal{S}$ in this definition, which avoids the trivialization of the notion of certain answers because of the use of an inconsistent knowledge base (if $\langle \mathcal{S}, \mathcal{A}_{\mathsf{s}} \rangle$ is inconsistent, $\mathit{cert}(q, \langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_{\mathsf{s}} \rangle)$ contains every possible tuple of constants). Below we provide a simple example of a UCQ-representation in the digital camera scenario.

Example 4.8. Consider $\mathcal{M}_{\mathsf{cam}} = (\Sigma_{\mathsf{cam}}, \Gamma_{\mathsf{cam}}, \mathcal{B}_{\mathsf{cam}})$ from Example 4.2 and $\mathcal{S}_{cam} = \{\mathit{DSLRCamera} \sqsubseteq \mathit{DigitalCamera}\}$. Then $\mathcal{T}_{cam} = \{\mathit{ReflexCamera} \sqsubseteq \mathit{DigitalPhotoCamera}\}$ is a UCQ-representation of $\mathcal{S}_{cam}$ under $\mathcal{M}_{\mathsf{cam}}$. ■

We would like to emphasize why we are interested in UCQ-representations. First of all, UCQ-representations are designed to preserve in the target the implicit information from the source, which conforms to the idea of knowledge base exchange as opposed to plain data exchange. Second, UCQ-representations allow to minimize the amount of extensional information that has to be transferred from the source (which can be very large in size). Third, if there exists a UCQ-representation $\mathcal{T}$ of a source TBox $\mathcal{S}$ under a mapping $\mathcal{M}$, then we obtain a straightforward algorithm to construct a universal UCQ-solution for a given source KB $\langle \mathcal{S}, \mathcal{A}_{\mathsf{s}} \rangle$: take a target ABox obtained by "translating" the source ABox $\mathcal{A}_{\mathsf{s}}$ with respect to $\mathcal{M}$ and denote it by $\mathcal{M}(\mathcal{A}_{\mathsf{s}})$,[2] then $\langle \mathcal{T}, \mathcal{M}(\mathcal{A}_{\mathsf{s}}) \rangle$ is a universal UCQ-solution for $\langle \mathcal{S}, \mathcal{A}_{\mathsf{s}} \rangle$ under $\mathcal{M}$ (see Figure 2). Finally, notice that UCQ-representations do not depend on the actual data, so if in the previous case ABox $\mathcal{A}_{\mathsf{s}}$ is updated, then it is sufficient to update $\mathcal{M}(\mathcal{A}_{\mathsf{s}})$ in order to obtain a universal UCQ-solution for $\langle \mathcal{S}, \mathcal{A}_{\mathsf{s}} \rangle$ under $\mathcal{M}$.

[1] If disjointness axioms are not allowed, then this new notion can be shown to be equivalent to the original formalization of UCQ-representation.

[2] Observe that $\mathcal{M}(\mathcal{A}_{\mathsf{s}})$ could be defined as a universal UCQ-solution for $\langle \emptyset, \mathcal{A}_{\mathsf{s}} \rangle$ under $\mathcal{M}$.

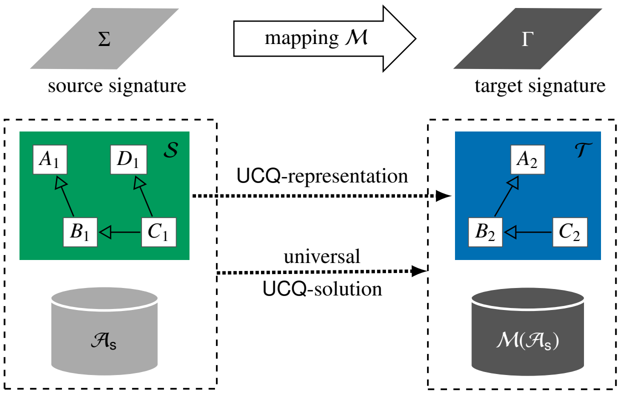


Figure 2: UCQ-representations in the context of knowledge base exchange.

A natural question at this point is why in knowledge base exchange the source KB is not mapped as it is, thus simplifying the problem of computing solutions (under any of the notions given before). Notice that this can be easily done by including some additional concept and role symbols in the target signature, which represent the corresponding concepts and roles in the source signature. We would like to conclude this section by providing evidence why this is not desirable, or it could even be impossible, in some scenarios. First, the target signature might be given and not under control of the user, therefore it might not be extensible. Second, there might be privacy issues that prevent the use of all the information in a source KB, so only the information about some concepts and roles have to be displayed. This problem can be viewed as a knowledge exchange problem where the target signature stores the symbols to be displayed, and which cannot include some new concepts or roles. Third, a source signature might be very large, hence the user would like to switch to a smaller target signature. In this case, it is not desirable to add new symbols that can make the target signature to grow. Finally, an instance of data exchange could be part of the more general problem of schema evolution [31, 32], where one needs to consider a sequence of several instances of data exchange. In this context, allowing for keeping existing symbols might result in an unacceptable (and undesired) growth of the signature.

4.2. On the problem of computing solutions

In this section, we present the space of reasoning problems that naturally arise in the framework introduced in this paper. The problem space has three dimensions and can be depicted as in Figure 3. First, one is interested in the task of computing a translation of a KB or a TBox according to a mapping, which is arguably the most important problem in knowledge exchange [25, 58], as well as in data exchange [16, 59]. Thus, the first dimension in Figure 3 defines the type of translation, which as mentioned in the previous section can be either: (1) a universal solution, or (2) a universal UCQ-solution, or (3) a UCQ-representation. Second, as it will become clear in Section 5, in order to be able to compute a translation, in some cases it is necessary to use extended ABoxes. Therefore, the second dimension is along the type of ABoxes allowed to be used in translations: (1) simple ABoxes, or (2) extended ABoxes. Finally, to study the computational complexity of knowledge exchange, we consider two classical decision problems: the membership problem and the non-emptiness problem, which constitute the third dimension.

As usual, the membership problem is concerned with deciding whether a particular instance (a target KB or target TBox, in our case) belongs to a class of instances (all solutions for a given source KB or TBox under a given mapping, in our case). Since we consider three classes of translations, we need to deal with three membership problems. The *membership* problem for universal solutions (resp. universal UCQ-solutions) has as input a mapping $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$ and KBs $\mathcal{K}_s$, $\mathcal{K}_t$ over Σ and Γ, respectively. Then the question to answer is whether $\mathcal{K}_t$ is a universal solution (resp. universal UCQ-solution) for $\mathcal{K}_s$ under $\mathcal{M}$. Moreover, the membership problem for UCQ-representations has as input a mapping $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$ and TBoxes $\mathcal{S}$, $\mathcal{T}$ over Σ and Γ, respectively, and the question to answer is whether $\mathcal{T}$ is a UCQ-representation of $\mathcal{S}$ under $\mathcal{M}$.

The non-emptiness problem corresponds to the existential version of the membership problem, and it is concerned with deciding whether a class has at least one instance (is there some solution for a given source KB or TBox under a

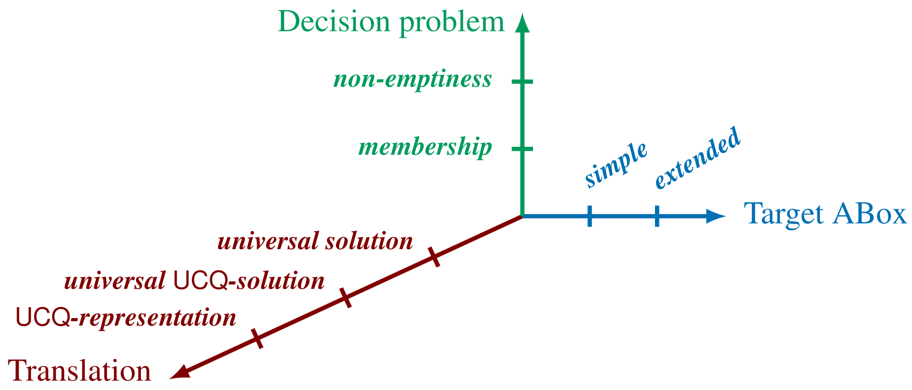


Figure 3: The space of reasoning problems.

given mapping?). Again, we consider three non-emptiness problems, one for each class of translation. Formally, the *non-emptiness* problem for universal solutions (resp. universal UCQ-solutions) has as input a mapping $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$ and a KB $\mathcal{K}_\mathsf{s}$ over Σ. Then the question to answer is whether there exists a universal solution (resp. universal UCQ-solution) for $\mathcal{K}_\mathsf{s}$ under $\mathcal{M}$. Moreover, the non-emptiness problem for UCQ-representations has as input a mapping $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$ and a TBox $\mathcal{S}$ over Σ, and the question to answer is whether there exists a UCQ-representation of $\mathcal{S}$ under $\mathcal{M}$. In the case it exists, we say that $\mathcal{S}$ is UCQ-*representable* under $\mathcal{M}$, otherwise, $\mathcal{S}$ is not UCQ-representable under this mapping.

Observe that UCQ-representations do not depend on target ABoxes, therefore, in total we have defined 10 different reasoning problems: 4 for universal solutions, 4 for universal UCQ-solutions, and 2 for UCQ-representations. We investigate in Sections 6 and 7 the computational complexity of the reasoning problems for universal solutions and UCQ-representations, respectively. As for universal UCQ-solutions, the main results, summarized in Table 1, have been established in [30]. We prove here only, in Section 6.4, that the non-emptiness problem for universal UCQ-solutions for extended ABoxes is PSpace-hard. A lower bound for simple ABoxes has yet not been established.

5. The shape of different notions of solutions

The goal of this section is to provide examples and some facts about universal solutions, universal UCQ-solutions and UCQ-representations, which can help the reader to understand their advantages and limitations.

5.1. Universal solutions

We start by giving some simple examples of universal solutions.

Example 5.1. Let $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$, where $\Sigma = \{A, B\}$, $\Gamma = \{A', B'\}$, and $\mathcal{B} = \{A \sqsubseteq A',\ B \sqsubseteq B'\}$. If $\mathcal{S} = \{\}$ and $\mathcal{A}_\mathsf{s} = \{A(a), B(a)\}$, then the KB $\mathcal{K}_\mathsf{t} = \langle \mathcal{T}, \mathcal{A}_\mathsf{t} \rangle$, where $\mathcal{T} = \emptyset$ and $\mathcal{A}_\mathsf{t} = \{A'(a), B'(a)\}$ is a universal solution for $\mathcal{K}_\mathsf{s} = \langle \mathcal{S}, \mathcal{A}_\mathsf{s} \rangle$ under $\mathcal{M}$. Moreover, if $\mathcal{S} = \{A \sqsubseteq B\}$ and $\mathcal{A}_\mathsf{s} = \{A(a)\}$, then $\mathcal{K}_\mathsf{t}$ is again a universal solution for $\mathcal{K}_\mathsf{s} = \langle \mathcal{S}, \mathcal{A}_\mathsf{s} \rangle$ under $\mathcal{M}$. ■

Universal solutions are the preferred solutions to materialize when exchanging relational databases [16, 60, 23, 17], even in the case of incomplete information [25]. However, universal solutions were not thought to take into account source data including implicit knowledge (in the form of TBoxes), which is demonstrated in the following example.

Example 5.2. Let $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$ be as in Example 5.1, and assume that $\mathcal{K}_\mathsf{s} = \langle \mathcal{S}, \mathcal{A}_\mathsf{s} \rangle$, where $\mathcal{S} = \{A \sqsubseteq B\}$ and $\mathcal{A}_\mathsf{s} = \{A(a)\}$. Furthermore, suppose that $\mathcal{K}_\mathsf{t} = \langle \mathcal{T}, \mathcal{A}_\mathsf{t} \rangle$, where $\mathcal{T} = \{A' \sqsubseteq B'\}$ and $\mathcal{A}_\mathsf{t} = \{A'(a)\}$. Then we have that $\mathcal{K}_\mathsf{t}$ is a solution for $\mathcal{K}_\mathsf{s}$ under $\mathcal{M}$. However, $\mathcal{K}_\mathsf{t}$ is not a universal solution for $\mathcal{K}_\mathsf{s}$ under $\mathcal{M}$. To see why this is the case, consider an interpretation $\mathcal{I}$ of Σ such that $a^\mathcal{I} = 1$, $A^\mathcal{I} = \{1\}$ and $B^\mathcal{I} = \{1\}$, and an interpretation $\mathcal{J}$ of Γ such that $a^\mathcal{J} = 1$, $B'^\mathcal{J} = \{1\}$ and $A'^\mathcal{J} = \{1, 2\}$. Then we have that $\mathcal{I}$ is a model of $\mathcal{K}_\mathsf{s}$ and $(\mathcal{I}, \mathcal{J}) \models \mathcal{B}$, and thus $\mathcal{J} \in \textsc{Sat}_\mathcal{M}(\textsc{Mod}(\mathcal{K}_\mathsf{s}))$. Thus, given that $\mathcal{J}$ is not a model of $\mathcal{K}_\mathsf{t}$ (since it does not satisfy inclusion $A' \sqsubseteq B'$), we conclude that $\mathcal{K}_\mathsf{t}$ is not a universal solution for $\mathcal{K}_\mathsf{s}$ under $\mathcal{M}$ as $\textsc{Sat}_\mathcal{M}(\textsc{Mod}(\mathcal{K}_\mathsf{s})) \neq \textsc{Mod}(\mathcal{K}_\mathsf{t})$. ■

All the universal solutions shown in the previous examples have empty TBoxes. In the following proposition, we prove that this is the case in general, which shows that universal solutions are not appropriate to represent implicit knowledge. We say that a TBox $\mathcal{O}$ over a signature Σ is *trivial* if for every interpretation $\mathcal{I}$ of Σ, it holds that $\mathcal{I} \models \mathcal{O}$ (or, in other words, if $\mathcal{O}$ is equivalent to the empty set of formulas).

Proposition 5.3. *Let $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$ be a mapping, $\mathcal{K}_\mathsf{s} = \langle \mathcal{S}, \mathcal{A}_\mathsf{s} \rangle$ a KB over Σ, and $\mathcal{K}_\mathsf{t} = \langle \mathcal{T}, \mathcal{A}_\mathsf{t} \rangle$ a KB over Γ. If $\langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s} \rangle$ is consistent and $\mathcal{K}_\mathsf{t}$ is a universal solution for $\mathcal{K}_\mathsf{s}$ under $\mathcal{M}$, then $\mathcal{T}$ is a trivial TBox.*

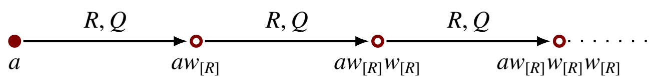
The proof of Proposition 5.3 can be found in the appendix. Notice that this proposition shows that universal solutions can be viewed as target ABoxes with empty TBoxes. We denote by $\mathcal{A}$ a KB of the form $\langle \emptyset, \mathcal{A} \rangle$.

We continue our investigation by showing that extended ABoxes are necessary to guarantee the existence of universal solutions in certain cases.

Example 5.4. Let $\mathcal{M} = (\{A, R\}, \{B\}, \{\exists R^- \sqsubseteq B\})$ and $\mathcal{K}_\mathsf{s} = \langle \mathcal{S}, \mathcal{A}_\mathsf{s} \rangle$, where $\mathcal{S} = \{A \sqsubseteq \exists R\}$ and $\mathcal{A}_\mathsf{s} = \{A(a)\}$. A natural way to construct a universal solution for $\mathcal{K}_\mathsf{s}$ under $\mathcal{M}$ is to "populate" the target with all the facts implied by $\mathsf{uni}(\mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s})$ (as it is usually done in data exchange [16, 17, 20]). In this case, we have that $A^{\mathsf{uni}(\mathcal{S}\cup\mathcal{B},\mathcal{A}_\mathsf{s})} = \{a\}$, $R^{\mathsf{uni}(\mathcal{S}\cup\mathcal{B},\mathcal{A}_\mathsf{s})} = \{(a, aw_{[R]})\}$ and $B^{\mathsf{uni}(\mathcal{S}\cup\mathcal{B},\mathcal{A}_\mathsf{s})} = \{aw_{[R]}\}$, where $aw_{[R]}$ is an object different from any of the constants in N_a, which is used to represent a null value. Thus, the ABox $\mathcal{A}_\mathsf{t} = \{B(n)\}$, where n is a labeled null, is a universal solution for $\mathcal{K}_\mathsf{s}$ under $\mathcal{M}$ if nulls are allowed, which can be readily checked using the definition of universal solution. Nevertheless, a universal solution with simple ABoxes does not exist in this case, as substituting n by any constant is too restrictive, ruining universality. ■

A natural question at this point is whether the use of null values guarantees the existence of universal solutions. Unfortunately, the following example shows that this is not the case. In fact, this example shows two different situations in which universal solutions do not exist; in the first case this is due to the impossibility of representing an infinite number of facts in a finite ABox, while in the second case this is due to the the use of disjointness axioms and the absence of the UNA (which has to be given up to comply with the OWL 2 QL standard).

Example 5.5. Let $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$, where $\Sigma = \{A, R\}$, $\Gamma = \{Q\}$, and $\mathcal{B} = \{R \sqsubseteq Q\}$. Furthermore, assume that $\mathcal{K}_\mathsf{s} = \langle \mathcal{S}, \mathcal{A}_\mathsf{s} \rangle$, where $\mathcal{A}_\mathsf{s} = \{A(a)\}$ and $\mathcal{S} = \{A \sqsubseteq \exists R,\ \exists R^- \sqsubseteq \exists R\}$. In this case, $\mathsf{uni}(\mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s})$ is infinite:



so in principle one would need an infinite number of labeled nulls to construct a universal solution. It can be easily proved that if $\mathcal{A}_\mathsf{t}$ is an (extended) ABox over Γ, then $\mathcal{A}_\mathsf{t}$ cannot be a universal solution for $\mathcal{K}_\mathsf{s}$ under $\mathcal{M}$. ■

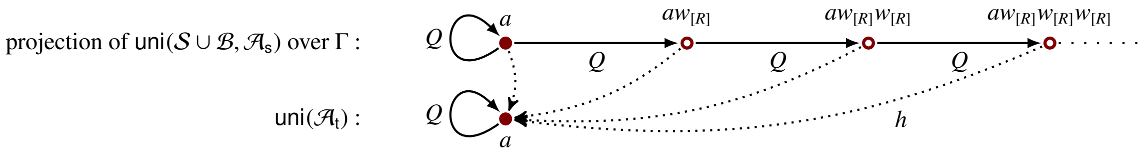
Example 5.6. Now let $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$ be defined as in Example 5.1. Moreover, assume that $\mathcal{K}_\mathsf{s} = \langle \mathcal{S}, \mathcal{A}_\mathsf{s} \rangle$, where $\mathcal{S} = \{A \sqsubseteq \neg B\}$ and $\mathcal{A}_\mathsf{s} = \{A(a), B(b)\}$, and assume that $\mathcal{A}_\mathsf{t} = \{A'(a), B'(b)\}$. As in Example 5.1, it is possible to show that $\mathcal{A}_\mathsf{t}$ is a universal solution for KB $\langle \emptyset, \mathcal{A}_\mathsf{s} \rangle$ under $\mathcal{M}$. However, with the addition of the disjointness axiom $A \sqsubseteq \neg B$, KB $\mathcal{A}_\mathsf{t}$ is no longer a universal solution (not even a solution) for $\mathcal{K}_\mathsf{s}$ under $\mathcal{M}$. The reason for this is the lack of the UNA on the one hand, and the presence of the disjointness axiom that forces a and b to be interpreted differently in the source, on the other hand. More precisely, for a model $\mathcal{J}$ of $\mathcal{A}_\mathsf{t}$ such that $a^\mathcal{J} = b^\mathcal{J}$, $A'^\mathcal{J} = B'^\mathcal{J} = \{a^\mathcal{J}\}$, there is no model $\mathcal{I}$ of $\mathcal{K}_\mathsf{s}$ such that $(\mathcal{I}, \mathcal{J}) \models \mathcal{B}$, as this forces $a^\mathcal{I} = a^\mathcal{J}$ and $b^\mathcal{I} = b^\mathcal{J}$, which is not possible since $a^\mathcal{I} \neq b^\mathcal{I}$. It can be straightforwardly proved that in this case there is no universal solution for $\mathcal{K}_\mathsf{s}$ under $\mathcal{M}$. ■

From the previous examples, we conclude that:

Proposition 5.7. *There exists a mapping $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$ and a KB $\mathcal{K}_\mathsf{s} = \langle \mathcal{S}, \mathcal{A}_\mathsf{s} \rangle$ over Σ such that there is no universal solution for $\mathcal{K}_\mathsf{s}$ under $\mathcal{M}$ (even if extended ABoxes are allowed).*

Let $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$ be a mapping and $\mathcal{K}_\mathsf{s} = \langle \mathcal{S}, \mathcal{A}_\mathsf{s} \rangle$ be a KB over Σ. As pointed out in the previous examples, a natural way to construct a universal solution for $\mathcal{K}_\mathsf{s}$ under $\mathcal{M}$ is to populate the target with all the facts implied by $\mathsf{uni}(\mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s})$. In Example 5.5, this procedure generates an infinite chain that cannot be represented in a finite ABox, which lead us to conclude that $\mathcal{K}_\mathsf{s}$ does not have a universal solution under $\mathcal{M}$ in this case. Thus, the reader may wonder whether the finiteness of $\mathsf{uni}(\mathcal{S} \cup \mathcal{B}, \mathcal{A}_s)$ is a necessary condition for the existence of universal solutions. The following example shows that this is not the case, and also gives evidence that checking whether a universal solution exists can be a computationally hard task (the complexity of this problem is studied in Section 6).

Example 5.8. Let $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$, where $\Sigma = \{A, R, S\}$, $\Gamma = \{Q\}$ and $\mathcal{B} = \{S \sqsubseteq Q, R \sqsubseteq Q\}$. Moreover, let $\mathcal{K}_s = \langle \mathcal{S}, \mathcal{A}_s \rangle$, where $\mathcal{S} = \{A \sqsubseteq \exists R, \exists R^- \sqsubseteq \exists R\}$ and $\mathcal{A}_s = \{A(a), S(a,a)\}$. Notice that $\mathsf{uni}(\mathcal{S} \cup \mathcal{B}, \mathcal{A}_s)$ as well as its projection over Γ are infinite. However, we can conclude that $\mathcal{A}_t = \{Q(a,a)\}$ is a universal solution for $\mathcal{K}_s$ under $\mathcal{M}$, as if the projection of $\mathsf{uni}(\mathcal{S} \cup \mathcal{B}, \mathcal{A}_s)$ over Γ is transformed into an infinite ABox, then the resulting ABox has the same interpretations as $\mathcal{A}_t$. Or, in other words, it is possible to conclude that $\mathcal{A}_t$ is a universal solution for $\mathcal{K}_s$ under $\mathcal{M}$ as $\mathsf{uni}(\mathcal{A}_t)$ is contained in the projection of $\mathsf{uni}(\mathcal{S} \cup \mathcal{B}, \mathcal{A}_s)$ over Γ, and there exists a homomorphism h from the projection of $\mathsf{uni}(\mathcal{S} \cup \mathcal{B}, \mathcal{A}_s)$ over Γ to $\mathsf{uni}(\mathcal{A}_t)$:



■

We conclude this section by demonstrating that universal solutions can be of exponential size, thus indicating that it can be difficult to deal with them in practice. We use $|\mathcal{M}|$ and $|\mathcal{K}|$ to denote the sizes (number of symbols) of a mapping $\mathcal{M}$ and a KB $\mathcal{K}$, respectively.

Example 5.9. We show that there exists a family of mappings $\{\mathcal{M}^n = (\Sigma^n, \Gamma^n, \mathcal{B}^n)\}_{n \geq 1}$ and a family of KBs $\{\mathcal{K}_s^n\}_{n \geq 1}$ such that every $\mathcal{K}_s^n$ is defined over Σ^n ($n \geq 1$), and the smallest universal solution for $\mathcal{K}_s^n$ under $\mathcal{M}^n$ is of size $2^{c(|\mathcal{M}^n| + |\mathcal{K}_s^n|)}$, for some constant $c > 0$.

Indeed, let $n \geq 1$ be a natural number. Then mapping $\mathcal{M}^n = (\Sigma^n, \Gamma^n, \mathcal{B}^n)$ is defined as follows:

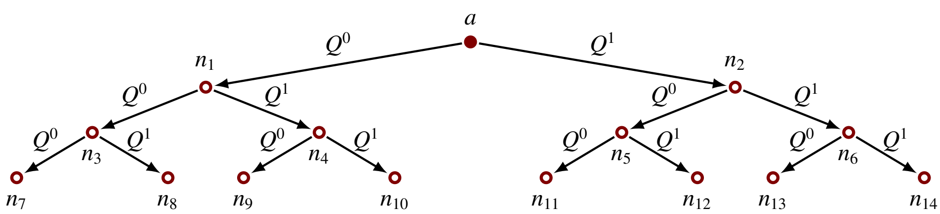
$$\begin{aligned}
\Sigma^n &= \{A\} \cup \{R_i^k \mid i \in \{1, \ldots, n\}, k \in \{0,1\}\}, \\
\Gamma^n &= \{Q^k \mid k \in \{0,1\}\} \\
\mathcal{B}^n &= \{R_i^k \sqsubseteq Q^k \mid i \in \{1, \ldots, n\}, k \in \{0,1\}\}.
\end{aligned}$$

Moreover, knowledge base $\mathcal{K}_s^n$ is defined as $\langle \mathcal{S}^n, \mathcal{A}_s^n \rangle$, where $\mathcal{S}^n$ is defined as:

$$\{A \sqsubseteq \exists R_1^k \mid k \in \{0,1\}\} \quad \cup \quad \{\exists R_i^{k-} \sqsubseteq \exists R_{i+1}^{\ell} \mid i \in \{1, \ldots, n-1\}, k \in \{0,1\} \text{ and } \ell \in \{0,1\}\},$$

and $\mathcal{A}_s^n$ is defined as $\{A(a)\}$.

For every $n \geq 1$, a universal solution $\mathcal{A}_t^n$ for $\mathcal{K}_s^n$ under $\mathcal{M}^n$ exists. This universal solution $\mathcal{A}_t^n$ is an edge-labeled full binary tree of depth n (containing 2^n leaves). Below we depict $\mathcal{A}_t^3$, where n_1, …, n_{14} are null values:



It can be proved that $|\mathcal{A}_t^n| \geq 2^{c(|\mathcal{M}^n| + |\mathcal{K}_s^n|)}$ (for every $n \geq 1$) for some $c > 0$. Moreover, it is straightforward to prove that $\mathcal{A}_t^n$ is the smallest universal solution for $\mathcal{K}_s^n$ under $\mathcal{M}^n$. ■

5.2. Universal UCQ*-Solutions*

Our first observation is that the notion of universal UCQ-solution is a relaxation of the notion of universal solution, as shown in the following proposition.

Proposition 5.10. *Let $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$ be a mapping, $\mathcal{K}_s$ a KB over Σ, and $\mathcal{K}_t$ a KB over Γ. If $\mathcal{K}_t$ is a universal solution for $\mathcal{K}_s$ under $\mathcal{M}$, then $\mathcal{K}_t$ is a universal* UCQ*-solution for $\mathcal{K}_s$ under $\mathcal{M}$.*

Proof. Let $\mathcal{K}_t$ be a universal solution for $\mathcal{K}_s = \langle \mathcal{S}, \mathcal{A}_s \rangle$ under $\mathcal{M}$ and q a UCQ over Γ.

First, we show $cert(q, \langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_s \rangle) \subseteq cert(q, \mathcal{K}_t)$. Assume $\mathcal{J}$ is a model of $\mathcal{K}_t$. Since $\mathcal{K}_t$ is a solution for $\mathcal{K}_s$ under $\mathcal{M}$, there exists a model $\mathcal{I}$ of $\mathcal{K}_s$ such that $(\mathcal{I}, \mathcal{J}) \models \mathcal{B}$. Let $\mathcal{H}$ be the interpretation of $\Sigma \cup \Gamma$ defined as the union of $\mathcal{I}$ and $\mathcal{J}$, that is, $\mathcal{H} = \langle \Delta^{\mathcal{H}}, \cdot^{\mathcal{H}} \rangle$, $\Delta^{\mathcal{H}} = \Delta^{\mathcal{I}} \cup \Delta^{\mathcal{J}}$, $a^{\mathcal{H}} = a^{\mathcal{I}}$ for each $a \in N_a$ such that $a^{\mathcal{I}}$ is defined, $A^{\mathcal{H}} = A^{\mathcal{I}}$ for each concept name $A \in \Sigma$, $A^{\mathcal{H}} = A^{\mathcal{J}}$ for each concept name $A \in \Gamma$, $P^{\mathcal{H}} = P^{\mathcal{I}}$ for each role name $P \in \Sigma$, and $P^{\mathcal{H}} = P^{\mathcal{J}}$ for each role name $P \in \Gamma$. Then $\mathcal{H}$ is a model of $\langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_s \rangle$. Suppose $\vec{a} \in cert(q, \langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_s \rangle)$, it implies $\mathcal{H} \models q(\vec{a})$. Next, as q is a target query, we have that $\mathcal{J} \models q(\vec{a})$. Given that $\mathcal{J}$ is an arbitrary model of $\mathcal{K}_t$, we conclude that $\vec{a} \in cert(q, \mathcal{K}_t)$.

Now, we show $cert(q, \mathcal{K}_t) \subseteq cert(q, \langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_s \rangle)$. Let $\mathcal{H}$ be a model of $\langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_s \rangle$. From $\mathcal{H}$ we can construct interpretations $\mathcal{I}$ and $\mathcal{J}$ of Σ and Γ, respectively, such that $\mathcal{H}$ is the union of $\mathcal{I}$ and $\mathcal{J}$. Then $\mathcal{I}$ is a model of $\mathcal{K}_s$ and $(\mathcal{I}, \mathcal{J}) \models \mathcal{B}$. Since $\mathcal{K}_t$ is a universal solution for $\mathcal{K}_s$ under $\mathcal{M}$, it follows that $\mathcal{J}$ is a model of $\mathcal{K}_t$. Suppose $\vec{a} \in cert(q, \mathcal{K}_t)$, it implies that $\mathcal{J} \models q(\vec{a})$, and since q is a target query, and $\mathcal{J}$ and $\mathcal{H}$ agree on the constants and target symbols, it follows that $\mathcal{H} \models q(\vec{a})$. Given that $\mathcal{H}$ is an arbitrary model of $\langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_s \rangle$, we have that $\vec{a} \in cert(q, \langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_s \rangle)$. □

However, the converse direction of Proposition 5.10 does not hold, as shown in the following example.

Example 5.11. Let $\mathcal{K}_s$, $\mathcal{M}$ and $\mathcal{K}_t$ be as in Example 5.2. As pointed out in that example, $\mathcal{K}_t$ is not a universal solution for $\mathcal{K}_s$ under $\mathcal{M}$. However, it is easy to see that $\mathcal{K}_t$ is a universal UCQ-solution for $\mathcal{K}_s$ under $\mathcal{M}$. ■

Notably, the previous example also shows that, as opposed to universal solutions, universal UCQ-solutions can have non-trivial TBoxes. As a consequence of this, we obtain that universal UCQ-solutions can be smaller than universal solutions, as there is no need to materialize all facts (since they can be derived using the target TBoxes).

In the following example, we show that there are cases where universal solutions do not exist but universal UCQ-solutions do. More precisely, we focus on the two cases provided in Example 5.5, and show that certain infinite chains that cannot be encoded in a universal solution can be finitely represented if the more relaxed notion of universal UCQ-solution is considered, and also show that disjointness axioms in the source or the mapping do not have any impact on universal UCQ-solutions.

Example 5.12. Let $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$, where $\Sigma = \{A, R\}$, $\Gamma = \{Q\}$, and $\mathcal{B} = \{R \sqsubseteq Q\}$. Furthermore, assume that $\mathcal{K}_s = \langle \mathcal{S}, \mathcal{A}_s \rangle$, where $\mathcal{A}_s = \{A(a)\}$ and $\mathcal{S} = \{A \sqsubseteq \exists R,\ \exists R^- \sqsubseteq \exists R\}$. It can be verified that KB $\mathcal{K}_t = \langle \mathcal{T}, \mathcal{A}_t \rangle$, where $\mathcal{T} = \{\exists Q^- \sqsubseteq \exists Q\}$ and $\mathcal{A}_t = \{\exists Q(a)\}$, is a universal UCQ-solution for $\mathcal{K}_s$ under $\mathcal{M}$, as the aforementioned infinite chain (c.f. Example 5.5) can be finitely represented by combining $\exists Q(a)$ with $\exists Q^- \sqsubseteq \exists Q$. ■

Example 5.13. Now let $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$ be defined as in Example 5.1. Moreover, assume that $\mathcal{K}_s = \langle \mathcal{S}, \mathcal{A}_s \rangle$, where $\mathcal{S} = \{A \sqsubseteq \neg B\}$ and $\mathcal{A}_s = \{A(a), B(b)\}$, and assume that $\mathcal{A}_t = \{A'(a), B'(b)\}$. In Example 5.6, we show that $\mathcal{A}_t$ is not a universal solution for KB $\mathcal{K}_s$ under $\mathcal{M}$. On the other hand, it can be shown that $\mathcal{A}_t$ is a universal UCQ-solution for $\mathcal{K}_s$ under $\mathcal{M}$. In fact, this holds independently of whether the unique name assumption is employed. ■

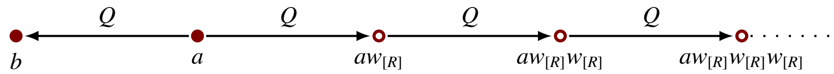
From the previous examples, we conclude that:

Proposition 5.14. *There exists a mapping $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$ and a KB $\mathcal{K}_s = \langle \mathcal{S}, \mathcal{A}_s \rangle$ over Σ such that, there is no universal solution for $\mathcal{K}_s$ under $\mathcal{M}$, but there exists a universal* UCQ*-solution for $\mathcal{K}_s$ under $\mathcal{M}$.*

Unfortunately, we show in the following example that there are cases where universal UCQ-solutions do not exist.

Example 5.15. Let $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$, where $\Sigma = \{A, R, S\}$, $\Gamma = \{Q\}$ and $\mathcal{B} = \{R \sqsubseteq Q, S \sqsubseteq Q\}$. Moreover, let $\mathcal{K}_s = \langle \mathcal{S}, \mathcal{A}_s \rangle$, where $\mathcal{S} = \{A \sqsubseteq \exists R, \exists R^- \sqsubseteq \exists R\}$ and $\mathcal{A}_s = \{A(a), S(a, b)\}$. Then the projection over Γ of the canonical model of $\langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_s \rangle$ can be depicted as follows:

projection of $\text{uni}(\mathcal{S} \cup \mathcal{B}, \mathcal{A}_s)$ over Γ :



In this case, the basic requirement for a KB $\mathcal{K}_t = \langle \mathcal{T}, \mathcal{A}_t \rangle$ to be a universal UCQ-solution for $\mathcal{K}_s$ under $\mathcal{M}$ is that $\mathcal{A}_t$ contain $\{\exists Q(a), Q(a,b)\}$. Thus, the approach in Example 5.12 to obtain a universal UCQ-solution cannot work, as having the axiom $\exists Q^- \sqsubseteq \exists Q$ in $\mathcal{T}$ would also make the query $\exists x. Q(b,x)$ evaluate to true over $\mathcal{K}_t$, while it evaluates to false over $\langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_s \rangle$. In general, a universal UCQ-solution for $\mathcal{K}_s$ under $\mathcal{M}$ does not exists, as every KB $\mathcal{K}_t = \langle \mathcal{T}, \mathcal{A}_t \rangle$ over Γ with $\{\exists Q(a), Q(a,b)\} \subseteq \mathcal{A}_t$ is not a universal UCQ-solution for $\mathcal{K}_s$ under $\mathcal{M}$. ■

We conclude this section by showing that, as in the case of universal solutions, there are some cases where only universal UCQ-solutions of exponential size exists.

Example 5.16. There exists a family of mappings $\{\mathcal{M}^n = (\Sigma^n, \Gamma^n, \mathcal{B}^n)\}_{n \geq 1}$ and a family of KBs $\{\mathcal{K}_s^n\}_{n \geq 1}$ such that every $\mathcal{K}_s^n$ is defined over Σ^n ($n \geq 1$), and the smallest universal UCQ-solution for $\mathcal{K}_s^n$ under $\mathcal{M}_n$ is of size $2^{\Omega(|\mathcal{M}_n| + |\mathcal{K}_s^n|)}$.

Indeed, let $n \geq 1$ be a natural number. Then mapping $\mathcal{M}^n = (\Sigma^n, \Gamma^n, \mathcal{B}^n)$ is defined as follows:

$$\begin{array}{rcl} \Sigma^n & = & \{A\} \cup \{R_i^k \mid i \in \{1, \ldots, n\}, k \in \{0,1\}\} \cup \{S^0, S^1\}, \\ \Gamma^n & = & \{Q^k \mid k \in \{0,1\}\} \\ \mathcal{B}^n & = & \{R_i^k \sqsubseteq Q^k \mid i \in \{1, \ldots, n\}, k \in \{0,1\}\} \cup \{S^k \sqsubseteq Q^k \mid k \in \{0,1\}\}. \end{array}$$

Moreover, knowledge base $\mathcal{K}_s^n$ is defined as $\langle \mathcal{S}^n, \mathcal{A}_s^n \rangle$, where $\mathcal{S}^n$ is defined as:

$$\{A \sqsubseteq \exists R_1^k \mid k \in \{0,1\}\} \quad \cup \quad \{\exists R_i^{k-} \sqsubseteq \exists R_{i+1}^{\ell} \mid i \in \{1, \ldots, n-1\}, k \in \{0,1\} \text{ and } \ell \in \{0,1\}\},$$

and $\mathcal{A}_s^n$ is defined as $\{A(a), S^0(b,c), S^1(d,e)\}$, where a,b,c,d,e are pairwise distinct constants.

For every $n \geq 1$, a universal solution $\mathcal{A}_t^n$ for $\mathcal{K}_s^n$ under $\mathcal{M}^n$ exists. This universal solution $\mathcal{A}_t^n$ consists of membership assertions $Q^0(b,c), Q^1(d,e)$ together with an edge-labeled full binary tree of depth n (that contains 2^n leaves). As in the case of Example 5.9, the root of this tree is a, the label of each edge is one of the role names Q^k ($k \in \{0,1\}$), and the tree contains labeled nulls in every node except for the root.

In this case, there exist no universal UCQ-solution distinct from the universal solutions for $\mathcal{K}_s^n$ under $\mathcal{M}^n$, as each of the non-trivial axioms over $\Gamma^n = \{Q^0, Q^1\}$ combined with $\mathcal{A}_t^n$ would produce more certain answers to some queries than $\mathcal{S}^n \cup \mathcal{B}^n$ combined with $\mathcal{A}_s^n$. Hence, as in the case of Example 5.9, we can conclude that $\mathcal{A}_t^n$ is the smallest universal UCQ-solution for $\mathcal{K}_s^n$ under $\mathcal{M}^n$, from which our initial claim follows. ■

5.3. UCQ-*representations*

In this section, we discuss several simple examples explaining various cases when a UCQ-representation exists and when it does not. We start by showing how the existence of UCQ-representations depends on the shape of the mappings. In the following example, we consider signatures consisting of concept names only, and TBoxes and mappings containing only positive axioms (i.e., no disjointness axioms).

Example 5.17. Assume that $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$, where $\Sigma = \{A, B, C\}$ and $\Gamma = \{A', B', C'\}$. Moreover, let $\mathcal{S} = \{A \sqsubseteq B\}$. Consider the following cases for TBox $\mathcal{B}$.

(1) If $\mathcal{B} = \{B \sqsubseteq B'\}$, then there exists no UCQ-representation: take ABox $\mathcal{A}_s = \{A(a)\}$, then query $q = B'(a)$ evaluates to true over $\langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_s \rangle$. However, for every target TBox $\mathcal{T}$, q evaluates to false over $\langle \mathcal{T} \cup \mathcal{B}, \mathcal{A}_s \rangle$.

(2) If $\mathcal{B} = \{A \sqsubseteq A', B \sqsubseteq B'\}$, then, as expected, $\mathcal{T} = \{A' \sqsubseteq B'\}$ is a UCQ-representation of $\mathcal{S}$ under $\mathcal{M}$.

(3) If $\mathcal{B} = \{A \sqsubseteq A', B \sqsubseteq B', A \sqsubseteq C'\}$, then there exist several UCQ-representations: $\mathcal{T} = \{A' \sqsubseteq B'\}$, $\mathcal{T}' = \{C' \sqsubseteq B'\}$ and their combination.

(4) If $\mathcal{B} = \{A \sqsubseteq A', B \sqsubseteq B', C \sqsubseteq A'\}$, then there exists no UCQ-representation: on one hand, if a target TBox contains $A' \sqsubseteq B'$, then for $\mathcal{A}_s = \{C(c)\}$, $q = B'(c)$ evaluates to true over $\langle \mathcal{T} \cup \mathcal{B}, \mathcal{A}_s \rangle$ and to false over $\langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_s \rangle$. On the other hand, if a target TBox does not imply $A' \sqsubseteq B'$, then for $\mathcal{A}_s = \{A(a)\}$, $q = B'(a)$ evaluates to true over $\langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_s \rangle$ and to false over $\langle \mathcal{T} \cup \mathcal{B}, \mathcal{A}_s \rangle$.

(5) If $\mathcal{B} = \{A \sqsubseteq A', B \sqsubseteq B', A \sqsubseteq C', C \sqsubseteq A'\}$, then $\mathcal{T}' = \{C' \sqsubseteq B'\}$ is a UCQ-representation of $\mathcal{S}$ under $\mathcal{M}$. Note that $\mathcal{T} = \{A' \sqsubseteq B'\}$ is not a UCQ-representation of $\mathcal{S}$ under $\mathcal{M}$ for the same reason as explained in item (4) above. ■

Roughly speaking, the previous example illustrates that there exists no UCQ-representation when the mapping is underspecified for the source concepts, as in (1) where A is not mapped to anything, or the mapping is overspecified for the target concepts, as in (4) where A' is the image of two source concepts. A "good" mapping is a mapping that is overspecified for the source concepts, as in (3) where A is mapped to two distinct target concepts and it is possible to construct two incomparable UCQ-representations.

In the next example, we also consider roles in the signatures. This examples shows that in some cases to ensure the existence of a UCQ-representation, it is necessary to map a complete role, that is, it must appear in a role inclusion in the mapping.

Example 5.18. Assume that $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$, where $\Sigma = \{A, R\}$ and $\Gamma = \{A', R', B'\}$. Moreover, let $\mathcal{S} = \{A \sqsubseteq \exists R\}$. Consider the following cases for TBox $\mathcal{B}$.

(1) If $\mathcal{B} = \{A \sqsubseteq A', \exists R^- \sqsubseteq B'\}$, then there exists no UCQ-representation of $\mathcal{S}$ under $\mathcal{M}$: take $\mathcal{A}_{\mathsf{s}} = \{A(a)\}$ and a Boolean target query $q = \exists x. (A'(a) \wedge B'(x))$. Then q evaluates to true over $\langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_{\mathsf{s}} \rangle$. Let us consider two target TBoxes $\mathcal{T}$ such that q also evaluates to true over $\langle \mathcal{T} \cup \mathcal{B}, \mathcal{A}_{\mathsf{s}} \rangle$:

   (a) $\mathcal{T} = \{A' \sqsubseteq B'\}$. Then for the query $q' = B'(a)$, it holds that $\langle \mathcal{T} \cup \mathcal{B}, \mathcal{A}_{\mathsf{s}} \rangle \models q'$, while $\langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_{\mathsf{s}} \rangle \not\models q'$. Hence $\mathcal{T}$ is not a UCQ-representation.

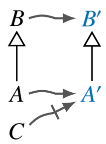
   (b) $\mathcal{T} = \{A' \sqsubseteq \exists R', \exists R'^- \sqsubseteq B'\}$. Then for the query $q' = \exists x. R'(a, x)$, it holds that $\langle \mathcal{T} \cup \mathcal{B}, \mathcal{A}_{\mathsf{s}} \rangle \models q'$, while $\langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_{\mathsf{s}} \rangle \not\models q'$. Hence $\mathcal{T}$ is not a UCQ-representation.

(2) If $\mathcal{B} = \{A \sqsubseteq A', \exists R^- \sqsubseteq B', R \sqsubseteq R'\}$, then, as opposed to the previous case, $\mathcal{T} = \{A' \sqsubseteq \exists R', \exists R'^- \sqsubseteq B'\}$ is a UCQ-representation of $\mathcal{S}$ under $\mathcal{M}$. ■

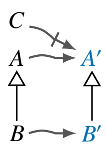
Finally, we provide an example involving disjointness axioms in the mapping. Now we will, however, fix the mapping, and see how the shape of UCQ-representations depends on the shape of the source TBox.

Example 5.19. Assume $\mathcal{M} = (\{A, B, C\}, \{A', B'\}, \mathcal{B})$, where $\mathcal{B} = \{A \sqsubseteq A', B \sqsubseteq B', C \sqsubseteq \neg A'\}$. In the following, to better illustrate the structure of TBoxes and mappings, we use a graphical notation in which basic concepts are represented as nodes in a graph, and we use different types of directed edges: (⟶▷) unlabeled edges to represent inclusion assertions between basic concepts, (⇝) unlabeled "wavy" edges to represent assertions in the mapping. The barred arrows represent disjointness axioms.

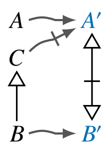
(1) If $\mathcal{S} = \{A \sqsubseteq B\}$, then TBox $\mathcal{T} = \{A' \sqsubseteq B'\}$ is a UCQ-representation of $\mathcal{S}$ under $\mathcal{M}$. First, notice that every source ABox $\mathcal{A}_{\mathsf{s}}$ is consistent with $\mathcal{S}$. It should be clear that for every $\mathcal{A}_{\mathsf{s}} = \{X(a)\}$ for $X \in \{A, B, C\}$ or $\mathcal{A}_{\mathsf{s}} = \{B(a), C(a)\}$, $\mathcal{A}_{\mathsf{s}}$ is consistent with $\mathcal{S} \cup \mathcal{B}$, and $\mathit{cert}(q, \langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_{\mathsf{s}} \rangle) = \mathit{cert}(q, \langle \mathcal{T} \cup \mathcal{B}, \mathcal{A}_{\mathsf{s}} \rangle)$ for each UCQ q.
Consider now $\mathcal{A}_{\mathsf{s}} = \{A(a), C(a)\}$, then $\mathcal{A}_{\mathsf{s}}$ is not consistent with $\mathcal{S} \cup \mathcal{B}$ (in fact, $\mathcal{A}_{\mathsf{s}}$ is not consistent already with $\mathcal{B}$), so $\mathit{cert}(q, \langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_{\mathsf{s}} \rangle) = \mathit{AllTup}(q)$ for each UCQ q. On the other hand, $\mathcal{A}_{\mathsf{s}}$ is not consistent with $\mathcal{T} \cup \mathcal{B}$ either, so as well, $\mathit{cert}(q, \langle \mathcal{T} \cup \mathcal{B}, \mathcal{A}_{\mathsf{s}} \rangle) = \mathit{AllTup}(q)$ for each UCQ q.



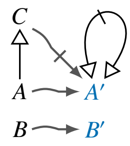
(2) If $\mathcal{S} = \{B \sqsubseteq A\}$, then similarly to the previous case, TBox $\mathcal{T} = \{B' \sqsubseteq A'\}$ is a UCQ-representation of $\mathcal{S}$ under $\mathcal{M}$, but now it is a bit more involved. Namely, in this case ABox $\mathcal{A}_{\mathsf{s}} = \{B(a), C(a)\}$ is not consistent with $\mathcal{S} \cup \mathcal{B}$, but consistent with $\mathcal{B}$ alone. But $\mathcal{A}_{\mathsf{s}}$ is not consistent with $\mathcal{T} \cup \mathcal{B}$ due to the axiom $B' \sqsubseteq A'$ in $\mathcal{T}$. So $\mathit{cert}(q, \langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_{\mathsf{s}} \rangle) = \mathit{cert}(q, \langle \mathcal{T} \cup \mathcal{B}, \mathcal{A}_{\mathsf{s}} \rangle)$ for each ABox $\mathcal{A}_{\mathsf{s}}$ and UCQ q over Γ.



(3) If $\mathcal{S} = \{B \sqsubseteq C\}$, then TBox $\mathcal{T} = \{B' \sqsubseteq \neg A'\}$ is a UCQ-representation of $\mathcal{S}$ under $\mathcal{M}$. This case is in some sense the opposite of (2). Consider ABox $\mathcal{A}_{\mathsf{s}} = \{A(a), B(a)\}$, then $\mathcal{A}_{\mathsf{s}}$ is inconsistent with $\mathcal{S} \cup \mathcal{B}$. Now the fact that $\mathcal{A}_{\mathsf{s}}$ is inconsistent with $\mathcal{T} \cup \mathcal{B}$ is achieved with the disjointness axiom $B' \sqsubseteq \neg A'$ in $\mathcal{T}$.



(4) If $\mathcal{S} = \{A \sqsubseteq C\}$, then TBox $\mathcal{T} = \{A' \sqsubseteq \neg A'\}$ is a UCQ-representation of $\mathcal{S}$ under $\mathcal{M}$. Observe, that every ABox $\mathcal{A}_{\mathsf{s}}$ such that $A(a) \in \mathcal{A}_{\mathsf{s}}$ for some constant a is inconsistent with $\mathcal{S} \cup \mathcal{B}$. So the axiom $A' \sqsubseteq \neg A'$ in $\mathcal{T}$ assures that every such $\mathcal{A}_{\mathsf{s}}$ is also inconsistent with $\mathcal{T} \cup \mathcal{B}$. One the other hand, it is easy to see that for every source ABox that does not contain assertions of the form $A(a)$, the required condition is satisfied.



■

Notice that in the previous example, the source TBox always contains exactly one inclusion of concept names, and depending on the concepts involved in it, this inclusion needs to be represented either by another inclusion of concept names, or by a disjointness axiom.

It is worth mentioning that in Section 7.1, Example 7.4 illustrates a case with disjointness axioms in the mapping where a UCQ-representation does not exist.

5.4. Comparison of solutions

Out of the three notions of solution discussed in the previous sections, none of them could be considered as the preferred one in all possible scenarios. Each one of them has its strengths and its weaknesses, which can be summarized as follows.

Universal solutions are the preferred translations if one is interested in preserving logical correctness of the knowledge stored in the target KB, as these solutions are the most precise model-theoretical translations. However, they present several limitations from the practical point of view: *(i)* if one considers extended ABoxes, then universal solutions can be of exponential size; *(ii)* universal solutions are sensitive to presence of disjointness axioms: in some cases one disjointness axiom is enough to ruin existence of a universal solution (see Example 5.6); and *(iii)* universal solutions are sensitive to whether the UNA is employed or not: there are examples when a universal solution exists under the UNA, but it does not exist without the UNA. This is illustrated, e.g., in Example 5.13.[3]

Universal UCQ-solutions are the preferred translations if one considers a scenario where the main reasoning task is query answering over the target KB. In this scenario, universal UCQ-solutions behave better than universal solutions, in particular they overcome the last two limitations of universal solutions mentioned in the previous paragraph. Besides, universal UCQ-solutions are, in general, more succinct than universal solutions (although in the worst case can be of the same size).

Finally, in a scenario where data is changing, or it is not known, and where the main reasoning task is query answering, UCQ-representations immediately stand out with their nice computational properties: it is shown in Section 7 that their existence is decidable in polynomial time, and their size is bound by a polynomial as they are TBoxes. Moreover, when a UCQ-representation exists, one has a straightforward polynomial-time algorithm for computing universal UCQ-solutions of polynomial size.

6. Complexity results on existence and membership of universal solutions

In this section, we study the membership and non-emptiness problems for universal solutions, in the cases where such solutions are required to be (simple) KBs, see Section 6.2, and where they are allowed to be extended KBs (i.e., nulls are allowed in the ABoxes), see Sections 6.3 and 6.4. We start by presenting in Section 6.1 a characterization of universal solutions in *DL-Lite*$_\mathcal{R}$.

6.1. Characterization of universal solutions

We define the notion of Γ-safeness required to deal with disjointness axioms in the source KB and mapping. Assume that $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$ is a mapping and $\mathcal{K}_{\mathsf{s}} = \langle \mathcal{S}, \mathcal{A}_{\mathsf{s}} \rangle$ is a KB over Σ. Let $\mathcal{K} = \langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_{\mathsf{s}} \rangle$ and let $\mathsf{uni}(\mathcal{K})$ be the canonical model of $\mathcal{K}$. We say that an element $o \in \Delta^{\mathsf{uni}(\mathcal{K})}$ is Γ-*invisible* if

$$o \notin N_a \quad \text{and} \quad \mathbf{t}_\Gamma^{\mathsf{uni}(\mathcal{K})}(o) = \emptyset.$$

[3]Note that standard reasoning and conjunctive query answering in *DL-Lite*$_\mathcal{R}$ is not sensitive to the presence of the UNA.

Then a basic concept B over Σ is said to be *safe in* $\mathsf{uni}(\mathcal{K})$ if for every $o \in B^{\mathsf{uni}(\mathcal{K})}$, o is Γ-invisible. Intuitively, safeness for B means no constant "associated" with B and no target concept "associated" with B via $\mathcal{S}$ and $\mathcal{B}$ can be mentioned in the target; in Example 5.6 neither A nor B is safe in $\mathsf{uni}(\mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s})$. Furthermore, a pair of basic concepts (B, C) is is said to be *safe* if B or C is safe. Intuitively, if a pair (B, C) is not safe and $(B \sqsubseteq \neg C) \in \mathcal{S}$, then universal solutions cannot exist, as explained in Example 5.6. Similarly, we say that a basic role R over Σ is *safe in* $\mathsf{uni}(\mathcal{K})$ if for every $(o, o') \in R^{\mathsf{uni}(\mathcal{K})}$, either o or o' is Γ-invisible. Then, a pair of basic roles (R, Q) is *safe in* $\mathsf{uni}(\mathcal{K})$ if *(1)* R or Q is safe in $\mathsf{uni}(\mathcal{K})$, and *(2)* for every $(o, o') \in R^{\mathsf{uni}(\mathcal{K})}$ and $(o, o'') \in Q^{\mathsf{uni}(\mathcal{K})}$, either o' or o'' is Γ-invisible.

Definition 6.1. *$\mathcal{K}_\mathsf{s} = \langle \mathcal{S}, \mathcal{A}_\mathsf{s} \rangle$ is* Γ-safe with respect to $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$ *if*

(cs) *each pair of concepts (B, C) is safe in $\mathsf{uni}(\mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s})$, whenever $(B \sqsubseteq \neg C) \in \mathcal{S}$,*

(rs) *each pair of roles (R, Q) is safe in $\mathsf{uni}(\mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s})$, whenever $(R \sqsubseteq \neg Q) \in \mathcal{S}$,*

(ce) *$B^{\mathsf{uni}(\mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s})} = \emptyset$, for each basic concept B such that $(B \sqsubseteq \neg B') \in \mathcal{B}$,*

(re) *$R^{\mathsf{uni}(\mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s})} = \emptyset$, for each basic role R such that $(R \sqsubseteq \neg R') \in \mathcal{B}$.*

Note that if $\mathcal{K}_\mathsf{s}$ and $\mathcal{B}$ do not contain disjointness axioms, $\mathcal{K}_\mathsf{s}$ is trivially Γ-safe with respect to $\mathcal{M}$.

We also define the canonical model of an extended ABox $\mathcal{A}$. Without loss of generality we may assume that $\mathcal{A}$ contains only membership assertions with atomic concepts and roles. Denote by $\mathsf{null}(\mathcal{A})$ the set of labeled nulls occurring in $\mathcal{A}$. Then the canonical model $\mathsf{uni}(\mathcal{A})$ is defined as follows:

$$\begin{array}{rclrcl} \Delta^{\mathsf{uni}(\mathcal{A})} & = & \mathsf{ind}(\mathcal{A}) \cup \mathsf{null}(\mathcal{A}), & A^{\mathsf{uni}(\mathcal{A})} & = & \{a \in \mathsf{ind}(\mathcal{A}) \cup \mathsf{null}(\mathcal{A}) \mid A(a) \in \mathcal{A}\}, \\ a^{\mathsf{uni}(\mathcal{A})} & = & a, \text{ for } a \in \mathsf{ind}(\mathcal{A}), & P^{\mathsf{uni}(\mathcal{A})} & = & \{(a, b) \in (\mathsf{ind}(\mathcal{A}) \cup \mathsf{null}(\mathcal{A})) \times (\mathsf{ind}(\mathcal{A}) \cup \mathsf{null}(\mathcal{A})) \mid P(a, b) \in \mathcal{A}\}. \end{array}$$

Now, we are ready to provide a characterization of universal solutions, where we already take into account Proposition 5.3, and therefore consider only target ABoxes as universal solutions. The proof can be found in the appendix.

Lemma 6.2. *An (extended) ABox $\mathcal{A}_\mathsf{t}$ over Γ is a universal solution for a KB $\mathcal{K}_\mathsf{s} = \langle \mathcal{S}, \mathcal{A}_\mathsf{s} \rangle$ under a mapping $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$ iff the following conditions hold:*

(safe) *$\mathcal{K}_\mathsf{s}$ is Γ-safe with respect to $\mathcal{M}$;*

(hom) *$\mathsf{uni}(\mathcal{A}_\mathsf{t})$ is Γ-homomorphically equivalent to $\mathsf{uni}(\mathcal{K}_\mathsf{sb})$, for $\mathcal{K}_\mathsf{sb} = \langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s} \rangle$.*

Below we show how checking whether a concept is safe can be done in NLogSpace by using TBox reasoning in *DL-Lite*$_\mathcal{R}$ [61]. The proof can be extended to show that condition **(safe)** can also be checked in NLogSpace.

Proposition 6.3. *Given a KB $\mathcal{K}$, it can be decided in* NLogSpace *whether a basic concept B is safe in $\mathsf{uni}(\mathcal{K})$.*

Proof. Checking whether B is safe in $\mathsf{uni}(\mathcal{K})$, for $\mathcal{K} = \langle \mathcal{O}, \mathcal{A} \rangle$, amounts to verifying whether *(i)* $\mathcal{K} \not\models B(a)$ for each $a \in \mathsf{ind}(\mathcal{A})$, and *(ii)* for each role R such that $w_{[R]} \in \Delta^{\mathsf{gen}(\mathcal{K})}$ and $\mathcal{O} \models \exists R^- \sqsubseteq B$, it holds that $\mathcal{O} \not\models \exists R^- \sqsubseteq B'$ for each basic concept B' over Γ. Then, given a role R, we can verify whether $w_{[R]} \in \Delta^{\mathsf{gen}(\mathcal{K})}$ in NLogSpace as follows. We use an algorithm for directed graph reachability, in a graph where the nodes are taken from the union of $\mathsf{ind}(\mathcal{K})$ and $\{w_{[S]} \mid S$ is a role in $\mathcal{K}\}$, and the edges correspond to the generating relation $\rightsquigarrow_\mathcal{K}$ (c.f. Section 3.2, the definition of the canonical model). Starting from some $a \in \mathsf{ind}(\mathcal{K})$, we "follow" a sequence of roles $R_1, \ldots, R_n = R$ (with $n \geq 1$) in such a way that, when we "guess" R_1 we check whether $a \rightsquigarrow_\mathcal{K} w_{[R_1]}$, and when, while "remembering" R_i, $i > 0$, we "guess" R_{i+1}, we check whether $w_{[R_i]} \rightsquigarrow_\mathcal{K} w_{[R_{i+1}]}$, and "forget" R_i. □

As for condition **(hom)**, we show how to check it in Section 6.2 for simple universal solutions, i.e., when we consider only simple target ABoxes, and in Section 6.3 for extended universal solutions, i.e., when we consider extended target ABoxes. Next, we provide a characterization of the cases when a universal solution exists.

Lemma 6.4. *Let $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$ be a mapping, and $\mathcal{K}_\mathsf{s} = \langle \mathcal{S}, \mathcal{A}_\mathsf{s} \rangle$ a KB over Σ. Then, a universal solution with extended ABoxes for $\mathcal{K}_\mathsf{s}$ under $\mathcal{M}$ exists iff the following conditions hold:* ***(safe)*** *and*

(core) *$\mathsf{uni}(\mathcal{K}_\mathsf{sb})$ is Γ-homomorphically embeddable into a finite subset of itself, for $\mathcal{K}_\mathsf{sb} = \langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s} \rangle$.*

Proof. ($\Leftarrow$) Let $\mathcal{A}_t$ be an ABox over Γ such that $\mathsf{uni}(\mathcal{A}_t)$ is a finite subset of $\mathsf{uni}(\mathcal{K}_{sb})$ and there exists a Γ-homomorphism h from $\mathsf{uni}(\mathcal{K}_{sb})$ to $\mathsf{uni}(\mathcal{A}_t)$. Then, $\mathsf{uni}(\mathcal{A}_t)$ is trivially homomorphically embeddable into $\mathsf{uni}(\mathcal{K}_{sb})$. Since, $\mathcal{K}_s$ is Γ-safe with respect to $\mathcal{M}$, by Lemma 6.2, we obtain that $\mathcal{A}_t$ is a universal solution for $\mathcal{K}_s$ under $\mathcal{M}$.

($\Rightarrow$) Let $\mathcal{A}_t$ be a universal solution for $\mathcal{K}_s$ under $\mathcal{M}$. By Lemma 6.2, it follows that $\mathcal{K}_s$ is Γ-safe with respect to $\mathcal{M}$ and that $\mathsf{uni}(\mathcal{A}_t)$ is Γ-homomorphically equivalent to $\mathsf{uni}(\mathcal{K}_{sb})$. Let h be a homomorphism from $\mathsf{uni}(\mathcal{A}_t)$ to $\mathsf{uni}(\mathcal{K}_{sb})$, and $h(\mathsf{uni}(\mathcal{A}_t))$ the image of h. Then, $h(\mathsf{uni}(\mathcal{A}_t))$ is a finite subset of $\mathsf{uni}(\mathcal{K}_{sb})$, moreover it is homomorphically equivalent to $\mathsf{uni}(\mathcal{A}_t)$ and to $\mathsf{uni}(\mathcal{K}_{sb})$. Therefore, $\mathsf{uni}(\mathcal{K}_{sb})$ is Γ-homomorphically embeddable to a finite subset of itself. □

It follows from the proof of Lemma 6.4 that the ABox $\mathcal{A}_t$ corresponding to the finite subset $\mathsf{uni}(\mathcal{A}_t)$ of $\mathsf{uni}(\mathcal{K}_{sb})$ in condition **(core)** is a universal solution. Hence, if we additionally require in condition **(core)** that the finite subset $\mathsf{uni}(\mathcal{A}_t)$ does not contain anonymous individuals, we obtain a characterization for universal solutions with simple ABoxes.

We introduce some additional notation needed in this section. For a KB $\mathcal{K}$ and $a \in \mathsf{ind}(\mathcal{K})$ define $\mathsf{gen}_a(\mathcal{K})$ to be an interpretation obtained from $\mathsf{gen}(\mathcal{K})$ by restricting it to the domain $\{a\} \cup \mathsf{wit}(\mathcal{K})$ and removing (a, a) from the interpretation $P^{\mathsf{gen}(\mathcal{K})}$ of every role name P. We denote by $\mathsf{uni}_a(\mathcal{K})$ the unraveling of $\mathsf{gen}_a(\mathcal{K})$. Observe that $\Delta^{\mathsf{uni}_a(\mathcal{K})} = \{a\sigma \mid a\sigma \in \Delta^{\mathsf{uni}(\mathcal{K})}\}$ and that $\mathsf{uni}_a(\mathcal{K})$ is a tree structure.

6.2. Universal solutions with simple ABoxes

In this section, we show that both the membership and the non-emptiness problems for universal solutions without null values are PTime-complete.

We start with tackling the membership problem: we are given a mapping $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$, a source KB $\mathcal{K}_s = \langle \mathcal{S}, \mathcal{A}_s \rangle$, and a simple target ABox $\mathcal{A}_t$, and the question to decide is whether $\mathcal{A}_t$ is a universal solution for $\mathcal{K}_s$ under $\mathcal{M}$. By Lemma 6.2, it is sufficient to check conditions **(safe)** and **(hom)**. The former condition does not depend on $\mathcal{A}_t$ and can be checked in polynomial time. As for the latter condition, denote by $\mathcal{K}_{sb}$ the KB $\langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_s \rangle$. First, checking the existence of a Γ-homomorphism from $\mathsf{uni}(\mathcal{A}_t)$ to $\mathsf{uni}(\mathcal{K}_{sb})$ for a simple ABox $\mathcal{A}_t$ amounts to checking,

$$\mathbf{t}_\Gamma^{\mathsf{uni}(\mathcal{A}_t)}(a) \subseteq \mathbf{t}_\Gamma^{\mathsf{uni}(\mathcal{K}_{sb})}(a) \quad \text{and} \quad \mathbf{r}_\Gamma^{\mathsf{uni}(\mathcal{A}_t)}(a,b) \subseteq \mathbf{r}_\Gamma^{\mathsf{uni}(\mathcal{K}_{sb})}(a,b) \quad \text{for all } a, b \in \mathsf{ind}(\mathcal{A}_t). \tag{2}$$

Second, a necessary condition for the existence of a Γ-homomorphism in the opposite direction, is that

$$\mathbf{t}_\Gamma^{\mathsf{uni}(\mathcal{K}_{sb})}(a) \subseteq \mathbf{t}_\Gamma^{\mathsf{uni}(\mathcal{A}_t)}(a) \quad \text{and} \quad \mathbf{r}_\Gamma^{\mathsf{uni}(\mathcal{K}_{sb})}(a,b) \subseteq \mathbf{t}_\Gamma^{\mathsf{uni}(\mathcal{A}_t)}(a,b) \quad \text{for all } a, b \in \mathsf{ind}(\mathcal{A}_s). \tag{3}$$

Clearly, these two conditions can be checked in PTime. In addition, we need to check for each $a \in \mathsf{ind}(\mathcal{A}_s)$, whether the tree $\mathsf{uni}_a(\mathcal{K}_{sb})$ can be Γ-homomorphically mapped to $\mathsf{uni}(\mathcal{A}_t)$. To do so, we make use of infinite reachability games on graphs [62]. Specifically, we show how this problem can be reduced to the problem of existence of a winning strategy for Duplicator in a reachability game, known to be solvable in polynomial time. For a short introduction to (reachability) games, we refer to Section B.6. Below we show how to construct the game G_c for a KB $\mathcal{K}$, an ABox $\mathcal{A}$, a signature Σ, and $c \in \mathsf{ind}(\mathcal{K})$.

The *reachability game* $G_c = (\mathsf{A}_c, F_c)$ is formally defined as follows: $\mathsf{A}_c = (\mathsf{S}, \mathsf{D}, \mathsf{T})$ is the *game arena*, where S and D are respectively the sets of *Spoiler* and *Duplicator* states defined next, and T is the transition relation defined below; F_c is the winning condition, i.e., the set of states that Spoiler wants to reach. Each state in S has the form $(u \mapsto a)$ with $\mathbf{t}_\Sigma^{\mathsf{gen}(\mathcal{K})}(u) \subseteq \mathbf{t}_\Sigma^{\mathsf{uni}(\mathcal{A})}(a)$, while each state in D has the form $(a,\ u \rightsquigarrow u')$ with $u \rightsquigarrow_\mathcal{K} u'$, where $u, u' \in \Delta^{\mathsf{gen}_c(\mathcal{K})}$ and $a \in \mathsf{ind}(\mathcal{A})$.Intuitively, the game proceeds as follows. Duplicator tries to construct a Σ-homomorphism from the tree $\mathsf{uni}_c(\mathcal{K})$ to $\mathsf{uni}(\mathcal{A})$, and Spoiler attempts to fail him by finding a path in $\mathsf{uni}_c(\mathcal{K})$ that does not have a homomorphic image in $\mathsf{uni}(\mathcal{A})$, given the partial homomorphism constructed so far. Spoiler starts in $(u_0 \mapsto a_0)$ for $u_0 = a_0 = c$ if $(c \mapsto c) \in \mathsf{S}$, which corresponds to mapping c to c, and at each of his turns chooses a successor u_{i+1} of u_i in $\mathsf{gen}_a(\mathcal{K})$: the “challenge” represented by the state $(a_i,\ u_i \rightsquigarrow u_{i+1})$. Then Duplicator tries to find a constant $a_{i+1} \in \mathsf{ind}(\mathcal{A})$ that could be the image of the “challenged” element $u_0 \cdots u_{i+1}$ of $\mathsf{uni}_c(\mathcal{K})$, i.e., he chooses a state $(u_{i+1} \mapsto a_{i+1})$ such that $\mathbf{r}_\Sigma^{\mathsf{gen}_c(\mathcal{K})}(u_i, u_{i+1}) \subseteq \mathbf{r}_\Sigma^{\mathsf{uni}(\mathcal{A})}(a_i, a_{i+1})$. Note that, if $\mathbf{r}_\Sigma^{\mathsf{gen}_c(\mathcal{K})}(u_i, u_{i+1})$ is empty, then Duplicator can respond with any a_{i+1} such that $(u_{i+1} \mapsto a_{i+1})$ is a Spoiler state, even if a_{i+1} is not connected to a_i in $\mathsf{uni}(\mathcal{A})$. Duplicator loses if he cannot find

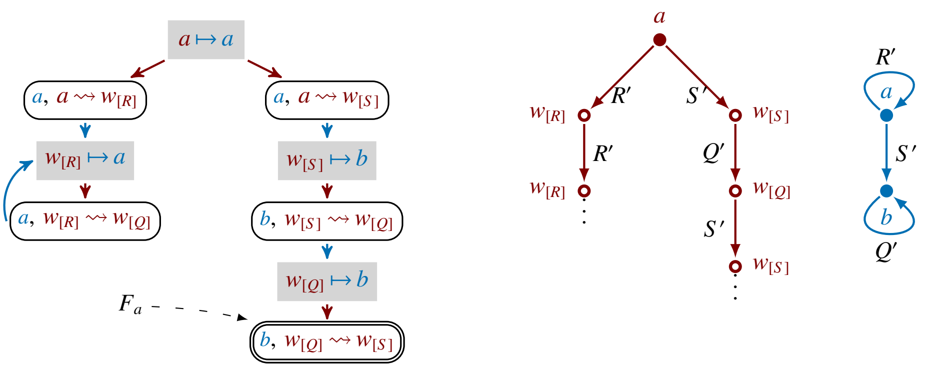


Figure 4: Example of a game: a) the game arena A_a, b) projection of $\mathsf{uni}(\mathcal{K})$ over Σ, c) $\mathsf{uni}(\mathcal{A})$.

where to map the challenged element, i.e., for all $a_{i+1} \in \mathsf{uni}(\mathcal{A})$ we have that either $\mathbf{r}_\Sigma^{\mathsf{gen}_c(\mathcal{K})}(u_i, u_{i+1}) \not\subseteq \mathbf{r}_\Sigma^{\mathsf{uni}(\mathcal{A})}(a_i, a_{i+1})$ or $(u_{i+1} \mapsto a_{i+1})$ is not a state in S. In other words, the game reaches a "dead-end" of Duplicator, i.e., $(a_i,\ u_i \rightsquigarrow u_{i+1}) \in F_c$. Otherwise, the game can reach a dead-end of Spoiler, or continue forever avoiding the dead-ends of Duplicator, hence Duplicator wins. Note that, if $(c \mapsto c) \notin \mathsf{S}$, then we assume that Spoiler "wins" the game immediately.

Formally, we define T and F_c as follows:

$$\begin{aligned}
\mathsf{T} &= \{\, ((u \mapsto a),\ (a,\ u \rightsquigarrow u')) \mid (u \mapsto a) \in \mathsf{S} \text{ and } (a,\ u \rightsquigarrow u') \in \mathsf{D} \,\} \cup \\
&\quad\ \{\, ((a,\ u \rightsquigarrow u'),\ (u' \mapsto a')) \mid (a,\ u \rightsquigarrow u') \in \mathsf{D},\ (u' \mapsto a') \in \mathsf{S},\ \text{and } \mathbf{r}_\Sigma^{\mathsf{gen}_c(\mathcal{K})}(u, u') \subseteq \mathbf{r}_\Sigma^{\mathsf{uni}(\mathcal{A})}(a, a') \,\} \\
F_c &= \{\, (a,\ u \rightsquigarrow u') \mid (u' \mapsto a') \notin \mathsf{S} \text{ or } \mathbf{r}_\Sigma^{\mathsf{gen}_c(\mathcal{K})}(u, u') \not\subseteq \mathbf{r}_\Sigma^{\mathsf{uni}(\mathcal{A})}(a, a'),\ \text{for all } a' \in \Delta^{\mathsf{uni}(\mathcal{A})} \,\}.
\end{aligned}$$

Notice that the size of A_c is $O(|\mathsf{gen}_c(\mathcal{K})| \times |\mathcal{A}|)$, and that A_c and F_c can be directly computed according to their definition in time that is linear in their size.

We illustrate such games in the following example.

Example 6.5. Assume $\Sigma = \{R', S', Q'\}$, $\mathcal{K} = \langle \mathcal{O}, \{\exists R(a), \exists S(a)\}\rangle$, where $\mathcal{O} = \{\exists R^- \sqsubseteq \exists R, \exists S^- \sqsubseteq \exists Q, \exists Q^- \sqsubseteq \exists S, R \sqsubseteq R', S \sqsubseteq S', Q \sqsubseteq Q'\}$, and $\mathcal{A} = \{R'(a,a), S'(a,b), Q'(b,b)\}$. Then $F_a = \{(b,\ w_{[Q]} \rightsquigarrow w_{[S]})\}$, and the game arena A_a can be depicted as in Figure 4(a), where the Duplicator states are shown as ovals and the Spoiler states are shown as boxes (we ignore the states that are not reachable from $(a \mapsto a)$). In Figure 4(b) and (c), we show the projection over Σ of $\mathsf{uni}(\mathcal{K})$ and $\mathsf{uni}(\mathcal{A})$, respectively.

The game starts in state $(a \mapsto a)$, which corresponds to setting the homomorphic image of $a \in \Delta^{\mathsf{uni}_a(\mathcal{K})}$ to $a \in \Delta^{\mathsf{uni}(\mathcal{A})}$. Then Spoiler can choose one of the two successors of a in $\mathsf{gen}_a(\mathcal{K})$: either $w_{[R]}$ or $w_{[S]}$. If he chooses $w_{[R]}$, it means he moves to the state $(a,\ a \rightsquigarrow w_{[R]})$. Now, Duplicator has to respond by finding where in $\mathsf{uni}(\mathcal{A})$ to map $aw_{[R]}$: he can map it only to a (note the role labels), so he moves to $(w_{[R]} \mapsto a)$. In this manner, the two players have to continue forever moving between the states $(a,\ w_{[R]} \rightsquigarrow w_{[R]})$ and $(w_{[R]} \mapsto a)$, which corresponds to mapping all elements of the form $aw_{[R]} \cdots w_{[R]} \in \Delta^{\mathsf{uni}_a(\mathcal{K})}$ to $a \in \Delta^{\mathsf{uni}(\mathcal{A})}$. Thus, this play is infinite: $(a \mapsto a) \cdot (a,\ a \rightsquigarrow w_{[R]}) \cdot (w_{[R]} \mapsto a) \cdot (a,\ w_{[R]} \rightsquigarrow w_{[R]}) \cdot (w_{[R]} \mapsto a) \cdots$ and it is a win for Duplicator.

Instead, if Spoiler at his first move chooses the successor $w_{[S]}$ of a, hence moves to the state $(a,\ a \rightsquigarrow w_{[S]})$, the game finishes soon in a dead-end of Duplicator. Hence, the second play is finite: $(a \mapsto a) \cdot (a,\ a \rightsquigarrow w_{[S]}) \cdot (w_{[S]} \mapsto b) \cdot (b,\ w_{[S]} \rightsquigarrow w_{[Q]}) \cdot (w_{[Q]} \mapsto b) \cdot (b,\ w_{[Q]} \rightsquigarrow w_{[S]})$, and it is a win for Spoiler, since the game reaches a state in F_a. ■

Having constructed the game $G_c = (\mathsf{A}_c, F_c)$, we prove that verifying whether $\mathsf{uni}_c(\mathcal{K})$ can be Σ-homomorphically mapped to $\mathsf{uni}(\mathcal{A})$ reduces to checking whether both $(c \mapsto c)$ is a state in the game arena A_c (i.e., $\mathbf{t}_\Sigma^{\mathsf{gen}(\mathcal{K})}(c) \subseteq \mathbf{t}_\Sigma^{\mathsf{uni}(\mathcal{A})}(c)$) and Duplicator has a winning strategy in G_c from $(c \mapsto c)$.

Lemma 6.6. *Let $\mathcal{K}$ be a KB, $\mathcal{A}$ an ABox, and Σ a signature. There exists a Σ-homomorphism from $\mathsf{uni}(\mathcal{K})$ to $\mathsf{uni}(\mathcal{A})$ iff*

(abox1) $\mathbf{r}_\Sigma^{\mathsf{uni}(\mathcal{K})}(a,b) \subseteq \mathbf{r}_\Sigma^{\mathsf{uni}(\mathcal{A})}(a,b)$*, for all* $a, b \in \mathsf{ind}(\mathcal{K})$*;*

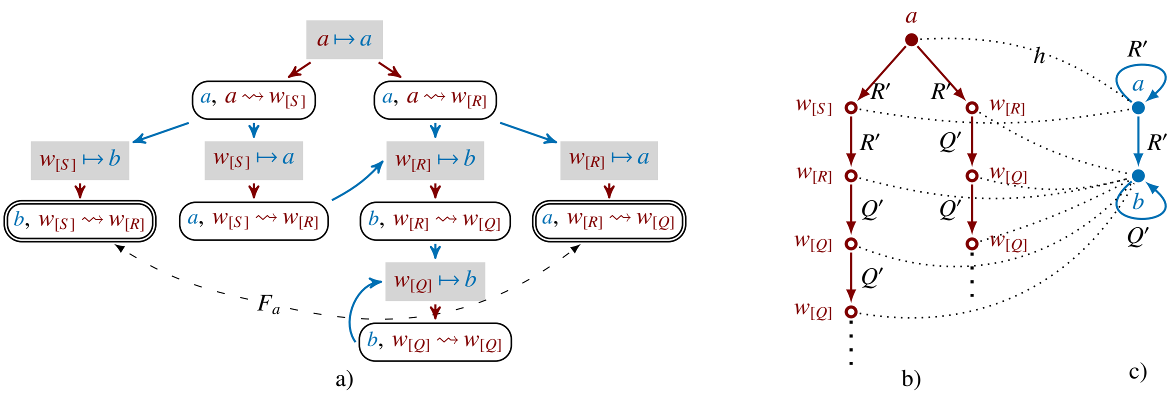


Figure 5: Example of a game: a) the game arena A_a, b) projection of $\mathsf{uni}(\mathcal{K})$ over Σ, c) $\mathsf{uni}(\mathcal{A})$.

(win) *$(c \mapsto c)$ is a state in A_c and Duplicator has a winning strategy in $G_c = (\mathsf{A}_c, F_c)$ from $(c \mapsto c)$, for each $c \in \mathsf{ind}(\mathcal{K})$.*

Proof. ($\Rightarrow$) Suppose h is a Σ-homomorphism from $\mathsf{uni}(\mathcal{K})$ to $\mathsf{uni}(\mathcal{A})$: clearly, **(abox1)**, and $\mathbf{t}_\Sigma^{\mathsf{gen}(\mathcal{K})}(a) \subseteq \mathbf{t}_\Sigma^{\mathsf{uni}(\mathcal{A})}(a)$ for each $a \in \mathsf{ind}(\mathcal{K})$ hold. Let $c \in \mathsf{ind}(\mathcal{K})$, then $(c \mapsto c)$ is a state of A_c. We describe a winning strategy f for Duplicator in G_c from $(c \mapsto c)$. Let $\pi = (u_0 \mapsto a_0) \cdot (a_0,\, u_0 \rightsquigarrow u_1) \cdots (u_k \mapsto a_k) \cdot (a_k,\, u_k \rightsquigarrow u_{k+1})$ be a finite sequence of states in A_c, where $k \geq 0$, $u_0 = a_0 = c$, and $a_i \in \mathsf{ind}(\mathcal{A})$, $u_i \in \Delta^{\mathsf{gen}_c(\mathcal{K})}$ for $i \geq 1$. Then we set $f(\pi) = (u_{k+1} \mapsto h(cu_1 \cdots u_{k+1}))$. Note that by construction of T, $cu_1 \cdots u_{k+1}$ is an element of $\Delta^{\mathsf{uni}_c(\mathcal{K})}$, and since h is defined for $\Delta^{\mathsf{uni}(\mathcal{K})}$, it follows that f is defined for each possible sequence π. Moreover, $f(\pi)$ is never a dead-end of Duplicator. Hence each play, either ends in a dead-end of Spoiler (i.e., Spoiler is in a leaf of the tree in $\mathsf{uni}(\mathcal{K})$), or continues infinitely long avoiding visits to the dead-ends of Duplicator. In any case Duplicator wins.

($\Leftarrow$) Assume that both **(abox1)** and **(win)** hold (in particular, $\mathbf{t}_\Sigma^{\mathsf{gen}(\mathcal{K})}(a) \subseteq \mathbf{t}_\Sigma^{\mathsf{uni}(\mathcal{A})}(a)$, for each $a \in \mathsf{ind}(\mathcal{K})$). Given $c \in \mathsf{ind}(\mathcal{K})$, we construct a Σ-homomorphism h_c from the tree $\mathsf{uni}_c(\mathcal{K})$ to $\mathsf{uni}(\mathcal{A})$. Let f be a winning strategy of Duplicator from $(c \mapsto c)$. Let $\pi = (u_0 \mapsto a_0) \cdot (a_0,\, u_0 \rightsquigarrow u_1) \cdots (u_k \mapsto a_k) \cdot (a_k,\, u_k \rightsquigarrow u_{k+1}) \cdots$ be a play conforming with f, where $u_0 = a_0 = c$, $u_i \in \Delta^{\mathsf{gen}_c(\mathcal{K})}$, and $a_i \in \mathsf{ind}(\mathcal{A})$. Then Duplicator wins π, and either

- $\pi = (u_0 \mapsto a_0) \cdot (a_0,\, u_0 \rightsquigarrow u_1) \cdots (u_k \mapsto a_k)$ is a *finite* play, $k \geq 0$, and $(u_k \mapsto a_k)$ is a dead-end of Spoiler. In this case, we set $h_c(cu_1 \cdots u_i) = a_i$, for $0 \leq i \leq k$.
- π is an *infinite* play such that no state from F_c occur in it. In this case, we set $h_c(cu_1 \cdots u_i) = a_i$, for $i \geq 0$.

The function h_c is well defined for all elements in $\Delta^{\mathsf{uni}_c(\mathcal{K})}$, and one can verify that it is a Σ-homomorphism from $\mathsf{uni}_c(\mathcal{K})$ to $\mathsf{uni}(\mathcal{A})$. Finally, we define a Σ-homomorphism from $\mathsf{uni}(\mathcal{K})$ to $\mathsf{uni}(\mathcal{A})$ as the union of h_c, for each $c \in \mathsf{ind}(\mathcal{K})$. □

The example below illustrates the presented reduction.

Example 6.7. Assume $\Sigma = \{R', Q'\}$, $\mathcal{K} = \langle \mathcal{O}, \{\exists R(a), \exists S(a)\}\rangle$, where $\mathcal{O} = \{\exists S^- \sqsubseteq \exists R, \exists R^- \sqsubseteq \exists Q, \exists Q^- \sqsubseteq \exists Q, R \sqsubseteq R', S \sqsubseteq R', Q \sqsubseteq Q'\}$ and $\mathcal{A} = \{R'(a,a), R'(a,b), Q'(b,b)\}$. Then $F_a = \{(b,\, w_{[S]} \rightsquigarrow w_{[R]}),\ (a,\, w_{[R]} \rightsquigarrow w_{[Q]})\}$. In Figure 5 we depict the game arena A_a and a Σ-homomorphism h from $\mathsf{uni}(\mathcal{K})$ to $\mathsf{uni}(\mathcal{A})$. Observe that in the game G_a Spoiler does not have a winning strategy from $(a \mapsto a)$, because there is a way for Duplicator to play (infinitely) so that the game never reaches F_a. It is not difficult to see that such strategy of Duplicator can be used to define the homomorphism h, and vice versa. ■

Finally, combining Lemma 6.2 and Lemma 6.6, and considering that **(win)** in Lemma 6.6 can be checked in polynomial time (see Section B.6), we obtain that the membership problem for universal solutions with simple ABoxes is in PTime. Below we show the matching lower bound.

Lemma 6.8. *Given a KB $\mathcal{K}_\mathsf{s} = \langle \mathcal{S}, \mathcal{A}_\mathsf{s}\rangle$, a mapping $\mathcal{M}$, and a simple target ABox $\mathcal{A}_\mathsf{t}$, checking whether $\mathcal{A}_\mathsf{t}$ is a universal solution for $\mathcal{K}_\mathsf{s}$ under $\mathcal{M}$ is PTime-hard.*

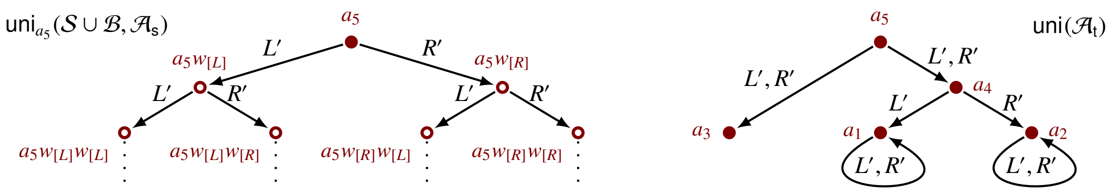
Proof. The proof is inspired by one in [30], but makes use of a reduction from the Circuit Value problem, known to be PTime-complete [63, Theorem 8.1], instead of a reduction from the Horn Satisfiability problem. Given a monotone Boolean circuit C consisting of a finite set of assignments to Boolean variables $P_1, \dots, P_n$ of the form $P_i = 0$, $P_i = 1$, $P_i = P_j \wedge P_k$, $j, k < i$, or $P_i = P_j \vee P_k$, $j, k < i$, where each P_i appears on the left-hand side of exactly one assignment, check whether the value P_n is 1 in C.

We fix signatures $\Sigma = \{P(\cdot), L(\cdot,\cdot), R(\cdot,\cdot)\}$ and $\Gamma = \{L'(\cdot,\cdot), R'(\cdot,\cdot)\}$. Let $a_1, \dots, a_n \in N_a$, and consider

$$\begin{aligned}
\mathcal{A}_s &= \{P(a_n)\} \cup \{L(a_i, a_i), R(a_i, a_i) \mid P_i = 1 \text{ in } C\} \cup \{L(a_i, a_j), R(a_i, a_k) \mid P_i = P_j \wedge P_k \text{ in } C\} \\
&\quad \cup \{L(a_i, a_j), R(a_i, a_j), L(a_i, a_k), R(a_i, a_k) \mid P_i = P_j \vee P_k \text{ in } C\} \\
\mathcal{S} &= \{P \sqsubseteq \exists L,\ P \sqsubseteq \exists R,\ \exists L^- \sqsubseteq P,\ \exists R^- \sqsubseteq P\}, \qquad \mathcal{B} = \{L \sqsubseteq L',\ R \sqsubseteq R'\} \\
\mathcal{A}_t &= \{L'(a_i, a_j) \mid L(a_i, a_j) \in \mathcal{A}_s\} \cup \{R'(a_i, a_j) \mid R(a_i, a_j) \in \mathcal{A}_s\}
\end{aligned}$$

Note that Σ, Γ, $\mathcal{S}$, and $\mathcal{B}$ do not depend on C, which is encoded by $\mathcal{A}_t$ only. Hence, the reduction provides a lower bound for data complexity [64]. In the appendix we show that the value of P_n in C is 1 if and only if $\mathcal{A}_t$ is a universal solution for $\mathcal{K}_s = \langle \mathcal{S}, \mathcal{A}_s \rangle$ under $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$. □

Example 6.9. For a circuit C containing assignments $P_1 = 1$, $P_2 = 1$, $P_3 = 0$, $P_4 = P_1 \wedge P_2$, and $P_5 = P_3 \vee P_4$, we depict the projections over Γ of $\mathsf{uni}_{a_5}(\mathcal{S} \cup \mathcal{B}, \mathcal{A}_s)$ and $\mathsf{uni}(\mathcal{A}_t)$:



We explain why the value of P_5 in C is 1 if and only if there is a Γ-homomorphism h from an infinite binary tree $\mathsf{uni}_{a_5}(\mathcal{S} \cup \mathcal{B}, \mathcal{A}_s)$ to a finite tree with loops on the leaves $\mathsf{uni}(\mathcal{A}_t)$. First, $h(a_5) = a_5$. Then, a_5 has two successors in $\mathsf{uni}_{a_5}(\mathcal{S} \cup \mathcal{B}, \mathcal{A}_s)$, $a_5 w_{[L]}$ and $a_5 w_{[R]}$, that could be mapped either to a_3 or to a_4. Intuitively, this corresponds to the fact that $P_5 = P_3 \vee P_4$, therefore in order for the value of P_5 to be 1, at least one of P_3, P_4 should evaluate to 1. The former option is not good because the value of P_3 is 0 and a_3 has no successors. Therefore we map both $a_5 w_{[L]}$ and $a_5 w_{[R]}$ to a_4: $h(a_5 w_{[L]}) = h(a_5 w_{[R]}) = a_4$. Intuitively, this corresponds to the fact that the value of P_4 is 1. Let σ be $a_5 w_{[L]}$ or $a_5 w_{[R]}$. Then $\sigma w_{[L]}$ has to be mapped to a_1 and $\sigma w_{[R]}$ has to be mapped to a_2. This corresponds to the fact that $P_4 = P_1 \wedge P_2$, therefore in order for the value of P_4 to be 1, the values of both P_1 and P_1 should be 1. Finally, since the values of P_1 and P_2 are, in fact, 1, there are loops on a_1 and a_2 labeled with L' and R'. So, all successors of $\sigma w_{[L]}$ and $\sigma w_{[R]}$ can be mapped to a_1 and a_2, respectively. ■

Theorem 6.10. *The membership problem for universal solutions with simple ABoxes is* PTime*-complete.*

We conclude this section by addressing the non-emptiness problem. It follows from what is observed after Lemma 6.4 that there exists a universal solution for $\mathcal{K}_s$ under $\mathcal{M}$ that is a simple ABox iff the (simple) ABox $\mathcal{A}_t$ over Γ is a universal solution for $\mathcal{K}_s$ under $\mathcal{M}$, where $\mathcal{A}_t$ satisfies equations (2) and (3). Obviously, we can construct the required $\mathcal{A}_t$ in PTime, then it remains to check if it is a universal solution. Moreover, we can adapt the reduction in Lemma 6.8 above to show that the PTime bound is tight. We obtain the following result.

Theorem 6.11. *The non-emptiness problem for universal solutions with simple ABoxes is* PTime*-complete. Moreover, there is an effective algorithm to compute a universal solution in polynomial time.*

6.3. The membership problem for universal solutions with extended ABoxes

In this section, we study the membership problem for universal solutions when extended ABoxes are allowed in the target, and show that it is NP-complete.

Assume given a mapping $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$, a KB $\mathcal{K}_s = \langle \mathcal{S}, \mathcal{A}_s \rangle$ over Σ, and an extended ABox $\mathcal{A}_t$ over Γ, and let $\mathcal{K} = \langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_s \rangle$. In this setting, existence of Γ-homomorphism from $\mathsf{uni}(\mathcal{K})$ to $\mathsf{uni}(\mathcal{A}_t)$ can be still checked in PTime using the technique of reachability games presented in Section 6.2 (note that for homomorphisms in this direction, there is no distinction made between the constants and the labeled nulls in $\mathcal{A}_t$). Instead, existence of a Γ-homomorphism in the opposite direction cannot be checked efficiently due to the nulls in $\mathcal{A}_t$. In fact, we show now that the membership problem for universal solutions with extended ABoxes is NP-hard in data complexity.

Lemma 6.12. *Given a KB $\mathcal{K}_s = \langle \mathcal{S}, \mathcal{A}_s \rangle$, a mapping $\mathcal{M}$, and an extended target ABox $\mathcal{A}_t$, checking whether $\mathcal{A}_t$ is a universal solution for $\mathcal{K}_s$ under $\mathcal{M}$ is* NP*-hard.*

Proof Sketch. The proof is by reduction from 3-colorability of undirected graphs, known to be NP-hard. Consider an undirected graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$, which we view as a symmetric directed graph, and fix signatures $\Sigma = \{E(\cdot, \cdot)\}$ and $\Gamma = \{E'(\cdot, \cdot)\}$. Further, let $r, g, b \in N_a$, $\mathsf{V} \subseteq N_l$ and

$$\mathcal{A}_s = \{E(r,g), E(g,r), E(r,b), E(b,r), E(g,b), E(b,g)\}, \qquad \mathcal{S} = \{\}, \qquad \mathcal{B} = \{E \sqsubseteq E'\},$$
$$\mathcal{A}_t = \{E'(r,g), E'(g,r), E'(r,b), E'(b,r), E'(g,b), E'(b,g)\} \cup \{E'(x,y) \mid (x,y) \in \mathsf{E}\}.$$

Note that the vertices in G become labeled nulls in $\mathcal{A}_t$. In the appendix we show that G is 3-colorable if and only if $\mathcal{A}_t$ is a universal solution for $\mathcal{K}_s = \langle \mathcal{S}, \mathcal{A}_s \rangle$ under $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$. □

We provide now a matching upper bound.

Lemma 6.13. *The membership problem for universal solutions with extended ABoxes is in* NP.

Proof. Given a KB $\mathcal{K}_s = \langle \mathcal{S}, \mathcal{A}_s \rangle$, a mapping $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$, and an extended target ABox $\mathcal{A}_t$, it suffices to show that the existence of a homomorphism from $\mathsf{uni}(\mathcal{A}_t)$ to $\mathsf{uni}(\mathcal{K})$, for $\mathcal{K} = \langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_s \rangle$, can be checked in NP in the size of $\mathcal{K}_s$, $\mathcal{M}$, and $\mathcal{A}_t$ (checking the existence of a Γ-homomorphism in the other direction is in PTime, as discussed above). For this, we use the fact that the image $W \subseteq \Delta^{\mathsf{uni}(\mathcal{K})}$ of the function h on $\Delta^{\mathsf{uni}(\mathcal{A}_t)}$ is bounded by the size of $\mathcal{A}_t$. Therefore, for each constant and null in $\mathcal{A}_t$, one needs to guess its homomorphic image in $\Delta^{\mathsf{uni}(\mathcal{K})}$, and then check whether the resulting function is a homomorphism.

First, if there exists a homomorphism h from $\mathsf{uni}(\mathcal{A}_t)$ to $\mathsf{uni}(\mathcal{K})$, then there exists witness W with a number of elements bounded by the size of $\mathcal{A}_t$, such that $W \subseteq \Delta^{\mathsf{uni}(\mathcal{K})}$ and h is a function from $\Delta^{\mathsf{uni}(\mathcal{A}_t)}$ to W: take $W = h(\Delta^{\mathsf{uni}(\mathcal{A}_t)})$.

Second, we show that there exists a witness W such that $W \subseteq \Delta^{\mathsf{uni}(\mathcal{K})}$ and every $x \in W$ is a path of length smaller or equal $2m$, where for $x = aw_{[S_1]} \cdots w_{[S_k]}$ the length of x is $k+1$, and m is the size of $\mathcal{S} \cup \mathcal{B} \cup \mathcal{A}_t$. To this end, let h be a homomorphism from $\mathsf{uni}(\mathcal{A}_t)$ to $\mathsf{uni}(\mathcal{K})$ and $W = h(\Delta^{\mathsf{uni}(\mathcal{A}_t)})$. Let $\mathcal{I}_W$ be the sub-interpretation of $\mathsf{uni}(\mathcal{K})$ induced by W. For $x, y \in W$, we say that x is connected to y in $\mathcal{I}_W$, if there exists $n \geq 0$ and a path $x_1, x_2, \ldots, x_n, x_{n+1}$ such that $x_i \in W$, $x_1 = x$, $x_{n+1} = y$, and $(x_i, x_{i+1}) \in R_i^{\mathcal{I}_W}$ for some role R_i, $i \in \{1, \ldots, n\}$. Assume that $x \in W$ and the length of x is more than $2m$. Then, since $W = h(\Delta^{\mathsf{uni}(\mathcal{A}_t)})$, we have that x is not connected to any element of $\mathsf{ind}(\mathcal{A}_s)$ in $\mathcal{I}_W$. Let C be the maximal connected subset of W with $x \in C$, i.e., for each $y \in C$, *(i)* y is connected to y' in $\mathcal{I}_W$, for each $y' \in C$, and *(ii)* y is not connected to any $z \in W \setminus C$. Note that $C \cap \mathsf{ind}(\mathcal{A}_s) = \emptyset$. Let y be the path in $\mathsf{path}(\mathcal{K})$ of minimal length in C, it exists and is unique since $C \subseteq \Delta^{\mathcal{I}_W}$ and there are no constants in C. Then for each $y' \in C$, we have that $y' = y \cdot w_{[R_1]} \cdots w_{[R_k]}$ for some roles $R_1, \ldots, R_k$. Further assume $\mathsf{tail}(y) = w_{[R]}$, and let z be a path of minimal length in $\Delta^{\mathsf{uni}(\mathcal{K})}$ with $\mathsf{tail}(z) = w_{[R]}$. Then the length of z is bounded by the size of $\mathcal{S} \cup \mathcal{B}$ and the length of each $z \cdot w_{[R_1]} \cdots w_{[R_k]}$ for some $y \cdot w_{[R_1]} \cdots w_{[R_k]} \in C$, is bounded by the size of $\mathcal{S} \cup \mathcal{B} \cup \mathcal{A}_t$. Now, define a new function $h' : \Delta^{\mathsf{uni}(\mathcal{A}_t)} \to \Delta^{\mathsf{uni}(\mathcal{K})}$ such that $h'(x) = h(x)$ if $h(x) \notin C$, and $h'(x) = z \cdot w_{[R_1]} \cdots w_{[R_k]}$ if $h(x) = y \cdot w_{[R_1]} \cdots w_{[R_k]}$. It is easy to see that h' is a Γ-homomorphism from $\mathsf{uni}(\mathcal{A}_t)$ to $\mathsf{uni}(\mathcal{K})$. Now we can take $W = h'(\Delta^{\mathsf{uni}(\mathcal{A}_t)})$, and repeat the above construction until the claim is satisfied.

Finally, to verify in NP whether a homomorphism h from $\mathsf{uni}(\mathcal{A}_t)$ to $\mathsf{uni}(\mathcal{K})$ exists, it is sufficient to guess W of polynomial size and check if $\mathsf{uni}(\mathcal{A}_t)$ can be homomorphically mapped to $\mathcal{I}_W$. □

Thus, we obtain the exact complexity of the membership problem with extended ABoxes.

Theorem 6.14. *The membership problem for universal solutions with extended ABoxes is* NP*-complete.*

6.4. The non-emptiness problem for universal solutions with extended ABoxes

We now turn to the non-emptiness problem for universal solutions with null values. This problem turns out to be harder than the membership problem as now candidate solutions, which can be of exponential size, are not part of the input. In fact, we show by reduction from the validity problem for quantified Boolean formulas that checking the existence of a universal solution is PSPACE-hard. We also show an EXPTIME upper bound by relying on techniques based on *two-way alternating automata on infinite trees (2ATA)*. 2ATAs are a generalization of non-deterministic automata on infinite trees whose non-emptiness problem is in EXPTIME [65]. They are at the basis of a variety of reasoning techniques for description and modal logics. In particular, due to their ability of traversing trees both downwards and upwards, they are well suited for handling inverse roles in *DL-Lite*$_{\mathcal{R}}$. We briefly introduce in Section B.7 the basic notions about infinite trees and 2ATAs and the notation that we use for them.

The lower bound can be shown (see the appendix) similarly to Theorem 11 in [55] by reduction from the validity problem for quantified Boolean formulas, known to be PSPACE-complete:

Lemma 6.15. *The non-emptiness problem for universal solutions with extended ABoxes in DL-Lite*$_{\mathcal{R}}$ *is* PSPACE-*hard.*

As a corollary, we obtain a PSPACE lower bound for the non-emptiness problem for universal UCQ-solutions with extended ABoxes by a straightforward reduction from the non-emptiness problem for universal solutions with extended ABoxes.

Lemma 6.16. *The non-emptiness problem for universal* UCQ-*solutions with extended ABoxes is* PSPACE-*hard.*

Proof. Let $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$ be a mapping, and $\mathcal{K}_{\mathsf{s}} = \langle \mathcal{S}, \mathcal{A}_{\mathsf{s}} \rangle$ a KB over Σ. We construct $\mathcal{K}'_{\mathsf{s}}$ and $\mathcal{M}'$ such that there exists a universal solution for $\mathcal{K}_{\mathsf{s}}$ under $\mathcal{M}$ iff there exists a universal UCQ-solution for $\mathcal{K}'_{\mathsf{s}}$ under $\mathcal{M}'$.

Define $\mathcal{M}'$ to be $(\Sigma', \Gamma', \mathcal{B}')$, where Σ' extends Σ with fresh concept and roles names $\{X_1 \mid X \in \Gamma\}$ and fresh role names Q_1, Q_2, Γ' extends Γ with a fresh role name Q, and $\mathcal{B}' = \mathcal{B} \cup \{X_1 \sqsubseteq X \mid X \in \Gamma\} \cup \{Q_1 \sqsubseteq Q, Q_2 \sqsubseteq Q\}$. Let $\mathcal{K}'_{\mathsf{s}} = \langle \mathcal{S}', \mathcal{A}'_{\mathsf{s}} \rangle$, where $\mathcal{A}'_{\mathsf{s}}$ is the union of $\mathcal{A}_{\mathsf{s}}$, assertions

$$\{X_1(a_X) \mid X \in \Gamma \text{ is a concept name}\} \cup \{X_1(a_X, b_X) \mid X \in \Gamma \text{ is a role name}\},$$

for fresh constants a_X, b_X for each symbol X, and assertions $\{\exists Q_1(a_Q), Q_2(a_Q, b_Q)\}$, for fresh constants a_Q, b_Q. If $\mathcal{K}_{\mathsf{s}}$ is not Γ-safe with respect to $\mathcal{M}$, then $\mathcal{S}' = \mathcal{S} \cup \{\exists Q_1^- \sqsubseteq \exists Q_1\}$, otherwise $\mathcal{S}' = \mathcal{S}$. In the appendix, we prove that $\mathcal{K}'_{\mathsf{s}}$ and $\mathcal{M}'$ are as required. □

As for the upper bound, we show how to check condition **(core)** of Lemma 6.4, i.e., whether there exists a finite subset D of $\Delta^{\mathsf{uni}(\mathcal{K}_{\mathsf{sb}})}$ and a Γ-homomorphism from $\mathsf{uni}(\mathcal{K}_{\mathsf{sb}})$ to its finite sub-interpretation induced by D. In the following, for an interpretation $\mathcal{U}$ and a finite subset D of $\Delta^{\mathcal{U}}$, we denote with $\mathcal{U}^D$ the sub-interpretation of $\mathcal{U}$ induced by D. We also write $\mathcal{U}^d$ if $D = \{d\}$. To simplify the presentation, in the rest of this section we tackle a more general problem: given two (simple) KBs $\mathcal{K}_1$ and $\mathcal{K}_2$ with canonical models $\mathcal{U}_1$ and $\mathcal{U}_2$, and a signature Σ, decide whether there exists a Σ-homomorphism from $\mathcal{U}_1$ to $\mathcal{U}_2^D$, for some finite subset D of $\Delta^{\mathcal{U}_2}$.

As in the case of the membership problem for simple universal solutions in Section 6.2, for such a homomorphism to exist, *(i)* an analog of condition **(abox1)** must hold (cf. Lemma 6.17), and *(ii)* for each $c \in \mathsf{ind}(\mathcal{K}_1)$, the tree $\mathcal{U}_1^c$ must be Σ-homomorphically embeddable into $\mathcal{U}_2^{D_c}$, for some finite subset D_c of $\Delta^{\mathcal{U}_2}$. To check condition *(ii)* we adopt 2ATAs; more precisely, we show how to construct for each constant $c \in \mathsf{ind}(\mathcal{K}_1)$, an automaton $\mathbb{A}_c$ (with Büchi acceptance condition) accepting (infinite) trees that correspond to (the finite) $\mathcal{U}_2^{D_c}$. Hence, to verify the existence of the required Σ-homomorphism, we solve the non-emptiness problem of $\mathbb{A}_c$, for each constant c. It follows that, if the language accepted by $\mathbb{A}_c$ for some $c \in \mathsf{ind}(\mathcal{K}_1)$ is empty, then there is no such homomorphism, otherwise we can obtain $\mathcal{U}_2^D$ from the trees accepted by $\mathbb{A}_c$. Below we show how to construct the automaton $\mathbb{A}_c$ for two KBs $\mathcal{K}_1, \mathcal{K}_2$, a signature Σ, and some constant $c \in \mathsf{ind}(\mathcal{K}_1)$.

In the following, we assume that $\mathsf{ind}(\mathcal{K}_2) = \{a_1, \ldots, a_{n_a}\}$, $\mathsf{wit}(\mathcal{K}_2) = \{w_1, \ldots, w_{n_w}\}$, and $n = \max(n_a, n_w)$. Denote by $\mathcal{U}_1$ and $\mathcal{U}_2$ the canonical models, and by $\mathcal{G}_1$ and $\mathcal{G}_2$ the generating structures of $\mathcal{K}_1$ and $\mathcal{K}_2$. We define the *automaton* $\mathbb{A}_c$ as the tuple $\langle \Gamma, Q, \delta, q_0, F \rangle$, where the *alphabet* Γ is the set

$$\Gamma = \{root, stop\} \cup \{\hat{a}_i \mid 1 \leq i \leq n_a\} \cup \{\hat{w}_i \mid 1 \leq i \leq n_w\}.$$

Hence, $\mathbb{A}_c$ accepts n-ary trees where each node either corresponds to a constant of $\mathcal{K}_2$, labeled with the symbol $\hat{a}_i$, or corresponds to a witness of $\mathcal{K}_2$, labeled with the symbol $\hat{w}_i$, or is the root of the tree, labeled with *root*, or is a node outside the finite part, labeled with *stop*. The set Q of *states* is partitioned into three sets:

$$Q = \{q_0\} \cup Q_f \cup Q_h,$$

where Q_f is the set of states responsible for labeling an input tree T as an appropriate *finite* substructure of $\mathcal{U}_2$, and Q_h is the set of states responsible for checking the existence of a homomorphism from $\mathcal{U}_1^c$ into a finite substructure of $\mathcal{U}_2$. We define

$$Q_f = \{\alpha_i \mid 1 \le i \le n_a\} \cup \{\omega_i \mid 1 \le i \le n_w\},$$

where the states α_i are responsible for labeling T with the constants of $\mathcal{K}_2$, and the states ω_i are responsible for labeling T with the witnesses of $\mathcal{K}_2$. We define the *transition function* δ for these states and for the *initial state* q_0 as follows:

$$\delta(q_0, L) = \begin{cases} \Big(\bigwedge_{i=1}^{n_a} (i, \alpha_i)\Big) \land (0, q_h), & \text{if } L = \mathit{root} \\ \bot, & \text{otherwise,} \end{cases} \tag{4}$$

$$\text{for } 1 \le i \le n_a, \quad \delta(\alpha_i, L) = \begin{cases} \bigwedge_{\substack{1 \le j \le n_w,\\ a_i \rightsquigarrow_{\mathcal{K}_2} w_j}} (j, \omega_j), & \text{if } L = \hat{a}_i \\ \bot, & \text{otherwise,} \end{cases} \tag{5}$$

$$\text{for } 1 \le i \le n_w, \quad \delta(\omega_i, L) = \begin{cases} \bigwedge_{\substack{1 \le j \le n_w,\\ w_i \rightsquigarrow_{\mathcal{K}_2} w_j}} (j, \omega_j), & \text{if } L = \hat{w}_i, \\ \top, & \text{if } L = \mathit{stop} \\ \bot, & \text{if } L \in \Gamma \setminus \{\hat{w}_i, \mathit{stop}\}, \end{cases} \tag{6}$$

where q_h is a state from Q_h, which we are going to define below. For now observe that due to the transitions above, a tree T accepted by $\mathbb{A}_c$ will have the symbol *root* in the root and the symbol $\hat{a}_i$ in the i-th successor of the root. Then, each of the i-th successors above will have its j-th successor marked with $\hat{w}_j$ whenever $a_i \rightsquigarrow_{\mathcal{K}_2} w_j$. Further, each of the j-th successors above will have its i-th successor marked with $\hat{w}_i$ whenever $w_j \rightsquigarrow_{\mathcal{K}_2} w_i$, and so on. Note that at some step, when $w_j \rightsquigarrow_{\mathcal{K}_2} w_i$, a node in T marked with $\hat{w}_j$ can have its i-th successor marked with *stop* (instead of $\hat{w}_i$). This should mean that this i-th successor is not inside the finite substructure of $\mathcal{U}_2$ to which the homomorphism will map $\mathcal{U}_1^c$, and $\mathbb{A}_c$ will stop going down T. Note that it is not yet guaranteed that each path in T from the root contains at some point a node labeled with *stop* instead of $\hat{w}_i$. However, if this is not the case, we would have an infinite path in T over which the automaton passes infinitely often through states ω_i. We rule this out by means of an appropriate acceptance condition of the automaton, which we present below.

Let $\mathsf{wit}(\mathcal{K}_1) = \{u_1, \ldots, u_m\}$, and assume that $u_0 = c$. Now, the set of states Q_h is defined as:

$$Q_h \;=\; \{q_h\} \;\cup\; \{\gamma_\ell,\ \chi_\ell \mid 0 \le \ell \le m\} \;\cup\; \{\kappa_\ell^i \mid 1 \le \ell \le m,\ 1 \le i \le n_a\},$$

and the transitions for theses states are defined as follows, where $1 \leq \ell \leq m$ and $1 \leq i \leq n_a$:

$$\delta(q_h, L) = \begin{cases}(j, \gamma_0), & \text{if } L = \mathit{root} \text{ and } c = a_j \text{ for some } j,\\ \bot, & \text{otherwise;}\end{cases} \tag{7}$$

$$\text{for } t \in \{a_1, \dots, a_{n_a}, w_1, \dots, w_{n_w}\}, \quad \delta(\chi_\ell, \hat{t}) = (0, \gamma_\ell) \vee \bigvee_{\substack{1 \leq j \leq n_w,\\ t \rightsquigarrow_{\mathcal{K}_2} w_j}} (j, \chi_\ell) \vee (-1, \chi_\ell); \tag{8}$$

$$\delta(\chi_\ell, \mathit{root}) = \bigvee_{j=1}^{n_a} (j, \chi_\ell); \tag{9}$$

$$\delta(\kappa_\ell^i, L) = \begin{cases}(i, \gamma_\ell), & \text{if } L = \mathit{root},\\ \bot, & \text{otherwise;}\end{cases} \tag{10}$$

$$\text{for } q \in Q_h, \quad \delta(q, \mathit{stop}) = \bot. \tag{11}$$

Next, for $0 \leq \ell \leq m$ and $b \in \{a_1, \dots, a_{n_a}\}$,

$$\delta(\gamma_\ell, \hat{b}) = \tau_b^{u_\ell} \wedge \bigwedge_{\substack{1 \leq k \leq m,\\ u_\ell \dashrightarrow_{\Sigma} u_k}} (0, \chi_k) \wedge \bigwedge_{\substack{1 \leq k \leq m,\\ u_\ell \xrightarrow[\Sigma]{} u_k}} \Big(\bigvee_{\substack{1 \leq j \leq n_w,\\ b \rightsquigarrow_{\mathcal{K}_2} w_j}} (\rho_{b,w_j}^{u_\ell,u_k} \wedge (j, \gamma_k)) \vee \bigvee_{i=1}^{n_a} (\rho_{b,a_i}^{u_\ell,u_k} \wedge (-1, \kappa_k^i)) \Big); \tag{12}$$

and for $1 \leq \ell \leq m$ and $v \in \{w_1, \dots, w_{n_w}\}$,

$$\delta(\gamma_\ell, \hat{v}) = \tau_v^{u_\ell} \wedge \bigwedge_{\substack{1 \leq k \leq m,\\ u_\ell \dashrightarrow_{\Sigma} u_k}} (0, \chi_k) \wedge \bigwedge_{\substack{1 \leq k \leq m,\\ u_\ell \xrightarrow[\Sigma]{} u_k}} \Big(\bigvee_{\substack{1 \leq j \leq n_w,\\ v \rightsquigarrow_{\mathcal{K}_2} w_j}} (\rho_{v,w_j}^{u_\ell,u_k} \wedge (j, \gamma_k)) \vee (\eta_v^{u_\ell,u_k} \wedge (-1, \gamma_k)) \Big), \tag{13}$$

where the relations $\xrightarrow[\Sigma]{}$ and $\dashrightarrow_{\Sigma}$ defined between elements $s, s' \in \{u_0, \dots, u_m\}$ indicate whether the edge between s and s' has a nonempty or empty Σ-role label, respectively:

$$s \xrightarrow[\Sigma]{} s' \quad \text{if} \quad s \rightsquigarrow_{\mathcal{K}_1} s' \text{ and } \mathbf{r}_\Sigma^{\mathcal{G}_1}(s, s') \neq \emptyset, \qquad \text{and} \qquad s \dashrightarrow_{\Sigma} s' \quad \text{if} \quad s \rightsquigarrow_{\mathcal{K}_1} s' \text{ and } \mathbf{r}_\Sigma^{\mathcal{G}_1}(s, s') = \emptyset,$$

the functions τ_t^s and $\rho_{t,t'}^{s,s'}$, encoding local homomorphism conditions, return true iff s can be mapped to t, and the edge (s, s') can be mapped to the edge (t, t'), respectively:

$$\tau_t^s = \begin{cases}\top, & \text{if } \mathbf{t}_\Sigma^{\mathcal{G}_1}(s) \subseteq \mathbf{t}_\Sigma^{\mathcal{G}_2}(t)\\ \bot, & \text{otherwise}\end{cases} \qquad \rho_{t,t'}^{s,s'} = \begin{cases}\top, & \text{if } \mathbf{r}_\Sigma^{\mathcal{G}_1}(s, s') \subseteq \mathbf{r}_\Sigma^{\mathcal{G}_2}(t, t')\\ \bot, & \text{otherwise}\end{cases}$$

and the function $\eta_w^{u,u'}$ returns true iff the edge (u, u') can be "inversely" mapped to the edge (w, s), for the predecessor s of w:

$$\eta_w^{u,u'} = \begin{cases}\top, & \text{if } \{R^- \mid R \in \mathbf{r}_\Sigma^{\mathcal{G}_1}(u, u')\} \subseteq \mathbf{r}_\Sigma^{\mathcal{G}_2}(s, w) \text{ for some } s \rightsquigarrow_{\mathcal{K}_2} w\\ \bot, & \text{otherwise,}\end{cases}$$

for s, s' as above, $t, t' \in \{a_1, \dots, a_{n_a}, w_1, \dots, w_{n_w}\}$, $u, u' \in \{u_1, \dots, u_m\}$ and $w \in \{w_1, \dots, w_{n_w}\}$. This concludes the definition of the transition function.

Observe that for each witness $u_\ell \in \mathsf{wit}(\mathcal{K}_1)$ there are two states in Q_h: γ_ℓ is responsible for checking the existence of a homomorphic image for the sub-tree generated by u_ℓ, and χ_ℓ is the "expecting state", which is responsible for non-deterministically finding a homomorphic image of u_ℓ; moreover for each witness $u_\ell \in \mathsf{wit}(\mathcal{K}_1)$ and constant $a_i \in \mathsf{ind}(\mathcal{K}_2)$, there is a state κ_ℓ^i used to move from the current constant in $\mathcal{K}_2$ via the root to a_i, to which u_ℓ is mapped. Intuitively, suppose an element $cu_{\ell_1} \cdots u_{\ell_k}$ of $\Delta^{\mathcal{U}_1}$ is homomorphically mapped to the element $a_{i_1} w_{i_2} \cdots w_{i_r}$ of $\Delta^{\mathcal{U}_2}$ and $u_{\ell_k} \rightsquigarrow_{\mathcal{K}_1} u_{\ell_{k+1}}$. If $u_{\ell_k} \xrightarrow[\Sigma]{} u_{\ell_{k+1}}$ then the element $cu_{\ell_1} \cdots u_{\ell_k} u_{\ell_{k+1}}$ of $\Delta^{\mathcal{U}_1}$ has to be mapped to an immediate successor or predecessor of the image of $cu_{\ell_1} \cdots u_{\ell_k}$ in $\mathcal{U}_2$. For $w_{i_r} \rightsquigarrow_{\mathcal{K}_2} w_{i_{r+1}}$, whenever $\tau_{w_{i_{r+1}}}^{u_{\ell_{k+1}}} = \top$ and $\rho_{w_{i_r}, w_{i_{r+1}}}^{u_{\ell_k}, u_{\ell_{k+1}}} = \top$, it is guaranteed that the edge $(cu_{\ell_1}\cdots u_{\ell_k},\ cu_{\ell_1}\cdots u_{\ell_k}u_{\ell_{k+1}})$ of $\mathcal{U}_1$ can be mapped to the edge $(a_{i_1}w_{i_2}\cdots w_{i_r},\ a_{i_1}w_{i_2}\cdots w_{i_r}w_{i_{r+1}})$ of $\mathcal{U}_2$. Alternatively, if $\eta^{u_{\ell_k},u_{\ell_{k+1}}}_{w_{i_r}} = \top$ then the edge $(cu_{\ell_1}\cdots u_{\ell_k},\ cu_{\ell_1}\cdots u_{\ell_k}u_{\ell_{k+1}})$ can be "inversely" mapped to the edge $(a_{i_1}w_{i_2}\cdots w_{i_r},\ a_{i_1}w_{i_2}\cdots w_{i_{r-1}})$. If, however, $u_{\ell_k} \dashrightarrow_{\Sigma} u_{\ell_{k+1}}$ then $cu_{\ell_1}\ldots u_{\ell_k}u_{\ell_{k+1}}$ can be mapped to any element of $\mathcal{U}_2$, which is reflected by switching to the state χ_ℓ.

For the *(Büchi) acceptance condition* we take $F = \{\gamma_i \mid 1 \le i \le m\}$. Observe that neither the states ω_i of Q_f nor χ_l of Q_h are in F. This implies that a tree is rejected if it has an infinite branch all of whose nodes are labeled with $\hat{w}_i$, or if all runs on it are such that the mapping of a "disconnected successor" (such as $u_{\ell_{k+1}}$ with $u_{\ell_k} \dashrightarrow_{\Sigma} u_{\ell_{k+1}}$ in the example above) is "infinitely postponed". On the other hand, each accepted tree represents a finite substructure of $\mathcal{U}_2$ to which $\mathcal{U}_1^c$ can be Σ-homomorphically mapped. The number of states of the automaton $\mathbb{A}_c$ is quadratic and the overall size of the automaton $\mathbb{A}_c$ is polynomial in the combined size of the two generating structures $\mathcal{G}_1$ and $\mathcal{G}_2$.

We prove that verifying whether $\mathcal{U}_1$ can be Σ-homomorphically mapped to $\mathcal{U}_2^D$ for some finite $D \subseteq \Delta^{\mathcal{U}_2}$ reduces to checking the non-emptiness problem of $\mathbb{A}_c$.

Lemma 6.17. *Let $\mathcal{K}_1, \mathcal{K}_2$ be KBs and Σ a signature. There exists a finite subset D of $\Delta^{\mathcal{U}_2}$ and a Σ-homomorphism from $\mathcal{U}_1$ to $\mathcal{U}_2^D$ if and only if*

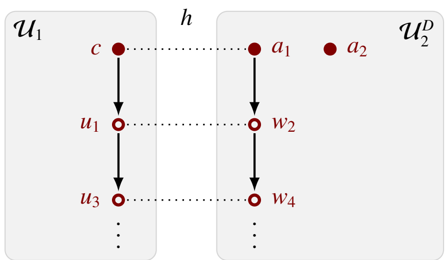
(abox2) $\mathbf{r}_\Sigma^{\mathcal{U}_1}(a,b) \subseteq \mathbf{r}_\Sigma^{\mathcal{U}_2}(a,b)$, *for all $a, b \in \mathsf{ind}(\mathcal{K}_1)$, and*

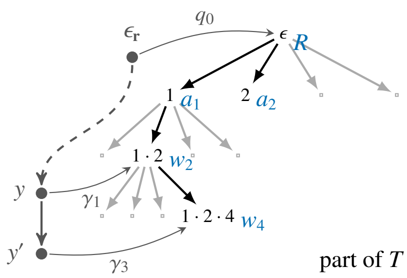
(aut) *the language of the automaton $\mathbb{A}_c$ is non-empty, for each $c \in \mathsf{ind}(\mathcal{K}_1)$.*

Proof Sketch. ($\Rightarrow$) Let $D \subseteq \Delta^{\mathcal{U}_2}$ be finite, and h a Σ-homomorphism from $\mathcal{U}_1$ to $\mathcal{U}_2^D$. We construct a labeled tree $T = (\{1,\ldots,n\}^*, V)$ where $n = \max(n_a, n_w)$ and show that $T \in \mathcal{L}(\mathbb{A}_c)$, for each $c \in \mathsf{ind}(\mathcal{K}_1)$. The labeling function V is defined as follows:

$V(\epsilon) = root$;
$V(i) = \hat{a}_i$, for each $a_i \in D \cap \mathsf{ind}(\mathcal{K}_2)$;
$V(i_1 i_2 \cdots i_r) = \hat{w}_{i_r}$, for each $a_{i_1}w_{i_2}\cdots w_{i_r} \in D$;
$V(x) = stop$, for each $x \in \{1,\ldots,n\}^*$ such that $V(x)$ is not otherwise defined.

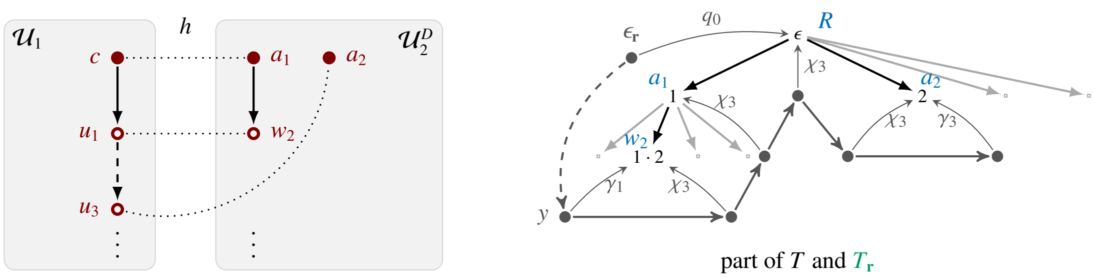
To show that $T \in \mathcal{L}(\mathbb{A}_c)$, we construct a run tree $(T_\mathbf{r}, \mathbf{r})$ of $\mathbb{A}_c$ on T. The idea behind this construction is the following. Assume that $y \in T_\mathbf{r}$ with $\mathbf{r}(y) = (x,q)$, $x \in \{1,\ldots,n\}^*$, and $V(x) = L$. Observe that the transition function can be viewed as a conjunction $\delta(q,L) = \bigwedge_i \Phi_i$, where each $\Phi_i = \bigvee_j \psi^i_j$. To satisfy $\delta(q,L)$, we construct exactly one child for each Φ_i, and we satisfy Φ_i by choosing exactly one ψ^i_j from Φ_i, making use of the given homomorphism h. Thus, for instance, if $\mathbf{r}(y) = (1\cdot 2, \gamma_1)$, $V(1\cdot 2) = \hat{w}_2$, the current path in $\mathcal{U}_1$ is cu_1 (this path can be obtained from the path from the root of $T_\mathbf{r}$ to y), $h(cu_1) = a_1w_2$, and $u_1 \rightarrow_{\Sigma} u_3$ and $h(cu_1u_3) = a_1w_2w_4$, then we satisfy $\psi^i_j = (4, \gamma_3)$, so y would have a child y' with $\mathbf{r}(y') = (1\cdot 2\cdot 4, \gamma_3)$.





part of T and $T_\mathbf{r}$

If, instead, $u_1 \dashrightarrow_{\Sigma} u_3$ and $h(cu_1u_3) = a_2$, we switch to the "expecting" state χ_3 and remain in this state while traversing the tree $\{1,\ldots,n\}^*$ from the node $1\cdot 2$ via the root to the node 2. Once node 2 is reached, we switch to the state γ_3. The choices for satisfying the transition function follow from that. Thus, the run from y continues as: $(1\cdot 2, \gamma_1)$, $(1\cdot 2, \chi_3), (1, \chi_3), (\epsilon, \chi_3), (2, \chi_3), (2, \gamma_3)$.



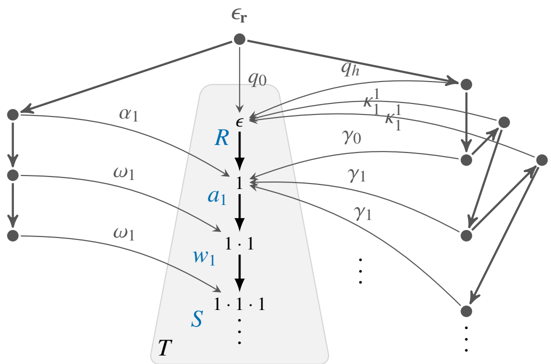
part of T and $T_{\mathbf{r}}$

For the formal definition of $(T_{\mathbf{r}}, \mathbf{r})$, we refer to the appendix.

($\Leftarrow$) If the language of $\mathbb{A}_c$ is non-empty, then there is a tree $T = (\{1, \ldots, n\}^*, V) \in \mathcal{L}(\mathbb{A}_c)$ and an accepting run $(T_{\mathbf{r}}, \mathbf{r})$ of $\mathbb{A}_c$ over T. We can construct a finite set $D_c \subseteq \Delta^{\mathcal{U}_2}$ by proving that T encodes a finite subset of $\Delta^{\mathcal{U}_2}$, extracting D_c from it, and defining a Σ-homomorphism h_c from $\mathcal{U}_1^c$ to $\mathcal{U}_2^D$ by induction, based on the choices in $T_{\mathbf{r}}$ to satisfy the transition function. A Σ-homomorphism from $\mathcal{U}_1$ to $\mathcal{U}_2^D$ for $D = \bigcup_c D_c$ is defined as the union of h_c for each $c \in \mathsf{ind}(\mathcal{K}_1)$. □

Example 6.18. Consider $\mathcal{M}$ and $\mathcal{K}_\mathsf{s}$ from Example 5.8, i.e., $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$, where $\Sigma = \{A, R, S\}$, $\Gamma = \{Q\}$, and $\mathcal{B} = \{R \sqsubseteq Q, S \sqsubseteq Q\}$, and $\mathcal{K}_\mathsf{s} = \langle \mathcal{S}, \mathcal{A}_\mathsf{s} \rangle$, where $\mathcal{A}_\mathsf{s} = \{A(a), S(a, a)\}$ and $\mathcal{S} = \{A \sqsubseteq \exists R,\ \exists R^- \sqsubseteq \exists R\}$.

We construct the automaton $\mathbb{A}_a$ for $\mathcal{K}_1$, $\mathcal{K}_2$ and Σ, where $\mathcal{K}_1 = \langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s} \rangle$, $\mathcal{K}_2 = \mathcal{K}_1$ and $\Sigma = \Gamma$. Moreover, $\mathsf{ind}(\mathcal{K}_2) = \{a_1\}$, $\mathsf{wit}(\mathcal{K}_2) = \{w_1\}$ and $\mathsf{wit}(\mathcal{K}_1) = \{u_1\}$, where $a_1 = a$, $w_1 = w_{[R]}$, and $u_1 = w_{[R]}$. Thus $n = 1$, so $\mathbb{A}_a$ accepts trees of the form $(\{1\}^*, V)$, where $V(x) \in \{root, stop, a_1, w_1\}$, and the set of accepting states is $F = \{\gamma_1\}$. Below we depict a tree $T \in \mathcal{L}(\mathbb{A}_a)$ with an accepting run over T that starts in $\epsilon_{\mathbf{r}}$ with $\mathbf{r}(\epsilon_{\mathbf{r}}) = (\epsilon, q_0)$.



From T we can extract the ABox $\mathcal{A}_\mathsf{t} = \{Q(a, a), Q(a, n)\}$, which is also a universal solution for $\mathcal{K}_\mathsf{s}$ under $\mathcal{M}$. ■

Summing up, we get:

Theorem 6.19. *If extended ABoxes are allowed in universal solutions, then the non-emptiness problem for universal solutions is* PSpace*-hard and in* ExpTime.

7. Complexity results on UCQ-representability

In this section, we develop techniques and complexity results for the problem of UCQ-representability. More precisely, we show in Section 7.1 that the membership problem for UCQ-representations is NLogSpace-complete, and then we prove in Section 7.2 that the same complexity bound holds also for the non-emptiness problem for UCQ-representations.

7.1. The membership problem

One can immediately notice some similarities between the membership problem for UCQ-representations and the membership problem for universal UCQ-solutions, which was shown to be ExpTime-complete in [30]. However, the universal quantification over ABoxes in the definition of UCQ-representations makes the former problem computationally simpler; in fact, we prove in this section that this problem is NLogSpace-complete, which coincides with the complexity of TBox reasoning in *DL-Lite*$_\mathcal{R}$ [61]. We now list several observations that help to understand this drop in complexity, and also provide an intuition for the characterization of UCQ-representations that is stated in Lemma 7.1, and which is used to pinpoint the complexity of the membership problem for UCQ-representations. In the following, assume fixed a mapping $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$, a source TBox $\mathcal{S}$, and a target TBox $\mathcal{T}$.

1) For simplicity, we assume first that $\mathcal{S}$, $\mathcal{B}$, and $\mathcal{T}$ do not contain disjointness axioms. Let $\mathcal{A}_\mathsf{s} = \{A(a)\}$ be a source ABox, for an atomic concept A, and assume that $\mathcal{S}\cup\mathcal{B} \models A \sqsubseteq B'$ for some basic concept B' over Γ. Then $\langle \mathcal{S}\cup\mathcal{B}, \mathcal{A}_\mathsf{s}\rangle \models B'(a)$ and, thus, $q = B'(a)$ evaluates to true over $\langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s}\rangle$. Hence, for $\mathcal{T}$ to be a UCQ-representation of $\mathcal{S}$ under $\mathcal{M}$, it should be the case that $\langle \mathcal{T} \cup \mathcal{B}, \mathcal{A}_\mathsf{s}\rangle \models q$. From Lemma 3.5 it then follows that $\mathsf{uni}(\mathcal{T} \cup \mathcal{B}, \mathcal{A}_\mathsf{s}) \models B'(a)$, thus, $\mathcal{T} \cup \mathcal{B} \models A \sqsubseteq B'$. The converse can be shown in the same way but starting with the assumption that $\mathcal{T} \cup \mathcal{B} \models A \sqsubseteq B'$. It is easy to extend the above reasoning to the case $\mathcal{A}_\mathsf{s} = \{B(a)\}$ for a basic concept B over Σ, or $\mathcal{A}_\mathsf{s} = \{R(a, b)\}$ for a basic role R over Σ. As we quantify over all possible source ABoxes, we are free to choose any such concept B or role R. Hence, if $\mathcal{T}$ is a UCQ-representation of $\mathcal{S}$ under $\mathcal{M}$, then for each basic concept or role X over Σ and each basic concept or role X' over Γ, it holds that $\mathcal{S} \cup \mathcal{B} \models X \sqsubseteq X'$ if and only if $\mathcal{T} \cup \mathcal{B} \models X \sqsubseteq X'$. This is the main intuition behind condition **(ii)** in Lemma 7.1.

2) For the sake of readability, below we denote by $\mathcal{U}_\mathsf{sb}$ and $\mathcal{U}_\mathsf{tb}$ the canonical models of $\langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s}\rangle$ and $\langle \mathcal{T} \cup \mathcal{B}, \mathcal{A}_\mathsf{s}\rangle$, respectively. Moreover, for a TBox $\mathcal{O}$, we say that a concept B *generates* a role R in $\mathcal{O}$, and we write

$$B \rightsquigarrow_{\mathcal{O}} R$$

if for every constant $a \in N_a$, it holds that $a \rightsquigarrow_{\langle \mathcal{O}, \{B(a)\}\rangle} w_{[R]}$.

Let $\mathcal{A}_\mathsf{s} = \{A(a)\}$ for an atomic concept $A \in \Sigma$, and assume that $A \rightsquigarrow_{\mathcal{S}} R$, $\mathcal{S} \cup \mathcal{B} \models \exists R^- \sqsubseteq B'$ and $\mathcal{S} \cup \mathcal{B} \models R \sqsubseteq R'$, for a role R over Σ, a concept B' over Γ, and a role R' over Γ. Then

$$aw_{[R]} \in \Delta^{\mathcal{U}_\mathsf{sb}}, \quad B' \in \mathbf{t}_\Gamma^{\mathcal{U}_\mathsf{sb}}(aw_{[R]}), \quad \text{and} \quad R' \in \mathbf{r}_\Gamma^{\mathcal{U}_\mathsf{sb}}(a, aw_{[R]}).$$

Next, for $\mathcal{T}$ to be a UCQ-representation of $\mathcal{S}$ under $\mathcal{M}$, by Lemma 3.7, it follows that $\mathcal{U}_\mathsf{sb}$ has to be finitely Γ-homomorphically equivalent to $\mathcal{U}_\mathsf{tb}$. Let Δ be the set containing a and all paths of the form $aw_{[Q]}$ in $\Delta^{\mathcal{U}_\mathsf{sb}}$, $\mathcal{I}$ the sub-interpretation of $\mathcal{U}_\mathsf{sb}$ induced by Δ, and h a Γ-homomorphism from $\mathcal{I}$ to $\mathcal{U}_\mathsf{tb}$. Then $h(a) = a$ and there exists $aw_{[S]} \in \Delta^{\mathcal{U}_\mathsf{tb}}$, for a basic role S over Γ, such that

$$h(aw_{[R]}) = aw_{[S]}, \quad B' \in \mathbf{t}_\Gamma^{\mathcal{U}_\mathsf{tb}}(aw_{[S]}), \quad \text{and} \quad R' \in \mathbf{r}_\Gamma^{\mathcal{U}_\mathsf{tb}}(a, aw_{[S]}),$$

since the image of $aw_{[R]}$ cannot be a constant as $\mathsf{ind}(\mathcal{A}_\mathsf{s}) = \{a\}$ and there are no loops on a in $\mathcal{A}_\mathsf{s}$. By construction of the canonical model and by the fact that $\mathcal{B}$ is a set of inclusions from Σ to Γ, it follows that $\mathcal{T} \cup \mathcal{B} \models A \sqsubseteq \exists S$, $\mathcal{T} \models \exists S^- \sqsubseteq B'$, and $\mathcal{T} \models S \sqsubseteq R'$. Clearly, given $\mathcal{T}$ and $\mathcal{B}$, one can check the existence of such S effectively. On the other hand, if we assume that $A \rightsquigarrow_{\mathcal{S}} R$, $\mathcal{S} \cup \mathcal{B} \models \exists R^- \sqsubseteq B'$, and $\mathcal{S} \cup \mathcal{B} \not\models R \sqsubseteq R'$ for any role R' over Γ (i.e., $\mathbf{r}_\Gamma^{\mathcal{U}_\mathsf{sb}}(a, aw_{[R]}) = \emptyset$), then the homomorphic image of $aw_{[R]}$ could be any element y in $\Delta^{\mathcal{U}_\mathsf{tb}}$ with $\mathbf{t}_\Gamma^{\mathcal{U}_\mathsf{sb}}(aw_{[R]}) \subseteq \mathbf{t}_\Gamma^{\mathcal{U}_\mathsf{tb}}(y)$. This example provides the intuition behind condition **(iii)** in Lemma 7.1.

Observe that it is sufficient to consider only chains of roles of length 1. Thus, for example, if $A \rightsquigarrow_{\mathcal{S}\cup\mathcal{B}} R$ and $\exists R^- \rightsquigarrow_{\mathcal{S}\cup\mathcal{B}} Q$, for some roles R,Q, then the fact that $\mathcal{T}$ is a UCQ-representation for $\mathcal{S}$ under $\mathcal{M}$ depends on whether $\mathcal{T}$ satisfies the condition **(iii)** for two separate cases:

- $\mathcal{A}_\mathsf{s} = \{A(a)\}$ and $A \rightsquigarrow_{\mathcal{S}\cup\mathcal{B}} R$,
- $\mathcal{A}_\mathsf{s} = \{\exists R^-(a)\}$ and $\exists R^- \rightsquigarrow_{\mathcal{S}\cup\mathcal{B}} Q$.

Condition **(iv)** is symmetric to condition **(iii)** if we start with the assumption $A \rightsquigarrow_{\mathcal{T}\cup\mathcal{B}} R'$ and $\mathcal{T} \models \exists R'^- \sqsubseteq B'$ for a role R' over Γ and a concept B' over Γ.

3) To conclude, we analyze the cases when $\mathcal{S}$, $\mathcal{B}$, and $\mathcal{T}$ contain disjointness axioms. First, notice that without loss of generality we can assume that there are no disjointness axioms in $\mathcal{S}$ as in the definition of UCQ-representations, we consider only ABoxes $\mathcal{A}_\mathsf{s}$ that are consistent with $\mathcal{S}$. So we will take into account only disjointness axioms in $\mathcal{B}$ and $\mathcal{T}$. Then for a source ABox $\mathcal{A}_\mathsf{s}$ consistent with $\mathcal{S}$, it is possible that $\langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s} \rangle$ is inconsistent due to the disjointness axioms in the mapping, which will make all possible tuples to be in the answer to every query.

Consider an ABox $\mathcal{A}_\mathsf{s} = \{A(a), C(a)\}$ for atomic concepts A, C over Σ, and assume that $\mathcal{A}_\mathsf{s}$ is consistent with $\mathcal{S}$. Furthermore, assume that the KB $\langle \mathcal{S} \cup \mathcal{B}, \{A(b), C(b)\} \rangle$, for an arbitrary constant b, is inconsistent. Then $\langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s} \rangle$ is inconsistent, and by definition of certain answers over an inconsistent KB, $cert(q, \langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s} \rangle) = AllTup(q)$ for each target UCQ q. Therefore, in order for $\mathcal{T}$ to be a UCQ-representation of $\mathcal{S}$ under $\mathcal{M}$, $\langle \mathcal{T} \cup \mathcal{B}, \mathcal{A}_\mathsf{s} \rangle$ has to be inconsistent as well. To ensure that this is the case, we need to check that (A, C) is also $(\mathcal{T} \cup \mathcal{B})$-inconsistent. Similarly but in the opposite direction, if we start with the assumption that $\langle \mathcal{T} \cup \mathcal{B}, \{A(b), C(b)\} \rangle$ is inconsistent, it should be verified that also $\{\mathcal{S} \cup \mathcal{B}, \{A(b), C(b)\}\}$ is inconsistent, for some arbitrary constant b. This is the intuition behind condition **(i)** in Lemma 7.1.

Finally, we are ready to characterize UCQ-representations. To capture the above intuitions, in the following, for a TBox O, we say that a pair (B, B') of basic concepts is *O-consistent*, if the KB $\langle O, \{B(a), B'(a)\} \rangle$ is consistent, where a is an arbitrary constant, and (B, B') is *O-inconsistent* otherwise. Similarly, a pair (R, R') of basic roles is *O-consistent*, if the KB $\langle O, \{R(a, b), R'(a, b)\} \rangle$ is consistent, where a, b are arbitrary distinct constants, and (R, R') is *O-inconsistent* otherwise. Moreover, a concept or role X is O-consistent if (X, X) is *O-consistent*, and *O-inconsistent* otherwise. Below, we abuse notation and write $\mathsf{gen}(O, B(o))$ instead of $\mathsf{gen}(\langle O, \{B(o)\} \rangle)$, and $\mathsf{uni}(O, B(o))$ instead of $\mathsf{uni}(\langle O, \{B(o)\} \rangle)$, for a TBox O, a concept B and $o \in N_a$.

Lemma 7.1. *Given a mapping $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$, a TBox $\mathcal{T}$ over Γ is a UCQ-representation of a TBox $\mathcal{S}$ over Σ under $\mathcal{M}$ if and only if the following conditions hold:*

- ***(i)*** *for each pair of $\mathcal{S}$-consistent concepts or roles X, X' over Σ, (X, X') is $(\mathcal{S} \cup \mathcal{B})$-consistent iff (X, X') is $(\mathcal{T} \cup \mathcal{B})$-consistent;*
- ***(ii)*** *for each $(\mathcal{S} \cup \mathcal{B})$-consistent concept or role X over Σ and each X' over Γ, $\mathcal{S} \cup \mathcal{B} \models X \sqsubseteq X'$ iff $\mathcal{T} \cup \mathcal{B} \models X \sqsubseteq X'$;*
- ***(iii)*** *for each $(\mathcal{S} \cup \mathcal{B})$-consistent concept B over Σ and each role R such that $B \rightsquigarrow_{\mathcal{S} \cup \mathcal{B}} R$, there exists $y \in \Delta^{\mathsf{gen}(\mathcal{T} \cup \mathcal{B}, B(o))}$, where o is an arbitrary constant, such that*

$$\mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{S} \cup \mathcal{B}, B(o))}(w_{[R]}) \subseteq \mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{T} \cup \mathcal{B}, B(o))}(y), \quad \textit{and} \quad \mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{S} \cup \mathcal{B}, B(o))}(o, w_{[R]}) \subseteq \mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{T} \cup \mathcal{B}, B(o))}(o, y);$$

- ***(iv)*** *for each $(\mathcal{S} \cup \mathcal{B})$-consistent concept B over Σ and each role R such that $B \rightsquigarrow_{\mathcal{T} \cup \mathcal{B}} R$, there exists $y \in \Delta^{\mathsf{gen}(\mathcal{S} \cup \mathcal{B}, B(o))}$, where o is an arbitrary constant, such that*

$$\mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{T} \cup \mathcal{B}, B(o))}(w_{[R]}) \subseteq \mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{S} \cup \mathcal{B}, B(o))}(y), \quad \textit{and} \quad \mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{T} \cup \mathcal{B}, B(o))}(o, w_{[R]}) \subseteq \mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{S} \cup \mathcal{B}, B(o))}(o, y).$$

Proof. ($\Leftarrow$) Let the conditions above hold for $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{B}$, and let $\mathcal{A}_\mathsf{s}$ be an ABox over Σ such that $\langle \mathcal{S}, \mathcal{A}_\mathsf{s} \rangle$ is consistent. Moreover, denote by $\mathcal{K}_\mathsf{sb}$ the KB $\langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s} \rangle$, and by $\mathcal{K}_\mathsf{tb}$ the KB $\langle \mathcal{T} \cup \mathcal{B}, \mathcal{A}_\mathsf{s} \rangle$, and let $\mathcal{U}_\mathsf{sb}$ and $\mathcal{U}_\mathsf{tb}$ be the canonical models of $\mathcal{K}_\mathsf{sb}$ and $\mathcal{K}_\mathsf{tb}$, respectively. Next we show that $\mathcal{K}_\mathsf{sb}$ and $\mathcal{K}_\mathsf{tb}$ are Γ-query inseparable.

Observe that condition **(i)** ensures that for every ABox $\mathcal{A}_\mathsf{s}$ over Σ that is consistent with $\mathcal{S}$, $\mathcal{K}_\mathsf{sb}$ is consistent iff $\mathcal{K}_\mathsf{tb}$ is consistent. Indeed, if $\mathcal{K}_\mathsf{sb}$ is consistent, then for each pair of basic concepts B, B' over Σ such that $\mathcal{A}_\mathsf{s} \models B(a)$ and $\mathcal{A}_\mathsf{s} \models B'(a)$ for some $a \in \mathsf{ind}(\mathcal{A}_\mathsf{s})$, the KB $\mathcal{K}'_\mathsf{sb} = \langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s} \cup \{B(a), B'(a)\} \rangle$ is consistent, and by monotonicity of first-order logic we obtain that the KB $\langle \mathcal{S} \cup \mathcal{B}, \{B(a), B'(a)\} \rangle$ is also consistent, and thus (B, B') is $\mathcal{S} \cup \mathcal{B}$-consistent. And similarly, for each pair of basic roles R, R' over Σ such that $\mathcal{A}_\mathsf{s} \models R(b, c)$ and $\mathcal{A}_\mathsf{s} \models R'(b, c)$ for some $b, c \in \mathsf{ind}(\mathcal{A}_\mathsf{s})$, we can derive that (R, R') is $\mathcal{S} \cup \mathcal{B}$-consistent. Then, by **(i)** for each pair B, B' as above, (B, B') is $\mathcal{T} \cup \mathcal{B}$-consistent, and likewise for each pair R, R' as above. To see that $\mathcal{K}_\mathsf{tb}$ is consistent, it suffices to observe that the interpretation $\mathcal{I}$ defined as the union of the canonical models $\mathsf{uni}(\mathcal{T} \cup \mathcal{B}, \{B(a), B'(a)\})$ and $\mathsf{uni}(\mathcal{T} \cup \mathcal{B}, \{R(b, c), R'(b, c)\})$ for B, B', R, R', and a, b, c as above, is a model $\mathcal{K}_\mathsf{tb}$. Note that in this paragraph, B and B' can denote the same concept, and R and R' can denote the same role. The proof can be inverted to show that consistency of $\mathcal{K}_\mathsf{tb}$ implies consistency of $\mathcal{K}_\mathsf{sb}$.

First, assume $\mathcal{K}_{sb}$ is inconsistent, it follows that $cert(q, \mathcal{K}_{sb}) = AllTup(q)$ for each UCQ q over Γ. By the argument above, $\mathcal{K}_{tb}$ is inconsistent, so $cert(q, \mathcal{K}_{tb}) = AllTup(q)$ for each UCQ q over Γ as well, hence $\mathcal{K}_{sb}$ and $\mathcal{K}_{tb}$ are Γ-query inseparable.

Now assume $\mathcal{K}_{sb}$ is consistent. One can show that from **(ii)** and **(iii)** it follows that $\mathcal{U}_{sb}$ is Γ-homomorphically embeddable into $\mathcal{U}_{tb}$ (see Proposition C.1). Since $\mathcal{K}_{tb}$ is consistent, we can apply Lemma 3.7 to obtain that $\mathcal{K}_{tb}$ Γ-query entails $\mathcal{K}_{sb}$. On the other hand, one can show that **(ii)** and **(iv)** imply that $\mathcal{U}_{tb}$ is Γ-homomorphically embeddable into $\mathcal{U}_{sb}$ (see Proposition C.2), hence $\mathcal{K}_{sb}$ Γ-query entails $\mathcal{K}_{tb}$ by Lemma 3.7. We obtain again that $\mathcal{K}_{sb}$ and $\mathcal{K}_{tb}$ are Γ-query inseparable.

($\Rightarrow$) Assume, by contradiction, that one of the conditions **(i)** – **(iv)** is not satisfied. We produce an $\mathcal{S}$-consistent ABox $\mathcal{A}_s$ over Σ and a Boolean CQ q over Γ such that it is not the case that $\mathcal{K}_{sb} \models q$ iff $\mathcal{K}_{tb} \models q$.

Assume, first, that condition **(i)** is violated. Then we take $\mathcal{A}_s = \{B_1(o), B_2(o)\}$ for concepts B_1 and B_2 violating it and $q = B_1(a)$ for some constant a distinct from o. If (B_1, B_2) are $\mathcal{S} \cup \mathcal{B}$-consistent, but $\mathcal{T} \cup \mathcal{B}$-inconsistent, it follows that $\mathcal{K}_{sb} \not\models q$ and $\mathcal{K}_{tb} \models q$, and the opposite holds if (B_1, B_2) are $\mathcal{T} \cup \mathcal{B}$-consistent, but $\mathcal{S} \cup \mathcal{B}$-inconsistent. If **(i)** is violated for roles, the proof is analogous.

Let now condition **(ii)** be violated for some $\mathcal{S} \cup \mathcal{B}$-consistent concept B over Σ. Assume there is B' such that $\mathcal{S} \cup \mathcal{B} \models B \sqsubseteq B'$ and $\mathcal{T} \cup \mathcal{B} \not\models B \sqsubseteq B'$, and consider $\mathcal{A}_s = \{B(o)\}$ and $q = B'(o)$. Then $B' \in \mathbf{t}_\Gamma^{\mathcal{U}_{sb}}(o)$ and $B' \notin \mathbf{t}_\Gamma^{\mathcal{U}_{tb}}(o)$, so it follows that $\mathcal{U}_{sb} \models q$ and $\mathcal{U}_{tb} \not\models q$; finally by Lemma 3.5 it follows $\mathcal{K}_{sb} \models q$ and $\mathcal{K}_{tb} \not\models q$. The opposite follows if we assume that $\mathcal{S} \cup \mathcal{B} \not\models B \sqsubseteq B'$ and $\mathcal{T} \cup \mathcal{B} \models B \sqsubseteq B'$, which completes the proof for this case. If condition **(ii)** is violated for some role, the proof is analogous.

Next, assume condition **(iii)** is violated, so there exists an $\mathcal{S} \cup \mathcal{B}$-consistent concept B over Σ and a role R such that $B \rightsquigarrow_{\mathcal{S}\cup\mathcal{B}} R$, and for $\mathcal{A}_s = \{B(o)\}$ there is no $y \in \Delta^{\mathsf{gen}(\mathcal{K}_{tb})}$ such that both $\mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{K}_{sb})}(w_{[R]}) \subseteq \mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{K}_{tb})}(y)$ and $\mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{K}_{sb})}(o, w_{[R]}) \subseteq \mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{K}_{tb})}(o, y)$. Let $\mathbf{B} = \mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{K}_{sb})}(w_{[R]})$, $\mathbf{R} = \mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{K}_{sb})}(o, w_{[R]})$, and consider

$$q = \exists x \left(\bigwedge_{R' \in \mathbf{R}} R'(o, x) \wedge \bigwedge_{B' \in \mathbf{B}} B'(x) \right),$$

where $B'(x)$ denotes atom $A(x)$ if $B' = A$ for an atomic concept A, and $B'(x)$ denotes formula $\exists x'.S(x, x')$ if $B' = \exists S$ for a role S. Then $\mathcal{U}_{sb} \models q$ by mapping the existentially quantified variable x to $ow_{[R]}$. On the other hand, $\mathcal{U}_{tb} \not\models q$ as there is no element of $\Delta^{\mathcal{U}_{tb}}$ to which x could be mapped. Using Lemma 3.5 we obtain that $\mathcal{K}_{sb} \models q$ and $\mathcal{K}_{tb} \not\models q$.

The case when condition **(iv)** is violated is analogous to the case above. This completes the proof. $\square$

Having devised a characterization of UCQ-representations, we discuss several examples of (non-)UCQ-representations.

Example 7.2. Assume that $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$, where $\Sigma = \{A, R\}$, $\Gamma = \{A', R', B'\}$, and $\mathcal{B} = \{A \sqsubseteq A', \exists R^- \sqsubseteq B'\}$. Moreover, let $\mathcal{S} = \{A \sqsubseteq \exists R\}$.

- **(a)** In Example 5.18 we showed that $\mathcal{T} = \{A' \sqsubseteq B'\}$ is not a UCQ-representation of $\mathcal{S}$ under $\mathcal{M}$. In fact, in this case, condition **(ii)** is not satisfied, as $\mathcal{T} \cup \mathcal{B} \models A \sqsubseteq B'$ while $\mathcal{S} \cup \mathcal{B} \not\models A \sqsubseteq B'$.
- **(b)** In the same example we showed that also $\mathcal{T} = \{A' \sqsubseteq \exists R', \exists R'^- \sqsubseteq B'\}$ is not a UCQ-representation of $\mathcal{S}$ under $\mathcal{M}$. In this case, condition **(iv)** is not satisfied, as $A \rightsquigarrow_{\mathcal{T}\cup\mathcal{B}} R'$, but there exists no $y \in \Delta^{\mathsf{gen}(\mathcal{S}\cup\mathcal{B}, A(o))}$ such that

  $$\mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{T}\cup\mathcal{B}, A(o))}(w_{[R']}) \subseteq \mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{S}\cup\mathcal{B}, A(o))}(y) \quad \text{and} \quad \mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{T}\cup\mathcal{B}, A(o))}(o, w_{[R']}) \subseteq \mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{S}\cup\mathcal{B}, A(o))}(o, y),$$

  since neither $y = o$, nor $y = w_{[R]}$ in $\Delta^{\mathsf{gen}(\mathcal{S}\cup\mathcal{B}, A(o))}$ satisfy $R' \in \mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{S}\cup\mathcal{B}, A(o))}(o, y)$. ■

Example 7.3. Assume that $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$, where

$$\begin{aligned} \Sigma &= \{A, R, S, Q\} \\ \Gamma &= \{A', B', S', Q'\} \\ \mathcal{B} &= \{A \sqsubseteq A', \exists R^- \sqsubseteq B', S \sqsubseteq S', Q \sqsubseteq Q', \exists Q^- \sqsubseteq B'\} \end{aligned}$$

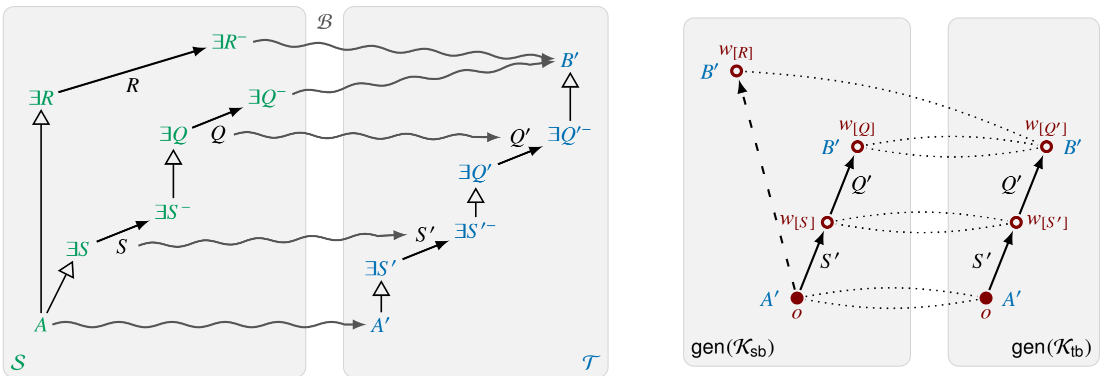
and let

$$\begin{aligned} \mathcal{S} &= \{A \sqsubseteq \exists R, A \sqsubseteq \exists S, \exists S^- \sqsubseteq \exists Q\} \\ \mathcal{T} &= \{A' \sqsubseteq \exists S', \exists S'^- \sqsubseteq \exists Q', \exists Q'^- \sqsubseteq B'\} \end{aligned}$$

Then $\mathcal{T}$ is a UCQ-representation of $\mathcal{S}$ under $\mathcal{M}$. We verify that conditions **(iii)** and **(iv)** are satisfied. First, $A \rightsquigarrow_{\mathcal{S}\cup\mathcal{B}} R$: we take $w_{[Q']} \in \Delta^{\mathsf{gen}(\mathcal{T}\cup\mathcal{B},A(o))}$ and it is easy to see that the following is satisfied:

$$\mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{S}\cup\mathcal{B},A(o))}(w_{[R]}) \subseteq \mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{T}\cup\mathcal{B},A(o))}(w_{[Q']}) \quad \text{and} \quad \mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{S}\cup\mathcal{B},A(o))}(o, w_{[R]}) \subseteq \mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{T}\cup\mathcal{B},A(o))}(o, w_{[Q']}),$$

as $\mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{S}\cup\mathcal{B},A(o))}(o, w_{[R]}) = \emptyset$. Next, $A \rightsquigarrow_{\mathcal{S}\cup\mathcal{B}} S$ and $\exists S^- \rightsquigarrow_{\mathcal{S}\cup\mathcal{B}} Q$. It should be clear that we take $w_{[S']}$ and $w_{[Q']}$ in $\Delta^{\mathsf{gen}(\mathcal{T}\cup\mathcal{B},A(o))}$ and $\Delta^{\mathsf{gen}(\mathcal{T}\cup\mathcal{B},\exists S^-(o))}$ respectively to satisfy condition **(iii)**. As for the opposite direction, now differently from Example 7.2, for both $w_{[S']}$ and $w_{[Q']}$ in $\Delta^{\mathsf{gen}(\mathcal{T}\cup\mathcal{B},A(o))}$ and $\Delta^{\mathsf{gen}(\mathcal{T}\cup\mathcal{B},\exists S^-(o))}$ respectively, there exist $w_{[S]}$ and $w_{[Q]}$ in $\Delta^{\mathsf{gen}(\mathcal{S}\cup\mathcal{B},A(o))}$ and $\Delta^{\mathsf{gen}(\mathcal{S}\cup\mathcal{B},\exists S^-(o))}$ that satisfy condition **(iv)**. Below we provide the graphical representation of $\mathcal{S}$, $\mathcal{B}$ and $\mathcal{T}$, and we illustrate the projections of $\mathsf{gen}(\mathcal{K}_{\mathsf{sb}})$ and $\mathsf{gen}(\mathcal{K}_{\mathsf{tb}})$ on Γ, for $\mathcal{K}_{\mathsf{sb}} = \langle \mathcal{S}\cup\mathcal{B}, A(o)\rangle$ and $\mathcal{K}_{\mathsf{tb}} = \langle \mathcal{T}\cup\mathcal{B}, A(o)\rangle$ (concept labels of the form $\exists P, \exists P^-$ for a role P are not shown). Notice that the dashed edge $(o, w_{[R]})$ represents the fact that the role type $\mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{K}_{\mathsf{sb}})}(o, w_{[R]})$ is empty.



■

Example 7.4. Assume that $\mathcal{M} = (\{A, B, C, D\}, \{A', B'\}, \mathcal{B})$, where $\mathcal{B} = \{A \sqsubseteq A', B \sqsubseteq B', C \sqsubseteq \neg A', D \sqsubseteq B'\}$, and let $\mathcal{S} = \{D \sqsubseteq C\}$. Then $\mathcal{T} = \{A' \sqsubseteq \neg B'\}$ is not a UCQ-representation of $\mathcal{S}$ under $\mathcal{M}$. To see that, consider source ABox $\mathcal{A}_{\mathsf{s}} = \{A(a), B(a)\}$: it is consistent with $\mathcal{S}\cup\mathcal{B}$, but inconsistent with $\mathcal{T}\cup\mathcal{B}$. So for $q = A'(b)$ where b is a constant distinct from a, $\langle \mathcal{S}\cup\mathcal{B}, \mathcal{A}_{\mathsf{s}}\rangle \not\models q$, and $\langle \mathcal{T}\cup\mathcal{B}, \mathcal{A}_{\mathsf{s}}\rangle \models q$. Let us verify that using the characterization. In fact, although, $\mathcal{T}$ satisfies condition **(i)** for the pair of concepts (A, D), which is both $\mathcal{S}\cup\mathcal{B}$-inconsistent and $\mathcal{T}\cup\mathcal{B}$-inconsistent, $\mathcal{T}$ violates this condition for the pair (A, B), which is clearly $\mathcal{S}\cup\mathcal{B}$-consistent, however $\mathcal{T}\cup\mathcal{B}$-inconsistent as $\mathcal{T}\cup\mathcal{B}$ entails both $A \sqsubseteq \neg B'$ and $B \sqsubseteq B'$. We note that in general, $\mathcal{S}$ is not UCQ-representable under $\mathcal{M}$. ■

Note that the proof of Lemma 7.1 implies an alternative characterization of UCQ-representations in terms of homomorphisms.

Lemma 7.5. *A TBox $\mathcal{T}$ over Γ is a* UCQ*-representation of a TBox $\mathcal{S}$ over Σ under a mapping $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$ if and only if the following conditions hold:*

- *for each ABox $\mathcal{A}_{\mathsf{s}}$ consistent with $\mathcal{S}$, $\langle \mathcal{S}\cup\mathcal{B}, \mathcal{A}_{\mathsf{s}}\rangle$ is consistent iff $\langle \mathcal{T}\cup\mathcal{B}, \mathcal{A}_{\mathsf{s}}\rangle$ is consistent;*
- *for each ABox $\mathcal{A}_{\mathsf{s}}$ consistent with $\mathcal{S}\cup\mathcal{B}$, $\mathsf{uni}(\mathcal{S}\cup\mathcal{B}, \mathcal{A}_{\mathsf{s}})$ is Γ-homomorphically equivalent to $\mathsf{uni}(\mathcal{T}\cup\mathcal{B}, \mathcal{A}_{\mathsf{s}})$.*

We can devise an efficient algorithm for checking the membership problem for UCQ-representations from the conditions in Lemma 7.1. Combining it with the complexity of reasoning in *DL-Lite*$_{\mathcal{R}}$, we obtain the following complexity bound, which provides the main result of this section.

Theorem 7.6. *The membership problem for* UCQ*-representations is* NLogSpace*-complete.*

Proof. The lower bound can be obtained by the following reduction from the directed graph reachability problem, which is known to be NLogSpace-hard: given a graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ and a pair of vertices $v_k, v_m \in \mathsf{V}$, decide if there is a directed path from v_k to v_m. To encode the problem, we need a source signature Σ of concept names $\{V_i \mid v_i \in \mathsf{V}\}$ and a target signature Γ of concept names $\{V'_i \mid v_i \in \mathsf{V}\}$. Consider $\mathcal{S} = \{V_k \sqsubseteq V_m\} \cup \{V_i \sqsubseteq V_j \mid (v_i, v_j) \in \mathsf{E}\}$, $\mathcal{B} = \{V_i \sqsubseteq V'_i \mid v_i \in \mathsf{V}\}$, and $\mathcal{T} = \{V'_i \sqsubseteq V'_j \mid (v_i, v_j) \in \mathsf{E}\}$. One can easily verify that the condition **(ii)** of Lemma 7.1 is satisfied iff there is a directed path from v_k to v_m in G, whereas the other conditions of Lemma 7.1 are satisfied trivially. Therefore, there is a directed path from v_k to v_m in G iff $\mathcal{T}$ is a UCQ-representation of $\mathcal{S}$ under $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$. This concludes the proof of the lower bound.

For the upper bound, we show that conditions **(i)** – **(iv)** of Lemma 7.1 can be verified in NLogSpace. It is well known (see, e.g., [61]), that given a pair B, B' of *DL-Lite*$_{\mathcal{R}}$ concepts, and a TBox $\mathcal{O}$, it can be verified in NLogSpace, if (B, B') is $\mathcal{O}$-consistent (using an algorithm for directed graph reachability); the same holds for a pair R, R' of *DL-Lite*$_{\mathcal{R}}$ roles. The same algorithm can be straightforwardly adopted to check, if $\mathcal{O} \models B \sqsubseteq B'$ or $\mathcal{O} \models R \sqsubseteq R'$. Therefore, clearly, conditions **(i)** and **(ii)** can be verified in NLogSpace. Conditions **(iii)** and **(iv)** can be checked similarly to the proof of Proposition 6.3. □

7.2. The non-emptiness problem

We start with examples that provide some intuition on how the non-emptiness problem is solved.

Example 7.7. Consider $\mathcal{M}$ and the UCQ-representable TBox $\mathcal{S}$ from Example 5.17-(3): $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$, where $\Sigma = \{A, B, C\}$, $\Gamma = \{A', B', C'\}$, and $\mathcal{B} = \{A \sqsubseteq A', B \sqsubseteq B', A \sqsubseteq C'\}$, and $\mathcal{S} = \{A \sqsubseteq B\}$. It follows that $\mathcal{S} \cup \mathcal{B} \models A \sqsubseteq B'$. A first and obvious requirement for a UCQ-representation $\mathcal{T}$ is that $\mathcal{T}$ should entail an axiom of the form $D' \sqsubseteq B'$ so that $\mathcal{T} \cup \mathcal{B} \models A \sqsubseteq B'$ (hence, $\mathcal{B} \models A \sqsubseteq D'$). On the other hand, it could be that $\mathcal{B} \models D \sqsubseteq D'$ for some D distinct from A, in which case it follows also $\mathcal{T} \cup \mathcal{B} \models D \sqsubseteq B'$. Since we want $\mathcal{T}$ to be a UCQ-representation, it should be the case that $\mathcal{S} \cup \mathcal{B} \models D \sqsubseteq B'$. In our case, we can take D' equal to A' or C', and there exists no such concept D (distinct from A). Hence, there are two UCQ-representations of $\mathcal{S}$ under $\mathcal{M}$, namely $\{A' \sqsubseteq B'\}$ and $\{C' \sqsubseteq B'\}$.

Consider now the slightly different $\mathcal{B} = \{A \sqsubseteq A', B \sqsubseteq B', C \sqsubseteq A'\}$ from Example 5.17-(4), where we showed that $\mathcal{S}$ is not UCQ-representable. As before, $\mathcal{S} \cup \mathcal{B} \models A \sqsubseteq B'$. However now, the only candidate for D' is A', and there exists a concept D distinct from A, namely C, such that $\mathcal{B} \models D \sqsubseteq A'$. So on the one hand, the only way to have a UCQ-representation $\mathcal{T}$ is to include axiom $A' \sqsubseteq B'$ in $\mathcal{T}$, but on the other hand since $\mathcal{S} \cup \mathcal{B} \not\models C \sqsubseteq B'$, this axiom cannot be in $\mathcal{T}$. In general, there is no way to "represent" the inclusion $A \sqsubseteq B'$ in the target, so in this case $\mathcal{S}$ is not UCQ-representable under $\mathcal{M}$. ■

Example 7.8. Consider $\mathcal{M}$, $\mathcal{S}$ and $\mathcal{B}$ from Example 7.4 such that $\mathcal{S}$ is not UCQ-representable under $\mathcal{M}$. It follows that the pair of concepts (A, D) is $\mathcal{S} \cup \mathcal{B}$-inconsistent as $\mathcal{S} \cup \mathcal{B} \models A \sqsubseteq A'$ and $\mathcal{S} \cup \mathcal{B} \models D \sqsubseteq \neg A'$. So a candidate UCQ-representation $\mathcal{T}$ should be such that (A, D) is $\mathcal{T} \cup \mathcal{B}$-inconsistent. One possible way to achieve that is by having $\mathcal{T} \cup \mathcal{B} \models D \sqsubseteq \neg A'$, and since D is transferred only to B' through the mapping, it means that $\mathcal{T}$ should entail $B' \sqsubseteq \neg A'$, or $B' \sqsubseteq \neg B'$, or $A' \sqsubseteq \neg A'$. In the first case, however, the pair (A, B) would be $\mathcal{T} \cup \mathcal{B}$-inconsistent as well, since $A \sqsubseteq A'$ and $B \sqsubseteq B'$ are in $\mathcal{B}$. Then, for $\mathcal{T}$ to be a UCQ-representation of $\mathcal{S}$ under $\mathcal{M}$, (A, B) should be $\mathcal{S} \cup \mathcal{B}$-inconsistent, which is not the case. In the second case, the pair (B, B) would be $\mathcal{T} \cup \mathcal{B}$-inconsistent, while it is $\mathcal{S} \cup \mathcal{B}$-consistent. Similarly, we obtain that it cannot be the case that $\mathcal{T} \models A' \sqsubseteq \neg A'$. In general, it is impossible to have a target TBox $\mathcal{T}$ such that (A, D) is $\mathcal{T} \cup \mathcal{B}$-inconsistent and $\mathcal{T}$ is a UCQ-representation of $\mathcal{S}$ under $\mathcal{M}$, i.e., it is impossible to enforce that concepts A and D "contradict" each other in the target. ■

We illustrated in the examples above that in order to check whether $\mathcal{S}$ is UCQ-representable under $\mathcal{M}$ one needs to verify whether the axioms implied by $\mathcal{S} \cup \mathcal{B}$ are "representable", and whether $\mathcal{S} \cup \mathcal{B}$-inconsistent pairs are "target contradictable". To formally define these notions, which are required for the characterization in Lemma 7.13, we first introduce the following notion. We say that a target TBox $\mathcal{T}$ is a *parsimonious* UCQ-*representation* of $\mathcal{S}$ under $\mathcal{M}$, if for every ABox $\mathcal{A}_s$ over Σ that is consistent with $\mathcal{S}$, $\langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_s \rangle$ Γ-query entails $\langle \mathcal{T} \cup \mathcal{B}, \mathcal{A}_s \rangle$. Observe that the empty TBox is a parsimonious UCQ-representation. In the definitions below, X and Y denote basic concepts or roles over Σ, and X' denotes a basic concept or role over Γ.

Definition 7.9. *Inclusion $X \sqsubseteq X'$ is* representable *in $\mathcal{S}$ and $\mathcal{M}$, if there exists a (possibly trivial) target axiom α such that, whenever $\mathcal{T}$ is a parsimonious* UCQ-*representation of $\mathcal{S}$ under $\mathcal{M}$, it holds that $\mathcal{T}' = \mathcal{T} \cup \{\alpha\}$ is also a parsimonious* UCQ-*representation of $\mathcal{S}$ under $\mathcal{M}$, and moreover $\mathcal{T}' \cup \mathcal{B} \models X \sqsubseteq X'$.*

In this case, we say that $X \sqsubseteq X'$ is representable via α.

Definition 7.10. *Pair (X, Y) is* target contradictable *in $\mathcal{S}$ and $\mathcal{M}$, if there exists a (possibly trivial) target axiom α such that, whenever $\mathcal{T}$ is a parsimonious* UCQ-*representation of $\mathcal{S}$ under $\mathcal{M}$, it holds that $\mathcal{T}' = \mathcal{T} \cup \{\alpha\}$ is also a parsimonious* UCQ-*representation of $\mathcal{S}$ under $\mathcal{M}$, and moreover (X, Y) is $\mathcal{T}' \cup \mathcal{B}$-inconsistent.*

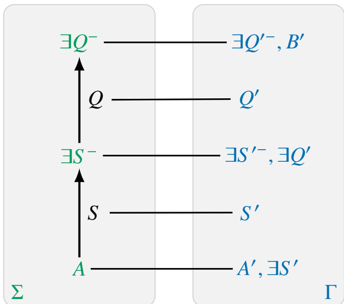
In this case, we say that (X, Y) is target contradictable via α.

Our last definition before we present a characterization of the cases when $\mathcal{S}$ is UCQ-representable under $\mathcal{M}$ is the notion of a *generating path*. In the case a concept B generates a role R in $\mathcal{S} \cup \mathcal{B}$, $B \rightsquigarrow_{\mathcal{S}\cup\mathcal{B}} R$, existence of a generating path for (B, R) ensures that there exists a parsimonious UCQ-representation $\mathcal{T}$ satisfying condition **(iii)** of Lemma 7.1 for B and R. For a TBox $\mathcal{O}$ and a concept B (resp., role R), denote by $\mathbf{sup}^{\mathcal{O}}_{\Sigma}(B)$ (resp., $\mathbf{sup}^{\mathcal{O}}_{\Sigma}(R)$) the set of all concepts B' (resp., roles R') over Σ such that $\mathcal{O} \models B \sqsubseteq B'$ (resp., $\mathcal{O} \models R \sqsubseteq R'$).

Definition 7.11. *Let B be a concept over Σ and R a role. A* generating path for (B, R) *in $\mathcal{S}$ and $\mathcal{M}$ is a sequence $\langle C_0, C_1, \dots C_n\rangle$ of concepts, with $n \geq 0$, such that $C_0 = B$, and such that for $1 \leq i \leq n$ and $0 \leq j \leq n$ the following holds:*

- ***(A)*** *$C_i = \exists Q_i^-$ for some role Q_i such that $\mathcal{S} \cup \mathcal{B} \models C_{i-1} \sqsubseteq \exists Q_i$ and $\mathbf{sup}^{\mathcal{S}\cup\mathcal{B}}_{\Gamma}(Q_i) \neq \emptyset$;*
- ***(B)*** *for each $D_j \in \mathbf{sup}^{\mathcal{S}\cup\mathcal{B}}_{\Gamma}(C_j)$, inclusion $C_j \sqsubseteq D_j$ is representable in $\mathcal{S}$ and $\mathcal{M}$;*
- ***(C)*** *for each $S_i \in \mathbf{sup}^{\mathcal{S}\cup\mathcal{B}}_{\Gamma}(Q_i)$, inclusion $Q_i \sqsubseteq S_i$ is representable in $\mathcal{S}$ and $\mathcal{M}$;*
- ***(D)*** *$\mathbf{sup}^{\mathcal{S}\cup\mathcal{B}}_{\Gamma}(\exists R^-) \subseteq \mathbf{sup}^{\mathcal{S}\cup\mathcal{B}}_{\Gamma}(C_n)$, and if $\mathbf{sup}^{\mathcal{S}\cup\mathcal{B}}_{\Gamma}(R) \neq \emptyset$, then $n = 1$ and $\mathbf{sup}^{\mathcal{S}\cup\mathcal{B}}_{\Gamma}(R) \subseteq \mathbf{sup}^{\mathcal{S}\cup\mathcal{B}}_{\Gamma}(Q_1)$.*

Example 7.12. Consider $\mathcal{M}$ and $\mathcal{S}$ from Example 7.3. Then $\langle A, \exists S^-, \exists Q^-\rangle$ is a generating path for (A, R) in $\mathcal{S}$ and $\mathcal{M}$. Below we represent it graphically, where the $\mathbf{sup}^{\mathcal{S}\cup\mathcal{B}}_{\Gamma}$ labels are shown to the right.



To the contrary, for $\mathcal{M}$ and $\mathcal{S}$ from Example 7.2, there exists no generating path for (A, R) in $\mathcal{S}$ and $\mathcal{M}$. ■

Having defined all notions above, we provide a characterization of the cases when $\mathcal{S}$ is UCQ-representable under $\mathcal{M}$, which has a similar structure to the characterization of UCQ-representations in Lemma 7.1.

Lemma 7.13. *Given a mapping $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$ and a TBox $\mathcal{S}$ over Σ, $\mathcal{S}$ is* UCQ*-representable under $\mathcal{M}$, if and only if the following conditions are satisfied:*

- ***(I)*** *For each $\mathcal{S}$-consistent pair of concepts or roles X, Y over Σ, such that (X, Y) is $\mathcal{S} \cup \mathcal{B}$-inconsistent, (X, Y) is target contradictable in $\mathcal{S}$ and $\mathcal{M}$.*
- ***(II)*** *For each $\mathcal{S} \cup \mathcal{B}$-consistent concept or role X over Σ and each X' over Γ such that $\mathcal{S} \cup \mathcal{B} \models X \sqsubseteq X'$, inclusion $X \sqsubseteq X'$ is representable in $\mathcal{S}$ and $\mathcal{M}$.*
- ***(III)*** *For each $\mathcal{S} \cup \mathcal{B}$-consistent concept B over Σ and each role R such that $B \rightsquigarrow_{\mathcal{S}\cup\mathcal{B}} R$, there exists a generating path for (B, R) in $\mathcal{S}$ and $\mathcal{M}$.*

Proof. ($\Leftarrow$) Assume that conditions **(I)** – **(III)** are satisfied, we construct a TBox $\mathcal{T}$ over Γ and prove that it is a UCQ-representation for $\mathcal{S}$ under $\mathcal{M}$. The required $\mathcal{T}$ will be given as the union of the three sets of axioms presented below. First, let (B, C) be an $\mathcal{S}$-consistent and $\mathcal{S} \cup \mathcal{B}$-inconsistent pair of concepts over Σ, then (B, C) is target contradictable by condition **(I)**: assume that (B, C) is target contradictable via α, then define set $ax_i(B, C)$ to be equal to $\{\alpha\}$. Similarly, we define $ax_i(R, Q) = \{\alpha\}$ for an $\mathcal{S}$-consistent and $\mathcal{S} \cup \mathcal{B}$-inconsistent pair of roles R, Q over Σ. Next, take an $\mathcal{S} \cup \mathcal{B}$-consistent concept B over Σ, and assume that $\mathcal{S} \cup \mathcal{B} \models B \sqsubseteq C'$ for C' over Γ, then by condition **(II)**, $B \sqsubseteq C'$ is representable in $\mathcal{S}$ and $\mathcal{M}$: let $ax_{ii}(B, C') = \{\alpha\}$ such that $B \sqsubseteq C'$ is representable via α. Similarly, for an $\mathcal{S} \cup \mathcal{B}$-consistent role R over Σ and Q' over Γ, such that $\mathcal{S} \cup \mathcal{B} \models R \sqsubseteq Q'$. Finally, for each $\mathcal{S} \cup \mathcal{B}$-consistent concept B over Σ and each role R such that $B \rightsquigarrow_{\mathcal{S}\cup\mathcal{B}} R$, define the set $ax_{iii}(B, R)$ from the generating path $\langle C_0, \dots, C_n\rangle$ for (B, R)

in $\mathcal{S}$ and $\mathcal{M}$ given by condition **(III)**. Take $ax_{iii}(B,R)$ equal to the set of all axioms α, where $C_i \sqsubseteq D_i$ is representable via α in **(B)**, or $Q_i \sqsubseteq S_i$ is representable via α in **(C)**. Finally we have:

$$\mathcal{T} = \bigcup_{\substack{X,Y \text{ conc. or roles over } \Sigma,\\ \mathcal{S}\text{-consistent and}\\ \mathcal{S}\cup\mathcal{B}\text{-inconsistent}}} ax_i(X,Y) \;\cup \bigcup_{\substack{X \text{ conc. or role over } \Sigma,\\ \mathcal{S}\cup\mathcal{B}\text{-consistent },\\ X' \text{ over } \Gamma,\, \mathcal{S}\cup\mathcal{B}\models X\sqsubseteq X'}} ax_{ii}(X,X') \;\cup \bigcup_{\substack{\mathcal{S}\cup\mathcal{B}\text{-cons. } B \text{ over } \Sigma,\\ B\rightsquigarrow_{\mathcal{S}\cup\mathcal{B}}R}} ax_{iii}(B,R)$$

Then it immediately follows that $\mathcal{T}$ is a UCQ-representation of $\mathcal{S}$ under $\mathcal{M}$: On the one hand, by construction, $\mathcal{T}$ is a parsimonious UCQ-representation. On the other hand, the $\Rightarrow$ directions of conditions **(i)** and **(ii)**, and condition **(iii)** of Lemma 7.1 are satisfied by construction of $\mathcal{T}$ and by definition of ax_i, ax_{ii}, ax_{iii}. From this it follows that for each ABox $\mathcal{A}_\mathsf{s}$ consistent with $\mathcal{S}$, $\langle\mathcal{T}\cup\mathcal{B},\mathcal{A}_\mathsf{s}\rangle$ Γ-query entails $\langle\mathcal{S}\cup\mathcal{B},\mathcal{A}_\mathsf{s}\rangle$. Hence, indeed, $\mathcal{T}$ is a UCQ-representation of $\mathcal{S}$ under $\mathcal{M}$.

($\Rightarrow$) Let $\mathcal{T}$ be a UCQ-representation for $\mathcal{S}$ under $\mathcal{M}$. It is easy to see that conditions **(I)** and **(II)** are satisfied.

We show that condition **(III)** is satisfied; assume that B is an $\mathcal{S}\cup\mathcal{B}$-consistent concept over Σ and $B \rightsquigarrow_{\mathcal{S}\cup\mathcal{B}} R$ for some role R. By condition **(iii)** of Lemma 7.1 it follows that there exists $y \in \Delta^{\mathsf{gen}(\mathcal{T}\cup\mathcal{B},B(o))}$ such that $\mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{S}\cup\mathcal{B},B(o))}(w_{[R]}) \subseteq \mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{T}\cup\mathcal{B},B(o))}(y)$, and $\mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{S}\cup\mathcal{B},B(o))}(o,w_{[R]}) \subseteq \mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{T}\cup\mathcal{B},B(o))}(o,y)$. Assume that $y = w_{[Q_n]}$ for $n \geq 0$, where $o \rightsquigarrow_{\langle\mathcal{T}\cup\mathcal{B},B(o)\rangle} w_{[Q_1]} \rightsquigarrow \cdots \rightsquigarrow w_{[Q_n]}$. Then $\mathcal{T}\cup\mathcal{B} \models \{B \sqsubseteq \exists Q_1\} \cup \bigcup_{i=1}^{n-1}\{\exists Q_i^- \sqsubseteq \exists Q_{i+1}\} \cup \{\exists Q_n^- \sqsubseteq B'\}$, for all $B' \in \mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{S}\cup\mathcal{B},B(o))}(w_{[R]})$, and $\mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{S}\cup\mathcal{B},B(o))}(o,w_{[R]}) \neq \emptyset$ implies $n = 1$ and $\mathcal{T}\cup\mathcal{B} \models Q_1 \sqsubseteq R'$ for all $R' \in \mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{S}\cup\mathcal{B},B(o))}(o,w_{[R]})$. One can show by induction on n that for each i, $1 \leq i \leq n$, there exist S_i over Σ such that $\mathcal{S}\cup\mathcal{B} \models S_i \sqsubseteq Q_i$ and $\mathcal{S}\cup\mathcal{B} \models \{B \sqsubseteq \exists S_1\} \cup \bigcup_{i=1}^{n-1}\{\exists S_i^- \sqsubseteq \exists S_{i+1}\} \cup \{\exists S_n^- \sqsubseteq B'\}$, for all $B' \in \mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{S}\cup\mathcal{B},B(o))}(w_{[R]})$. We define the sequence $\langle C_0,\ldots,C_n\rangle$ as $C_0 = B$, and $C_i = \exists S_i^-$, for $1 \leq i \leq n$: it can be straightforwardly verified that $\langle C_0,\ldots,C_n\rangle$ is a generating path for (B,R) in $\mathcal{S}$ and $\mathcal{M}$. □

We now use the above characterization to verify UCQ-representability in the following examples.

Example 7.14. Consider $\mathcal{M}$ and $\mathcal{S}$ from Example 7.3, that is, $\mathcal{M} = (\Sigma,\Gamma,\mathcal{B})$, where

$$\begin{array}{ll} \Sigma = \{A,R,S,Q\} & \mathcal{B} = \{A \sqsubseteq A', \exists R^- \sqsubseteq B', S \sqsubseteq S', Q \sqsubseteq Q', \exists Q^- \sqsubseteq B'\} \\ \Gamma = \{A',B',S',Q'\} & \mathcal{S} = \{A \sqsubseteq \exists R, A \sqsubseteq \exists S, \exists S^- \sqsubseteq \exists Q\} \end{array}$$

Then one can see that conditions **(I)** – **(III)** are satisfied. Thus, for instance, $\mathcal{S}\cup\mathcal{B} \models A \sqsubseteq \exists S'$ and $\mathcal{S}\cup\mathcal{B} \models \exists S^- \sqsubseteq \exists Q'$: clearly both inclusions are representable in $\mathcal{S}$ and $\mathcal{M}$. Then, $A \rightsquigarrow_{\mathcal{S}\cup\mathcal{B}} R$ and $A \rightsquigarrow_{\mathcal{S}\cup\mathcal{B}} S$, and in both cases there exist generating paths: $\langle A, \exists S^-, \exists Q^-\rangle$ from Example 7.12 and $\langle A, \exists S^-\rangle$, respectively. This confirms that $\mathcal{S}$ is UCQ-representable under $\mathcal{M}$. ■

Example 7.15. Consider $\mathcal{M}$ and $\mathcal{S}$ from Example 7.2, that is, $\mathcal{M} = (\Sigma,\Gamma,\mathcal{B})$, where $\Sigma = \{A,R\}$, $\Gamma = \{A',R',B'\}$, $\mathcal{B} = \{A \sqsubseteq A', \exists R^- \sqsubseteq B'\}$, and $\mathcal{S} = \{A \sqsubseteq \exists R\}$.

In contrast with the previous example, condition **(III)** is not satisfied. In fact, $A \rightsquigarrow_{\mathcal{S}\cup\mathcal{B}} R$, however there exists no generating path for (A,R) in $\mathcal{S}$ and $\mathcal{M}$ as we mentioned in Example 7.12. So indeed, $\mathcal{S}$ is not UCQ-representable under $\mathcal{M}$. ■

Example 7.16. Consider $\mathcal{M}$ and $\mathcal{S}$ from Example 5.19–(3), that is, $\mathcal{M} = (\Sigma,\Gamma,\mathcal{B})$, where $\Sigma = \{A,B,C\}$, $\Gamma = \{A',B'\}$, $\mathcal{B} = \{A \sqsubseteq A', B \sqsubseteq B', C \sqsubseteq \neg A'\}$, and $\mathcal{S} = \{B \sqsubseteq C\}$. We show that condition **(I)** is satisfied: the pairs (A,C) and (A,B) are $\mathcal{S}\cup\mathcal{B}$-inconsistent. As the former pair is already $\mathcal{B}$-inconsistent, this case is not interesting. For the latter pair, one can easily verify that (A,B) is target contradictable in $\mathcal{S}$ and $\mathcal{M}$ via $B' \sqsubseteq \neg A'$: in particular, $\mathcal{T} = \{B' \sqsubseteq \neg A'\}$ is a parsimonious UCQ-representation, and (A,B) is $\mathcal{T}\cup\mathcal{B}$-inconsistent. ■

Finally, we obtain the complexity bound of the non-emptiness problem for UCQ-representations.

Theorem 7.17. *The non-emptiness problem for* UCQ*-representations is* NLogSpace*-complete.*

Proof. As in the case of Theorem 7.6, the lower bound is shown by a reduction from the directed graph reachability problem, however, we need a slightly more involved encoding.To encode the graph $\mathsf{G} = (\mathsf{V},\mathsf{E})$, we use a set $\{V_i \mid v_i \in \mathsf{V}\} \cup \{S_1,F_1,S_2,F_2\}$ of Σ-concept names and a set $\{V_i' \mid v_i \in \mathsf{V}\} \cup \{S',F'\}$ of Γ-concept names. Consider the TBox

$$\mathcal{S} = \{V_i \sqsubseteq V_j \mid (v_i,v_j) \in \mathsf{E}\} \cup \{S_1 \sqsubseteq V_k, V_m \sqsubseteq F_1, S_2 \sqsubseteq F_2\},$$

where v_k and v_m are, respectively, the initial and final vertices. Then, let

$$\mathcal{B} = \{V_i \sqsubseteq V_i' \mid v_i \in \mathsf{V}\} \cup \{S_j \sqsubseteq S', F_j \sqsubseteq F' \mid j = 1, 2\};$$

we show:

– there is a directed path from v_k to v_m in G iff there exists a UCQ-representation for $\mathcal{S}$ under $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$.

Indeed, using Lemma 7.13, there exists a representation iff condition **(II)** is satisfied. By the structure of $\mathcal{S} \cup \mathcal{B}$ one can see that this is the case iff inclusion $S_2 \sqsubseteq F'$ is representable in $\mathcal{S}$ and $\mathcal{M}$ via $S' \sqsubseteq F'$, i.e., iff $\mathcal{S} \cup \mathcal{B} \models S_1 \sqsubseteq S'$ implies $\mathcal{S} \cup \mathcal{B} \models S_1 \sqsubseteq F'$, and this holds iff $\mathcal{S} \models S_1 \sqsubseteq F_1$. The latter is the case iff there exists a path from v_k to v_m in G. This completes the proof of the lower bound.

To show the upper bound, we prove that conditions **(I)** – **(III)** of Lemma 7.13 can be checked in NLogSpace. First, one can derive syntactic conditions that allow one to check whether an inclusion is representable in $\mathcal{S}$ and $\mathcal{M}$, and whether a pair is target contradictable in $\mathcal{S}$ and $\mathcal{M}$ (see Propositions D.1, D.2, D.3 and D.4). In fact, these conditions can be checked using a directed graph reachability algorithm, similar to what is done in the proof of Theorem 7.6. The new case is condition **(III)**; to verify for an $\mathcal{S} \cup \mathcal{B}$-consistent concept B over Σ and a role R such that $B \rightsquigarrow_{\mathcal{S}\cup\mathcal{B}} R$, that there exists a generating path $\pi = \langle C_0, \dots C_n \rangle$ for (B, R) in $\mathcal{S}$ and $\mathcal{M}$, we can use the following procedure, which runs in NLogSpace. First, we take $C_0 = B$ and guess whether the path should end here (i.e., $n = 0$). If we guessed so, it only remains to verify condition **(D)**. This verification can be performed in NLogSpace, similarly to the method described in the proof of Theorem 7.6. If, on the other hand, we guessed that the path should continue, we guess $C_1 = \exists Q^-$ for some role Q, and verify that conditions **(A)**, **(B)** and **(C)** are satisfied. Now, if we guess that the path should stop, it remains to verify condition **(D)**. If, on the contrary, we guess that the path should continue, we can forget C_0, guess C_2, and proceed with it in the same way as we did with C_1. Finally, when we reach the concept C_n, such that the algorithm guesses to stop, it remains to verify condition **(D)**. It should be clear that whenever a generating path $\pi = \langle C_0, \dots C_n \rangle$ for (B, R) in $\mathcal{S}$ and $\mathcal{M}$ exists, we can find it by the above non-deterministic procedure.Note that n is bounded by the number of roles in $\mathcal{S} \cup \mathcal{B}$, since every generating path in which a role appears more than once can be shortened to one in which the subpath between the first and last occurrence of the role is removed (in fact, if $\langle C_0, \dots, C_i, \dots, C_j, \dots, C_n \rangle$ is a generating path for (B, R) in $\mathcal{S}$ and $\mathcal{M}$, for $0 \leq i < j$ and $C_i = C_j$, then it is easy to see that $\langle C_0, \dots, C_{i-1}, C_j, \dots, C_n \rangle$ is also a generating path for (B, R) in $\mathcal{S}$ and $\mathcal{M}$). □

We conclude this section by observing that the proof of Lemma 7.13 contained an algorithm for computing a UCQ-representation in the case $\mathcal{S}$ is UCQ-representable under $\mathcal{M}$.

8. Concluding Remarks and Future Work

In this article, we have defined the problem of exchanging knowledge between a source and a target KB connected through a mapping. In particular, we have considered source KBs, target KBs, and mappings specified in the Description Logic *DL-Lite*$_\mathcal{R}$, which is the logic underlying OWL 2 QL (one of the three profiles of the standard Web Ontology Language OWL 2), and we have studied some fundamental problems related to the exchange of knowledge in this context. We have developed novel game- and automata-theoretic techniques, and have provided complexity results for these problems that range from NLogSpace to ExpTime.

As future work, we first note that the complexity of the non-emptiness problem has not been pinpointed in all cases (see Table 1). In particular, it would be interesting to close the gap between the lower and upper bounds for the complexity of this problem for universal solutions and extended ABoxes, as we currently know it to be PSpace-hard and included in ExpTime. Moreover, it would also be interesting to establish a lower bound for this problem for the case of universal UCQ-solutions and simple ABoxes, and to prove it to be decidable for the case of universal UCQ-solutions and extended ABoxes. Second, the target signature in the non-emptiness problem is allowed to include new concepts or roles neither in universal solutions nor in universal UCQ-solutions nor in UCQ-representations. The problem of allowing such additional symbols in these constructions is certainly interesting and worth investigating in the future. Third, it is interesting to study the problem of knowledge exchange for richer ontology formalisms, such as the DLs of the $\mathcal{ALC}$-family, DLs with number restrictions or functionality, or existential rule languages/Datalog$^\pm$ [66, 67, 68]. The aim would be to understand for which variants of such formalisms the existing techniques can be extended, and which variants instead would require a novel approach. For example, the techniques based on reachability games and two-way alternating tree automata, both of which heavily rely on the canonical model property, can be extended to other Horn DLs, such as $\textit{DL-Lite}^{\mathcal{H}}_{horn}$, $\mathcal{ELH}$, and Horn-$\mathcal{ALCHI}$, similarly to the approach in [30]. Finally, in this work we have not dealt with other standard data exchange reasoning tasks, such as composition and inversion of mappings [69, 70, 31, 20, 21]. These problems are certainly of interest in the KB exchange framework, and will be the subject of further investigation.

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A. Proofs in Section 5

A.1. Proof of Proposition 5.3

Proof. For the sake of contradiction, assume that $\mathcal{T}$ is not trivial, that is, there exists an interpretation $\mathcal{J}^\star = \langle \Delta^{\mathcal{J}^\star}, \cdot^{\mathcal{J}^\star} \rangle$ of Γ such that $\mathcal{J}^\star \not\models \mathcal{T}$.

Given that $\langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s} \rangle$ is consistent, there exists an interpretation $\mathcal{I}^\star = \langle \Delta^{\mathcal{I}^\star}, \cdot^{\mathcal{I}^\star} \rangle$ of $(\Sigma \cup \Gamma)$ such that $\mathcal{I}^\star \models \langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s} \rangle$. Then define interpretations $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ of Σ and $\mathcal{J} = \langle \Delta^{\mathcal{J}}, \cdot^{\mathcal{J}} \rangle$ of Γ as follows: (1) $\Delta^{\mathcal{I}} = \Delta^{\mathcal{J}} = \Delta^{\mathcal{I}^\star}$; (2) $a^{\mathcal{I}} = a^{\mathcal{J}} = a^{\mathcal{I}^\star}$, for every constant $a \in N_a$; (3) $A_1^{\mathcal{I}} = A_1^{\mathcal{I}^\star}$ and $A_2^{\mathcal{J}} = A_2^{\mathcal{I}^\star}$, for every pair of concept names $A_1 \in \Sigma$ and $A_2 \in \Gamma$; and (4) $P_1^{\mathcal{I}} = P_1^{\mathcal{I}^\star}$ and $P_2^{\mathcal{J}} = P_2^{\mathcal{I}^\star}$, for every pair of role names $P_1 \in \Sigma$ and $P_2 \in \Gamma$. By definition of $\mathcal{I}, \mathcal{J}$ and given that $\mathcal{I}^\star \models \langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s} \rangle$, we conclude that $\mathcal{I} \in \textsc{Mod}(\mathcal{K}_\mathsf{s})$ and $(\mathcal{I}, \mathcal{J}) \models \mathcal{B}$.

Without loss of generality, we assume that $\Delta^{\mathcal{I}^\star} \cap \Delta^{\mathcal{J}^\star} = \emptyset$. Then define an interpretation $\mathcal{J}'$ of Γ as follows: (1) $\Delta^{\mathcal{J}'} = \Delta^{\mathcal{I}^\star} \cup \Delta^{\mathcal{J}^\star}$; (2) $a^{\mathcal{J}'} = a^{\mathcal{I}^\star}$, for every constant $a \in N_a$; (3) $A^{\mathcal{J}'} = A^{\mathcal{I}^\star} \cup A^{\mathcal{J}^\star}$, for every concept name $A \in \Gamma$; and (4) $P^{\mathcal{J}'} = P^{\mathcal{I}^\star} \cup P^{\mathcal{J}^\star}$, for every role name $P \in \Gamma$. Given that $(\mathcal{I}, \mathcal{J}) \models \mathcal{B}$, we conclude that $(\mathcal{I}, \mathcal{J}') \models \mathcal{B}$. In fact, for every concept inclusion $B_1 \sqsubseteq B_2 \in \mathcal{B}$, where B_1, B_2 are basic concepts, we have that $B_1^{\mathcal{I}} \subseteq B_2^{\mathcal{J}'}$ given that $B_1^{\mathcal{I}} \subseteq B_2^{\mathcal{J}}$, $B_2^{\mathcal{J}} = B_2^{\mathcal{I}^\star}$ and $B_2^{\mathcal{J}'} = B_2^{\mathcal{I}^\star} \cup B_2^{\mathcal{J}^\star}$. Moreover, for every concept inclusion $B_1 \sqsubseteq \neg B_2 \in \mathcal{B}$, where B_1, B_2 are basic concepts, we have that $B_1^{\mathcal{I}} \subseteq (\neg B_2)^{\mathcal{J}'}$ given that $B_1^{\mathcal{I}} \subseteq (\neg B_2)^{\mathcal{J}}$, $(\neg B_2)^{\mathcal{J}} = (\neg B_2)^{\mathcal{I}^\star}$ and $(\neg B_2)^{\mathcal{J}'} = (\neg B_2)^{\mathcal{I}^\star} \cup (\neg B_2)^{\mathcal{J}^\star}$ (since $B_2^{\mathcal{J}'} = B_2^{\mathcal{I}^\star} \cup B_2^{\mathcal{J}^\star}$ and $\Delta^{\mathcal{I}^\star} \cap \Delta^{\mathcal{J}^\star} = \emptyset$). Finally, for role inclusions $R_1 \sqsubseteq R_2$ and $R_1 \sqsubseteq \neg R_2$ in $\mathcal{B}$, where R_1, R_2 are basic roles, we conclude that $R_1^{\mathcal{I}} \subseteq R_2^{\mathcal{J}'}$ and $R_1^{\mathcal{I}} \subseteq (\neg R_2)^{\mathcal{J}'}$ as in the previous two cases.

From the results in the previous paragraph, we conclude that $\mathcal{J}' \in \textsc{Sat}_{\mathcal{M}}(\textsc{Mod}(\mathcal{K}_\mathsf{s}))$ (since $\mathcal{J}' \in \textsc{Sat}_{\mathcal{M}}(\mathcal{I})$ and $\mathcal{I} \in \textsc{Mod}(\mathcal{K}_\mathsf{s})$). On the other hand, we have that $\mathcal{J}' \not\models \mathcal{T}$, by definition of $\mathcal{J}'$ and given that $\mathcal{J}^\star \not\models \mathcal{T}$. Thus, we have that $\mathcal{J}' \not\models \mathcal{K}_\mathsf{t}$ and, thus, $\mathcal{J}' \notin \textsc{Mod}(\mathcal{K}_\mathsf{t})$. Therefore, we conclude that $\textsc{Sat}_{\mathcal{M}}(\textsc{Mod}(\mathcal{K}_\mathsf{s})) \neq \textsc{Mod}(\mathcal{K}_\mathsf{t})$, which contradicts the fact that $\mathcal{K}_\mathsf{t}$ is a universal solution for $\mathcal{K}_\mathsf{s}$ under $\mathcal{M}$. This concludes the proof of the proposition. □

B. Proofs in Section 6

B.1. Proof of Lemma 6.2

Proof. In this proof we assume that $\mathcal{K}_\mathsf{s} = \langle \mathcal{S}, \mathcal{A}_\mathsf{s} \rangle$ and we denote by $\mathcal{K}_\mathsf{sb}$ the KB $\langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s} \rangle$.

($\Rightarrow$) Let $\mathcal{A}_\mathsf{t}$ be a universal solution for $\mathcal{K}_\mathsf{s}$ under $\mathcal{M}$. Then $\mathsf{uni}(\mathcal{A}_\mathsf{t})$ is Γ-homomorphically equivalent to $\mathsf{uni}(\mathcal{K}_\mathsf{sb})$: since $\mathcal{A}_\mathsf{t}$ is a solution, there exists $\mathcal{I}$, a model of $\mathcal{K}_\mathsf{s}$, such that $(\mathcal{I}, \mathsf{uni}(\mathcal{A}_\mathsf{t})) \models \mathcal{B}$. Then $\mathcal{I} \cup \mathsf{uni}(\mathcal{A}_\mathsf{t})$ is a model of $\mathcal{K}_\mathsf{sb}$, therefore there is a homomorphism h from $\mathsf{uni}(\mathcal{K}_\mathsf{sb})$ to $\mathcal{I} \cup \mathsf{uni}(\mathcal{A}_\mathsf{t})$. As Σ and Γ are disjoint signatures it follows that h is a Γ-homomorphism from $\mathsf{uni}(\mathcal{K}_\mathsf{sb})$ to $\mathsf{uni}(\mathcal{A}_\mathsf{t})$. On the other hand, as $\mathcal{A}_\mathsf{t}$ is a universal solution, $\mathcal{J}$, the interpretation of Γ obtained from $\mathsf{uni}(\mathcal{K}_\mathsf{sb})$ is a model of $\mathcal{A}_\mathsf{t}$ with a substitution h'. This h' is exactly a homomorphism from $\mathsf{uni}(\mathcal{A}_\mathsf{t})$ to $\mathsf{uni}(\mathcal{K}_\mathsf{sb})$. Thus, we showed **(hom)**.

For the sake of contradiction, assume that **(safe)** does not hold, i.e., $\mathcal{K}_\mathsf{s}$ is not Γ-safe with respect to $\mathcal{M}$, and e.g., **(cs)** does not hold, i.e., there is a disjointness axiom in $\mathcal{S}$ of the form $B \sqsubseteq \neg C$, such that (B, C) is not safe. Then both B and C are not safe in $\mathsf{uni}(\mathcal{K}_\mathsf{sb})$: for some $b \in B^{\mathsf{uni}(\mathcal{K}_\mathsf{sb})}$ and $c \in C^{\mathsf{uni}(\mathcal{K}_\mathsf{sb})}$,

$$\mathbf{t}_\Gamma^{\mathsf{uni}(\mathcal{K}_\mathsf{sb})}(b) \neq \emptyset \quad \text{or} \quad b \in N_a, \qquad \text{and} \qquad \mathbf{t}_\Gamma^{\mathsf{uni}(\mathcal{K}_\mathsf{sb})}(c) \neq \emptyset \quad \text{or} \quad c \in N_a.$$

Let h be a Γ-homomorphism from $\mathsf{uni}(\mathcal{K}_\mathsf{sb})$ to $\mathsf{uni}(\mathcal{A}_\mathsf{t})$ (it exists by **(hom)**), and $h(b) = t$ and $h(c) = s$. Then it follows that

$$\mathbf{t}_\Gamma^{\mathsf{uni}(\mathcal{A}_\mathsf{t})}(t) \neq \emptyset \quad \text{or} \quad b \in N_a, \qquad \text{and} \qquad \mathbf{t}_\Gamma^{\mathsf{uni}(\mathcal{A}_\mathsf{t})}(s) \neq \emptyset \quad \text{or} \quad c \in N_a.$$

Take a model $\mathcal{J}$ of $\mathcal{A}_\mathsf{t}$ with a substitution $h_{\mathcal{J}}$ such that $\Delta^{\mathcal{J}} = \{d\}$ (hence, $t^{\mathcal{J}} = s^{\mathcal{J}}$). Such a model exists because $\mathcal{A}_\mathsf{t}$ does not assert any negative information and the UNA does *not* hold. First, assume that both b and c are constants (i.e., $b^{\mathcal{J}} = c^{\mathcal{J}}$). Then, obviously there exists no model $\mathcal{I}$ of Σ such that $\mathcal{I} \models \mathcal{K}_\mathsf{s}$ and $(\mathcal{I}, \mathcal{J}) \models \mathcal{B}$: in every such $\mathcal{I}$, $b^{\mathcal{I}}$ must be equal to $c^{\mathcal{I}}$ which contradicts $B \sqsubseteq \neg C$, and $b^{\mathcal{I}} \in B^{\mathcal{I}}$ and $c^{\mathcal{I}} \in C^{\mathcal{I}}$. Now, assume that at least b is not a constant and $\mathsf{tail}(b) = w_{[R]}$ for some role R over Σ (hence, $b \in (\exists R^-)^{\mathsf{uni}(\mathcal{K}_\mathsf{sb})}$ and $\mathcal{S} \models \exists R^- \sqsubseteq B$). Let $B' \in \mathbf{t}_\Gamma^{\mathsf{uni}(\mathcal{K}_\mathsf{sb})}(b)$, then by construction of the canonical model, $\mathcal{S} \cup \mathcal{B} \models \exists R^- \sqsubseteq B'$, by homomorphism, $B'(t) \in \mathcal{A}_\mathsf{t}$, and by construction of $\mathcal{J}$, $B'^{\mathcal{J}} = \{d\}$. As $\mathcal{A}_\mathsf{t}$ is a universal solution, let $\mathcal{I}$ be a model of $\mathcal{K}_\mathsf{s}$ such that $(\mathcal{I}, \mathcal{J}) \models \mathcal{B}$. Then $(\exists R^-)^{\mathcal{I}}$ is non-empty and

$(\exists R^-)^{\mathcal{I}} \subseteq B'^{\mathcal{J}}$. It immediately follows that $d \in (\exists R^-)^{\mathcal{I}}$, hence $d \in B^{\mathcal{I}}$. By a similar argument, it can be shown that d must be in $C^{\mathcal{I}}$, which contradicts that $\mathcal{I}$ is a model of $B \sqsubseteq \neg C$. Contradiction with $\mathcal{A}_\mathsf{t}$ being a universal solution.

Similar to **(cs)** we can derive a contradiction if assume that **(rs)** does not hold.

Now, assume **(re)** does not hold, i.e., $B \sqsubseteq \neg B' \in \mathcal{B}$ and $B^{\mathsf{uni}(\mathcal{K}_\mathsf{sb})} \neq \emptyset$. Note that $\mathcal{A}_\mathsf{t}$ is an extended ABox, i.e., it contains only assertions of the form $A(u)$, $P(u, v)$ for $u, v \in N_a \cup N_l$. Take a model $\mathcal{J}$ of $\mathcal{A}_\mathsf{t}$ such that $B'^{\mathcal{J}} = \Delta^{\mathcal{J}}$. Such $\mathcal{J}$ exists as $\mathcal{A}_\mathsf{t}$ contains only positive facts. Since $\mathcal{A}_\mathsf{t}$ is a universal solution, there exist a model $\mathcal{I}$ of $\mathcal{K}_\mathsf{s}$ such that $(\mathcal{I}, \mathcal{J}) \models \mathcal{B}$. Then, $B^{\mathcal{I}} \neq \emptyset$, and it is easy to see that $(\mathcal{I}, \mathcal{J}) \not\models B \sqsubseteq \neg B'$ because $\Delta^{\mathcal{J}} \setminus B'^{\mathcal{J}} = \emptyset$ and $B^{\mathcal{I}} \not\subseteq \Delta^{\mathcal{J}} \setminus B'^{\mathcal{J}}$.

Similar to **(ce)** we can derive a contradiction if assume that **(re)** does not hold.

In every case we derive a contradiction, hence $\mathcal{K}_\mathsf{s}$ is Γ-safe with respect to $\mathcal{M}$.

($\Leftarrow$) Assume **(hom)** and **(safe)** hold. We show that $\mathcal{A}_\mathsf{t}$ is a universal solution for $\mathcal{K}_\mathsf{s}$ under $\mathcal{M}$.

First, $\mathcal{A}_\mathsf{t}$ is a solution for $\mathcal{K}_\mathsf{s}$ under $\mathcal{M}$. Let $\mathcal{J}$ be a model of $\mathcal{A}_\mathsf{t}$, and h_1 a homomorphism from $\mathsf{uni}(\mathcal{A}_\mathsf{t})$ to $\mathcal{J}$. Furthermore, let h be a Γ-homomorphism from $\mathsf{uni}(\mathcal{K}_\mathsf{sb})$ to $\mathsf{uni}(\mathcal{A}_\mathsf{t})$. Then $h_2(x) = h_1(h(x))$ is a Γ-homomorphism from $\mathsf{uni}(\mathcal{K}_\mathsf{sb})$ to $\mathcal{J}$. Let $\mathcal{I}$ be the interpretation of Σ defined as the image of h_2 applied to $\mathsf{uni}(\mathcal{K}_\mathsf{s})$, i.e., $\mathcal{I} = h_2(\mathsf{uni}(\mathcal{K}_\mathsf{s}))$. Next, define a new function $h' : \Delta^{\mathsf{uni}(\mathcal{K}_\mathsf{s})} \to \Delta \cup \Delta^{\mathcal{I}}$, where Δ is an infinite set of domain elements disjoint from $\Delta^{\mathcal{I}}$, as follows:

- $h'(x) = h_2(x)$ if $\mathbf{t}_\Gamma^{\mathsf{uni}(\mathcal{K}_\mathsf{sb})}(x) \neq \emptyset$ or $x \in N_a$.
- $h'(x) = d_x$, a fresh domain element from Δ, otherwise.

We show that interpretation $\mathcal{I}'$ defined as the image of h' applied to $\mathsf{uni}(\mathcal{K}_\mathsf{sb})$, is a model of $\mathcal{K}_\mathsf{s}$ and $(\mathcal{I}', \mathcal{J}) \models \mathcal{M}$. It is straightforward to verify that $\mathcal{I}'$ is a model of the positive inclusions in $\mathcal{S}$ and $(\mathcal{I}', \mathcal{J})$ satisfy the positive inclusions from $\mathcal{B}$. In what follows we prove that $\mathcal{I}'$ is a model of the disjointness axioms in $\mathcal{S}$.

Let $\mathcal{S} \models B \sqsubseteq \neg C$ for basic concepts B, C. By contradiction, assume $\mathcal{I}' \not\models B \sqsubseteq \neg C$, i.e., for some $d \in \Delta^{\mathcal{I}'}$, $d \in B^{\mathcal{I}'} \cap C^{\mathcal{I}'}$. We defined $\mathcal{I}'$ as the image of h' on $\mathsf{uni}(\mathcal{K}_\mathsf{s})$, hence there must exist $b, c \in \Delta^{\mathsf{uni}(\mathcal{K}_\mathsf{s})}$ such that $b \in B^{\mathsf{uni}(\mathcal{K}_\mathsf{s})}$, $c \in C^{\mathsf{uni}(\mathcal{K}_\mathsf{s})}$, and $h'(b) = h'(c) = d$. Then, since $\mathcal{K}_\mathsf{s}$ is Γ-safe with respect to $\mathcal{M}$, it follows that (B, C) is safe and it cannot be the case that

$$\mathbf{t}_\Gamma^{\mathsf{uni}(\mathcal{K}_\mathsf{sb})}(b) \neq \emptyset \quad \text{or} \quad b \in N_a, \qquad \text{and} \qquad \mathbf{t}_\Gamma^{\mathsf{uni}(\mathcal{K}_\mathsf{sb})}(c) \neq \emptyset \quad \text{or} \quad c \in N_a.$$

Assume b is a null and $\mathbf{t}_\Gamma^{\mathsf{uni}(\mathcal{K}_\mathsf{sb})}(b) = \emptyset$. Then by definition of h', $h'(b) = d_b \in \Delta$ (hence $d = d_b$). In either case c is a constant, or $\mathbf{t}_\Gamma^{\mathsf{uni}(\mathcal{K}_\mathsf{sb})}(c) \neq \emptyset$, or $\mathbf{t}_\Gamma^{\mathsf{uni}(\mathcal{K}_\mathsf{sb})}(c) = \emptyset$, we obtain contradiction with $h'(b) = d_b = h'(c)$ (recall, Δ and $\Delta^{\mathcal{I}}$ are disjoint). Contradiction rises from the assumption $\mathcal{I} \not\models B \sqsubseteq \neg C$.

Next, assume $\mathcal{S} \models R \sqsubseteq \neg Q$ for roles R, Q, and $\mathcal{I}' \not\models R \sqsubseteq \neg Q$, i.e., for some $d_1, d_2 \in \Delta^{\mathcal{I}'}$, $(d_1, d_2) \in R^{\mathcal{I}'} \cap Q^{\mathcal{I}'}$. We defined $\mathcal{I}'$ as the image of h' on $\mathsf{uni}(\mathcal{K}_\mathsf{s})$, hence there must exist $b_1, b_2, c_1, c_2 \in \Delta^{\mathsf{uni}(\mathcal{K}_\mathsf{s})}$ such that $(b_1, b_2) \in R^{\mathsf{uni}(\mathcal{K}_\mathsf{s})}$, $(c_1, c_2) \in Q^{\mathsf{uni}(\mathcal{K}_\mathsf{s})}$, and $h'(b_i) = h'(c_i) = d_i$ for $i = 1, 2$. Then, since $\mathcal{K}_\mathsf{s}$ is Γ-safe with respect to $\mathcal{M}$, it follows that (R, Q) is safe and it cannot be the case that 1) R and Q are not safe, i.e.,

$$\mathbf{t}_\Gamma^{\mathsf{uni}(\mathcal{K}_\mathsf{sb})}(b_i) \neq \emptyset \quad \text{or} \quad b_i \in N_a, \qquad \text{and} \qquad \mathbf{t}_\Gamma^{\mathsf{uni}(\mathcal{K}_\mathsf{sb})}(c_i) \neq \emptyset \quad \text{or} \quad c_i \in N_a,$$

or 2) $\mathbf{t}_\Gamma^{\mathsf{uni}(\mathcal{K}_\mathsf{sb})}(b_2) \neq \emptyset$ and $\mathbf{t}_\Gamma^{\mathsf{uni}(\mathcal{K}_\mathsf{sb})}(c_2) \neq \emptyset$ if $b_1 = c_1$. Consider the following possible cases:

- b_1 is a null and $\mathbf{t}_\Gamma^{\mathsf{uni}(\mathcal{K}_\mathsf{sb})}(b_1) = \emptyset$. Then by definition of h', $h'(b_1) = d_{b_1} \in \Delta$ (and $d_1 = d_{b_1}$).
  - c_1 is a null and $\mathbf{t}_\Gamma^{\mathsf{uni}(\mathcal{K}_\mathsf{sb})}(c_1) = \emptyset$, then $h'(c_1) = d_{c_1} = d_1$, hence $c_1 = b_1$ and $(b_1, b_2) \in R^{\mathsf{uni}(\mathcal{K}_\mathsf{s})}$, $(b_1, c_2) \in Q^{\mathsf{uni}(\mathcal{K}_\mathsf{s})}$. Assume b_2 is a null and $\mathbf{t}_\Gamma^{\mathsf{uni}(\mathcal{K}_\mathsf{sb})}(b_2) = \emptyset$. Then $h'(b_2) = d_{b_2} \in \Delta$ and in either case c_2 is a constant, or $\mathbf{t}_\Gamma^{\mathsf{uni}(\mathcal{K}_\mathsf{sb})}(c_2) \neq \emptyset$, or $\mathbf{t}_\Gamma^{\mathsf{uni}(\mathcal{K}_\mathsf{sb})}(c_2) = \emptyset$, we obtain contradiction with $h'(b_2) = d_{b_2} = h'(c_2)$.
  - otherwise we obtain contradiction with $h'(b_1) = d_{b_1} = h'(c_1)$.

The cases b_2 or c_i are nulls with the empty Γ-type are covered by swapping R and Q or by taking their inverses.

Finally, assume $B \sqsubseteq \neg B' \in \mathcal{B}$ and $(\mathcal{I}', \mathcal{J}) \not\models B \sqsubseteq \neg B'$, i.e., for some $d \in B^{\mathcal{I}'}$, $d \notin \Delta^{\mathcal{J}} \setminus C^{\mathcal{J}}$. Then there must exist $b \in B^{\mathsf{uni}(\mathcal{K}_\mathsf{s})}$ such that $h'(b) = d$. Contradiction with **(ce)**. Similarly, we derive a contradiction with **(re)** if assume that $R \sqsubseteq \neg R' \in \mathcal{B}$ and $(\mathcal{I}', \mathcal{J}) \not\models R \sqsubseteq \neg R'$.

Therefore, indeed, $\mathcal{I}$ is a model of $\mathcal{K}_\mathsf{s}$ and $(\mathcal{I}, \mathcal{J}) \models \mathcal{B}$. This concludes the proof $\mathcal{A}_\mathsf{t}$ is a solution for $\mathcal{K}_\mathsf{s}$ under $\mathcal{M}$.

Second, $\mathcal{A}_\mathsf{t}$ is a universal solution. Let $\mathcal{I}$ be a model of $\mathcal{K}_\mathsf{s}$ and $\mathcal{J}$ an interpretation of Γ such that $(\mathcal{I}, \mathcal{J}) \models \mathcal{M}$. Then, since $\mathsf{uni}(\mathcal{K}_\mathsf{sb})$ is the canonical model of $\mathcal{K}_\mathsf{sb}$, there exists a homomorphism h from $\mathsf{uni}(\mathcal{K}_\mathsf{sb})$ to $\mathcal{I} \cup \mathcal{J}$ ($\mathcal{I} \cup \mathcal{J}$ is a model of $\mathcal{K}_\mathsf{sb}$). In turn, there is a homomorphism h_1 from $\mathsf{uni}(\mathcal{A}_\mathsf{t})$ to $\mathsf{uni}(\mathcal{K}_\mathsf{sb})$, therefore $h' = h \circ h_1$ is a homomorphism from $\mathsf{uni}(\mathcal{A}_\mathsf{t})$ to $\mathcal{I} \cup \mathcal{J}$, and a Γ-homomorphism from $\mathsf{uni}(\mathcal{A}_\mathsf{t})$ to $\mathcal{J}$. Hence, $\mathcal{J}$ is a model of $\mathcal{A}_\mathsf{t}$: take h' as the substitution for the labeled nulls. By definition of universal solution, $\mathcal{A}_\mathsf{t}$ is a universal solution for $\mathcal{K}_\mathsf{s}$ under $\mathcal{M}$. □

B.2. Proof of Lemma 6.8

Proof. The proof is inspired by one in [30], but makes use of a reduction from the Circuit Value problem, known to be PTime-complete [63, Theorem 8.1], instead of a reduction from the Horn Satisfiability problem. Given a monotone Boolean circuit C consisting of a finite set of assignments to Boolean variables $P_1, \ldots, P_n$ of the form $P_i = 0$, $P_i = 1$, $P_i = P_j \wedge P_k$, $j, k < i$, or $P_i = P_j \vee P_k$, $j, k < i$, where each P_i appears on the left-hand side of exactly one assignment, check whether the value P_n is 1 in C.

We fix signatures $\Sigma = \{P, L, R\}$ and $\Gamma = \{L', R'\}$. Let $a_1, \ldots, a_n \in N_a$, and consider

$$\begin{aligned}
\mathcal{A}_\mathsf{s} &= \{P(a_n)\} \cup \{L(a_i, a_i), R(a_i, a_i) \mid P_i = 1 \text{ in } C\} \cup \{L(a_i, a_j), R(a_i, a_k) \mid P_i = P_j \wedge P_k \text{ in } C\} \\
&\quad \cup \{L(a_i, a_j), R(a_i, a_j), L(a_i, a_k), R(a_i, a_k) \mid P_i = P_j \vee P_k \text{ in } C\} \\
\mathcal{S} &= \{P \sqsubseteq \exists L,\ P \sqsubseteq \exists R,\ \exists L^- \sqsubseteq P,\ \exists R^- \sqsubseteq P\}, \qquad \mathcal{B} = \{L \sqsubseteq L',\ R \sqsubseteq R'\} \\
\mathcal{A}_\mathsf{t} &= \{L'(a_i, a_j) \mid L(a_i, a_j) \in \mathcal{A}_\mathsf{s}\} \cup \{R'(a_i, a_j) \mid R(a_i, a_j) \in \mathcal{A}_\mathsf{s}\}
\end{aligned}$$

Note that Σ, Γ, $\mathcal{S}$, and $\mathcal{B}$ do not depend on C, hence the reduction provides a lower bound for data complexity. We show that the value of P_n in C is 1 if and only if $\mathcal{A}_\mathsf{t}$ is a universal solution for $\mathcal{K}_\mathsf{s} = \langle \mathcal{S}, \mathcal{A}_\mathsf{s} \rangle$ under $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$. Denote by $\mathcal{K}_\mathsf{sb}$ the KB $\langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s} \rangle$. Clearly, $\mathsf{uni}(\mathcal{A}_\mathsf{t}) \subseteq \mathsf{uni}(\mathcal{K}_\mathsf{sb})$ (independently of the value of P_n in C). So, it suffices to show that the value of P_n in C is 1 if and only if $\mathsf{uni}(\mathcal{K}_\mathsf{sb})$ is Γ-homomorphically embeddable into $\mathsf{uni}(\mathcal{A}_\mathsf{t})$.

($\Rightarrow$) Suppose P_n evaluates to 1 in C. Observe that the projection of $\mathsf{uni}(\mathcal{K}_\mathsf{sb})$ over Γ contains an infinite binary tree whose root is a_n, and in which each left edge is labeled with L' and each right edge is labeled with R'. We define a Γ-homomorphism h from $\mathsf{uni}(\mathcal{K}_\mathsf{sb})^{a_n}$ to $\mathsf{uni}(\mathcal{A}_\mathsf{t})$ by induction on the length of $\sigma \in \Delta^{\mathsf{uni}(\mathcal{K}_\mathsf{sb})^{a_n}}$. Note that, since Γ contains only role names, the local homomorphism condition is trivially satisfied.

For the base case, we set $h(a_n) = a_n$. For the inductive step, assume the value of P_i is 1 and we already defined $h(\sigma) = a_i$ for $\sigma \in \Delta^{\mathsf{uni}(\mathcal{K}_\mathsf{sb})^{a_n}}$. Consider the following three cases. First, if $P_i = P_j \wedge P_k$ in C, then $\mathcal{A}_\mathsf{t}$ contains assertions $L'(a_i, a_j)$ and $R'(a_i, a_k)$, moreover, P_j and P_k both evaluate to 1: we set $h(\sigma w_{[L]}) = a_j$ and $h(\sigma w_{[R]}) = a_k$. Second, if $P_i = P_j \vee P_k$ in C, then $\mathcal{A}_\mathsf{t}$ contains assertions $L'(a_i, a_j), R'(a_i, a_j)$ and $L'(a_i, a_k), R'(a_i, a_k)$, and at least one of P_j and P_k evaluates to 1, assume it is P_j: we set $h(\sigma w_{[L]}) = a_j$ and $h(\sigma w_{[R]}) = a_j$. Finally, if $P_i = 1$ in C, then $\mathcal{A}_\mathsf{t}$ contains assertions $L'(a_i, a_i)$ and $R'(a_i, a_i)$: we set $h(\sigma w_{[L]}) = a_i$ and $h(\sigma w_{[R]}) = a_i$. Hence, by construction, h is a Γ-homomorphism.

($\Leftarrow$) Suppose $\mathcal{A}_\mathsf{t}$ is a universal solution for $\mathcal{K}_\mathsf{s}$ under $\mathcal{M}$. Then $\mathsf{uni}(\mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s})$ is Γ-homomorphically embeddable in $\mathsf{uni}(\mathcal{A}_\mathsf{t})$. We prove that the value of P_n is 1 in C.

Let h be a Γ-homomorphism from $\mathsf{uni}(\mathcal{K}_\mathsf{sb})$ to $\mathsf{uni}(\mathcal{A}_\mathsf{t})$. Since $\mathsf{uni}(\mathcal{K}_\mathsf{sb})^{a_n}$ is an infinite tree, and the only role cycles that $\mathcal{A}_\mathsf{t}$ contains are loops of the form $L'(a_i, a_i)$ and $R'(a_i, a_i)$, there exists a bound m such that for each $\sigma = a_n w_{[S_1]} \cdots w_{[S_m]} \in \Delta^{\mathsf{uni}(\mathcal{K}_\mathsf{sb})^{a_n}}$ with $S_j \in \{L, R\}$, it holds $h(\sigma) = a_i$ for some i such that $P_i = 1$ in C.

Assume $1 \leq \ell \leq m$ and for each $\sigma = a_n w_{[S_1]} \cdots w_{[S_\ell]}$ with $S_j \in \{L, R\}$ and each $1 \leq i \leq n$, the value of P_i is 1 in C whenever $h(\sigma) = a_i$. We verify by induction on ℓ that for each $\delta = a_n w_{[S_1]} \cdots w_{[S_{\ell-1}]}$ and each $1 \leq i \leq n$, the value of P_i is 1 in C whenever $h(\delta) = a_i$. Assume that $h(\delta w_{[L]}) = a_j$, $h(\delta w_{[R]}) = a_k$ and the values of P_j and P_k are 1 in C, moreover $h(\delta) = a_i$. If $i = j = k$, then obviously the value of P_i is 1 in C. Otherwise $i \neq j$ and $i \neq k$. If $j = k$, then given that h is a Γ-homomorphism, $\mathcal{A}_\mathsf{t}$ contains assertions $L'(a_i, a_j)$ and $R'(a_i, a_j)$ (hence, $\mathcal{A}_\mathsf{s}$ contains assertions $L(a_i, a_j)$ and $R(a_i, a_j)$). By construction of $\mathcal{A}_\mathsf{s}$, it follows that there is an assignment $P_i = P_j \vee P_{j'}$ in C for some j'. As P_j is 1, we obtain that also P_i evaluates to 1. If $j \neq k$, then $\mathcal{A}_\mathsf{t}$ contains assertions $L'(a_i, a_j)$ and $R'(a_i, a_k)$, so by construction of $\mathcal{A}_\mathsf{s}$ there is an assignment $P_i = P_j \wedge P_k$ or $P_i = P_j \vee P_k$ in C. Again it follows that P_i evaluates to 1 in C. By induction, P_n evaluates to 1 in C. □

B.3. Proof of Lemma 6.12

Proof. The proof is by reduction from 3-colorability of undirected graphs known to be NP-hard. Consider an undirected graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$, which we view as a symmetric directed graph, and fix signatures $\Sigma = \{E(\cdot,\cdot)\}$ and $\Gamma = \{E'(\cdot,\cdot)\}$. Further, let $r, g, b \in N_a$, $\mathsf{V} \subseteq N_l$ and

$$
\begin{array}{rcl}
\mathcal{A}_\mathsf{s} & = & \{E(r,g), E(g,r), E(r,b), E(b,r), E(g,b), E(b,g)\},\\
\mathcal{S} & = & \{\},\\
\mathcal{B} & = & \{E \sqsubseteq E'\},\\
\mathcal{A}_\mathsf{t} & = & \{E'(r,g), E'(g,r), E'(r,b), E'(b,r), E'(g,b), E'(b,g)\} \cup\\
& & \{E'(x,y) \mid (x,y) \in \mathsf{E}\}.
\end{array}
$$

Note that the nodes in G become labeled nulls in $\mathcal{A}_\mathsf{t}$. We show that G is 3-colorable if and only if $\mathcal{A}_\mathsf{t}$ is a universal solution for $\mathcal{K}_\mathsf{s} = \langle \mathcal{S}, \mathcal{A}_\mathsf{s} \rangle$ under $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$.

($\Rightarrow$) Suppose G is 3-colorable. Then it follows that there exists a function h that assigns to each vertex from V one of the colors $\{r, g, b\}$ such that if $(x,y) \in \mathsf{E}$, then $h(x) \neq h(y)$. Hence h is a homomorphism from G to the undirected graph $(\{r,g,b\}, \{(r,g),(g,b),(b,r)\})$.

We prove that $\mathcal{A}_\mathsf{t}$ is a universal solution for $\mathcal{K}_\mathsf{s}$ under $\mathcal{M}$ by employing Lemma 6.2. Obviously, $\mathcal{K}_\mathsf{s}$ is Γ-safe with respect to $\mathcal{M}$. Thus, it remains to verify that $\mathsf{uni}(\mathcal{A}_\mathsf{t})$ is Γ-homomorphically equivalent to $\mathsf{uni}(\mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s})$. First, it is easy to see that $\mathsf{uni}(\mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s})$ is Γ-homomorphically embeddable into $\mathsf{uni}(\mathcal{A}_\mathsf{t})$. Second, h is also a homomorphism from $\mathsf{uni}(\mathcal{A}_\mathsf{t})$ to $\mathsf{uni}(\mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s})$. Thus $\mathcal{A}_\mathsf{t}$ is indeed a universal solution for $\mathcal{K}_\mathsf{s}$ under $\mathcal{M}$.

($\Leftarrow$) Suppose now $\mathcal{A}_\mathsf{t}$ is a universal solution for $\mathcal{K}_\mathsf{s}$ under $\mathcal{M}$. Then by Lemma 6.2 it follows that $\mathsf{uni}(\mathcal{A}_\mathsf{t})$ is Γ-homomorphically equivalent to $\mathsf{uni}(\mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s})$. Let h be a homomorphism from $\mathsf{uni}(\mathcal{A}_\mathsf{t})$ to $\mathsf{uni}(\mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s})$. Notice that $\Delta^{\mathsf{uni}(\mathcal{S}\cup\mathcal{B},\mathcal{A}_\mathsf{s})} = \mathsf{ind}(\mathcal{A}_\mathsf{s})$, hence h assigns to each labeled null $x \in \Delta^{\mathsf{uni}(\mathcal{A}_\mathsf{t})}$ some constant $a \in \mathsf{ind}(\mathcal{A}_\mathsf{s})$, and it is easy to see that h is an assignment for the vertices in V that is a 3-coloring of G. ☐

B.4. Proof of Lemma 6.15

Proof. The proof is by reduction from the validity problem for Quantified Boolean Formulas (QBF), known to be PSpace-complete. Consider a QBF

$$\varphi = \mathsf{Q}_1 X_1 \cdots \mathsf{Q}_n X_n \bigwedge_{j=1}^{m} C_j$$

where $\mathsf{Q}_i \in \{\forall, \exists\}$ and C_j, $1 \leq j \leq m$, are clauses over the variables X_i, $1 \leq i \leq n$.

Let $\Sigma = \{A,\ Q_0, Q_i, Q_i^k, R_j,\ P_0, P_i, P_i^k, R_j^0, R_j^i \mid j \in \{1,\ldots,m\},\ i \in \{1,\ldots,n\},\ k \in \{0,1\}\}$ where A is a concept name and the rest are role names. Let $\mathcal{S}$ be the following TBox over Σ for $j \in \{1,\ldots,m\}$, $i \in \{1,\ldots,n\}$ and $k \in \{0,1\}$:

$$
\begin{array}{lllll}
A \sqsubseteq \exists Q_0^- & \exists Q_{i-1}^- \sqsubseteq \exists Q_i^k, \quad \text{if } \mathsf{Q}_i = \forall & Q_i^k \sqsubseteq Q_i & \exists Q_n^- \sqsubseteq \exists R_j & \exists R_j^- \sqsubseteq \exists R_j\\
& \exists Q_{i-1}^- \sqsubseteq \exists Q_i, \quad \text{if } \mathsf{Q}_i = \exists & & &
\end{array}
\tag{B.1}
$$

$$
\begin{array}{lllll}
A \sqsubseteq \exists P_0^- & \exists P_{i-1}^- \sqsubseteq \exists P_i^k & P_i^k \sqsubseteq P_i & \exists (P_i^0)^- \sqsubseteq \exists R_j^i, \quad \text{if } \neg X_i \in C_j & \exists (R_j^i)^- \sqsubseteq \exists R_j^{i-1}\\
& & & \exists (P_i^1)^- \sqsubseteq \exists R_j^i, \quad \text{if } X_i \in C_j &
\end{array}
\tag{B.2}
$$

and $\mathcal{A}_\mathsf{s} = \{A(a)\}$.

Further, let $\Gamma = \{X_i^0, X_i^1, T, S_j\}$ where X_i^0, X_i^1 are concept names and T, S_j are role names, $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$, and $\mathcal{B}$ the following set of inclusions:

$$
\begin{array}{llll}
Q_i \sqsubseteq T & \exists (Q_i^k)^- \sqsubseteq X_i^k & R_j \sqsubseteq S_j & R_j^i \sqsubseteq S_j\\
P_i \sqsubseteq T & \exists (P_i^k)^- \sqsubseteq X_i^k & P_i \sqsubseteq S_j^- & R_j^0 \sqsubseteq S_j^-
\end{array}
$$

Then, $\models \varphi$ if and only if $\mathsf{uni}(\mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s})$ is Γ-homomorphically embeddable into a finite subset of itself, i.e., if and only if a universal solution for $\mathcal{K}_\mathsf{s} = \langle \mathcal{S}, \mathcal{A}_\mathsf{s} \rangle$ under $\mathcal{M}$ exists. We show this following the line of the proof of Theorem 11 in the full version of [55].

(⇒) Suppose $\models \varphi$. We show that the canonical model $\mathsf{uni}(\mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s})$ is Γ-homomorphically embeddable into a finite subset of itself. More precisely, let us denote with $\mathcal{S}^{inf}$ the subset of $\mathcal{S}$ consisting of the first 6 axioms (B.1), and $\mathcal{S}^{fin}$ the subset of $\mathcal{S}$ consisting of the last 6 axioms (B.2). Then $\mathsf{uni}(\mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s}) = \mathsf{uni}(\mathcal{S}^{inf} \cup \mathcal{B}, \mathcal{A}_\mathsf{s}) \cup \mathsf{uni}(\mathcal{S}^{fin} \cup \mathcal{B}, \mathcal{A}_\mathsf{s})$. In the following we use $\mathcal{U}_{inf}$ to denote $\mathsf{uni}(\mathcal{S}^{inf} \cup \mathcal{B}, \mathcal{A}_\mathsf{s})$, and $\mathcal{U}_{fin}$ to denote $\mathsf{uni}(\mathcal{S}^{fin} \cup \mathcal{B}, \mathcal{A}_\mathsf{s})$, and show how to construct a Γ-homomorphism $h : \mathcal{U}_{inf} \to \mathcal{U}_{fin}$.

We begin by setting $h(a) = a$. Then we define h in such a way that, for each path π in $\mathcal{U}_{inf}$ of length $i + 1 \leq n$, $h(\pi)$ is a path of the form $aw_{[P_1^{k_1}]} \cdots w_{[P_i^{k_i}]}$ in $\mathcal{U}_{fin}$ and it defines an assignment $\alpha_{h(\pi)}$ to the variables $X_1, \ldots, X_i$ by taking $\alpha_{h(\pi)}(X_{i'}) = \top$ if $k_{i'} = 1$ and $\alpha_{h(\pi)}(X_{i'}) = \bot$ if $k_{i'} = 0$, for all $1 \leq i' \leq i$. Such assignments $\alpha_{h(\pi)}$ will satisfy the following:

$$\text{the QBF obtained from } \varphi \text{ by removing } \mathsf{Q}_1X_1 \ldots \mathsf{Q}_iX_i \text{ from its prefix is true under } \alpha_{h(\pi)}. \qquad (\alpha)$$

For the paths of length 1 the Γ-homomorphism h has been defined and (α) trivially holds. Suppose that we have defined h for all paths in $\mathcal{U}_{inf}$ of length $i + 1 \leq n$. We extend h to all paths of length $i + 2$ in $\mathcal{U}_{inf}$ such that (α) holds. Let π be a path of length $i + 1$. Observe that $h(\pi)$ has two successors in $\mathcal{U}_{fin}$: $h(\pi) \cdot w_{[P^0_{i+1}]}$ and $h(\pi) \cdot w_{[P^1_{i+1}]}$. Now,

- if $\mathsf{Q}_i = \forall$ then π has two successors in $\mathcal{U}_{inf}$: $\pi \cdot w_{[Q^0_{i+1}]}$ and $\pi \cdot w_{[Q^1_{i+1}]}$. Thus, we set $h(\pi \cdot w_{[Q^k_{i+1}]}) = h(\pi) \cdot w_{[P^k_{i+1}]}$, for $k = 0, 1$. Clearly, (α) holds.
- if $\mathsf{Q}_i = \exists$ then π has one successor in $\mathcal{U}_{inf}$: $\pi \cdot w_{[Q_{i+1}]}$. Since φ is valid, by (α) the QBF obtained from φ by removing $\mathsf{Q}_1X_1 \ldots \mathsf{Q}_iX_i$ is true under either $\alpha_{h(\pi)} \cup \{X_i \mapsto \top\}$ or $\alpha_{h(\pi)} \cup \{X_i \mapsto \bot\}$. We set $h(\pi \cdot w_{[Q_{i+1}]}) = h(\pi) \cdot w_{[P^k_{i+1}]}$ where $k = 1$ in the former case, and $k = 0$ in the latter case. Either way, (α) holds.

Let now π be a path of length $n + 1$ in $\mathcal{U}_{inf}$. By construction, we have that $h(\pi) = a \cdot w_{[P_1^{k_1}]} \cdots w_{[P_n^{k_n}]}$. Next, on the one hand, in $\mathcal{U}_{inf}$ the path π has m infinite extensions of the form $\pi \cdot w_{[R_j]} \cdot w_{[R_j]} \cdots$, for $1 \leq j \leq m$. On the other hand, by (α), $\alpha_{h(\pi)} \models C_j$ for each clause C_j, i.e., there is some $1 \leq i \leq n$ such that $k_i = 1$ if $X_i \in C_j$, or $k_i = 0$ if $\neg X_i \in C_j$. For $l \geq 1$, denote by π_l the path $\pi \cdot w_{[R_j]} \cdot \ldots \cdot w_{[R_j]}$ where $w_{[R_j]}$ is repeated l times. We now set

$$\begin{aligned}
h(\pi_l) &= a \cdot w_{[P_1^{k_1}]} \cdot \ldots \cdot w_{[P_{n-l}^{k_{n-l}}]}, && \text{for } 1 \leq l \leq n - i,\\
h(\pi_l) &= a \cdot w_{[P_1^{k_1}]} \cdot \ldots \cdot w_{[P_i^{k_i}]} \cdot w_{[R_j^i]} \cdot \ldots \cdot w_{[R_j^{n-l+1}]}, && \text{for } n - i < l \leq n + 1,\\
h(\pi_l) &= a \cdot w_{[P_1^{k_1}]} \cdot \ldots \cdot w_{[P_i^{k_i}]} \cdot w_{[R_j^i]} \cdot w_{[R_j^{i-1}]} \cdot \ldots \cdot w_{[R_j^{i^\star}]}, && \text{for } n + 1 < l \text{ and } i^\star = (n - l + 1) \bmod 2.
\end{aligned}$$

It is immediate to verify that h is a Γ-homomorphism from $\mathcal{U}_{inf}$ to $\mathcal{U}_{fin}$. Since $\mathcal{K}_1$ is Γ-safe with respect to $\mathcal{M}$, by Lemma 6.4 we obtain that a universal solution for $\mathcal{K}_1$ under $\mathcal{M}$ exists.

(⇐) Let h be a Γ-homomorphism from $\mathcal{U}_{inf}$ to $\mathcal{U}_{fin}$. We show that $\models \varphi$.

Let $\pi = a \cdot w_1 \cdots w_n$ be a path of length $n + 1$ in $\mathcal{U}_{inf}$. Then its homomorphic image $h(\pi)$ must be of the form $a \cdot w_{[P_1^{k_1}]} \cdots w_{[P_n^{k_n}]}$. This implies a variable assignment α_π: $\alpha_\pi(X_i) = \top$ if $k_i = 1$ and $\alpha_\pi(X_i) = \bot$ if $k_i = 0$, for $1 \leq i \leq n$. It is sufficient to show that $\alpha_\pi \models C_j$ for every $1 \leq j \leq m$, i.e, the clause C_j contains at least one of the literals X_i with $\alpha_\pi(X_i) = \top$, or $\neg X_i$ with $\alpha_\pi(X_i) = \bot$.

Consider a path $\pi \cdot w_{[R_j]} \cdot \ldots \cdot w_{[R_j]}$ of length $2n + 2$ in $\mathcal{U}_{inf}$ (i.e., $w_{[R_j]}$ is repeated $n + 1$ times). Then its h-image in $\mathcal{U}_{fin}$ must be of the form $a \cdot w_{[P_1^{k_1}]} \cdot \ldots \cdot w_{[P_i^{k_i}]} \cdot w_{[R_j^i]} \cdot w_{[R_j^{i-1}]} \cdot \ldots \cdot w_{[R_j^0]}$ for some $1 \leq i \leq n$. Now, by construction of $\mathcal{S}$, if $k_i = 0$ (hence, $\alpha_\pi(X_i) = \bot$), then C_j must contain $\neg X_i$, otherwise C_j must contain X_i. □

We illustrate the above reduction with the following example.

Example B.1. Let us consider the QBF $\phi = \exists X_1 \forall X_2 \exists X_3 (X_1 \wedge (X_2 \vee \neg X_3))$, which is valid. A finite portion of the projection of $\mathsf{uni}(\mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s})$ over Γ is depicted in Figure B.6, where each edge -·-·▸ is labeled with T, each edge - - ▸ is labeled with T, S_1^-, S_2^-, and the labels of edges ⟶ are shown to the left of each infinite and finite path. The concept labels of the individuals (if any) are shown next to them.

Let $\mathcal{U}_{inf}$ be the projection over Γ of the part of $\mathsf{uni}(\mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s})$ generated using the axiom templates (B.1) of $\mathcal{S}$; similarly, for $\mathcal{U}_{fin}$ and the axiom templates (B.2). Note that $\mathcal{U}_{inf}$ is infinite, while $\mathcal{U}_{fin}$ is finite. Intuitively, in $\mathcal{U}_{fin}$, the dashed part is a full binary tree representing all possible assignments to the variables X_1, X_2, X_3, where edges whose target node is labeled with X_i^0 (resp., X_i^1) represent the assignment of 0 (resp., 1) to variable X_i. Moreover, each solid

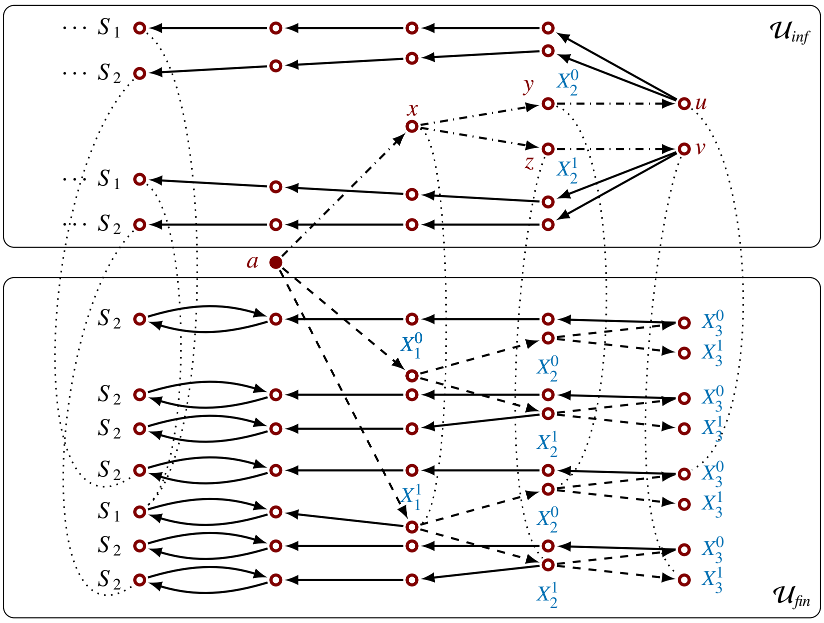


Figure B.6: The projection of $\mathsf{uni}(\mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s})$ over Γ for $\phi = \exists X_1 \forall X_2 \exists X_3 (X_1 \land (X_2 \lor \neg X_3))$.

part (ending in a loop) starting at a node labeled X_i^0 (resp., X_i^1) and labeled with S_1 represents the fact that literal $\neg X_i$ (resp., X_i) appears in clause C_1; analogously for S_2 and C_2. As for $\mathcal{U}_{inf}$, the dash-dotted part represents the quantifier prefix of ϕ: if quantifier Q_i is $\exists$, then there is a single edge at level i (counting from individual a); instead, if quantifier Q_i is $\forall$, then there are two distinct edges at level i, one whose target node is labeled with X_i^0 and one whose target node is labeled with X_i^1. For each clause C_j, each node at level 3 is the origin of an infinite chain, all of whose edges are labeled with S_j.

The QBF ϕ is valid, and we show that there is indeed a Γ-homomorphism h from $\mathcal{U}_{inf}$ to $\mathcal{U}_{fin}$ (hence, $\mathsf{uni}(\mathcal{S} \cup \mathcal{B}, \mathcal{A}_\mathsf{s})$ is Γ-homomorphically embeddable into $\mathcal{U}_{fin}$). Therefore, the ABox obtained from $\mathcal{U}_{fin}$ is a universal solution for $\mathcal{K}_\mathsf{s}$ under $\mathcal{M}$. First, by considering the assignment of 1 to X_1, we obtain the formula $\forall X_2 \exists X_3 (1 \land (X_2 \lor \neg X_3))$, which is valid. Hence h maps $x = aw_{[Q_1]}$ to the node in $\mathcal{U}_{fin}$ labeled with X_1^1. Then, for assignment of 0 to X_2 we obtain $\exists X_3 (0 \lor \neg X_3)$, and for assignment of 1 to X_2 we obtain $\exists X_3 (1 \lor \neg X_3)$, which are both valid. Hence h maps $y = xw_{[Q_2^0]}$ to the successor of $h(x)$ labeled with X_2^0 and $z = xw_{[Q_2^1]}$ to the successor of $h(x)$ labeled with X_2^1. Finally, in the case where $X_2 = 0$, for the assignment of 0 to X_3, we obtain $0 \lor \neg 0$, which is valid; instead, in the case where $X_2 = 1$, any assignment to X_3, e.g., 1 can be used. Hence h maps $u = yw_{[Q_3]}$ to the successor of $h(y)$ labeled with X_3^0 and $v = zw_{[Q_3]}$ to the successor of $h(z)$ labeled with X_3^1. Since for all considered assignments the clauses of ϕ are satisfied, h can indeed map each infinite chain starting from u and v, to a chain in $\mathcal{U}_{fin}$ ending in a loop. For example, the infinite chain starting from v and labeled with S_1 is mapped to the path in $\mathcal{U}_{fin}$ that starts with the dashed edges connecting $h(v)$ to $h(z)$ and $h(x)$, and continues with the edge and loop labeled with S_1. ∎

B.5. Proof of Lemma 6.16

Proof. Let $\mathcal{M} = (\Sigma, \Gamma, \mathcal{B})$ be a mapping, and $\mathcal{K}_\mathsf{s} = \langle \mathcal{S}, \mathcal{A}_\mathsf{s} \rangle$ a KB over Σ. We construct $\mathcal{K}'_\mathsf{s}$ and $\mathcal{M}'$ such that there exists a universal solution for $\mathcal{K}_\mathsf{s}$ under $\mathcal{M}$ iff there exists a universal UCQ-solution for $\mathcal{K}'_\mathsf{s}$ under $\mathcal{M}'$.

Define $\mathcal{M}'$ to be equal to $(\Sigma', \Gamma', \mathcal{B}')$, where Σ' extends Σ with fresh concept and roles names $\{X_1 \mid X \in \Gamma\}$ and fresh role names Q_1, Q_2, Γ' extends Γ with a fresh role name Q, and $\mathcal{B}' = \mathcal{B} \cup \{X_1 \sqsubseteq X \mid X \in \Gamma\} \cup \{Q_1 \sqsubseteq Q, Q_2 \sqsubseteq Q\}$. Let $\mathcal{K}'_\mathsf{s} = \langle \mathcal{S}', \mathcal{A}'_\mathsf{s} \rangle$, where $\mathcal{A}'_\mathsf{s}$ is the union of $\mathcal{A}_\mathsf{s}$, assertions

$$\{X_1(a_X) \mid X \in \Gamma \text{ is a concept name}\} \cup \{X_1(a_X, b_X) \mid X \in \Gamma \text{ is a role name}\},$$

for fresh constants a_X, b_X for each symbol X, and assertions $\{\exists Q_1(a_Q), Q_2(a_Q, b_Q)\}$, for fresh constants a_Q, b_Q. If $\mathcal{K}_s$ is not Γ-safe with respect to $\mathcal{M}$, then $\mathcal{S}' = \mathcal{S} \cup \{\exists Q_1^- \sqsubseteq \exists Q_1\}$, otherwise $\mathcal{S}' = \mathcal{S}$. We prove $\mathcal{K}'_s$ and $\mathcal{M}'$ are as required.

Assume $\mathcal{K}_s$ and $\mathcal{M}$ are inconsistent, that is, the KB $\langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_s \rangle$ is inconsistent. Then each inconsistent target KB is a universal solution for $\mathcal{K}_s$ under $\mathcal{M}$. On the other hand, $\mathcal{K}'_s$ and $\mathcal{M}'$ are inconsistent, and, again, each inconsistent target KB is a universal UCQ-solution for $\mathcal{K}'_s$ under $\mathcal{M}'$. In what follows, we assume $\mathcal{K}_s$ and $\mathcal{M}$ are consistent, and $\mathcal{K}'_s$ and $\mathcal{M}'$ are consistent.

Assume there exists a universal solution $\mathcal{A}_t$ for $\mathcal{K}_s$ under $\mathcal{M}$. Then $\mathcal{K}_s$ is Γ-safe with respect to $\mathcal{M}$, and it is easy to see that $\mathcal{A}_t \cup \{X(a_X) \mid X \in \Gamma \text{ is a concept name}\} \cup \{X(a_X, b_X) \mid X \in \Gamma \text{ is a role name}\} \cup \{Q(a_Q, b_Q)\}$ is a universal UCQ-solution for $\mathcal{K}'_s$ under $\mathcal{M}'$.

Now, assume there exists a universal UCQ-solution $\mathcal{K}_t = \langle \mathcal{T}, \mathcal{A}_t \rangle$ for $\mathcal{K}'_s$ under $\mathcal{M}'$. First, it follows that $\mathsf{uni}(\mathcal{S}' \cup \mathcal{B}', \mathcal{A}'_s)$ does not contain an infinite Q-chain starting from a_Q, hence $\mathcal{S}'$ does not contain the axiom $\exists Q_1^- \sqsubseteq \exists Q_1$ and $\mathcal{K}_s$ is Γ-safe with respect to $\mathcal{M}$. Second, without loss of generality, we may assume that $\mathcal{T}$ does not contain disjointness axioms and $\mathcal{A}_t$ is closed with respect to $\mathcal{T}$. Finally, $\mathsf{uni}(\mathcal{K}_t)$ is finitely Γ-homomorphically equivalent to $\mathsf{uni}(\mathcal{S}' \cup \mathcal{B}', \mathcal{A}'_s)$, so for each concept name $A \in \Gamma$, $A(a_A) \in \mathcal{A}_t$ and for each role name $P \in \Gamma$, $P(a_P, b_P) \in \mathcal{A}_t$. We show that $\mathcal{T}$ is a trivial TBox. By contradiction, assume $\alpha \in \mathcal{T}$ is a non-trivial axiom. Consider various cases of α:

$\alpha = A \sqsubseteq B$, for concept name B distinct from concept name A. Then $\mathcal{K}_t \models B(a_A)$, however $\langle \mathcal{S}' \cup \mathcal{B}', \mathcal{A}'_s \rangle \not\models B(a_A)$, hence it is not the case $\mathsf{uni}(\mathcal{K}_t)$ is finitely Γ-homomorphically equivalent to $\mathsf{uni}(\mathcal{S} \cup \mathcal{B}', \mathcal{A}'_s)$. Contradiction.

$\alpha = \exists P \sqsubseteq A$, for role name P. Then $\mathcal{K}_t \models A(a_P)$, however $\langle \mathcal{S}' \cup \mathcal{B}', \mathcal{A}'_s \rangle \not\models A(a_P)$, hence it is not the case $\mathsf{uni}(\mathcal{K}_t)$ is finitely Γ-homomorphically equivalent to $\mathsf{uni}(\mathcal{S}' \cup \mathcal{B}', \mathcal{A}'_s)$. Contradiction.

$\alpha = \exists P^- \sqsubseteq A$, for role name P. As above, but in this case $\mathcal{K}_t \models A(b_P)$ and $\langle \mathcal{S}' \cup \mathcal{B}', \mathcal{A}'_s \rangle \not\models A(b_P)$.

$\alpha = P \sqsubseteq R$, for role R distinct from role name P. Then $\mathcal{K}_t \models R(a_P, b_P)$, however $\langle \mathcal{S}' \cup \mathcal{B}', \mathcal{A}'_s \rangle \not\models R(a_P, b_P)$, hence it is not the case $\mathsf{uni}(\mathcal{K}_t)$ is finitely Γ-homomorphically equivalent to $\mathsf{uni}(\mathcal{S}' \cup \mathcal{B}', \mathcal{A}'_s)$. Contradiction.

$\alpha = A \sqsubseteq \exists R$, for role R. Then there exists $\sigma \in \Delta^{\mathsf{uni}(\mathcal{K}_t)}$ distinct from a_A such that $R \in \mathbf{r}^{\mathsf{uni}(\mathcal{K}_t)}(a_A, \sigma)$. Since in $\mathsf{uni}(\mathcal{S}' \cup \mathcal{B}', \mathcal{A}'_s)$, a_A is not connected to anything, $\mathsf{uni}(\mathcal{K}_t)$ is not finitely Γ-homomorphically embeddable into $\mathsf{uni}(\mathcal{S}' \cup \mathcal{B}', \mathcal{A}'_s)$. Contradiction.

$\alpha = \exists P \sqsubseteq \exists R$, for role R distinct from role name P. Then there exists $\sigma \in \Delta^{\mathsf{uni}(\mathcal{K}_t)}$ distinct from a_P such that $R \in \mathbf{r}^{\mathsf{uni}(\mathcal{K}_t)}(a_P, \sigma)$. If $\sigma = b_P$ then we get a contradiction similar to the case $\alpha = P \sqsubseteq R$. If $\sigma \neq b_P$ then we get a contradiction as above.

$\alpha = \exists P^- \sqsubseteq \exists R$, for role R distinct from P^-. As above.

$\alpha = \exists P^- \sqsubseteq \exists P$, for role name P. Then in $\mathsf{uni}(\mathcal{K}_t)$ there exists an infinite P-chain starting from b_P, and obviously, it is not finitely Γ-homomorphically embeddable into $\mathsf{uni}(\mathcal{S}' \cup \mathcal{B}', \mathcal{A}'_s)$. Contradiction.

Therefore, $\mathcal{T}$ is a trivial TBox, so we obtain that $\mathsf{uni}(\mathcal{A}_t)$ is finitely Γ-homomorphically equivalent to $\mathsf{uni}(\mathcal{S}' \cup \mathcal{B}', \mathcal{A}'_s)$. Since $\mathsf{uni}(\mathcal{A}_t)$ is finite, it follows $\mathsf{uni}(\mathcal{A}_t)$ is Γ-homomorphically equivalent to $\mathsf{uni}(\mathcal{S}' \cup \mathcal{B}', \mathcal{A}'_s)$. Let $\mathcal{A}_t^-$ be the subset of $\mathcal{A}_t$ such that $\mathsf{ind}(\mathcal{A}'_t) = \mathsf{ind}(\mathcal{A}_s)$. It is easy to see that $\mathsf{uni}(\mathcal{A}_t^-)$ is Γ-homomorphically equivalent to $\mathsf{uni}(\mathcal{S} \cup \mathcal{B}, \mathcal{A}_s)$, and as $\mathcal{K}_s$ is Γ-safe with respect to $\mathcal{M}$, we conclude that $\mathcal{A}_t^-$ is a universal solution for $\mathcal{K}_s$ under $\mathcal{M}$. □

B.6. Reachability Games on Graphs

Reachability games are two-person infinite games. Here we employ the "Spoiler vs. Duplicator" terminology instead of the standard "Player 0 vs. Player 1" terminology used for instance in [62], as we find it more intuitive.

A game is played by two players: Spoiler and Duplicator, and defined by a game arena (or playground) and a winning condition. A *(game) arena* is a triple $\mathsf{A} = (\mathsf{S}, \mathsf{D}, \mathsf{T})$, where $\mathsf{P} = \mathsf{S} \cup \mathsf{D}$ is a finite set of states, $\mathsf{S} \cap \mathsf{D} = \emptyset$, and $\mathsf{T} \subseteq \mathsf{P} \times \mathsf{P}$ is a transition relation. The game starts in some state $s_0 \in \mathsf{P}$, and it is played in turns. In each turn, if the current state s is in S, then Spoiler chooses some state $s' \in \mathsf{P}$ such that $(s, s') \in \mathsf{T}$, and if the current state s is in D, then Duplicator chooses some state $s' \in \mathsf{P}$ such that $(s, s') \in \mathsf{T}$. Thus, each play in the game is viewed as a path π, which can be infinite (i.e., $\pi = s_0 \cdot s_1 \cdot s_2 \cdots$, where $s_i \in \mathsf{P}$ and $(s_i, s_{i+1}) \in \mathsf{T}$ for every $i \geq 0$) or finite (i.e., $\pi = s_0 \cdot s_1 \cdot s_2 \cdots s_k \in \mathsf{P}^{k+1}$, where $(s_i, s_{i+1}) \in \mathsf{T}$ for every $i \in \{0, \ldots, k-1\}$ and $\{s \mid (s_k, s) \in \mathsf{T}\} = \emptyset$).

The winning condition characterizes the plays won by Spoiler. We consider a *reachability condition* specified as a set $F \subseteq \mathsf{P}$ of accepting states. Given a winning condition F, a play π is a *win* for Spoiler iff some state from F occurs in π. Finally, a *reachability game* is a pair $G = (\mathsf{G}, F)$ where G is a game arena and F is a reachability condition.

A *strategy for Spoiler from* state s is a (partial) function $f_S : \mathsf{P}^*\mathsf{S} \to \mathsf{P}$ that assigns to each finite sequence of states $s_0 \cdot s_1 \cdots s_k$ with $s_0 = s$ and $s_k \in \mathsf{S}$, a successor state s_{k+1} such that $(s_k, s_{k+1}) \in \mathsf{T}$. A play $\pi = s_0 \cdot s_1 \cdots$ is said to *conform* with strategy f_S if $s_{i+1} = f_0(s_0 s_1 \ldots s_i)$ for every $i \geq 0$ such that $s_i \in \mathsf{S}$. Then, a strategy f_S is a *winning strategy* for Spoiler *from* $s \in \mathsf{P}$, if every play that conforms with f_S and starts in s is a win for Spoiler. The corresponding notions for Duplicator are defined analogously.

Proposition B.2 ([62],[71]). *Given a reachability game $G = (\mathsf{A}, F)$ and a state s in A, it can be checked in* PTIME *whether Spoiler (or Duplicator) has a winning strategy from s.*

B.7. Two-way Alternating Automata

Infinite trees are represented as prefix closed (infinite) sets of words over $\mathbb{N}$ (the set of positive natural numbers). Formally, an infinite tree is a set of words $T \subseteq \mathbb{N}^*$, such that if $x \cdot c \in T$, where $x \in \mathbb{N}^*$ and $c \in \mathbb{N}$, then also $x \in T$. The elements of T are called nodes, the empty word ϵ is the root of T, and for every $x \in T$, the nodes $x \cdot c$, with $c \in \mathbb{N}$, are the successors of x. By convention we take $x \cdot 0 = x$, and $x \cdot i \cdot -1 = x$. The branching degree $d(x)$ of a node x denotes the number of successors of x. If the branching degree of all nodes of a tree is bounded by k, we say that the tree has branching degree k. An infinite path P of T is a prefix closed set $P \subseteq T$ such that for every $i \geq 0$ there exists a unique node $x \in P$ with $|x| = i$. A labeled tree over an alphabet Σ is a pair (T, V), where T is a tree and $V : T \to \Sigma$ maps each node of T to an element of Σ.

Alternating automata on infinite trees are a generalization of nondeterministic automata on infinite trees, introduced in [72]. They allow for an elegant reduction of decision problems for temporal and program logics [73, 74]. Let $\mathcal{B}(I)$ be the set of positive boolean formulae over I, built inductively by applying $\wedge$ and $\vee$ starting from $\top$ (denoting true), $\bot$ (denoting false), and elements of I. For a set $J \subseteq I$ and a formula $\phi \in \mathcal{B}(I)$, we say that J satisfies ϕ if and only if, assigning true to the elements in J and false to those in $I \setminus J$, makes ϕ true. For a positive integer k, let $[k] = \{-1, 0, 1, \ldots, k\}$. A *two-way alternating tree automaton (2ATA)* running over infinite trees with branching degree k, is a tuple $\mathbb{A} = \langle \Sigma, Q, \delta, q_0, F \rangle$, where Σ is the input alphabet, Q is a finite set of states, $\delta : Q \times \Sigma \to \mathcal{B}([k] \times Q)$ is the transition function, $q_0 \in Q$ is the initial state, and F specifies the acceptance condition.

The transition function maps a state $q \in Q$ and an input letter $\sigma \in \Sigma$ to a positive boolean formula over $[k] \times Q$. Intuitively, if $\delta(q, \sigma) = \phi$, then each pair (c, q') appearing in ϕ corresponds to a new copy of the automaton going to the direction suggested by c and starting in state q'. For example, if $k = 2$ and $\delta(q_1, \sigma) = ((1, q_2) \wedge (1, q_3)) \vee ((-1, q_1) \wedge (0, q_3))$, when the automaton is in the state q_1 and is reading the node x labeled by the letter σ, it proceeds either by sending off two copies, in the states q_2 and q_3 respectively, to the first successor of x (i.e., $x \cdot 1$), or by sending off one copy in the state q_1 to the predecessor of x (i.e., $x \cdot -1$) and one copy in the state q_3 to x itself (i.e., $x \cdot 0$).

A run of a 2ATA $\mathbb{A}$ over a labeled tree (T, V) is a labeled tree $(T_{\mathbf{r}}, \mathbf{r})$ in which every node is labeled by an element of $T \times Q$. A node in $T_{\mathbf{r}}$ labeled by (x, q) describes a copy of A that is in the state q and reads the node x of T. The labels of adjacent nodes have to satisfy the transition function of $\mathbb{A}$. Formally, a run $(T_{\mathbf{r}}, \mathbf{r})$ is a $T \times Q$-labeled tree satisfying:

– $\epsilon \in T_{\mathbf{r}}$ and $\mathbf{r}(\epsilon) = (\epsilon, q_0)$.
– Let $y \in T_{\mathbf{r}}$, with $\mathbf{r}(y) = (x, q)$ and $\delta(q, V(x)) = \phi$. Then there is a (possibly empty) set $S = \{(c_1, q_1), \ldots, (c_n, q_n)\} \subseteq [k] \times Q$ such that:
  – S satisfies ϕ and
  – for all $1 \leq i \leq n$, we have that $y \cdot i \in T_{\mathbf{r}}$, $x \cdot c_i$ is defined ($x \cdot c_i \in T$), and $\mathbf{r}(y \cdot i) = (x \cdot c_i, q_i)$.

A run $(T_{\mathbf{r}}, \mathbf{r})$ is accepting if all its infinite paths satisfy the acceptance condition. Given an infinite path $P \in T_{\mathbf{r}}$, let $\mathit{inf}(P) \subseteq Q$ be the set of states that appear infinitely often in P (as second components of node labels). We consider here Büchi acceptance conditions. A Büchi condition over a state set Q is a subset F of Q, and an infinite path P satisfies F if $\mathit{inf}(P) \cap F \neq \emptyset$.

The non-emptiness problem for 2ATAs consists in determining, for a given 2ATA, whether the set of trees it accepts is nonempty. It is known that this problem can be solved in exponential time in the number of states of the input automaton $\mathbb{A}$, but in linear time in the size of the alphabet as well as in the size of the transition function of $\mathbb{A}$.

B.8. Proof of Lemma 6.17

Proof. (⇒) Let $D \subseteq \Delta^{\mathcal{U}_2}$ be a finite set, and h a Σ-homomorphism from $\mathcal{U}_1^b$ to $\mathcal{U}_2^D$. We construct a labeled tree $T = (\{1, \ldots, n\}^*, V)$ where $n = \max(n_a, n_w)$ and show that $T \in \mathcal{L}(\mathbb{A}_b)$. The labeling function V is defined as follows: $V(\epsilon) = R$ and

$$
\begin{array}{ll}
V(i) = \hat{a}_i, & \text{for each } a_i \in D \cap \mathsf{ind}(\mathcal{K}_2) \\
V(i_1 i_2 \cdots i_r) = \hat{w}_{i_r}, & \text{for each } a_{i_1} w_{i_2} \cdots w_{i_r} \in D \\
V(x) = S, & \text{for each } x \in \{1, \ldots, n\}^* \text{ s.t. } V(x) \text{ was not defined above.}
\end{array}
$$

To show that $T \in \mathcal{L}(\mathbb{A}_b)$, we construct a run tree $(T_\mathbf{r}, \mathbf{r})$ of $\mathbb{A}$ on T. The tree structure $T_\mathbf{r}$ and the labeling function $\mathbf{r}$ are defined inductively as follows, where for $(x, q) \in \{1, \ldots, n\}^* \times Q$, $f((x, q))$ denotes x, and $(z)^q$ denotes $z \cdots z$, where z is repeated q times:

- $\epsilon \in T_\mathbf{r}$ is the root of $T_\mathbf{r}$ and $\mathbf{r}(\epsilon) = (\epsilon, q_0)$,
- ϵ has two children 0_f and 0_h such that $\mathbf{r}(0_f) = (\epsilon, q_f)$ and $\mathbf{r}(0_h) = (\epsilon, q_h)$,
- 0_f has children $c_1, \ldots, c_{n_a}$ such that $\mathbf{r}(c_i) = (i, \alpha_i)$,
- for $i \in \{1, \ldots, n_a\}$ and each w_j such that $a_i \rightsquigarrow_{\mathcal{K}_2} w_j$, c_i has a child $c_i \cdot w_j$ with $\mathbf{r}(c_i \cdot w_j) = (i \cdot j, \omega_j)$,
- for each node in $T_\mathbf{r}$ of the form $x = c_{i_1} w_{i_2} \cdots w_{i_r}$, such that $r \geq 2$ and $a_{i_1} w_{i_2} \cdots w_{i_r} \in D$, and each w_j such that $w_{i_r} \rightsquigarrow_{\mathcal{K}_2} w_j$, x has a child $x \cdot w_j$ with $\mathbf{r}(x \cdot w_j) = (i_1 i_2 \cdots i_r j, \omega_j)$,
- 0_h has one child y_0 with $\mathbf{r}(y_0) = (i, \gamma_0)$ where $i \in \{1, \ldots, n_a\}$ is such that $b = a_i$,
- for each node of the form $x = y_0 \cdot (z_{l_1})^{q_1} \cdot y_{l_1} \cdot (z_{l_2})^{q_2} \cdot y_{l_2} \cdots (z_{l_k})^{q_k} \cdot y_{l_k}$, where $k \geq 0$, $q_i \geq 0$, z_{l_i} denotes x_{l_i} or $k^j_{l_i}$, and $f(\mathbf{r}(x)) = j_1 \cdots j_s$ with $s \geq 1$, and for each v_l such that $v_{l_k} \underset{\Sigma}{\dashrightarrow} v_l$ and $h(b v_{l_1} \cdots v_{l_k} v_l) = a_{i_1} w_{i_2} \cdots w_{i_r}$,
  - x has a child $x \cdot x_l$ with $\mathbf{r}(x \cdot x_l) = (j_1 \cdots j_s, \chi_l)$;
  - if $j_1 = i_1$, let t be the number s.t. $j_1 = i_1, \ldots, j_t = i_t$, and if $j_1 \neq i_1$, let $t = 0$, then
    - * every node of the form $x' = x(x_l)^q$, $1 \leq q \leq s - t$, has one child $x' \cdot x_l$ with $\mathbf{r}(x' \cdot x_l) = (j_1 \cdots j_{s-q}, \chi_l)$,
    - * every node of the form $x' = x(x_l)^q$, $s - t + 1 \leq q \leq s - t + r - t$ has one child $x' \cdot x_l$ with $\mathbf{r}(x' \cdot x_l) = (j_1 \cdots j_t i_{t+1} \cdots i_{t+q-(s-t)}, \gamma_l)$, and
    - * node $x' = x(x_l)^{s-t+r-t+1}$ has one child $x' \cdot y_l$ with $\mathbf{r}(x' \cdot y_l) = (i_1 \cdots i_r, \gamma_l)$.
- for each node of the form $x = y_0 \cdot (z_{l_1})^{q_1} \cdot y_{l_1} \cdot (z_{l_2})^{q_2} \cdot y_{l_2} \cdots (z_{l_k})^{q_k} \cdot y_{l_k}$, where $k \geq 0$, $q_i \geq 0$, z_{l_i} denotes x_{l_i} or $k^j_{l_i}$, and $f(\mathbf{r}(x)) = i$, and for each v_l such that $v_{l_k} \xrightarrow[\Sigma]{} v_l$, x has a child
  - $x \cdot y_l$ with $\mathbf{r}(x \cdot y_l) = (i \cdot j, \gamma_l)$, if $h(b v_{l_1} \cdots v_{l_k} v_l) = a_i w_j$,
  - $x \cdot k^j_l$ with $\mathbf{r}(x \cdot k^j_l) = (\epsilon, \kappa^j_l)$, if $h(b v_{l_1} \cdots v_{l_k} v_l) = a_j$.
- for each node of the form $x = y_0 \cdot (z_{l_1})^{q_1} \cdot y_{l_1} \cdot (z_{l_2})^{q_2} \cdot y_{l_2} \cdots (z_{l_k})^{q_k} \cdot y_{l_k}$, where $k \geq 0$, $q_i \geq 0$, z_{l_i} denotes x_{l_i} or $k^j_{l_i}$, and $f(\mathbf{r}(x)) = i_1 \cdots i_{r'}$ for $r' \geq 2$, and for each v_l such that $v_{l_k} \xrightarrow[\Sigma]{} v_l$ and $h(b v_{l_1} \cdots v_{l_k} v_l) = a_{i_1} w_{i_2} \cdots w_{i_r}$, x has a child $x \cdot y_l$ with $\mathbf{r}(x \cdot y_l) = (i_1 \cdots i_r, \gamma_l)$.
- for each node of the form $x = y_0 \cdot z_1 \cdots z_q \cdot k^j_l$, $q \geq 0$ and $z_i \in \{y_i, x_i, k^{i'}_i\}$, x has one child $x \cdot y_l$ with $\mathbf{r}(x \cdot y_l) = (j, \gamma_k)$.

It is easy to see that $(T_\mathbf{r}, \mathbf{r})$ is an accepting run of $\mathbb{A}_b$.

(⇐) Assume that the language of $\mathbb{A}_b$ is non-empty and $T = (\{1, \ldots, n\}^*, V) \in \mathcal{L}(\mathbb{A}_b)$. Let $(T_\mathbf{r}, \mathbf{r})$ be an accepting run of $\mathbb{A}_b$ over T. Denote by $\mathcal{U}_1$ and $\mathcal{U}_2$, and by $\mathcal{G}_1$ and $\mathcal{G}_2$ the canonical and the generating models of $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. We construct a finite set $D \subseteq \Delta^{\mathcal{U}_2}$ and a homomorphism h from $\mathcal{U}_1^b$ to $\mathcal{U}_2^D$ using T and $(T_\mathbf{r}, \mathbf{r})$.

Firstly, we prove that T encodes a finite subset of $\Delta^{\mathcal{U}_2}$. We show

(a) for each $i \in \{1, \ldots, n_a\}$, $V(i) = \hat{a}_i$;

(b) for each $k \geq 2$, such that $a_{i_1} w_{i_2} \cdots w_{i_k} \in \Delta^{\mathcal{U}_2}$, and for each $2 \leq j < k$, $V(i_1 \cdots i_j) = \hat{w}_{i_j}$, then $V(i_1 \cdots i_k) = \hat{w}_{i_k}$ or $V(i_1 \cdots i_k) = S$;

(c) for each infinite path $a_{i_1} \cdots w_{i_j} \cdots \in \Delta^{\mathcal{U}_2}$, there exists $j \geq 2$, s.t. $V(i_1 \cdots i_j) = S$.

Proof of **(a)**: by definition of $\delta(\alpha_i, L)$.
Proof of **(b)**: for the sake of contradiction, assume for some $a_{i_1}w_{i_2}\cdots w_{i_k} \in \Delta^{\mathcal{U}_2}$, $k \geq 2$, for each $2 \leq j < k$, $V(i_1 \cdots i_j) = \hat{w}_{i_j}$, but $V(i_1 \cdots i_k) = R$ or $V(i_1 \cdots i_k) = \hat{a}_i$. Since $(T_\mathbf{r}, \mathbf{r})$ is a run over T there exists a path in $T_\mathbf{r}$ of the form

$$(\epsilon, q_0), (\epsilon, q_f), (i_1, \alpha_{i_1}), (i_1 i_2, \omega_{i_2}), \ldots, (i_1 \cdots i_k, \omega_{i_k}).$$

Then by definition of the transition function, both $\delta(\omega_{i_k}, R) = \bot$ and $\delta(\omega_{i_k}, \hat{a}_i) = \bot$, which contradicts the assumption $(T_\mathbf{r}, \mathbf{r})$ is a run.
Proof of **(c)**: By contradiction, assume that there exists an infinite path $a_{i_1}\cdots w_{i_j}\cdots$ in $\Delta^{\mathcal{U}_2}$, such that for each $j \geq 2$, $V(i_1 \cdots i_j) \neq S$. Now, since $(T_\mathbf{r}, \mathbf{r})$ is a run of $\mathbb{A}_b$ over T, there must exist an infinite path π in $T_\mathbf{r}$ of the form

$$(\epsilon, q_0), (\epsilon, q_f), (i_1, \alpha_{i_1}), (i_1 i_2, \omega_{i_2}), \ldots, (i_1 \cdots i_j, \omega_{i_j}), \ldots.$$

Since $\mathit{inf}(\pi) \cap \{\gamma_1, \ldots, \gamma_{n_w}\} = \emptyset$ we obtain a contradiction with the assumption that $(T_\mathbf{r}, \mathbf{r})$ is an accepting run. Therefore, let $d \geq 2$ be the depth of S, i.e., for each $a_{i_1}\cdots w_{i_j}\cdots \in \Delta^{\mathcal{U}_2}$, for some $j \leq d$, $V(i_1 \cdots i_j) = S$. The finite set D is given by $\{a_{i_1}w_{i_2}\cdots w_{i_{d-1}} \in \Delta^{\mathcal{U}_2}\}$.

Next, we show there exists a Γ-homomorphism from $\mathcal{U}_1^b$ to $\mathcal{U}_2^D$ by constructing that h. By induction of k, we build $h(bv_{l_1}\cdots v_{l_k})$ for each $bv_{l_1}\cdots v_{l_k} \in \Delta^{\mathcal{U}_1^b}$.

Base of induction. First, in $T_\mathbf{r}$ there must exist a path $(\epsilon, q_0), (\epsilon, q_h)$, and as $T_\mathbf{r}$ is a run, for some i, $b = a_i$, hence this path continues with (i, γ_0) (and the current path is $(\epsilon, q_0), (\epsilon, q_h), (i, \gamma_0)$). Then, $\delta(\gamma_0, \hat{a}_i)$ is satisfied, which means that $\tau^b_{a_i} = \top$ and, in turn, $\mathbf{t}^{\mathcal{U}_1}_\Sigma(b) \subseteq \mathbf{t}^{\mathcal{U}_2}_\Sigma(a_i)$, so we can set $h(b) = a_i$.

Inductive step. Assume h is defined for each path of length $k+1$ in $\Delta^{\mathcal{U}_1}$, $k \geq 0$, let $bv_{l_1}\cdots v_{l_k} \in \Delta^{\mathcal{U}_1^b}$ (v_{l_0} denotes b), and $h(bv_{l_1}\cdots v_{l_k}) = a_{i_0}w_{i_1}\cdots w_{i_r}$, and assume the current path π in $T_\mathbf{r}$ is of the form

$$(\epsilon, q_0), (\epsilon, q_h), (i_0, \gamma_0), (x, q)^*, \ldots, (i_0 \cdots i_r, \gamma_k),$$

where $(x, q)^*$ denotes a finite (possibly empty) sequence of tuples (x, q) with $x \in \{1, \ldots, n\}^*$ and $q \in \{\gamma_l, \chi_l, \kappa^i_l \mid 1 \leq l \leq m, 1 \leq i \leq n_a\}$. Then $\delta(\gamma_k, \hat{w}_{i_r})$ (recall, that $i_0 \cdots i_r \in T$ is labeled with $\hat{w}_{i_r}$) is satisfied. Now, let $v_{l_k} \rightsquigarrow_{\mathcal{K}_1} v_{l_{k+1}}$. If $v_{l_k} \xrightarrow[\Sigma]{} v_{l_{k+1}}$, then at least one of the formulas

$$\begin{array}{rcll} \psi_j & = & \rho^{v_{l_k}, v_{l_{k+1}}}_{w_{i_r}, w_j} \wedge (j, \gamma_{l_{k+1}}), & \text{for } w_{i_r} \rightsquigarrow_{\mathcal{K}_2} w_j \ (w_{i_0} \text{ denotes } a_{i_0}), \\ \psi_i & = & \rho^{v_{l_k}, v_{l_{k+1}}}_{a_{i_0}, a_i} \wedge (-1, \kappa^i_{l_{k+1}}), & \text{if } r = 0, \\ \psi_{-1} & = & \eta^{v_{l_k}, v_{l_{k+1}}}_{w_{i_r}} \wedge (-1, \gamma_{l_{k+1}}), & \text{if } r > 0, \end{array}$$

is satisfied. Assume ψ_j is satisfied for some $j \in \{1, \ldots, n_w\}$: then $\rho^{v_{l_k}, v_{l_{k+1}}}_{w_{i_r}, w_j} = \top$, hence $\mathbf{r}^{\mathcal{G}_1}_\Sigma(v_{l_k}, v_{l_{k+1}}) \subseteq \mathbf{r}^{\mathcal{G}_2}_\Sigma(w_{i_r}, w_j)$, and the run is continued with $(i_0 \cdots i_r j, \gamma_{l_{k+1}})$. Moreover, $\delta(\gamma_{l_{k+1}}, \hat{w}_j)$ is satisfied, so $\tau^{v_{l_{k+1}}}_{w_j} = \top$, i.e., $\mathbf{t}^{\mathcal{G}_1}_\Sigma(v_{l_{k+1}}) \subseteq \mathbf{t}^{\mathcal{G}_2}_\Sigma(w_j)$. Therefore, we can set $h(bv_{l_1}\cdots v_{l_{k+1}})$ to be equal to $a_{i_0}w_{i_1}\cdots w_{i_r}w_j$.

In the case $r = 0$ and ψ_i is satisfied for some $i \in \{1, \ldots, n_a\}$, we have that $\rho^{v_{l_k}, v_{l_{k+1}}}_{a_{i_0}, a_i} = \top$, hence $\mathbf{r}^{\mathcal{G}_1}_\Sigma(v_{l_k}, v_{l_{k+1}}) \subseteq \mathbf{r}^{\mathcal{G}_2}_\Sigma(a_{i_0}, a_i)$, and the run is continued with $(\epsilon, \kappa^i_{l_{k+1}}), (i, \gamma_{l_{k+1}})$. Moreover, $\delta(\gamma_{l_{k+1}}, \hat{a}_i)$ is satisfied, so $\tau^{v_{l_{k+1}}}_{a_i} = \top$, i.e., $\mathbf{t}^{\mathcal{G}_1}_\Sigma(v_{l_{k+1}}) \subseteq \mathbf{t}^{\mathcal{G}_2}_\Sigma(a_i)$. Therefore, we set $h(bv_{l_1}\cdots v_{l_{k+1}})$ to be equal a_i.

Alternatively, if for $r > 0$, ψ_{-1} is satisfied, it follows that $\eta^{v_{l_k}, v_{l_{k+1}}}_{w_{i_r}} = \top$, hence $\{R^- \mid R \in \mathbf{r}^{\mathcal{G}_1}_\Sigma(v_{l_k}, v_{l_{k+1}})\} \subseteq \mathbf{r}^{\mathcal{G}_2}_\Sigma(w_{i_{r-1}}, w_{i_r})$, and the run is continued with $(i_0 \cdots i_{r-1}, \gamma_{l_{k+1}})$. Moreover, $\delta(\gamma_{l_{k+1}}, \hat{w}_{i_{r-1}})$ is satisfied, so $\tau^{v_{l_{k+1}}}_{w_{i_{r-1}}} = \top$, i.e., $\mathbf{t}^{\mathcal{G}_1}_\Sigma(v_{l_{k+1}}) \subseteq \mathbf{t}^{\mathcal{G}_2}_\Sigma(w_{i_{r-1}})$. Therefore, we can set $h(bv_{l_1}\cdots v_{l_{k+1}})$ to be equal to $a_{i_0}w_{i_1}\cdots w_{i_{r-1}}$. It concludes the inductive step for the case $v_{l_k} \xrightarrow[\Sigma]{} v_{l_{k+1}}$.

Consider now, $v_{l_k} \dashrightarrow_{\Sigma} v_{l_{k+1}}$. Then the run continues with $(i_1 \cdots i_r, \chi_{l_{k+1}})$. Let

$$(x_1, \chi_{l_{k+1}}), \ldots, (x_j, \chi_{l_{k+1}}), (x_j, \gamma_{l_{k+1}})$$

be a continuation of the current path $\pi \cdot (i_1 \cdots i_r, \chi_{l_{k+1}})$ in $T_\mathbf{r}$, and $x_j = j_0 \cdots j_s$. Then $\delta(\gamma_{l_{k+1}}, \hat{w}_{j_s})$ is satisfied, so $\tau^{v_{l_{k+1}}}_{w_{j_s}} = \top$, and $\mathbf{t}^{\mathcal{G}_1}_\Sigma(v_{l_{k+1}}) \subseteq \mathbf{t}^{\mathcal{G}_2}_\Sigma(w_{j_s})$. Since $\mathbf{r}^{\mathcal{U}_1}_\Sigma(bv_{l_1}\cdots v_{l_k}, bv_{l_1}\cdots v_{l_{k+1}}) = \emptyset$, we can set $h(bv_{l_1}\cdots v_{l_{k+1}})$ to be equal to $a_{j_0}w_{j_1}\cdots w_{j_s}$.

Note that the runs considered in the induction never visit a node labeled with S, otherwise it contradicts the definition of a run. Therefore, in such a manner, we can define h, a Σ-homomorphism from $\mathcal{U}_1^b$ to $\mathcal{U}_2^D$. □

C. Membership Problem for UCQ-representability

Let $\mathcal{K} = \langle \mathcal{O}, \mathcal{A} \rangle$ be a consistent KB, $a, b \in N_a$, $\sigma \in \Delta^{\mathsf{uni}(\mathcal{K})}$, and $\mathsf{tail}(\sigma) \rightsquigarrow_{\mathcal{K}} w_{[R]}$. We make use of the following properties:

(A) $B' \in \mathbf{t}^{\mathsf{uni}(\mathcal{K})}(a)$ iff $\mathcal{A} \models B(a)$ and $\mathcal{O} \models B \sqsubseteq B'$, and
$R' \in \mathbf{r}^{\mathsf{uni}(\mathcal{K})}(a, b)$ iff $\mathcal{A} \models R(a, b)$ and $\mathcal{O} \models R \sqsubseteq R'$;

Proof: first, by definition of the canonical model, $B' \in \mathbf{t}^{\mathsf{uni}(\mathcal{K})}(a)$ if and only if $\mathcal{K} \models B'(a)$. Next, assume $\mathcal{A} \not\models B'(a)$, i.e., neither $B'(a) \in \mathcal{A}$, nor $S(a, b) \notin \mathcal{A}$ for $B' = \exists S$ and some $b \in N_a$. Obviously, $a \in \mathsf{ind}(\mathcal{A})$, so for some concept A, $A(a) \in \mathcal{A}$, or for some role S, $S(a, b) \in \mathcal{A}$. By contradiction, assume that $\mathcal{O} \not\models A \sqsubseteq B'$ for each $A(a) \in \mathcal{A}$, and $\mathcal{O} \not\models \exists S \sqsubseteq B'$ for each $S(a, b) \in \mathcal{A}$. Then there exists a model $\mathcal{I}$ of $\mathcal{K}$ such that $a^{\mathcal{I}} \notin B'^{\mathcal{I}}$, which contradicts $\mathcal{K} \models B'(a)$. Hence, $\mathcal{O} \models A \sqsubseteq B'$ for some $A(a) \in \mathcal{A}$ or $\mathcal{O} \models \exists S \sqsubseteq B'$ for some $S(a, b) \in \mathcal{A}$. The opposite direction is obvious. The proof for $R \in \mathbf{r}^{\mathsf{uni}(\mathcal{K})}(a, b)$ is analogous.

(B) $B \in \mathbf{t}^{\mathsf{uni}(\mathcal{K})}(\sigma w_{[R]})$ iff $\mathcal{O} \models \exists R^- \sqsubseteq B$, and
$R \in \mathbf{r}^{\mathsf{uni}(\mathcal{K})}(\sigma, \sigma w_{[R']})$ iff $\mathcal{O} \models R' \sqsubseteq R$.

Proof: Follows from the definition of the canonical model and the types.

(C) Let $a \rightsquigarrow_{\mathcal{K}} w_{[R]}$ for some basic role R. Then there exists a basic concept B, such that $\mathcal{A} \models B(a)$ and $B \rightsquigarrow_{\mathcal{O}} R$.

Proof: by definition of $a \rightsquigarrow_{\mathcal{K}} w_{[R]}$ it follows that $\mathcal{K} \models \exists R(a)$ and R is a minimal with respect to $\leq_{\mathcal{O}}$ role among all $\{R' \mid \mathcal{K} \models \exists R'(a)\}$. By **(A)** we have that $\mathcal{A} \models B(a)$ for some concept B, and $\mathcal{O} \models B \sqsubseteq \exists R$. Now, consider KB $\mathcal{B} = \langle \mathcal{O}, \{B(o)\} \rangle$ for some $o \in N_a$. Obviously, $\mathcal{B} \models \exists R(o)$, $\mathcal{B} \not\models R(o, o)$, and R is a minimal with respect to $\leq_{\mathcal{O}}$ role among all $\{R' \mid \mathcal{B} \models \exists R'(o)\}$. Therefore, $o \rightsquigarrow_{\mathcal{B}} w_{[R]}$, and $B \rightsquigarrow_{\mathcal{O}} R$.

(D) Let $w_S \rightsquigarrow_{\mathcal{K}} w_{[R]}$ for basic roles S and R. Then $\exists S^- \rightsquigarrow_{\mathcal{O}} R$.

Proof: by definition of $w_{[S]} \rightsquigarrow_{\mathcal{K}} w_{[R]}$ it follows that $\mathcal{O} \models \exists S^- \sqsubseteq \exists R$, $[S^-] \neq [R]$, and R is a minimal with respect to $\leq_{\mathcal{O}}$ role among all $\{R' \mid \mathcal{O} \models \exists S^- \sqsubseteq \exists R'\}$. Consider KB $\mathcal{B} = \langle \mathcal{O}, \{\exists S^-(o)\} \rangle$ for some $o \in N_a$. The rest of the proof is similar to the proof of **(C)**.

(E) Let $\{B_1, \dots, B_n\}$ be a set of basic concepts and $\mathcal{O}'$ a TBox such that $\mathcal{K}_B = \langle \mathcal{O}, \{B_1(o), \dots, B_n(o)\} \rangle$ and $\langle \mathcal{O} \cup \mathcal{O}', \mathcal{A} \rangle$ are consistent. Assume $y \in \Delta^{\mathsf{uni}(\mathcal{K}_B)}$. If for some $\delta_o \in \Delta^{\mathsf{uni}(\mathcal{O} \cup \mathcal{O}', \mathcal{A})}$, $\{B_1, \dots, B_n\} \subseteq \mathbf{t}^{\mathsf{uni}(\mathcal{O} \cup \mathcal{O}', \mathcal{A})}(\delta_o)$, then there exists $\delta_y \in \Delta^{\mathsf{uni}(\mathcal{O} \cup \mathcal{O}', \mathcal{A})}$ such that

$$\mathbf{t}^{\mathsf{uni}(\mathcal{K}_B)}(y) \subseteq \mathbf{t}^{\mathsf{uni}(\mathcal{O} \cup \mathcal{O}', \mathcal{A})}(\delta_y) \quad \text{and} \quad \mathbf{r}^{\mathsf{uni}(\mathcal{K}_B)}(o, y) \subseteq \mathbf{r}^{\mathsf{uni}(\mathcal{O} \cup \mathcal{O}', \mathcal{A})}(\delta_o, \delta_y) \tag{C.1}$$

Proof: consider the cases of $y \in \Delta^{\mathsf{uni}(\mathcal{K}_B)}$. If $y = o$, then $\delta_y = \delta_o$. Let $y = o w_{[R_1]} \cdots w_{[R_m]}$ for $m \geq 1$: then for some $1 \leq i \leq n$, $\mathcal{O} \models B_i \sqsubseteq \exists R_1$, and for $1 \leq j < m$, $\mathcal{O} \models \exists R_j^- \sqsubseteq \exists R_{j+1}$. Obviously, these entailments are valid in $\mathcal{O} \cup \mathcal{O}'$, so $\exists R_1 \in \mathbf{t}^{\mathsf{uni}(\mathcal{O} \cup \mathcal{O}', \mathcal{A})}(\delta_o)$ and there exists $\delta_1 \in \Delta^{\mathsf{uni}(\mathcal{O} \cup \mathcal{O}', \mathcal{A})}$ s.t. $R_1 \in \mathbf{r}^{\mathsf{uni}(\mathcal{O} \cup \mathcal{O}', \mathcal{A})}(\delta_o, \delta_1)$ and $\exists R_1^- \in \mathbf{t}^{\mathsf{uni}(\mathcal{O} \cup \mathcal{O}', \mathcal{A})}(\delta_1)$. Moreover, for each $1 \leq j < m$, we have that $\exists R_{j+1} \in \mathbf{t}^{\mathsf{uni}(\mathcal{O} \cup \mathcal{O}', \mathcal{A})}(\delta_j)$ and there exists $\delta_{j+1} \in \Delta^{\mathsf{uni}(\mathcal{O} \cup \mathcal{O}', \mathcal{A})}$ such that $R_{j+1} \in \mathbf{r}^{\mathsf{uni}(\mathcal{O} \cup \mathcal{O}', \mathcal{A})}(\delta_j, \delta_{j+1})$ and $\exists R_{j+1}^- \in \mathbf{t}^{\mathsf{uni}(\mathcal{O} \cup \mathcal{O}', \mathcal{A})}(\delta_{j+1})$. So we take δ_y to be equal to δ_m. It is easy to see that (C.1) is satisfied.

(F) concept B is $\mathcal{O}$-inconsistent iff $\mathcal{O} \models B \sqsubseteq C \sqcap D$ for some concept disjointness $C \sqsubseteq \neg D \in \mathcal{O}$, or there exist $n \geq 1$ and roles $R_1, \dots, R_n$ such that $B \rightsquigarrow_{\mathcal{O}} R_1$, $\exists R_i^- \rightsquigarrow_{\mathcal{O}} R_{i+1}$, and

- $\mathcal{O} \models \exists R_n^- \sqsubseteq C \sqcap D$, for some concept disjointness $C \sqsubseteq \neg D \in \mathcal{O}$, or
- $\mathcal{O} \models R_n \sqsubseteq S \sqcap Q$ or $\mathcal{O} \models R_n \sqsubseteq S^- \sqcap Q^-$, for some role disjointness $S \sqsubseteq \neg Q \in \mathcal{O}$.

(G) role R is $\mathcal{O}$-inconsistent iff $\mathcal{O} \models R \sqsubseteq S \sqcap Q$ or $\mathcal{O} \models R \sqsubseteq S^- \sqcap Q^-$ for some role disjointness $S \sqsubseteq \neg Q \in \mathcal{O}$, or one of $\exists R, \exists R^-$ is $\mathcal{O}$-inconsistent.

Proposition C.1. *Let conditions **(ii)** and **(iii)** of Lemma 7.1 hold. Further, let $\mathcal{A}_{\mathsf{s}}$ be an ABox over Σ such that $\langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_{\mathsf{s}} \rangle$ and $\langle \mathcal{T} \cup \mathcal{B}, \mathcal{A}_{\mathsf{s}} \rangle$ are consistent, and let $\mathcal{U}_{\mathsf{sb}}$ and $\mathcal{U}_{\mathsf{tb}}$ be their respective canonical models. Then $\mathcal{U}_{\mathsf{sb}}$ is Γ-homomorphically embeddable into $\mathcal{U}_{\mathsf{tb}}$.*

Proof. We build a function h from $\Delta^{\mathcal{U}_{\mathsf{sb}}}$ to $\Delta^{\mathcal{U}_{\mathsf{tb}}}$, which is a Γ-homomorphism from $\mathcal{U}_{\mathsf{sb}}$ to $\mathcal{U}_{\mathsf{tb}}$.

Base of induction. Initially, for each $a \in \mathsf{ind}(\mathcal{A}_{\mathsf{s}})$ we define $h(a) = a$. Let us immediately verify that $\mathbf{t}_{\Gamma}^{\mathcal{U}_{\mathsf{sb}}}(a) \subseteq \mathbf{t}_{\Gamma}^{\mathcal{U}_{\mathsf{tb}}}(a)$. Let $B' \in \mathbf{t}_{\Gamma}^{\mathcal{U}_{\mathsf{sb}}}(a)$, it follows by **(A)** there exists B over Σ such that $\mathcal{A}_{\mathsf{s}} \models B(a)$ and $\mathcal{S} \cup \mathcal{B} \models B \sqsubseteq B'$. Note that B is $\mathcal{S} \cup \mathcal{B}$-consistent, then by **(ii)**, $\mathcal{T} \cup \mathcal{B} \models B \sqsubseteq B'$, therefore we obtain $B' \in \mathbf{t}^{\mathcal{U}_{\mathsf{tb}}}(a)$. The proof of $\mathbf{r}_{\Gamma}^{\mathcal{U}_{\mathsf{sb}}}(a, b) \subseteq \mathbf{r}_{\Gamma}^{\mathcal{U}_{\mathsf{tb}}}(a, b)$ is analogous.

Next, assume that $\sigma \in \Delta^{\mathcal{U}_{\mathsf{sb}}}$ and $\sigma = aw_{[R]}$. We show how to define $h(\sigma)$. It follows that $a \rightsquigarrow_{\mathcal{K}_{\mathsf{sb}}} w_{[R]}$ and by **(C)** we obtain a concept B over Σ such that $\mathcal{A}_{\mathsf{s}} \models B(a)$, and $B \rightsquigarrow_{\mathcal{S} \cup \mathcal{B}} R$. Then B is $\mathcal{S} \cup \mathcal{B}$-consistent, and by **(iii)** there exists $y \in \Delta^{\mathsf{gen}(\mathcal{T} \cup \mathcal{B}, B(o))}$ such that

$$\mathbf{t}_{\Gamma}^{\mathsf{gen}(\mathcal{S} \cup \mathcal{B}, B(o))}(w_{[R]}) \subseteq \mathbf{t}^{\mathsf{gen}(\mathcal{T} \cup \mathcal{B}, B(o))}(y), \quad \text{and} \quad \mathbf{r}_{\Gamma}^{\mathsf{gen}(\mathcal{S} \cup \mathcal{B}, B(o))}(o, w_{[R]}) \subseteq \mathbf{r}^{\mathsf{gen}(\mathcal{T} \cup \mathcal{B}, B(o))}(o, y).$$

Since $\{B\} \subseteq \mathbf{t}^{\mathcal{U}_{\mathsf{tb}}}(a)$, by **(E)** there exists $\delta \in \Delta^{\mathcal{U}_{\mathsf{tb}}}$ such that

$$\mathbf{t}^{\mathsf{gen}(\mathcal{T} \cup \mathcal{B}, B(o))}(y) \subseteq \mathbf{t}^{\mathcal{U}_{\mathsf{tb}}}(\delta), \quad \text{and} \quad \mathbf{r}^{\mathsf{gen}(\mathcal{T} \cup \mathcal{B}, B(o))}(o, y) \subseteq \mathbf{r}^{\mathcal{U}_{\mathsf{tb}}}(a, \delta).$$

As for a TBox $\mathcal{O}$, ABoxes $\mathcal{A}$ and $\mathcal{A}'$, and $x \in \Delta^{\mathsf{gen}(\mathcal{O}, \mathcal{A})}$, $z \in \Delta^{\mathsf{uni}(\mathcal{O}, \mathcal{A}')}$ with $x = \mathsf{tail}(z)$, the concept and role types of x and z coincide, it follows now by transitivity of '$\subseteq$' that

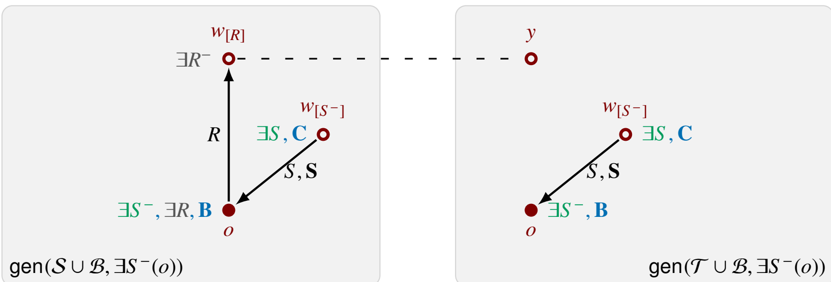
$$\mathbf{t}_{\Gamma}^{\mathcal{U}_{\mathsf{sb}}}(aw_{[R]}) \subseteq \mathbf{t}_{\Gamma}^{\mathcal{U}_{\mathsf{tb}}}(\delta), \quad \text{and} \quad \mathbf{r}_{\Gamma}^{\mathcal{U}_{\mathsf{sb}}}(a, aw_{[R]}) \subseteq \mathbf{r}_{\Gamma}^{\mathcal{U}_{\mathsf{tb}}}(a, \delta).$$

Hence, we assign $h(\sigma) = \delta$.

Inductive step. We show now how to define homomorphism for $\sigma w_{[R]} \in \Delta^{\mathcal{U}_{\mathsf{sb}}}$ with $\sigma = \sigma' w_{[S]}$ given that $h(\sigma)$ and $h(\sigma')$ are defined. It follows $w_{[S]} \rightsquigarrow_{\mathcal{K}_{\mathsf{sb}}} w_{[R]}$ and S is a basic role over Σ by the structure of $\mathcal{S} \cup \mathcal{B}$. Moreover, $\exists S^-$ is $\mathcal{S} \cup \mathcal{B}$-consistent, and by **(D)**, $\exists S^- \rightsquigarrow_{\mathcal{S} \cup \mathcal{B}} R$. So **(iii)** is triggered, and there exists $y \in \Delta^{\mathsf{gen}(\mathcal{T} \cup \mathcal{B}, \exists S^-(o))}$ satisfying

$$\mathbf{t}_{\Gamma}^{\mathsf{gen}(\mathcal{S} \cup \mathcal{B}, \exists S^-(o))}(w_{[R]}) \subseteq \mathbf{t}_{\Gamma}^{\mathsf{gen}(\mathcal{T} \cup \mathcal{B}, \exists S^-(o))}(y), \quad \text{and} \quad \mathbf{r}_{\Gamma}^{\mathsf{gen}(\mathcal{S} \cup \mathcal{B}, \exists S^-(o))}(o, w_{[R]}) \subseteq \mathbf{r}_{\Gamma}^{\mathsf{gen}(\mathcal{T} \cup \mathcal{B}, \exists S^-(o))}(o, y).$$

Let $\mathbf{B} = \mathbf{sup}_{\Gamma}^{\mathcal{B}}(\exists S^-)$, $\mathbf{C} = \mathbf{sup}_{\Gamma}^{\mathcal{B}}(\exists S)$, and $\mathbf{S} = \mathbf{sup}_{\Gamma}^{\mathcal{B}}(S)$ ($\mathbf{sup}_{\Sigma}^{\mathcal{O}}$ was defined in Section 7.2). Then $\mathsf{uni}(\mathcal{S} \cup \mathcal{B}, \exists S^-(o))$ and $\mathsf{uni}(\mathcal{T} \cup \mathcal{B}, \exists S^-(o))$ can be partially depicted as follows. Note that here the presented concept and role labels are not the exact concept and role types. Moreover, we depict only those individuals and links between them that are guaranteed to exist given the information at hand. Note also that in the pictures further in this proof, we depict only the necessary bits of information.



Denote by $\mathbf{B}(o)$ assertions $B_1(o), \dots, B_m(o)$ for $B_i \in \mathbf{B}$, and similarly for $\mathbf{C}(a)$. Moreover, denote by $\mathbf{S}(a, o)$ assertions $S_1(a, o), \dots, S_k(a, o)$ for $S_i \in \mathbf{S}$. There are two possible cases considering that $\mathcal{B}$ is a set of inclusions from Σ to Γ, $\mathcal{T}$ is a TBox over Γ, and S is a role over Σ.

(I) $o \rightsquigarrow_{\langle \mathcal{T} \cup \mathcal{B}, \exists S^-(o) \rangle} w_{[Q_1]} \rightsquigarrow \cdots \rightsquigarrow w_{[Q_n]}$, $n \geq 0$ and Q_i are roles over Γ.
Then, if $n = 0$, $y = o$, otherwise $y = w_{[Q_n]}$.

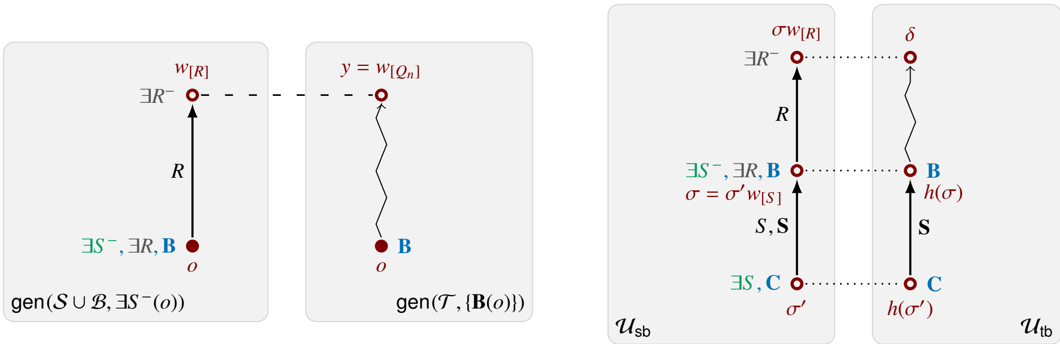
Consider KB $\langle \mathcal{T}, \{\mathbf{B}(o)\} \rangle$, then we obtain that $y \in \Delta^{\mathsf{gen}(\mathcal{T}, \{\mathbf{B}(o)\})}$ and

$$\mathbf{t}_{\Gamma}^{\mathsf{gen}(\mathcal{T} \cup \mathcal{B}, \exists S^-(o))}(y) \subseteq \mathbf{t}_{\Gamma}^{\mathsf{gen}(\mathcal{T}, \{\mathbf{B}(o)\})}(y), \quad \text{and} \quad \mathbf{r}_{\Gamma}^{\mathsf{gen}(\mathcal{T} \cup \mathcal{B}, \exists S^-(o))}(o, y) \subseteq \mathbf{r}_{\Gamma}^{\mathsf{gen}(\mathcal{T}, \{\mathbf{B}(o)\})}(o, y).$$

Observe that $\mathbf{B} \subseteq \mathbf{t}_\Gamma^{\mathcal{U}_{\mathsf{tb}}}(h(\sigma))$, since obviously $\mathbf{B} \subseteq \mathbf{t}_\Gamma^{\mathcal{U}_{\mathsf{sb}}}(\sigma)$ and h is a homomorphism on σ. Therefore, by **(E)** we obtain $\delta \in \Delta^{\mathcal{U}_{\mathsf{tb}}}$ such that

$$\mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{T},\{\mathbf{B}(o)\})}(y) \subseteq \mathbf{t}_\Gamma^{\mathcal{U}_{\mathsf{tb}}}(\delta), \quad \text{and} \quad \mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{T},\{\mathbf{B}(o)\})}(o,y) \subseteq \mathbf{r}_\Gamma^{\mathcal{U}_{\mathsf{tb}}}(h(\sigma),\delta).$$

As above, it follows $\mathbf{t}_\Gamma^{\mathcal{U}_{\mathsf{sb}}}(\sigma w_{[R]}) \subseteq \mathbf{t}_\Gamma^{\mathcal{U}_{\mathsf{tb}}}(\delta)$, and $\mathbf{r}_\Gamma^{\mathcal{U}_{\mathsf{sb}}}(\sigma, \sigma w_{[R]}) \subseteq \mathbf{r}_\Gamma^{\mathcal{U}_{\mathsf{tb}}}(h(\sigma),\delta)$. Hence, we assign $h(\sigma w_{[R]}) = \delta$. This case can be depicted as follows:



(II) $o \rightsquigarrow_{\langle \mathcal{T} \cup \mathcal{B}, \exists S^-(o) \rangle} w_{[S^-]} \rightsquigarrow w_{[Q_1]} \rightsquigarrow \cdots \rightsquigarrow w_{[Q_n]}$, $n \geq 0$, Q_i are roles over Γ.
Then, if $n = 0$, $y = w_{[S^-]}$, otherwise $y = w_{[Q_n]}$.

Consider KB $\langle \mathcal{T}, \{\mathbf{C}(a), \mathbf{S}(a,o)\} \rangle$. Then $a \rightsquigarrow_{\langle \mathcal{T}, \{\mathbf{C}(a), \mathbf{S}(a,o)\} \rangle} w_{[Q_1]} \rightsquigarrow \cdots \rightsquigarrow w_{[Q_n]}$, $y' \in \Delta^{\mathsf{gen}(\mathcal{T}, \{\mathbf{C}(a), \mathbf{S}(a,o)\})}$: if $n = 0$, $y' = a$, otherwise $y' = w_{[Q_n]}$, and

$$\mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{T} \cup \mathcal{B}, \exists S^-(o))}(y) \subseteq \mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{T}, \{\mathbf{C}(a), \mathbf{S}(a,o)\})}(y'), \quad \text{and} \quad \mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{T} \cup \mathcal{B}, \exists S^-(o))}(o,y) \subseteq \mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{T}, \{\mathbf{C}(a), \mathbf{S}(a,o)\})}(o,y').$$

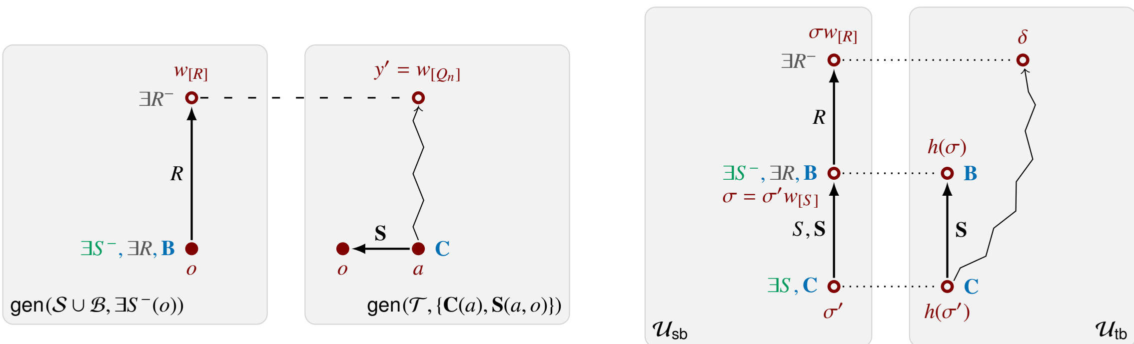
As above, $\mathbf{C} \subseteq \mathbf{t}_\Gamma^{\mathcal{U}_{\mathsf{tb}}}(h(\sigma'))$, therefore by **(E)** we obtain $\delta \in \Delta^{\mathcal{U}_{\mathsf{tb}}}$ such that $\mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{T}, \{\mathbf{C}(a), \mathbf{S}(a,o)\})}(y') \subseteq \mathbf{t}_\Gamma^{\mathcal{U}_{\mathsf{tb}}}(\delta)$.

Observe that if $\mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{S} \cup \mathcal{B}, \exists S^-(o))}(o, w_{[R]}) \neq \emptyset$, it has to be the case that

$$y = w_{[S^-]}, \quad y' = a, \quad \text{and} \quad \delta = h(\sigma').$$

Let $R' \in \mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{S} \cup \mathcal{B}, \exists S^-(o))}(o, w_{[R]})$, it follows $R' \in \mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{T}, \{\mathbf{C}(a), \mathbf{S}(a,o)\})}(o,a)$, and from the latter, $\mathcal{T} \models S_i^- \sqsubseteq R'$ for some $S_i \in \mathbf{S}$. As $S_i \subseteq \mathbf{r}_\Gamma^{\mathcal{U}_{\mathsf{tb}}}(h(\sigma'), h(\sigma))$, we obtain that $R' \in \mathbf{t}_\Gamma^{\mathcal{U}_{\mathsf{tb}}}(h(\sigma), h(\sigma'))$.

All in all, it follows that $\mathbf{t}_\Gamma^{\mathcal{U}_{\mathsf{sb}}}(\sigma w_{[R]}) \subseteq \mathbf{t}_\Gamma^{\mathcal{U}_{\mathsf{tb}}}(\delta)$, and $\mathbf{r}_\Gamma^{\mathcal{U}_{\mathsf{sb}}}(\sigma, \sigma w_{[R]}) \subseteq \mathbf{r}_\Gamma^{\mathcal{U}_{\mathsf{tb}}}(h(\sigma), \delta)$. Hence, we set $h(\sigma w_{[R]}) = \delta$. We conclude with a graphical representation of this case:



In such a way we can define $h(\sigma)$ for each $\sigma \in \Delta^{\mathcal{U}_{\mathsf{sb}}}$, hence h is a Γ-homomorphism from $\mathcal{U}_{\mathsf{sb}}$ to $\mathcal{U}_{\mathsf{tb}}$. □

Proposition C.2. *Let conditions* ***(ii)*** *and* ***(iv)*** *of Lemma 7.1 hold. Further, let $\mathcal{A}_{\mathsf{s}}$ be an ABox over Σ such that $\langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_{\mathsf{s}} \rangle$ and $\langle \mathcal{T} \cup \mathcal{B}, \mathcal{A}_{\mathsf{s}} \rangle$ are consistent, and let $\mathcal{U}_{\mathsf{sb}}$ and $\mathcal{U}_{\mathsf{tb}}$ be their respective canonical models. Then $\mathcal{U}_{\mathsf{tb}}$ is Γ-homomorphically embeddable into $\mathcal{U}_{\mathsf{sb}}$.*

Proof. We build a function h from $\Delta^{\mathcal{U}_{\mathsf{tb}}}$ to $\Delta^{\mathcal{U}_{\mathsf{sb}}}$, a Γ-homomorphism from $\mathcal{U}_{\mathsf{tb}}$ to $\mathcal{U}_{\mathsf{sb}}$.

Base of induction. Initially, for each $a \in \mathsf{ind}(\mathcal{A}_{\mathsf{s}})$ we define $h(a) = a$. Let us immediately verify that $\mathbf{t}_\Gamma^{\mathcal{U}_{\mathsf{tb}}}(a) \subseteq \mathbf{t}_\Gamma^{\mathcal{U}_{\mathsf{sb}}}(a)$. Let $B' \in \mathbf{t}_\Gamma^{\mathcal{U}_{\mathsf{tb}}}(a)$, it follows by **(A)** there exists B over Σ such that $\mathcal{A}_{\mathsf{s}} \models B(a)$ and $\mathcal{T} \cup \mathcal{B} \models B \sqsubseteq B'$. Then B is $\mathcal{S} \cup \mathcal{B}$-consistent (recall that $\mathcal{K}_{\mathsf{sb}} = \langle \mathcal{S} \cup \mathcal{B}, \mathcal{A}_{\mathsf{s}} \rangle$ is consistent), so by **(ii)**, $\mathcal{S} \cup \mathcal{B} \models B \sqsubseteq B'$, therefore we obtain $B' \in \mathbf{t}^{\mathcal{U}_{\mathsf{sb}}}(a)$. The proof of $\mathbf{r}_\Gamma^{\mathcal{U}_{\mathsf{tb}}}(a,b) \subseteq \mathbf{r}_\Gamma^{\mathcal{U}_{\mathsf{sb}}}(a,b)$ is analogous.

Next, assume $\sigma \in \Delta^{\mathcal{U}_{\mathsf{tb}}}$ and $\sigma = aw_{[R]}$, we show how to define $h(\sigma)$. It follows that $a \rightsquigarrow_{\mathcal{K}_{\mathsf{tb}}} w_{[R]}$ and by **(C)** we obtain B over Σ such that $\mathcal{A}_{\mathsf{s}} \models B(a)$, and $B \rightsquigarrow_{\mathcal{T} \cup \mathcal{B}} R$. We are going to show now there exists $y \in \Delta^{\mathsf{gen}(\mathcal{S} \cup \mathcal{B}, B(o))}$ such that

$$\mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{T} \cup \mathcal{B}, B(o))}(w_{[R]}) \subseteq \mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{S} \cup \mathcal{B}, B(o))}(y), \text{ and} \tag{C.2}$$

$$\mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{T} \cup \mathcal{B}, B(o))}(o, w_{[R]}) \subseteq \mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{S} \cup \mathcal{B}, B(o))}(o, y). \tag{C.3}$$

Assume, first, R is a role over Γ, and observe that B is $\mathcal{S} \cup \mathcal{B}$-consistent, then by **(iv)** there exists $y \in \Delta^{\mathsf{gen}(\mathcal{S} \cup \mathcal{B}, B(o))}$ satisfying (C.2) and (C.3).

Assume now R is a role over Σ, then it follows $B = \exists R$. Let $o \rightsquigarrow_{\langle \mathcal{S} \cup \mathcal{B}, \exists R(o) \rangle} w_{[Q]}$ for a role Q over Σ such that $\mathcal{S} \models Q \sqsubseteq R$ (such Q always exists, for instance R itself if it does not have proper subroles). Then we choose y to be $w_{[Q]}$, and show first that (C.2) is satisfied. Let $B \in \mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{T} \cup \mathcal{B}, \exists R(o))}(w_{[R]})$, then by **(B)**, $\mathcal{T} \cup \mathcal{B} \models \exists R^- \sqsubseteq B$, and as $\exists R^- \in \mathbf{t}_\Sigma^{\mathsf{gen}(\mathcal{S} \cup \mathcal{B}, \exists R(o))}(w_{[Q]})$, by **(ii)** we obtain that $B \in \mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{S} \cup \mathcal{B}, \exists R(o))}(w_{[Q]})$. In a similar way, we can show that (C.3) is satisfied.

To continue the proof consider $\{B\} \subseteq \mathbf{t}^{\mathcal{U}_{\mathsf{sb}}}(a)$, then by **(E)** there exists $\delta \in \Delta^{\mathcal{U}_{\mathsf{sb}}}$ such that $\mathbf{t}^{\mathsf{gen}(\mathcal{S} \cup \mathcal{B}, B(o))}(y) \subseteq \mathbf{t}^{\mathcal{U}_{\mathsf{sb}}}(\delta)$ and $\mathbf{r}^{\mathsf{gen}(\mathcal{S} \cup \mathcal{B}, B(o))}(o,y) \subseteq \mathbf{r}^{\mathcal{U}_{\mathsf{sb}}}(a,\delta)$. It follows now using (C.2) that $\mathbf{t}_\Gamma^{\mathcal{U}_{\mathsf{tb}}}(aw_{[R]}) \subseteq \mathbf{t}_\Gamma^{\mathcal{U}_{\mathsf{sb}}}(\delta)$. Analogously using (C.3) one obtains $\mathbf{r}_\Gamma^{\mathcal{U}_{\mathsf{tb}}}(a, aw_{[R]}) \subseteq \mathbf{r}_\Gamma^{\mathcal{U}_{\mathsf{sb}}}(a,\delta)$.

Inductive step. We show how to define homomorphism for $\sigma w_{[R]} \in \Delta^{\mathcal{U}_{\mathsf{tb}}}$ with $\sigma = \sigma' w_{[S]}$ given that $h(\sigma)$ is defined. It follows $w_{[S]} \rightsquigarrow_{\mathcal{K}_{\mathsf{tb}}} w_{[R]}$, therefore $\mathcal{T} \cup \mathcal{B} \models \exists S^- \sqsubseteq \exists R$, and R is a role over Γ *distinct from* S^-. By **(B)** it also follows $\exists R \in \mathbf{t}^{\mathcal{U}_{\mathsf{tb}}}(\sigma)$, and since h is a Γ-homomorphism, $\exists R \in \mathbf{t}_\Gamma^{\mathcal{U}_{\mathsf{sb}}}(h(\sigma))$. As $\mathcal{A}_{\mathsf{s}}$ is an ABox over Σ and $\mathcal{S}$ is a TBox over Σ, there exists a concept B over Σ such that $B \in \mathbf{t}^{\mathcal{U}_{\mathsf{sb}}}(h(\sigma))$ and $\mathcal{B} \models B \sqsubseteq \exists R$. Next, assume that $o \rightsquigarrow_{\langle \mathcal{T} \cup \mathcal{B}, B(o) \rangle} w_{[Q]}$ for some role Q such that $\mathcal{T} \cup \mathcal{B} \models Q \sqsubseteq R$. Then B is $\mathcal{S} \cup \mathcal{B}$-consistent and $B \rightsquigarrow_{\mathcal{T} \cup \mathcal{B}} Q$. As above for $\sigma = aw_{[R]}$, by **(iv)** there exists $y \in \Delta^{\mathsf{gen}(\mathcal{S} \cup \mathcal{B}, B(o))}$ such that

$$\mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{T} \cup \mathcal{B}, B(o))}(w_{[Q]}) \subseteq \mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{S} \cup \mathcal{B}, B(o))}(y), \quad \text{and} \quad \mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{T} \cup \mathcal{B}, B(o))}(o, w_{[Q]}) \subseteq \mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{S} \cup \mathcal{B}, B(o))}(o, y).$$

Again, by **(E)** we obtain δ in $\Delta^{\mathcal{U}_{\mathsf{sb}}}$ such that $\mathbf{t}^{\mathsf{gen}(\mathcal{S} \cup \mathcal{B}, B(o))}(y) \subseteq \mathbf{t}^{\mathcal{U}_{\mathsf{sb}}}(\delta)$ and $\mathbf{r}^{\mathsf{gen}(\mathcal{S} \cup \mathcal{B}, B(o))}(o,y) \subseteq \mathbf{r}^{\mathcal{U}_{\mathsf{sb}}}(h(\sigma),\delta)$. Observe that $\mathcal{T} \cup \mathcal{B} \models Q \sqsubseteq R$, so the concept and role types of $w_{[R]}$ and $(o, w_{[R]})$ are subsumed by those of $w_{[Q]}$ and $(o, w_{[Q]})$ in $\mathsf{gen}(\mathcal{T} \cup \mathcal{B}, B(o))$. Finally, we obtain that $\mathbf{t}^{\mathcal{U}_{\mathsf{tb}}}(\sigma w_{[R]}) \subseteq \mathbf{t}^{\mathcal{U}_{\mathsf{sb}}}(\delta)$ and $\mathbf{r}^{\mathcal{U}_{\mathsf{tb}}}(\sigma, \sigma w_{[R]}) \subseteq \mathbf{r}^{\mathcal{U}_{\mathsf{sb}}}(h(\sigma),\delta)$. Hence, we assign $h(\sigma w_{[R]}) = \delta$.

In such a way we can define $h(\sigma)$ for each $\sigma \in \Delta^{\mathcal{U}_{\mathsf{tb}}}$, hence h is a Γ-homomorphism from $\mathcal{U}_{\mathsf{tb}}$ to $\mathcal{U}_{\mathsf{sb}}$. □

D. Non-emptiness Problem for UCQ-representability

Proposition D.1. *For a concept B over Σ and C' over Γ, inclusion $B \sqsubseteq C'$ is representable in $\mathcal{S}$ and $\mathcal{M}$ if and only if there exists B' over Γ such that $\mathcal{B} \models B \sqsubseteq B'$, and for each $\mathcal{S}$-consistent concept D over Σ:*

- ***(H)*** *$\mathcal{S} \cup \mathcal{B} \models D \sqsubseteq B'$ implies $\mathcal{S} \cup \mathcal{B} \models D \sqsubseteq C'$,*
- ***(I)*** *if $B' = \exists Q'^-$ for some role Q' over Γ, then $\mathcal{S} \cup \mathcal{B} \models D \sqsubseteq \exists Q'$ implies $D \rightsquigarrow_{\mathcal{S} \cup \mathcal{B}} Q$ for some role Q such that $\mathcal{S} \cup \mathcal{B} \models \{Q \sqsubseteq Q', \exists Q^- \sqsubseteq C'\}$.*

In this case, $B \sqsubseteq C'$ is representable by $B' \sqsubseteq C'$.

Proof. ($\Leftarrow$) Let B be a concept over Σ and C' over Γ, $B' \neq C'$, and conditions **(H)** and **(I)** are satisfied. We show inclusion $B \sqsubseteq C'$ is representable in $\mathcal{S}$ and $\mathcal{M}$ by $B' \sqsubseteq C'$. Take $\mathcal{T}$ a parsimonious UCQ-representation for $\mathcal{S}$ under $\mathcal{M}$: we prove $\mathcal{T}' = \mathcal{T} \cup \{B' \sqsubseteq C'\}$ is a parsimonious UCQ-representation by showing the following is satisfied:

– for each $\mathcal{S}$-consistent and $\mathcal{T}' \cup \mathcal{B}$-inconsistent pair of concepts or roles (X, Y), it follows (X, Y) is $\mathcal{S} \cup \mathcal{B}$-inconsistent, which corresponds to the $\Leftarrow$ direction of condition **(i)** of Lemma 7.1,

– for each $\mathcal{S} \cup \mathcal{B}$-consistent concept or role X over Σ and each X' over Γ, $\mathcal{T}' \cup \mathcal{B} \models X \sqsubseteq X'$ implies $\mathcal{S} \cup \mathcal{B} \models X \sqsubseteq X'$, which corresponds to the $\Leftarrow$ direction of condition **(ii)** of Lemma 7.1, and

– condition **(iv)** of Lemma 7.1.

Observe that from $\mathcal{T}$ is a parsimonious UCQ-representation of $\mathcal{S}$ under $\mathcal{M}$, it follows the above conditions are already satisfied for $\mathcal{T}$, $\mathcal{S}$ and $\mathcal{M}$.

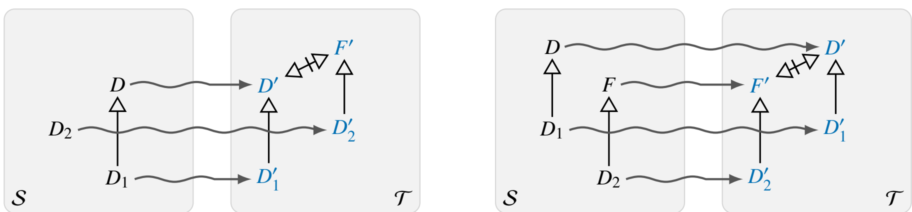
First, for condition **(ii)** of Lemma 7.1, let D be an $\mathcal{S} \cup \mathcal{B}$-consistent concept over Σ and E' a concept over Γ such that $\mathcal{T}' \cup \mathcal{B} \models D \sqsubseteq E'$ and $\mathcal{T} \cup \mathcal{B} \not\models D \sqsubseteq E'$. Hence, there exists D' over Γ such that $\mathcal{T} \models \{D' \sqsubseteq B', C' \sqsubseteq E'\}$ and $\mathcal{B} \models D \sqsubseteq D'$. Since $\mathcal{T}$ is a parsimonious UCQ-representation and $\mathcal{T} \cup \mathcal{B} \models D \sqsubseteq B'$, it follows $\mathcal{S} \cup \mathcal{B} \models D \sqsubseteq B'$, so there exists B_1 over Σ such that $\mathcal{S} \models D \sqsubseteq B_1$ and $\mathcal{B} \models B_1 \sqsubseteq B'$. Next, B', C' satisfy condition **(H)**, therefore $\mathcal{S} \cup \mathcal{B} \models B_1 \sqsubseteq C'$, so there exists C over Σ such that $\mathcal{S} \models B_1 \sqsubseteq C$ and $\mathcal{B} \models C \sqsubseteq C'$. And we can continue by analogy. To summarize, there exist B_1, C and E over Σ such that

$$\mathcal{S} \models \{D \sqsubseteq B_1, B_1 \sqsubseteq C, C \sqsubseteq E\} \tag{D.1}$$

and $\mathcal{B} \models \{B_1 \sqsubseteq B', C \sqsubseteq C', E \sqsubseteq E'\}$. Finally, we obtain that $\mathcal{S} \cup \mathcal{B} \models D \sqsubseteq E'$.

Next, for condition **(i)**, let (D_1, D_2) be a pair of $\mathcal{S}$-consistent, $\mathcal{T} \cup \mathcal{B}$-consistent and $\mathcal{T}' \cup \mathcal{B}$-inconsistent concepts. For the sake of contradiction, assume (D_1, D_2) is $\mathcal{S} \cup \mathcal{B}$-consistent (hence, each D_i is $\mathcal{S} \cup \mathcal{B}$-consistent).

Suppose both D_i are $\mathcal{T}' \cup \mathcal{B}$-consistent. Without loss of generality, we may assume that for some D' over Γ, $\mathcal{T}' \cup \mathcal{B} \models \{D_1 \sqsubseteq D',\ D_2 \sqsubseteq \neg D'\}$. From condition **(ii)**, it follows there exists D over Σ such that $\mathcal{S} \models D_1 \sqsubseteq D$ and $\mathcal{B} \models D \sqsubseteq D'$. Consider the following cases:
1) $\mathcal{T} \cup \mathcal{B} \models D_2 \sqsubseteq \neg D'$ (and $\mathcal{T} \cup \mathcal{B} \not\models D_1 \sqsubseteq D'$). Then, either there exist D'_2, F' over Γ such that $\mathcal{T} \models \{D'_2 \sqsubseteq F', F' \sqsubseteq \neg D'\}$ and $\mathcal{B} \models D_2 \sqsubseteq D'_2$ (see the diagram below), or $\mathcal{B} \models D_2 \sqsubseteq \neg D'$. In both cases, (D, D_2) is $\mathcal{T} \cup \mathcal{B}$-inconsistent, so it follows (D, D_2) is $\mathcal{S} \cup \mathcal{B}$-inconsistent. In view of $\mathcal{S} \models D_1 \sqsubseteq D$, we obtain contradiction with the assumption (D_1, D_2) is $\mathcal{S} \cup \mathcal{B}$-consistent.
2) $\mathcal{T} \cup \mathcal{B} \not\models D_2 \sqsubseteq \neg D'$. Then, there exists F' over Γ such that $\mathcal{T}' \cup \mathcal{B} \models D_2 \sqsubseteq F'$ and $\mathcal{T} \models F' \sqsubseteq \neg D'$ (note, $\mathcal{T} \cup \mathcal{B} \not\models D_2 \sqsubseteq F'$). From condition **(ii)**, it follows there exists F over Σ such that $\mathcal{S} \models D_2 \sqsubseteq F$ and $\mathcal{B} \models F \sqsubseteq F'$. Now, as (D, F) is $\mathcal{T} \cup \mathcal{B}$-inconsistent, it follows (D, F) is $\mathcal{S} \cup \mathcal{B}$-inconsistent, which in view of $\mathcal{S} \models \{D_1 \sqsubseteq D, D_2 \sqsubseteq F\}$ contradicts the assumption (D_1, D_2) is $\mathcal{S} \cup \mathcal{B}$-consistent.



(1) (2)

Suppose one of D_i is $\mathcal{T}' \cup \mathcal{B}$-inconsistent. Consider the following two cases by **(F)**:
1) for some D' over Γ, $\mathcal{T}' \cup \mathcal{B} \models \{D_i \sqsubseteq D',\ D_i \sqsubseteq \neg D'\}$. The contradiction is obtained similarly as in the case both D_i are $\mathcal{T}' \cup \mathcal{B}$-consistent.
2) there exist $n \geq 1$ and distinct roles $S'_1, \dots, S'_n$ such that $D_i \rightsquigarrow_{\mathcal{T}' \cup \mathcal{B}} S'_1$, $\exists S'^{-}_j \rightsquigarrow_{\mathcal{T}' \cup \mathcal{B}} S'_{j+1}$ and $\mathcal{T}' \cup \mathcal{B} \models S'_n \sqsubseteq R' \sqcap Q'$ for $R' \sqsubseteq \neg Q' \in \mathcal{T}$, or $\mathcal{T}' \cup \mathcal{B} \models \exists S'^{-}_n \sqsubseteq E' \sqcap F'$ for $E' \sqsubseteq \neg F' \in \mathcal{T}$.

If $n = 1$ and S'_1 is a role over Σ (i.e., $D_i = \exists S'_1$ and S'_1 is $\mathcal{S} \cup \mathcal{B}$-consistent), then from condition **(ii)**, it follows $\mathcal{S} \cup \mathcal{B} \models S'_1 \sqsubseteq R' \sqcap Q'$ or $\mathcal{S} \cup \mathcal{B} \models \exists S'^{-}_1 \sqsubseteq E' \sqcap F'$. In the former case, there exist roles R, Q over Σ such that $\mathcal{S} \models S'_1 \sqsubseteq R \sqcap Q$ and $\mathcal{B} \models \{R \sqsubseteq R',\ Q \sqsubseteq Q'\}$; then (R, Q) is $\mathcal{T} \cup \mathcal{B}$-inconsistent, since $\mathcal{T}$ is a parsimonious UCQ-representation, it follows (R, Q) is $\mathcal{S} \cup \mathcal{B}$-inconsistent. In the latter case, there exist concepts E, F over Σ such that $\mathcal{S} \models \exists S'^{-}_1 \sqsubseteq E \sqcap F$ and $\mathcal{B} \models \{E \sqsubseteq E',\ F \sqsubseteq F'\}$; then (E, F) is $\mathcal{T} \cup \mathcal{B}$-inconsistent, hence (E, F) is $\mathcal{S} \cup \mathcal{B}$-inconsistent. In any case we obtain S'_1 is $\mathcal{S} \cup \mathcal{B}$-inconsistent, which contradicts the assumption D_i is $\mathcal{S} \cup \mathcal{B}$-consistent.

If $n = 1$ and S'_1 is a role over Γ, assume $\mathcal{T} \cup \mathcal{B} \not\models D_i \sqsubseteq \exists S'_1$. From condition **(ii)** it follows $\mathcal{S} \cup \mathcal{B} \models D_i \sqsubseteq \exists S'_1$, so there exists D over Σ such that $\mathcal{S} \models D_i \sqsubseteq D$ and $\mathcal{B} \models D \sqsubseteq \exists S'_1$. Then $D \rightsquigarrow_{\mathcal{T}\cup\mathcal{B}} T'$ for some role T' (possibly coinciding with S'_1) such that $\mathcal{T} \cup \mathcal{B} \models T' \sqsubseteq S'_1$. In the case $\mathcal{T} \cup \mathcal{B} \models S'_1 \sqsubseteq R' \sqcap Q'$ or $\mathcal{T} \cup \mathcal{B} \models \exists S'^-_1 \sqsubseteq E' \sqcap F'$, since $\mathcal{T}$ is a parsimonious UCQ-representation, from condition **(iv)** it follows there exists a role T such that $D \rightsquigarrow_{\mathcal{S}\cup\mathcal{B}} T$, and $\mathcal{S} \cup \mathcal{B} \models T \sqsubseteq R' \sqcap Q'$ or $\mathcal{S} \cup \mathcal{B} \models \exists T^- \sqsubseteq E' \sqcap F'$. Again, we obtain that D is $\mathcal{S} \cup \mathcal{B}$-inconsistent, which contradicts the assumption D_i is $\mathcal{S} \cup \mathcal{B}$-consistent.

Assume now $\mathcal{T} \cup \mathcal{B} \not\models \exists S'^-_1 \sqsubseteq E' \sqcap F'$ (the case $\mathcal{T} \cup \mathcal{B} \not\models S'_1 \sqsubseteq R' \sqcap Q'$ is not possible). Then it follows $\mathcal{T} \models \{\exists S'^-_1 \sqsubseteq B',\ C' \sqsubseteq E'\}$ and/or $\mathcal{T} \models \{\exists S'^-_1 \sqsubseteq B',\ C' \sqsubseteq F'\}$, and the role T above is such that $\mathcal{S} \cup \mathcal{B} \models \exists T^- \sqsubseteq B'$. If T is over Σ, then $\mathcal{S} \models \exists T^- \sqsubseteq B_1$ and $\mathcal{B} \models B_1 \sqsubseteq B'$ for some concept B_1 over Σ, next we have that $\mathcal{T}' \cup \mathcal{B} \models B_1 \sqsubseteq E' \sqcap F'$, so from condition **(ii)** it follows $\mathcal{S} \cup \mathcal{B} \models B_1 \sqsubseteq E' \sqcap F'$, and as before B_1 is $\mathcal{S} \cup \mathcal{B}$-inconsistent, which contradicts the assumption D_i is $\mathcal{S} \cup \mathcal{B}$-consistent. If T is over Γ, then $B' = \exists T^- = \exists S^-_1$, and by **(I)** it follows there exists S_1 such that $D \rightsquigarrow_{\mathcal{S}\cup\mathcal{B}} S_1$ and $\mathcal{S} \cup \mathcal{B} \models \{S_1 \sqsubseteq S'_1,\ \exists S^-_1 \sqsubseteq C'\}$. Since $\exists S'^-_1 \neq C'$, it follows S_1 is over Σ, and there exists C over Σ such that $\mathcal{S} \models \exists S^-_1 \sqsubseteq C$ and $\mathcal{B} \models C \sqsubseteq C'$. Now, we have that $\mathcal{T}' \cup \mathcal{B} \models C \sqsubseteq E' \sqcap F'$, from condition **(ii)** it follows $\mathcal{S} \cup \mathcal{B} \models C \sqsubseteq E' \sqcap F'$, so as before C is $\mathcal{S} \cup \mathcal{B}$-inconsistent, which contradicts the assumption D_i is $\mathcal{S} \cup \mathcal{B}$-consistent.

For $n > 1$, we can continue reasoning as for the case $n = 1$ to obtain a contradiction. Finally, we conclude that D_i is $\mathcal{S} \cup \mathcal{B}$-inconsistent, hence (D_1, D_2) is $\mathcal{S} \cup \mathcal{B}$-inconsistent.

Let (S_1, S_2) be a pair of $\mathcal{S}$-consistent, $\mathcal{T} \cup \mathcal{B}$-consistent and $\mathcal{T}' \cup \mathcal{B}$-inconsistent roles (this is the only non-trivial case). Since $\mathcal{T}'$ extends $\mathcal{T}$ with a concept inclusion, we have that there exist D_1, D_2 covering $\{\exists S_1, \exists S_2\}$ or $\{\exists S^-_1, \exists S^-_2\}$ such (D_1, D_2) is $\mathcal{T}' \cup \mathcal{B}$-inconsistent and $\mathcal{T} \cup \mathcal{B}$-consistent. By reasoning as above, we obtain (D_1, D_2) is $\mathcal{S} \cup \mathcal{B}$-inconsistent, therefore (S_1, S_2) is $\mathcal{S} \cup \mathcal{B}$-inconsistent.

To show condition **(iv)** of Lemma 7.1 assume an $\mathcal{S} \cup \mathcal{B}$-consistent concept D over Σ and a role R such that $D \rightsquigarrow_{\mathcal{T}'\cup\mathcal{B}} R$ and it is not the case that $D \rightsquigarrow_{\mathcal{T}\cup\mathcal{B}} R$. Hence, R is a role over Γ, and there exists D' over Γ such that $\mathcal{T} \models \{D' \sqsubseteq B', C' \sqsubseteq \exists R\}$ and $\mathcal{B} \models D \sqsubseteq D'$. As before, we can conclude there exists (an $\mathcal{S} \cup \mathcal{B}$-consistent) C over Σ such that $\mathcal{B} \models C \sqsubseteq C'$ (and $\mathcal{S} \models D \sqsubseteq C$). It means $\mathcal{T} \cup \mathcal{B} \models C \sqsubseteq \exists R$, therefore either $C \rightsquigarrow_{\mathcal{T}\cup\mathcal{B}} R$, or $C = \exists Q$ for some role Q over Σ such that $\mathcal{T} \cup \mathcal{B} \models Q \sqsubseteq R$, and $C \rightsquigarrow_{\mathcal{T}\cup\mathcal{B}} Q$. Since $\mathcal{T}$ is a parsimonious UCQ-representation, it follows there exists $z \in \Delta^{\mathsf{gen}(\mathcal{S}\cup\mathcal{B},C(o))}$ such that

$$\mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{T}\cup\mathcal{B},C(o))}(x) \subseteq \mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{S}\cup\mathcal{B},C(o))}(z) \text{ and } \mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{T}\cup\mathcal{B},C(o))}(o,x) \subseteq \mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{S}\cup\mathcal{B},C(o))}(o,z),$$

with $x = w_{[R]}$ or $x = w_{[Q]}$. Observe that $R \in \mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{T}\cup\mathcal{B},C(o))}(o,x)$, which implies that $z = w_{[S]}$ for some role S such that $\mathcal{S} \cup \mathcal{B} \models C \sqsubseteq \exists S$. Now, notice that $\mathcal{S} \cup \mathcal{B} \models D \sqsubseteq \exists S$: we obtain that $o \rightsquigarrow_{\langle\mathcal{S}\cup\mathcal{B},D(o)\rangle} w_{[T]}$ for some role T (possibly coinciding with S) such that $\mathcal{S} \cup \mathcal{B} \models T \sqsubseteq S$. Finally, we have that

$$\mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{T}'\cup\mathcal{B},D(o))}(w_{[R]}) \subseteq \mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{S}\cup\mathcal{B},D(o))}(w_{[T]}) \text{ and } \mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{T}'\cup\mathcal{B},D(o))}(o,w_{[R]}) \subseteq \mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{S}\cup\mathcal{B},D(o))}(o,w_{[T]}),$$

so we take y in condition **(ii)** to be equal to $w_{[T]}$.

Assume now $B' = \exists R^-$ for some role R over Γ, and D is an $\mathcal{S} \cup \mathcal{B}$-consistent concept over Σ such that $D \rightsquigarrow_{\mathcal{T}\cup\mathcal{B}} R$. By condition **(ii)**, it follows $\mathcal{S} \cup \mathcal{B} \models D \sqsubseteq \exists R$. The interesting case to consider is $\mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{T}\cup\mathcal{B},D(o))}(w_{[R]}) = \{\exists R^-\}$ (hence, $\mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{T}\cup\mathcal{B},D(o))}(o, w_{[R]}) = \{R\}$), as for $\mathcal{T}$ it is enough to take $y \in \Delta^{\mathsf{gen}(\mathcal{S}\cup\mathcal{B},D(o))}$ equal to $w_{[S]}$ such that $D \rightsquigarrow_{\mathcal{S}\cup\mathcal{B}} S$ and $\mathcal{S} \cup \mathcal{B} \models S \sqsubseteq R$ (such S exists: we take S equal to R if $D \rightsquigarrow_{\mathcal{S}\cup\mathcal{B}} R$). However, given the axiom $\exists R^- \sqsubseteq C'$ in $\mathcal{T}'$, we have $\mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{T}'\cup\mathcal{B},D(o))}(w_{[R]}) \supseteq \{\exists R^-, C'\}$ (note, still $\mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{T}'\cup\mathcal{B},D(o))}(o, w_{[R]}) = \{R\}$). As B' and C' satisfy **(I)** and $\mathcal{S} \cup \mathcal{B} \models D \sqsubseteq \exists R$, it follows there exists S such that $D \rightsquigarrow_{\mathcal{S}\cup\mathcal{B}} S$ and $\mathcal{S} \cup \mathcal{B} \models \{S \sqsubseteq R,\ \exists S^- \sqsubseteq C'\}$; moreover by $C' \neq \exists R^-$ and the structure of $\mathcal{S} \cup \mathcal{B}$ it follows S is over Σ. From the latter we obtain a role Q over Σ such that $\mathcal{S} \models S \sqsubseteq Q$ and $\mathcal{B} \models Q \sqsubseteq R$, moreover $\exists Q^-$ and Q are $\mathcal{S} \cup \mathcal{B}$-consistent. Now, assume $\mathcal{T} \models \exists R^- \sqsubseteq E'$; then $\mathcal{T} \cup \mathcal{B} \models \exists Q^- \sqsubseteq E'$, and since $\mathcal{T}$ satisfies condition **(ii)** it follows $\mathcal{S} \cup \mathcal{B} \models \exists Q^- \sqsubseteq E'$, therefore $E' \in \mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{S}\cup\mathcal{B},D(o))}(w_{[S]})$. Thus $\mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{T}'\cup\mathcal{B},D(o))}(w_{[R]}) \subseteq \mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{S}\cup\mathcal{B},D(o))}(w_{[S]})$, and we take $y = w_{[S]}$ to satisfy condition **(iv)** of Lemma 7.1.

($\Rightarrow$) Suppose inclusion $B \sqsubseteq C'$ is representable in $\mathcal{S}$ and $\mathcal{M}$ by a target axiom α. Then $\mathcal{T} = \{\alpha\}$ is a parsimonious UCQ-representation and $\mathcal{T} \cup \mathcal{B} \models B \sqsubseteq C'$. If $\mathcal{B} \models B \sqsubseteq C'$, we take B' equal to C': obviously, **(H)** and **(I)** are satisfied. Now, assume $\mathcal{B} \not\models B \sqsubseteq C'$. Then it must be the case α is of the form $D' \sqsubseteq C'$ and $\mathcal{B} \models B \sqsubseteq D'$ for some concept D' over Γ. So we take B' equal to D', and prove below **(H)** and **(I)** are satisfied.

For **(H)**, let $\mathcal{S} \cup \mathcal{B} \models D \sqsubseteq B'$ for a $\mathcal{S} \cup \mathcal{B}$-consistent concept D over Σ. It follows $\mathcal{S} \models D \sqsubseteq B_1$ and $\mathcal{B} \models B_1 \sqsubseteq B'$ for some concept B_1 over Σ. Consequently, $\mathcal{T} \cup \mathcal{B} \models B_1 \sqsubseteq C'$, and as $\mathcal{T}$ is a parsimonious UCQ-representation, we obtain that $\mathcal{S} \cup \mathcal{B} \models B_1 \sqsubseteq C'$. Finally, we proved that $\mathcal{S} \cup \mathcal{B} \models D \sqsubseteq C'$.

For **(I)**, assume B' is of the form $\exists Q'^-$ for some role Q' over Γ, and $\mathcal{S} \cup \mathcal{B} \models D \sqsubseteq \exists Q'$. As above, there exists B_1 over Σ such that $\mathcal{B} \models B_1 \sqsubseteq \exists Q'$. Then, $B_1 \rightsquigarrow_{\mathcal{T}\cup\mathcal{B}} S'$ for some role S' (possibly coinciding with Q') such that $\mathcal{T} \cup \mathcal{B} \models S' \sqsubseteq Q'$. By condition **(iv)** of Lemma 7.1 and $Q' \in \mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{T}\cup\mathcal{B},B_1(o))}(o, w_{[S']})$, there exists a role S such that $\mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{T}\cup\mathcal{B},B_1(o))}(w_{[S']}) \subseteq \mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{S}\cup\mathcal{B},B_1(o))}(w_{[S]})$ and $\mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{T}\cup\mathcal{B},B_1(o))}(o, w_{[S']}) \subseteq \mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{S}\cup\mathcal{B},B_1(o))}(o, w_{[S]})$. It implies, $B_1 \rightsquigarrow_{\mathcal{S}\cup\mathcal{B}} S$. Further, since $\mathcal{S} \cup \mathcal{B} \models D \sqsubseteq B_1$, we have that $D \rightsquigarrow_{\mathcal{S}\cup\mathcal{B}} Q$ for some role Q (possibly coinciding with S) such that $\mathcal{T} \cup \mathcal{B} \models Q \sqsubseteq S$. It is straightforward to verify that $\mathcal{S} \cup \mathcal{B} \models \{Q \sqsubseteq Q',\ \exists Q^- \sqsubseteq C'\}$. □

Proposition D.2. *For a role R over Σ and Q' over Γ, inclusion $R \sqsubseteq Q'$ is* representable *in $\mathcal{S}$ and $\mathcal{M}$ if and only if there exists R' over Γ such that $\mathcal{B} \models R \sqsubseteq R'$, and*

- ***(J)*** *for each $\mathcal{S}$-consistent role S over Σ, $\mathcal{S} \cup \mathcal{B} \models S \sqsubseteq R'$ implies $\mathcal{S} \cup \mathcal{B} \models S \sqsubseteq Q'$;*
- ***(K)*** *B', C' satisfy conditions **(H)** and **(I)** for $B' = \exists R'$, $C' = \exists Q'$, and $B' = \exists R'^-$, $C' = \exists Q'^-$.*

Then, $R \sqsubseteq Q'$ is representable by *$R' \sqsubseteq Q'$.*

Proof. ($\Leftarrow$) Let R be a role over Σ and Q' over Γ, $R' \neq Q'$, and conditions **(J)** and **(K)** are satisfied. We show inclusion $R \sqsubseteq Q'$ is representable in $\mathcal{S}$ and $\mathcal{M}$ by $R' \sqsubseteq Q'$. Similarly, to the proof of Proposition D.1, take $\mathcal{T}$ a parsimonious UCQ-representation for $\mathcal{S}$ under $\mathcal{M}$: we prove $\mathcal{T}' = \mathcal{T} \cup \{R' \sqsubseteq Q'\}$ is a parsimonious UCQ-representation by showing the direction of condition **(i)** stating that for each $\mathcal{S}$-consistent and $\mathcal{T}' \cup \mathcal{B}$-inconsistent pair of concepts or roles (X, Y), (X, Y) is $\mathcal{S} \cup \mathcal{B}$-inconsistent, the $\Leftarrow$ direction of condition **(ii)**, and condition **(iv)** of Lemma 7.1 are satisfied.

Satisfaction of conditions **(ii)** and **(i)** of Lemma 7.1 can be shown by analogy with the corresponding proofs in Proposition D.1. Note, here for concept inclusions/disjointness axioms we use the fact that $\exists R', \exists Q'$ and $\exists R'^-, \exists Q'^-$ satisfy **(H)**, and for role inclusions/disjointness axioms we use the fact R', Q' satisfy **(J)**.

For condition **(iv)**, the interesting case to consider is $D \rightsquigarrow_{\mathcal{T}\cup\mathcal{B}} R'$, with D an $\mathcal{S} \cup \mathcal{B}$-consistent concept over Σ, $\mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{T}\cup\mathcal{B},D(o))}(w_{[R]}) = \{\exists R'^-\}$ and $\mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{T}\cup\mathcal{B},D(o))}(o, w_{[R]}) = \{R'\}$. Now, given $R' \sqsubseteq Q' \in \mathcal{T}'$, $\mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{T}'\cup\mathcal{B},D(o))}(w_{[R]}) \supseteq \{\exists R'^-, \exists Q'^-\}$ and $\mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{T}'\cup\mathcal{B},D(o))}(o, w_{[R]}) \supseteq \{R', Q'\}$. By condition **(ii)**, it follows $\mathcal{S} \cup \mathcal{B} \models D \sqsubseteq \exists R'$. As $\exists R'^-$ and $\exists Q'^-$ satisfy **(I)** and $\mathcal{S} \cup \mathcal{B} \models D \sqsubseteq \exists R'$, it follows there exists S such that $D \rightsquigarrow_{\mathcal{S}\cup\mathcal{B}} S$ and $\mathcal{S} \cup \mathcal{B} \models \{S \sqsubseteq R',\ \exists S^- \sqsubseteq \exists Q'^-\}$; moreover by $\exists Q'^- \neq \exists R'^-$ and the structure of $\mathcal{S} \cup \mathcal{B}$ it follows S is over Σ. From the latter we obtain a role Q over Σ such that $\mathcal{S} \models S \sqsubseteq Q$ and $\mathcal{B} \models Q \sqsubseteq R$, moreover $\exists Q^-$ and Q are $\mathcal{S} \cup \mathcal{B}$-consistent. Now, assume $\mathcal{T} \models \exists R'^- \sqsubseteq E'$; then $\mathcal{T} \cup \mathcal{B} \models \exists Q^- \sqsubseteq E'$, and since $\mathcal{T}$ satisfies condition **(ii)** it follows $\mathcal{S} \cup \mathcal{B} \models \exists Q^- \sqsubseteq E'$, therefore $E' \in \mathbf{t}_\Gamma^{\mathsf{gen}(\mathcal{S}\cup\mathcal{B},D(o))}(w_{[S]})$. Similarly, for T' such that $\mathcal{T} \models R' \sqsubseteq T'$, we can show $T' \in \mathbf{r}_\Gamma^{\mathsf{gen}(\mathcal{S}\cup\mathcal{B},D(o))}(o, w_{[S]})$. Thus, we take $y = w_{[S]}$ to satisfy condition **(iv)** of Lemma 7.1.

($\Rightarrow$) Suppose inclusion $R \sqsubseteq Q'$ is representable in $\mathcal{S}$ and $\mathcal{M}$ by a target axiom α. Then $\mathcal{T} = \{\alpha\}$ is a parsimonious UCQ-representation and $\mathcal{T} \cup \mathcal{B} \models R \sqsubseteq Q'$. If $\mathcal{B} \models R \sqsubseteq Q'$, we take R' equal to Q': obviously, **(J)** and **(K)** are satisfied. Now, assume $\mathcal{B} \not\models R \sqsubseteq Q'$. Then it must be the case α is of the form $S' \sqsubseteq Q'$ and $\mathcal{B} \models R \sqsubseteq S'$ for some role S' over Γ. So we take R' equal to S', then **(J)** is shown similarly to **(H)** in the proof of Proposition D.1, and satisfaction of **(K)** is shown exactly as in the proof of $\Rightarrow$ of Proposition D.1 for $B' = \exists R'$, $C' = \exists Q'$, and $B' = \exists R'^-$, $C' = \exists Q'^-$. □

Proposition D.3. *For roles R_1, R_2 over Σ, (R_1, R_2) is target contradictable in $\mathcal{S}$ and $\mathcal{M}$ iff **either** for $\{R, Q\} \subseteq \{R_1, R_2\}$ there exists R' over Γ such that*

- ***(L)*** *$\mathcal{B} \models R \sqsubseteq R'$, and either $Q \sqsubseteq \neg R' \in \mathcal{B}$, or there exists Q' over Γ s.t. $\mathcal{B} \models Q \sqsubseteq Q'$ and*
  - ***(a)*** *for each $\mathcal{S} \cup \mathcal{B}$-consistent pair of roles S_1, S_2 over Σ it is* not the case *$\mathcal{S} \cup \mathcal{B} \models \{S_1 \sqsubseteq R',\ S_2 \sqsubseteq Q'\}$;*
  - ***(b)*** *for each $\mathcal{S} \cup \mathcal{B}$-consistent concept D over Σ and each role S such that $D \rightsquigarrow_{\mathcal{S}\cup\mathcal{B}} S$, it is* neither the case *$\mathcal{S} \cup \mathcal{B} \models S \sqsubseteq R' \sqcap Q'$,* nor *$\mathcal{S} \cup \mathcal{B} \models S \sqsubseteq R'^- \sqcap Q'^-$,*
- ***(M)*** *or $\mathcal{B} \models R \sqsubseteq \neg R'$ and inclusion $Q \sqsubseteq R'$ is representable in $\mathcal{S}$ and $\mathcal{M}$;*

***or** for $\{B, C\} \subseteq \{\exists R_1, \exists R_2\}$ or $\{\exists R_1^-, \exists R_2^-\}$ there exists B' over Γ such that*

(N) *$\mathcal{B} \models B \sqsubseteq B'$, and either $C \sqsubseteq \neg B' \in \mathcal{B}$, or there exists C' over Γ s.t. $\mathcal{B} \models C \sqsubseteq C'$ and*

- ***(c)*** *for each $\mathcal{S} \cup \mathcal{B}$-consistent pair of concepts D_1, D_2 over Σ it is* not the case $\mathcal{S} \cup \mathcal{B} \models \{D_1 \sqsubseteq B',\ D_2 \sqsubseteq C'\}$;
- ***(d)*** *for each $\mathcal{S} \cup \mathcal{B}$-consistent concept D over Σ and each role S such that $D \rightsquigarrow_{\mathcal{S}\cup\mathcal{B}} S$ it is* not the case $\mathcal{S} \cup \mathcal{B} \models \exists S^- \sqsubseteq B' \sqcap C'$,

(O) *or $\mathcal{B} \models B \sqsubseteq \neg B'$ and inclusion $C \sqsubseteq B'$ is representable in $\mathcal{S}$ and $\mathcal{M}$;*

*Then (R_1, R_2) is target contradictable by either $R' \sqsubseteq R'$, or $Q' \sqsubseteq \neg R'$ in **(L)**, by axiom α, where $Q \sqsubseteq R'$ is representable by α in **(M)**, by either $B' \sqsubseteq B'$, or $C' \sqsubseteq \neg B'$ in **(N)**, and by axiom α, where $C \sqsubseteq B'$ is representable by α in **(O)**.*

Proof. ($\Leftarrow$) Let R_1, R_2 be roles over Σ and one of the conditions **(L)**, **(M)**, **(N)**, or **(O)** is satisfied. We show (R_1, R_2) is target contradictable by α given by each of the conditions. Take $\mathcal{T}$ a parsimonious UCQ-representation for $\mathcal{S}$ under $\mathcal{M}$: we prove $\mathcal{T}' = \mathcal{T} \cup \{\alpha\}$ is a parsimonious UCQ-representation, by showing conditions **(i)**, **(ii)**, and **(iv)** of Lemma 7.1 are satisfied (only the required directions, see the proof of Proposition D.1). That (R_1, R_2) is $\mathcal{T}' \cup \mathcal{B}$-inconsistent, follows immediately from the shape of α and $\mathcal{B}$ in each of the cases. Observe that if α is given by one of the conditions **(M)** or **(O)**, then $\mathcal{T}'$ is a parsimonious UCQ-representation follows from the proof of Propositions D.1 and D.2. As for α given by conditions **(L)** or **(N)**, it should be clear that conditions **(ii)** and **(iv)** of Lemma 7.1 are satisfied, as disjointness axioms do not affect entailments of the concept and role inclusions. Therefore, below we show $\mathcal{T}'$ satisfies condition **(i)**.

Assume condition **(L)** is satisfied, and $\alpha = Q' \sqsubseteq \neg R'$ (the case $\alpha = R' \sqsubseteq R'$ is trivial), hence $\mathcal{B} \not\models Q \sqsubseteq \neg R'$. Let (D_1, D_2) be a pair of $\mathcal{S}$-consistent, $\mathcal{T} \cup \mathcal{B}$-consistent and $\mathcal{T}' \cup \mathcal{B}$-inconsistent concepts. The case both D_i is $\mathcal{T}' \cup \mathcal{B}$-consistent is not possible due to the shape of α. Then some D_i is $\mathcal{T}' \cup \mathcal{B}$-inconsistent, and by **(F)** it follows there exist $n \geq 1$ and distinct roles $S'_1, \ldots, S'_n$ such that $D_i \rightsquigarrow_{\mathcal{T}\cup\mathcal{B}} S'_1$, $\exists S'^-_j \rightsquigarrow_{\mathcal{T}\cup\mathcal{B}} S'_{j+1}$ and $\mathcal{T} \cup \mathcal{B} \models S'_n \sqsubseteq R' \sqcap Q'$ or $\mathcal{T} \cup \mathcal{B} \models S'_n \sqsubseteq R'^- \sqcap Q'^-$. In the following, we consider only $\mathcal{T} \cup \mathcal{B} \models S'_n \sqsubseteq R' \sqcap Q'$.

For the sake of contradiction, assume D_i is $\mathcal{S} \cup \mathcal{B}$-consistent. If $n = 1$ and S'_1 is a role over Σ (i.e., $D_i = \exists S'_1$ and S'_1 is $\mathcal{S} \cup \mathcal{B}$-consistent), then we obtain contradiction with **(a)** rised from the assumption D_i is $\mathcal{S} \cup \mathcal{B}$-consistent. If $n = 1$ and S'_1 is a role over Γ, then since $\mathcal{T}$ is a parsimonious UCQ-representation and $D_i \rightsquigarrow_{\mathcal{T}\cup\mathcal{B}} S'_1$, by condition **(iv)**, we obtain a role S_1 such that $D_i \rightsquigarrow_{\mathcal{S}\cup\mathcal{B}} S_1$, and $\mathcal{S} \cup \mathcal{B} \models S_1 \sqsubseteq R' \sqcap Q'$: contradiction with **(b)**.

For $n > 1$, inductively using condition **(iv)**, we obtain roles $S_1, \ldots, S_{n-1}$ over Σ and S_n s.t. $D_i \rightsquigarrow_{\mathcal{S}\cup\mathcal{B}} S_1$, $\exists S^-_j \rightsquigarrow_{\mathcal{S}\cup\mathcal{B}} S_{j+1}$, and $\mathcal{S} \cup \mathcal{B} \models S_n \sqsubseteq R' \sqcap Q'$. Then **(b)** implies that $\exists S^-_{n-1}$ is $\mathcal{S} \cup \mathcal{B}$-inconsistent, which contradicts the assumption D_i is $\mathcal{S} \cup \mathcal{B}$-consistent. Finally, we conclude that D_i is $\mathcal{S} \cup \mathcal{B}$-inconsistent, hence (D_1, D_2) is $\mathcal{S} \cup \mathcal{B}$-inconsistent.

Let (S_1, S_2) be a pair of $\mathcal{S}$-consistent, $\mathcal{T}\cup\mathcal{B}$-consistent and $\mathcal{T}'\cup\mathcal{B}$-inconsistent roles. For the sake of contradiction, assume (S_1, S_2) is $\mathcal{S} \cup \mathcal{B}$-consistent (and each of S_i is $\mathcal{S} \cup \mathcal{B}$-consistent).

Suppose both S_i is $\mathcal{T}' \cup \mathcal{B}$-consistent. From the shape of α, without loss of generality, we may assume that $\mathcal{T}' \cup \mathcal{B} \models \{S_1 \sqsubseteq R',\ S_2 \sqsubseteq Q'\}$. From condition **(ii)**, we obtain $\mathcal{S} \cup \mathcal{B} \models \{S_1 \sqsubseteq R',\ S_2 \sqsubseteq Q'\}$, which contradicts **(a)**.

Suppose one of S_i is $\mathcal{T}' \cup \mathcal{B}$-inconsistent. Then by **(G)** either $\mathcal{T} \cup \mathcal{B} \models S_i \sqsubseteq R' \sqcap Q'$ or $\mathcal{T} \cup \mathcal{B} \models S_i \sqsubseteq R'^- \sqcap Q'^-$, or D is $\mathcal{T}' \cup \mathcal{B}$-inconsistent for $D = \exists S_i$ or $D = \exists S^-_i$. In the latter case, we obtain contradiction as in the case (D_1, D_2) is $\mathcal{T}' \cup \mathcal{B}$-inconsistent. In the former case, from condition **(ii)**, it follows $\mathcal{S}\cup\mathcal{B} \models S_i \sqsubseteq R' \sqcap Q'$ or $\mathcal{S}\cup\mathcal{B} \models S_i \sqsubseteq R'^- \sqcap Q'^-$, which contradicts **(a)**. Finally, we conclude (S_1, S_2) is $\mathcal{S} \cup \mathcal{B}$-inconsistent.

Assume condition **(N)** is satisfied, and $\alpha = C' \sqsubseteq \neg B'$ (the case $\alpha = B' \sqsubseteq B'$ is trivial), hence $\mathcal{B} \not\models C \sqsubseteq \neg B'$. Let (D_1, D_2) be a pair of $\mathcal{S}$-consistent, $\mathcal{T} \cup \mathcal{B}$-consistent and $\mathcal{T}' \cup \mathcal{B}$-inconsistent concepts. For the sake of contradiction, assume (D_1, D_2) is $\mathcal{S} \cup \mathcal{B}$-consistent (and each of D_i is $\mathcal{S} \cup \mathcal{B}$-consistent).

Suppose both D_i is $\mathcal{T}' \cup \mathcal{B}$-consistent. From the shape of α, without loss of generality, we may assume that $\mathcal{T} \cup \mathcal{B} \models \{D_1 \sqsubseteq B',\ D_2 \sqsubseteq C'\}$. From condition **(ii)**, it follows $\mathcal{S} \cup \mathcal{B} \models \{D_1 \sqsubseteq B',\ D_2 \sqsubseteq C'\}$: contradiction with **(c)**.

Suppose one of D_i is $\mathcal{T}' \cup \mathcal{B}$-inconsistent. By **(F)**, consider $\mathcal{T} \cup \mathcal{B} \models D_i \sqsubseteq B' \sqcap C'$. From condition **(ii)**, it follows $\mathcal{S} \cup \mathcal{B} \models D_i \sqsubseteq B' \sqcap C'$: contradiction with **(c)**. Now, consider the case there exist $n \geq 1$ and distinct roles $S'_1, \ldots, S'_n$ such that $D_i \rightsquigarrow_{\mathcal{T}\cup\mathcal{B}} S'_1$, $\exists S'^-_j \rightsquigarrow_{\mathcal{T}\cup\mathcal{B}} S'_{j+1}$ and $\mathcal{T} \cup \mathcal{B} \models \exists S'^-_n \sqsubseteq B' \sqcap C'$. Inductively using condition **(iv)**, we obtain roles $S_1, \ldots, S_{n-1}$ over Σ and S_n s.t. $D_i \rightsquigarrow_{\mathcal{S}\cup\mathcal{B}} S_1$, $\exists S^-_j \rightsquigarrow_{\mathcal{S}\cup\mathcal{B}} S_{j+1}$, and $\mathcal{S} \cup \mathcal{B} \models \exists S^-_n \sqsubseteq B' \sqcap C'$. Then **(d)** implies that $\exists S^-_{n-1}$ (or D_i if $n = 1$) is $\mathcal{S} \cup \mathcal{B}$-inconsistent, which contradicts the assumption D_i is $\mathcal{S} \cup \mathcal{B}$-consistent. Finally, we conclude that D_i is $\mathcal{S} \cup \mathcal{B}$-inconsistent, hence (D_1, D_2) is $\mathcal{S} \cup \mathcal{B}$-inconsistent.

Let (S_1, S_2) be a pair of $\mathcal{S}$-consistent, $\mathcal{T} \cup \mathcal{B}$-consistent and $\mathcal{T}' \cup \mathcal{B}$-inconsistent roles. From the shape of α, it follows D is $\mathcal{T}' \cup \mathcal{B}$-inconsistent, for $D = \exists S_i$ or $D = \exists S^-_i$ and $i \in \{1, 2\}$. It can be shown D is $\mathcal{S} \cup \mathcal{B}$-inconsistent as above.

($\Rightarrow$) Suppose pair (R_1, R_2) is target contradictable in $\mathcal{S}$ and $\mathcal{M}$ by a target axiom α. If (R_1, R_2) is $\mathcal{B}$-inconsistent, then there exist $R, Q \in \{R_1, R_2\}$ and R' over Γ such that $\mathcal{B} \models \{R \sqsubseteq R',\ Q \sqsubseteq \neg R'\}$ (hence, **(L)** is satisfied), or there exist B, C in $\{\exists R_1, \exists R_2\}$ or in $\{\exists R_1, \exists R_2\}$ and B' over Γ such that $\mathcal{B} \models \{B \sqsubseteq B',\ C \sqsubseteq \neg B'\}$ (hence, **(N)** is satisfied).

Assume (R_1, R_2) is $\mathcal{B}$-consistent. Then α is a non-trivial axiom, $\mathcal{T} = \{\alpha\}$ is a parsimonious UCQ-representation, and (R_1, R_2) is $\mathcal{T} \cup \mathcal{B}$-inconsistent.

Suppose α is a role disjointness axiom $S_1 \sqsubseteq \neg S_2$. Then it follows there exist $R, Q \in \{R_1, R_2\}$ and $S, T \in \{S_1, S_2\}$ such that $\mathcal{B} \models \{R \sqsubseteq S,\ Q \sqsubseteq T\}$. So we set R' equal to S and Q' equal to T. We prove **(a)** and **(b)** are satisfied. For **(a)**, assume an $\mathcal{S} \cup \mathcal{B}$-consistent pair of roles S_1, S_2 over Σ such that $\mathcal{S} \cup \mathcal{B} \models \{S_1 \sqsubseteq R',\ S_2 \sqsubseteq Q'\}$. It follows there exist S_{11}, S_{22} over Σ such that $\mathcal{S} \models \{S_1 \sqsubseteq S_{11},\ S_2 \sqsubseteq S_{22}\}$ and $\mathcal{B} \models \{S_{11} \sqsubseteq R',\ S_{22} \sqsubseteq Q'\}$. Next, (S_{11}, S_{22}) is $\mathcal{T} \cup \mathcal{B}$-inconsistent, and since $\mathcal{T}$ is a parsimonious UCQ-representation, it follows (S_{11}, S_{22}) is $\mathcal{S} \cup \mathcal{B}$-inconsistent, which contradicts (S_1, S_2) is $\mathcal{S} \cup \mathcal{B}$-consistent. Hence, it cannot be the case $\mathcal{S} \cup \mathcal{B} \models \{S_1 \sqsubseteq R',\ S_2 \sqsubseteq Q'\}$. For **(b)**, assume an $\mathcal{S} \cup \mathcal{B}$-consistent concept D over Σ such that $D \rightsquigarrow_{\mathcal{S} \cup \mathcal{B}} S$ and $\mathcal{S} \cup \mathcal{B} \models S \sqsubseteq R' \sqcap Q'$. If S is over Σ, then as above, we obtain a contradiction with D being $\mathcal{S} \cup \mathcal{B}$-consistent. If S is over Γ, it follows $S = R' = Q'$, and there exists a concept D_1 over Σ such that $\mathcal{S} \models D \sqsubseteq D_1$ and $\mathcal{B} \models D_1 \sqsubseteq \exists S$. As above, (D_1, D_1) is $\mathcal{T} \cup \mathcal{B}$-inconsistent, and since $\mathcal{T}$ is a parsimonious UCQ-representation, it follows (D_1, D_1) is $\mathcal{S} \cup \mathcal{B}$-inconsistent, which contradicts D is $\mathcal{S} \cup \mathcal{B}$-consistent. Hence, it cannot be the case $\mathcal{S} \cup \mathcal{B} \models S \sqsubseteq R' \sqcap Q'$. In a similar way we obtain a contradiction if assume $\mathcal{S} \cup \mathcal{B} \models S \sqsubseteq R'^- \sqcap Q'^-$. Thus, **(L)** is satisfied.

Suppose α is a role inclusion assertion $S_1 \sqsubseteq S_2$. Then it follows there exist $R, Q \in \{R_1, R_2\}$ such that $\mathcal{B} \models \{R \sqsubseteq \neg S_2,\ Q \sqsubseteq S_1\}$. So we set R' equal to S_2, the proof $Q \sqsubseteq R'$ is representable by $S_1 \sqsubseteq R'$ is similar to the proof of $\Rightarrow$ of Proposition D.2. Thus, **(M)** is satisfied.

Suppose α is a concept disjointness axiom $D_1 \sqsubseteq \neg D_2$. Then it follows there exist B, C in $\{\exists R_1, \exists R_2\}$ or $\{\exists R^-_1, \exists R^-_2\}$ and $D, E \in \{D_1, D_2\}$ such that $\mathcal{B} \models \{B \sqsubseteq D,\ C \sqsubseteq E\}$. So we set B' equal to D and C' equal to E. We can prove **(c)** and **(d)** are satisfied by analogy with the proof of **(a)** and **(b)**. Thus, **(N)** is satisfied.

Suppose α is a concept inclusion assertion $D_1 \sqsubseteq D_2$. Then it follows there exist B, C in $\{\exists R_1, \exists R_2\}$ or $\{\exists R^-_1, \exists R^-_2\}$ such that $\mathcal{B} \models \{B \sqsubseteq \neg D_2,\ C \sqsubseteq D_1\}$. So we set B' equal to D_2, the proof $C \sqsubseteq B'$ is representable by $D_1 \sqsubseteq B'$ is similar to the proof of $\Rightarrow$ of Proposition D.1. Thus, **(O)** is satisfied. □

Proposition D.4. *For concepts B_1, B_2 over Σ, (B_1, B_2) is* target contradictable *in $\mathcal{S}$ and $\mathcal{M}$ if **either** for $\{B, C\} \subseteq \{B_1, B_2\}$ there exists B' over Γ such that*

- ***(P)*** *$\mathcal{B} \models B \sqsubseteq B'$, and either $C \sqsubseteq \neg B' \in \mathcal{B}$, or there exists C' over Γ s.t. $\mathcal{B} \models C \sqsubseteq C'$ and*
  - ***(c)*** *for each $\mathcal{S} \cup \mathcal{B}$-consistent pair of concepts D_1, D_2 over Σ it is* not the case *$\mathcal{S} \cup \mathcal{B} \models \{D_1 \sqsubseteq B',\ D_2 \sqsubseteq C'\}$;*
  - ***(d)*** *for each $\mathcal{S} \cup \mathcal{B}$-consistent concept D over Σ and each role S such that $D \rightsquigarrow_{\mathcal{S} \cup \mathcal{B}} S$ it is* not the case *$\mathcal{S} \cup \mathcal{B} \models \exists S^- \sqsubseteq B' \sqcap C'$,*
- ***(Q)*** *or $\mathcal{B} \models B \sqsubseteq \neg B'$ and inclusion $C \sqsubseteq B'$ is representable in $\mathcal{S}$ and $\mathcal{M}$;*

***or** $B_1 = \exists R$ or $B_2 = \exists R$ for a role R, and*

- ***(R)*** *(R, R) is* target contradictable *in $\mathcal{S}$ and $\mathcal{M}$.*

*Then (B_1, B_2) is target contradictable by either $B' \sqsubseteq B'$ or $C' \sqsubseteq \neg B'$ in **(P)**, by axiom α, where $C \sqsubseteq B'$ is representable by α in **(Q)**, and by axiom α such that (R, R) is target contradictable by α in **(R)**.*

Proof. The proof is similar to the proof of Proposition D.3. □