Birkbeck Working Papers in Economics & Finance

ISSN 1745-8587

School of Economics, Mathematics and Statistics

BWPEF 0716

# A Dynamic Mechanism and Surplus Extraction Under Ambiguity

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August 2007

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**Abstract**

In the standard independent private values (IPV) model, each bidder's beliefs about the values of any other bidder is represented by a unique prior. In this paper we relax this assumption and study the question of auction design in an IPV setting characterized by ambiguity: bidders have an imprecise knowledge of the distribution of values of others, and are faced with a set of priors. We also assume that their preferences exhibit ambiguity aversion. We show that a simple variation of a discrete Dutch auction can extract almost all surplus. This contrasts with optimal auctions under IPV without ambiguity as well as with optimal static auctions with ambiguity - in all of these, types other than the lowest participating type obtain a positive surplus. And, unlike the well-known Cremer-McLean mechanism, our modified Dutch mechanism satisfies limited liability. An important point of departure is that the modified Dutch mechanism we consider is dynamic rather than static, establishing that under ambiguity aversion–even when the setting is IPV in all other respects–a dynamic mechanism could have additional bite over its static counterparts.



KEYWORDS: Ambiguity Aversion, Modified Dutch Auction, Surplus Extraction

JEL CLASSIFICATION: D44



[1]We thank Sujoy Mukerji for very useful comments.

# 1 Introduction

In the standard independent private values (IPV) setting bidders draw privately known valuations from a given distribution. Each bidder is assumed to maximize subjective expected utility, so that each bidder's beliefs about the values of any other bidder is represented by a unique prior (i.e. a unique distribution over the domain of values). In this setting Dutch auctions coincide with First Price Sealed Bid auctions,[2] and optimal auctions leave all but the lowest participating type with a surplus. This is true whether bidders are risk neutral or risk averse.[3]

In this paper we relax the unique prior assumption and study the question of auction design in an IPV setting characterized by ambiguity: bidders have an imprecise knowledge of the distribution of values of others, and are faced with a set of priors. We also assume that their preferences exhibit ambiguity aversion.

Previous work by Bose, Ozdenoren, and Pape (2006) has shown that in the setting of ambiguity that we consider, the optimal *static* mechanism also leaves buyer types with information rent (the amount of rent varies with the extent of the ambiguity). In contrast, our main result shows that in this setting of ambiguity averse buyers, the seller can use a simple variation of a discrete Dutch auction and extract almost all surplus. The important point of departure is that the modified Dutch mechanism we consider is *dynamic* rather than static, establishing that a dynamic mechanism can present the seller with additional surplus extraction opportunities under ambiguity aversion even in a setting that is captured by the IPV model in all other respects.

In a seminal paper, Ellsberg (1961) showed that lack of knowledge about the distribution over states, often referred to as ambiguity, can affect the choice of a decision maker in a fundamental way that cannot be captured by a framework that assumes a unique prior. Several subsequent studies have underlined the importance of ambiguity aversion in understanding decision making behavior,[4] and models taking such aversion

[2] As far as we are aware, Karni (1988) is the first to show that the equivalence between Dutch and First Price Sealed Bid auctions breaks down under non-expected utility preferences.

[3] Myerson (1981) and Riley and Samuelson (1981) analyze optimal auctions with risk neutral bidders. Under risk neutrality, all standard auctions are optimal given an appropriate choice of reserve price. Matthews (1983) and Maskin and Riley (1984) characterize the optimal auction with risk averse bidders. The optimal auction in this case is quite complex, involving payments by some losing bidders.

[4] See, for example, Camerer and Weber (1992).

into account have provided important insights in a variety of economic applications including auctions.[5]

We model ambiguity aversion using the maxmin expected utility (MMEU) model of Gilboa and Schmeidler (1989). The MMEU model is a generalization of the subjective expected utility model, and provides a natural and tractable framework to study ambiguity aversion. In MMEU agents have a set of priors (instead of a single prior) on the underlying state space, and the payoff from any action is the minimum expected utility over the set of priors. In our setting, each buyer considers a set of distributions that contain the distribution from which the other buyer's valuation is drawn and each action (from the mechanism proposed by the seller) is evaluated based on the minimum expected utility over the set of distributions. The buyer then chooses the best action from the set of actions. To make the sharpest contrast with the standard model, we assume that the seller is ambiguity neutral and both the buyers and the seller are risk neutral. In other words, apart from relaxing the unique prior assumption, our framework is as close to the standard IPV model as possible.[6]

In this paper we use a version of MMEU known as "epsilon-contamination."[7] The model we consider has a seller whose valuation of the object is (normalized to) zero. There are two potential buyers and the seller does not know either buyer's valuation but believes that the valuations are determined based on independent draws from the distribution $F(v)$ having support $[0, 1]$. Each buyer knows his own valuation but has ambiguity regarding the valuation of the other buyer. We model this by using the epsilon-contamination model. Intuitively, for some $\varepsilon > 0$, the buyer puts $1 - \varepsilon$ weight

[5] For example, using such preferences Mukerji (1998) explains the incompleteness of contracts and Mukerji and Tallon (2004) explain the puzzling absence of wage indexation. An application to auction theory is developed by Lo (1998), who shows that if bidders are ambiguity averse, the revenue equivalence theorem (which holds in the standard IPV setting) is violated - sealed bid first price auctions raise more revenue than sealed bid second price auctions.

[6] With multiple priors, the terms "independent" and "correlated" need to be used carefully. For the most part we avoid using these terms. The important point is that in the standard model, even with risk-neutrality, full surplus extraction is not possible when the beliefs do not depend on one's own valuation (i.e., in the independent case). Hence it is worth emphasizing that we consider the case where the sets of probability distributions are the same for every buyer and *do not depend* on a buyer's own valuations. As shown by Bose et al. (2006), the optimal *static* mechanism does not extract full surplus in this setting.

[7] An interesting paper by Kopylov (2007) shows that this specification can be obtained by adding an axiom requiring affinity for ambiguity hedging to those proposed by Gilboa and Schmeidler (1989) for axiomatizing the MMEU representation. See also Nishimura and Ozaki (2006).

that the other buyer's valuation is drawn from the distribution $F$, but puts $\varepsilon$ weight that the valuation could be drawn from some other distribution. Formally, letting $\mathcal{P}$ denote the set of all distributions on $[0,1]$ and $\mathcal{P}_B$ the set that represents the set from which a buyer thinks the other buyer's valuation is drawn, a distribution $G(v)$ is in $\mathcal{P}_B$ if

$$G(v) = (1-\varepsilon)F(v) + \varepsilon L(v)$$

for some distribution $L$ belonging to $\mathcal{P}$ and $\varepsilon \in (0,1]$. This specification–known in the literature as epsilon contamination–is in widespread use for its intuitive qualities and analytical tractability.[8] The parameter $\varepsilon$ captures the extent of the ambiguity; the buyer puts a weight $\varepsilon$ on the possibility that the distribution could be something other than $F(v)$.[9]

Let us now describe our Modified Dutch Mechanism (MDM). The seller declares a decreasing sequence of prices $\{p_1, .., p_n\}$ at the beginning. In stage $t$, provided the item has not been sold up to that point, the seller randomly approaches a buyer and offers the item at price $p_t$. This offer is secret in the sense that the other buyer is not made aware of this. If the approached buyer passes, the seller approaches the other buyer (also in secret) and offers the item at the same price $p_t$. If the second buyer refuses as well then the game goes to stage $t+1$. If both buyers refuse at stage $n$, the seller keeps the item.

Since the seller approaches the buyers in secret, they do not know whether they are in first or second place in that period. We assume that the seller approaches the buyer at random each period and the randomization is independent across periods. This, along with the fact that the buyers do not know their own place in the que, helps keep the mechanism symmetric.[10]

The crucial feature of the mechanism is that in each period if a buyer is approached he has the opportunity to buy the item with certainty, and therefore get an ex post surplus of $v - p_t$ *for sure*, or to wait, and face the outcome of some *lottery*. This feature

[8] The specification is used extensively in the literature on robust statistics, starting with (as far as we are aware) Huber (1973). Examples from the economics literature include Chen and Epstein (2002), Chu and Liu (2002), Mukerji (1998), Nishimura and Ozaki (2004).

[9] Epsilon contamination is used for all the results below. However, $F$ being focal is *inessential*; any other distribution in place of $F$ to generate the set $\mathcal{P}_B$ would suffice just as well. We use the same $F$ to represent the seller's beliefs as well as to generate $\mathcal{P}_B$ to save on notation.

[10] Note that the seller commits to a price sequence. Hence what is (random and) secret in each period is the order in which the seller approaches the buyers, and not the price that is offered.

is important in extracting surplus from buyers.

Our surplus extraction result states the following. Fix a preference parameter $\varepsilon > 0$. There is a $\delta^*(\varepsilon)$ such that for any given $\delta < \delta^*(\varepsilon)$ and any $\eta > 0$, the seller can construct an MDM (i.e., choose a price sequence $p_n$) such that the mass of buyer types who do not buy is at most $[0, \eta]$ (i.e. the reserve type is at most $\eta$), and the types who buy do so at a price such that their ex post surplus is at most $\delta$. Since both $\delta$ and $\eta$ can be arbitrarily small, the seller can therefore extract almost full surplus. The crucial element is the construction of the price sequence. In any period $t$, provided the item has not been sold already, a buyer type $v$ who has been offered the item can purchase and ensure a surplus $v - p_t$ Suppose $v - p_t > 0$. Consider the alternative strategy where the buyer plans to buy the item at price $p_{t+1}$ in the next period if it is still available. Because the buyer considers the worst conditional distribution while calculating the benefit and cost of purchasing versus waiting, epsilon contamination preference implies that the resulting equilibrium when the price sequence has been chosen appropriately makes every type buy at a price which is not more than $\delta$ below its true valuation. Put differently, even though gain from buying is only $\delta$, gain from waiting is made still smaller so that the buyer buys at the "right" price and obtains a surplus of at most $\delta$.

As noted earlier, in the standard unique prior IPV model, the optimal auction does not extract full surplus. Using a discrete type space, Crémer and McLean (1988) show that if types are correlated, under certain conditions full surplus can be extracted.[11] Following this result, McAfee and Reny (1992) show that when the type space is a continuum, full surplus cannot be extracted if beliefs are independent of valuations but when beliefs do depend on valuation, a lottery (random participation fee) can be used in conjunction to standard auctions to extract almost full surplus.[12],[13]

Our mechanism does not require any such extraneous lotteries, and considers valuations independent of beliefs, the case in which full surplus extraction is not possible in the standard subjective expected utility framework. Further, unlike the Cremer-

[11] Crémer and McLean (1985) provide conditions under a finite state space for a price discriminating monopolist to extract full surplus.

[12] The result requires a condition that ensures that the beliefs have enough variation (usually referred to as the "full rank" or "convex hull" conditions).

[13] See also McAfee et al. (1989) for surplus extraction results in the special case of common value auctions.

McLean mechanism, our mechanism–which is a variation of the Dutch auction–also satisfies limited liability.[14]

Several papers have studied auctions (or auction-like environments) when bidders have non-expected utility preferences (e.g. Karni and Safra 1986, 1989a,1989b; Karni 1988; Lo 1998; Nakajima 2004; Ozdenoren 2002; Volij 2002). The closest intellectual antecedents appear in the paper by Bose et al. (2006), who study optimal auction design for the same environment that we consider in this paper. They use the revelation principle and study essentially static mechanisms, and show that in the epsilon contamination case, the optimal static mechanism leaves types with information rents which approach those found in the unique prior case as $\varepsilon \to 0$. Thus the correspondence mapping $\varepsilon$ to the optimal (static) mechanism is upper semi continuous (though not lower semi continuous). Our results show that the upper semi continuity result depends crucially on the mechanism being a static mechanism. Our dynamic mechanism extracts almost all rents for arbitrarily small $\varepsilon$ - and is thus discontinuously different from the unique prior case.

Earlier work in the area of robust Bayesian statistics have studied dynamic inference problems facing a decision maker with maxmin preferences. The literature shows that the juxtaposition of maxmin preferences with full Bayesian updating can give rise to surprising results.[15] However, as far as we are aware, this paper is the first to study the question of dynamic mechanism design under such non-EU preferences.

The rest of the paper is organized as follows. The next section presents the model. Section 3 presents our mechanism, and characterizes equilibria in the induced game. The main (surplus extraction) result of the paper appears in section 4, and section 5 presents a numerical example. Section 6 discusses some aspects of the model, and section 7 concludes.

[14] Robert (1991) shows that the Cremer-McLean result relies crucially on risk neutrality as well as limited liability. While we do not explicitly consider risk-aversion, it is easy to show that our basic result is unchanged if bidders are risk averse.

[15] See, for example, Augustin (2003), Grunwald and Halpern (2004), Seidenfeld (2004).

## 2 The Model

There is a seller with one indivisible object for sale. The seller's valuation of the item is (normalized to) zero. There are two potential buyers with valuations of the object lying in the interval $[0, 1]$.[16], [17] Own valuation is private information of each buyer. Each buyer believes that the other's valuation is drawn from some distribution from a *set* of distributions on $[0, 1]$. The preferences of the buyers is represented by the maxmin expected utility (MMEU, henceforth) model of Gilboa and Schmeidler (1989). Briefly, if $\Omega$ is a set, $\mathcal{P}$ is a set of distributions on $\Omega$, and $\mathcal{F}$ is a set of acts from $\Omega$ to the real line $\mathbb{R}$, then an act $f \in \mathcal{F}$ is evaluated according to the rule

$$\min_{p \in \mathcal{P}} \int u(f) dp$$

where $u$ is some real valued function. In our context, we assume buyers are risk-neutral.

The seller is (risk and) ambiguity neutral and has a prior over a buyer's valuation given by the distribution $F(v)$ with a continuous density $f(v) > 0$. We model the set of priors representing buyer's ambiguous beliefs using the epsilon contamination model. Let $\mathcal{P}$ denote the set of all distributions on $[0, 1]$. The set of distributions, $\mathcal{P}_B$, representing the buyers' beliefs, is given by the following: $G(\cdot) \in \mathcal{P}_B$ if for any $v \in [0, 1]$, $G(v) \equiv (1 - \varepsilon)F(v) + \varepsilon L(v)$ for some $L(\cdot) \in \mathcal{P}$.[18] Note that other than non-unique priors, the rest of the model conforms as closely as possible to the IPV model standard in auction theory.

The Gilboa-Schmeidler model is atemporal. Since our mechanism is dynamic, we need to extend the basic model to suit the specific context of our dynamic mechanism. We discuss this in section 3.2 after specifying the mechanism.

[16] We could, for the sake of generality, represent the buyer's possible valuations to be the set $[\underline{v}, \overline{v}]$. However, we do allow the seller to have a non-trivial reserve price, and, as the result below shows, the normalization to the space $[0, 1]$ is harmless, and reduces algebraic clutter.

[17] Generalization to arbitrary $N > 2$ buyers is straightforward.

[18] Use of $F$ to generate both the seller's beliefs as well as the set $\mathcal{P}_B$ representing the buyers' beliefs is not essential. See footnote (9).

# 3 The Modified Dutch Mechanism

We now describe the Modified Dutch Mechanism (MDM). The mechanism works as follows. At the beginning, the seller declares a price sequence $\{p_1, p_2, ...p_n\}$ where $p_t$ is the asking price in period $t$. In period $t = 1$, the seller randomly chooses a buyer to approach first and offers the object at price $p_1$. If the buyer buys at that price the game is over; otherwise the seller approaches the other buyer and offers the same price $p_1$. Again, if the second buyer accepts, the game is over; otherwise we go to period 2. The game continues in this manner: provided the item remains unsold after period $t - 1$, in period $t$ the seller again randomly chooses a buyer to approach first and offers the item at price $p_t$. If this buyer refuses, the seller offers the item to the other buyer at (the same) price $p_t$. If the item is unsold at the end of period $n$, the seller keeps the item. Note that the seller randomly chooses the order of approaching the buyers each period and this randomization is independent across periods. Also, an important feature of the mechanism is that the buyers themselves do not know the outcome of the seller's randomization and therefore they do not know their own place in the que.[19] This keeps the mechanism symmetric.

The mechanism is a modification of a discrete price Dutch auction; in particular we assume–as is standard in dynamic auctions–that there is no discounting between periods. The seller's ex post payoff is the price at which the item is sold if it is sold and zero otherwise. The ex post payoff of a buyer of type $v$ is $v - p$ if it obtains the item at price $p$ and is zero otherwise.

We maintain the standard assumption of mechanism design literature that the seller, the mechanism designer in our context, can *commit* to the mechanism. In particular this means that the price sequence declared at the beginning of the game and the random procedure of approaching buyers every period is adhered to as the game progresses. Put differently, once a mechanism is chosen, only the two buyers - and not the seller - are the players in the game induced by the mechanism. We also make the standard assumption that all of the above is common knowledge.

[19] In any period when the item is offered to a buyer, he knows that this could be because he is the first in the que for that period, or it could be because the seller had first approached the other buyer who passed on the offer.

## 3.1 The Price Sequence

We now describe the price sequence associated with the MDM. For $\delta > 0$, let $\{p_0, p_1, ..., p_n\}$ be the price sequence where

$$\begin{aligned} p_0 &= 1 \text{ and} \\ p_k &= \frac{(1-\delta)^k}{(1-\delta+\varepsilon\delta)^{k-1}} \text{ for any } k > 0 \end{aligned} \tag{3.1}$$

We remind the reader that $\varepsilon$ is a preference parameter; the role of $\delta$ will become clear shortly. Note that $p_k$ is a decreasing sequence. Let $\Delta_k$ denote the "price gap" $p_k - p_{k+1}$, where

$$\begin{aligned} p_0 - p_1 \equiv \Delta_0 &= \delta, \text{ and} \\ p_k - p_{k+1} \equiv \Delta_k &= \frac{(1-\delta)^k \delta\varepsilon}{(1-\delta+\varepsilon\delta)^k} \text{ for any } k > 0 \end{aligned} \tag{3.2}$$

Note that $\Delta_k$ is also decreasing in $k$. It also follows directly that

$$\lim_{n\to\infty} \sum_{k=0}^{n} \Delta_k = 1$$

Since in the limit the prices cover the entire unit interval, we have the following property, which is important for later results:

**Property:** *Given any $\eta \in (0,1)$, there exists an integer $T$ such that $\sum_{k=1}^{T} \Delta_k \geqslant 1 - \eta$.*

Given any $\eta \in (0,1)$, let $T_*$ be the smallest integer for which the above inequality is satisfied. We set $n = T_*$, which defines the last offered price $p_n$.

### 3.2 Strategies and equilibria

As explained above, MDM results in a sequential (extensive form) game of incomplete information. A strategy of a type in this game is a plan to accept or reject the seller's offer at every information set (i.e. at every instance where the seller makes the offer) given the history of the game so far. An equilibrium is a pair of strategies, one for each buyer, satisfying the standard conditions: the pair is commonly known and each is a best response with respect to the other. Further restrictions on the structure of behavior of buyers are discussed below.

First, we make the standard assumption that the game itself is common knowledge. Each buyer faces ambiguity about the type of the other buyer, but, as in the standard models, knows how each type behaves in equilibrium.[20]

Second, the buyers have maxmin preferences throughout the game. In other words, at every stage, they behave by choosing actions to maximize the minimum expected payoff from a set of (updated) distributions. Note that both buyers start with the same set of priors and use the same updating rule every period.

Third, the dynamic behavior of the buyers is sophisticated. They form their decisions based on the entire game tree, and correctly anticipate their own behavior at future dates. Recently, Siniscalchi (2006) has provided an axiomatic foundation of such sophisticated dynamic choice for ambiguity-sensitive decision makers. We follow the same idea here and posit that the (conditional) preferences are defined over trees, rather than acts; we comment more on this in section 6 below.[21] The equilibrium strategy of a buyer is perfect in the sense that just like in the standard case, the same consistency requirement is imposed on the off-the-equilibrium-path information sets as well. A type's equilibrium decision in any period (i.e. to accept or to reject the

[20] Previous research has studied static mechanisms in exactly this context. Our objective is to have a dynamic mechanism while preserving other aspects of the framework. Note that we are ruling out strategic ambiguity: players do not doubt each other's rationality. Of course, the scenario where a player has ambiguous beliefs not only about the other player's *valuation* but also about what the other player *might do* is potentially interesting. However, since presence of ambiguity regarding the strategies or even rationality of others can only "worsen" the minimizing distribution, we conjecture that under maxmin preferences such additional ambiguity can only enhance the incentive to buy at earlier prices, which strengthens our results.

[21] Note that since we consider finite period games, the entire set of these conditional preferences can be recovered through backward induction.

seller's offer) is optimal not only with respect to the other buyer's strategy and the history of the game but also with respect to the knowledge of its own behavior at all future information sets, including those that will not occur if the type is to carry out its own equilibrium plan. We discuss this issue further in section 6 .

Finally, the **updating rule**. Several rules have been proposed in the literature when beliefs are sets of distributions. Perhaps the most prominent is the *full Bayesian* rule where all the original distributions are retained (except, of course, those under which it is impossible for the observed event to have occurred) and updated according to Bayes' rule. This is widely used in the extensive literature on robust statistics.[22] A second well known rule is the *generalized maximum likelihood* rule, axiomatized by Gilboa and Schmeidler (1993). Under this, a subset of distributions from the original set is retained and Bayes' rule is applied to these only. The retained distributions are the ones that assign maximum probability to the event that is known to have occurred. Our results apply under both rules. We clarify this in section 6 after presenting the results.

### 3.3 Characterizing Strategies

In this section, we discuss a particularly convenient way of representing strategies in the game induced by the MDM.

Recall that at each price $p_k$, $k \in \{1, \ldots, n\}$, a buyer, if asked by the seller, must choose one of two actions: accept or reject the seller's offer. A strategy of a type of buyer $i$ is therefore a plan to accept or reject the seller's offer at each price given the profile of actions up to price $p_{k-1}$. We assume that a buyer type accepts when indifferent between accepting and rejecting and buys at the earlier period if indifferent between buying in two different periods.[23]

[22] See, for example, Rios and Ruggeri (2000), Walley (1991). References from the economics literature include Epstein and Schneider (2003), Pires (2002).

[23] Note that this need not be an entirely innocuous assumption. The problem is as follows. Since at every node, a buyer's feasible action space is accept or reject, the strategies are pure. If we now introduce randomized strategies, in a non-EU setting , even if an agent is indifferent between two pure actions, he might "strictly" prefer a randomization over them to either pure action. If this were the case here, our assumption would require us to rule out randomizations. However, as we show in section 6, the assumption is in fact without loss of generality in our model - our results are unaffected if the action

An important feature of the strategies is that the decisions to buy by different types must have a certain monotonicity property. Specifically, suppose that $p_k$ is the highest price that a buyer of type $v$ accepts. This means that the payoff $v - p_k$ is better than the best (maxmin) expected payoff from either not accepting the seller's offer at all or accepting some future price. Since all types start with the same set of priors and use the same rule to update the set, any type $v' > v$ must then also optimally accept the offer $p_k$ rather than to continue. If $p_k$ is the first price at which type $v$ plans to accept, then the highest price that all higher types plan to accept must be at least as high as $p_k$.

Suppose $p_k$ is the highest price accepted by $v$. Then the highest price that types above $v$ accept is either $p_k$ or a higher price, and the highest price that types below $v$ accept is either $p_k$ or a lower price.

For each price $p_k$ there is a set of types (possibly empty) who buy at $p_k$. Note that monotonicity implies that if $p_k$ is the highest price accepted by types $v$ and $v'$, where $v > v'$, then the same is true of any type $v'' \in (v', v)$. Therefore such a strategy gives rise to a vector of $n$ cut-offs $\{v_1, \ldots, v_n\}$ where $1 \geqslant v_1 \geqslant v_2 \geqslant \ldots \geqslant v_n \geqslant 0$, and where types in the (possibly degenerate) interval $[v_1, 1]$ accept $p_1$, and the highest accepted price for those in the (again, possibly degenerate) interval $[v_k, v_{k-1})$ is $p_k$, $k \in \{2, \ldots, n\}$.

The arguments above establish that any strategy satisfying monotonicity must give rise to a vector of $n$ cut-offs as described above. Thus without loss of generality we can restrict attention to such strategies, and refer to these as "cut-off strategies." Note that any such cutoff-strategy currently places no restriction on the parts of the strategies which specify actions at prices below the highest acceptable price. For a strategy to be part of a perfect equilibrium, further restrictions are required and we clarify these once we establish the next result.

Next, we define an "interior cut-off strategy."

**Definition 1** ***Interior Cut-off Strategy:*** *A strategy of buyer $i$, $i \in \{1, 2\}$, is called an interior cut-off strategy if there exists a vector $\mathbf{v}^i = (v_1^i, \ldots, v_n^i)$, $0 \leqslant v_n^i < v_{n-1}^i < \ldots < v_1^i < 1$, such that for $k \geq 1$, the highest price accepted by the non-degenerate interval of types $[v_k^i, v_{k-1}^i)$ is $p_k$, where $v_0^i \equiv 1$.*

space is extended to allow for randomizations.

## 3.4 Characterizing Equilibria

In this section we discuss the properties of equilibria that results from the game induced by MDM. We show that when the price sequence $\{p_1, ..., p_n\}$ is chosen appropriately, any equilibrium has the property that for every price, there are sets of types of positive measure for both buyers who plan to buy at that price. (For the rest of the paper, the phrase "positive measure" is used with respect to the distribution $F$.) We also define perfect cut-off strategy, i.e., cut-off strategies that are part of a perfect equilibrium, and show existence of a symmetric equilibrium where both buyers follow the same cut-off strategy.

For the rest of the section, we fix the preference parameter $\varepsilon > 0$.

The first result calculates the difference between the payoffs from buying at the current price and waiting for the next lower price. This calculation is useful later when we show that exactly such a calculation features in deriving equilibrium cut-off vectors.

**Lemma 1** *Suppose the item has not been sold in periods* $1, \ldots, k-1$ *and in period* $k < n$ *the seller offers the item to buyer i at price* $p_k$ *(given by equation* (3.1)*). Suppose the strategy of j gives rise to a vector of cut-offs* $\mathbf{v}^j = (v_1^j, \ldots, v_n^j)$*. For any type v of i the difference in payoff from buying immediately versus waiting one period to buy at price* $p_{k+1}$ *is*

$$G_k^i(v) = v - p_k - (1-\varepsilon)(v - p_{k+1})H_k^i \tag{3.3}$$

*where*

$$H_k^i \equiv \frac{F(v_k^j) + F(v_{k+1}^j)}{F(v_k^j) + F(v_{k-1}^j)} \tag{3.4}$$

*where* $v_0^j \equiv 1$.

We give a concise proof below. The detailed derivation of the conditional probabilities used in the proof below is provided in section **A**.1 in the appendix.

**Proof:** If buyer $i$ accepts the price $p_k$, the payoff is $v - p_k$. If the buyer waits to buy in period $k+1$ and if he manages to obtain the item then the ex post payoff is $v - p_{k+1}$.

Let $H_k^i$ denote the probability under the distribution $F$ that $i$ wins the item at $p_{k+1}$ given that he refuses the current offer of $p_k$ Under epsilon contamination preference,

the buyer's expected payoff from waiting one period is therefore given by $(1-\varepsilon)(v - p_{k+1})H^i(k)$. Therefore $G^i_k(v)$ as specified. It remains to derive the expression for $H^i_k$. We do this in two steps.

First, under the distribution $F$ (i.e., if there were no ambiguity) the probability that $i$ wins the item at $p_{k+1}$ conditional on the item not being sold at $p_k$ is given by

$$\phi^i_k = \frac{1}{2} + \frac{1}{2}\frac{F(v^j_{k+1})}{F(v^j_k)}$$

Second, under distribution $F$, if $i$ passes at $p_k$, the probability that the item is left unsold at the end of the period $k$ is given by:

$$\pi^i_k = \frac{2F(v^j_k)}{F(v^j_{k-1}) + F(v^j_k)}$$

where $v^j_0 \equiv 1$. Clearly $H^i_k = \pi^i_k\, \phi^i_k = \dfrac{F(v^j_k) + F(v^j_{k+1})}{F(v^j_k) + F(v^j_{k-1})}$, where $v^j_0 \equiv 1$. This completes the proof.‖

A strategy of a buyer is said to have a "gap" at $p_k$ if there are no types of that buyer who buy at $p_k$. The next result shows that there are no such gaps in equilibrium strategies – a positive measure of types of both buyers buy at each price. This is crucial in characterizing all equilibria.

**Proposition 1** *There exists $\overline{\delta} > 0$ such that for all $\delta < \overline{\delta}$, in equilibrium, for each price $p_k$, $k \in \{1, \ldots, n\}$, there is a positive measure of types of each bidder who plan to buy at $p_k$.*

The formal proof is relegated to the appendix (section **A**.2). Here we provide an outline of the proof. Suppose the strategy followed by buyer $j$ has a gap , so that he does not plan to buy at some prices.

For example, suppose there are no types of $j$ who would accept offers of $p_{k-\ell}$ through $p_k$ (but that there are types of $j$ who plan to buy at $p_{k-\ell-1}$ and some who plan to buy at price $p_{k+1}$). Let $v^j_{k-\ell-1}$ be the lowest type of $j$ who buys at price $p_{k-\ell-1}$. By definition this type is indifferent between buying at $p_{k-\ell-1}$ and waiting till the price drops to $p_{k+1}$.

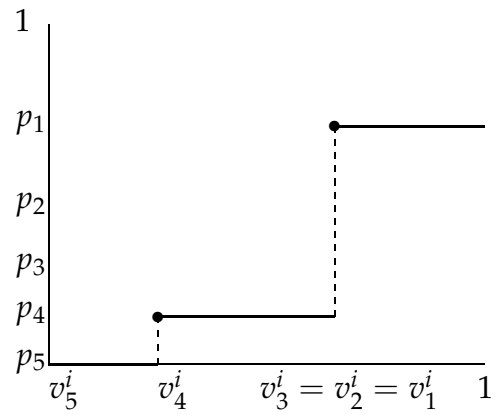


Figure 1: A cut-off strategy for buyer $i$ under $n = 5$ with gaps at $p_2$ and $p_3$ - there are no types of bidder $i$ who buys at $p_2$ or $p_3$. Our results rule out all gaps in equilibrium.

Now, according to the supposed equilibrium, all types in $(p_{k+1}, v^j_{k-\ell-1})$ refuse price offers $p_{k-\ell}$ through $p_k$. Note first that if $\ell$ is at least 1, and $j$ does not plan to buy at prices $p_{k-\ell}$ through $p_k$, the best response from $i$ should be to not buy at prices $p_{k-\ell}$ through $p_{k-1}$. (It is possible that some type of $i$ may want to buy at price $p_k$; however, the important point is that a gap from $j$ will give rise to a corresponding gap from $i$.)

It is useful to consider two separate cases. First, suppose that $\ell$ is some fixed number (so that informally, part of the description of the strategy of $j$ is of the form "does not buy at the next $\ell$ prices"). We show that as $\delta$ becomes small so that the gap between prices decrease, $H^j_{k-\ell-1}$, the conditional probability (under distribution $F$) that $j$ obtains the item in period $k+1$ if he passes in period $k-\ell-1$ is approximately the same as $H^j_k$, the conditional probability (again, under $F$) that $j$ obtains the item in period $k+1$ if he passes in period $k$.

But now consider type $v^j_{k-\ell-1}$. This type is indifferent between $p_{k-\ell-1}$ and $p_{k+1}$ and therefore the lowest type who buys at $p_{k-\ell-1}$. Therefore a type just below (but arbitrarily close to) $v^j_{k-\ell-1}$ is approximately indifferent between $p_{k-\ell-1}$ and $p_{k+1}$. But since $H^j_{k-\ell-1} \approx H^j_k$, and $p_k < p_{k-\ell-1}$, in period $k$ when the seller actually offers the price $p_k$, such a type must strictly prefer to buy at $p_k$ rather than wait till period $k+1$.

For the other case, consider now the situation where $\ell$ is not a fixed integer but varies as $\delta$ (and hence $n$) varies. Intuitively, this is where the strategy of $j$ is such that he does not plan to buy at some prices (no matter how many there are) as long as these

prices fall in some interval. Again, when $\delta$ is small, there is not much of a difference between $p_k$ and $p_{k+1}$, but since now there is some fixed finite gap between $p_{k-\ell-1}$ and $p_k$, if type $v^j_{k-\ell-1}$ is indifferent in period $k-\ell-1$ between $p_{k-\ell-1}$ and $p_{k+1}$, types just below $v^j_{k-\ell-1}$ must strictly prefer to buy at price $p_k$ when offered rather than wait till price drops to $p_{k+1}$, contradicting the supposed equilibrium behavior.[24]

It is essentially this argument that rules out any gaps in the strategies adopted by either player in equilibrium, proving the stated result. While this is the basic intuition, the formal proof has to carefully check several cases, (and go through several other steps to make the above informal argument rigorous) and is somewhat lengthy. We have relegated it to the appendix.

Recall that our definition of an interior cut-off strategy above did not impose any out-of-equilibrium restrictions. We now impose such restrictions and define a perfect cut-off strategy. Note that for a strategy to be part of an equilibrium which is perfect, it must specify behavior that is optimal at every information set given the (correct) assessment of the behavior of the other player as well as one's own behavior at every continuation information set. Specifically, if, say $p_k(v)$ is the highest acceptable price for type $v$, it must be better for $v$ to accept $p_k(v)$ than to reject and act optimally at every (off-equilibrium-path) future occasion if asked by the seller. The following result shows that for the appropriately chosen price sequence, such optimality simply implies that $v$ must accept all subsequent (off-equilibrium-path) offers by the seller as well.

**Lemma 2** *Let $p_k(v)$ be the highest acceptable price for type $v$ of buyer $i$, $i \in \{1,2\}$. For $\delta < \overline{\delta}$, optimal behavior at any subsequent information set requires that type $v$ also accepts all prices lower than $p_k(v)$.*

**Proof:** From Proposition 1 we know that for $\delta$ low enough, a positive measure of types of each buyer buy at each price in equilibrium. Thus for every price $p_k$, there is a set

[24] As the proof in the Appendix **A**.2 shows, the proof for second case is therefore easier than the first. In the first, since $p_{k-\ell-1} - p_k$ also becomes smaller as $\delta$ decreases, the sure gain $v^j_{k-\ell-1} - p_k$ is not too much greater than the sure gain of $v^j_{k-\ell-1} - p_{k-\ell-1}$ as $\delta$ decreases, so more work is needed to show that the respective ambiguities in the two periods (as reflected in the terms $(1-\varepsilon)H^j_{k-\ell-1}$ and $(1-\varepsilon)H^j_k$) change in such a way that the types slightly below $v^j_{k-\ell-1}$ prefer to accept the price $p_k$ than to wait till period $k+1$.

of types for whom $p_k$ is the highest price at which they buy. Therefore equilibrium strategies are interior cut-off strategies.

Next, suppose $p_k$ is the highest price at which type $v$ buys. Suppose also that $v$ does not buy at some price $p_{k+\ell}$, $\ell \geqslant 1$. From Proposition 1, there is some type $v'$ for whom $p_{k+\ell}$ is the highest acceptable price. The monotonicity property then immediately gives a contradiction. If $v' > v$, then the higher price $p_k$ cannot be acceptable to $v$. On the other hand, if $v' < v$, then since $v'$ finds it optimal to accept when offered $p_{k+\ell}$, the same must be true of the higher type $v$.

This shows that if $p_k$ is the highest price type $v$ accepts in equilibrium, then the (off-equilibrium-path) strategy of type $v$ is to accept every lower price as well.‖

Therefore, a **perfect cut-off strategy** is an interior cut-off strategy, with the additional requirement that if a type accepts any price, it must also accept all subsequent prices. From definition 1, the highest price accepted by a non-degenerate interval of types $(v^i_{k-1}, v^i_k]$ is $p_k$. It follows that for a perfect cut-off strategy, $v^i_k$ is the lowest type of $i$ who is indifferent between accepting $p_k$ or continuing for just one more period and accepting the next available price $p_{k+1}$.

We now use the results above to characterize perfect cut-off strategies. Since any equilibrium involves such strategies, this characterizes all equilibria.

**Proposition 2** *For $\delta < \overline{\delta}$, in any equilibrium the strategy of any bidder $i$ is a perfect cut-off strategy $\mathbf{v}^i = (v^i_1, \ldots, v^i_n)$ where $v_n = p_n$. Further, for $1 \leqslant k \leqslant (n-1)$, $v^i_k \in (p_k, v^i_{k-1})$, where $v_0 \equiv 1$, and $v^i_k$ is given by*

$$v^i_k = p_k + \Delta_k \frac{(1-\varepsilon)H^i_k}{1-(1-\varepsilon)H^i_k}$$

*where $H^i_k$ is given by equation* (3.4). *For any given $\mathbf{v}^j$, $v^i_k$ is unique.*

**Proof:** The fact that when $\delta$ is small, in any equilibrium bidders must use a perfect cut-off strategy follows directly from Lemma 2 above. From Lemma 3 (in Appendix **A**.2), we have $v^i_n = p_n$. From Lemma 1 we know that if the strategy of $j$ gives rise to the cut-off vector $\mathbf{v}^j = (v^j_1, \ldots, v^j_n)$, then for any type $v$ of $i$ the difference in payoff from buying immediately versus waiting one period to buy at price $p_{k+1}$ is given by $G^i_k(v)$.

Since the type $v_k^i$ is the lowest type that buys at $k$, it must be that $v_k^i$ is determined by solving $G_k^i(v) = 0$ for $v$.

Now, clearly, $G_k^i(p_k) < 0$. Therefore $v_k^i > p_k$. Since (as shown by Proposition 1) a positive measure of types of $i$ plan to buy at each price, we also have $v_k^i < v_{k-1}^i$. Thus it must be that $G_k^i(v_{k-1}) > 0$. Further, $G_k^i(v)$ is strictly increasing and continuous in $v$. Therefore if an equilibrium $(\mathbf{v}^i, \mathbf{v}^j)$ exists, for any given $\mathbf{v}^j$ there exists a unique $v_k^i \in (p_k, v_{k-1}^i)$ such that $G_k^i(v_k^i) = 0$.

Finally, $G_k^i(v_k^i) = 0$ implies (from equation (3.3))

$$\begin{aligned} v_k^i - p_k &= (1-\varepsilon)(v_k^i - p_{k+1})H_k^i \\ &= (1-\varepsilon)(v_k^i - p_k + \Delta_k)H_k^i \end{aligned}$$

Solving, we get the stated equation.‖

The result above characterizes all equilibria. Note that any equilibrium has the standard "skimming" property: a higher type buys earlier (at a higher price) than a lower type. Finally, we prove existence.

**Proposition 3** *There is $\underline{\delta} > 0$ such that for any $\delta < \underline{\delta}$, a symmetric equilibrium exists.*

The proof is essentially an application of Brouwer's fixed point theorem and has been relegated to the appendix.

# 4 The Main Result

We now present the main result of the paper which follows directly from the characterization results derived in the last section. For any preference parameter $\varepsilon > 0$, the seller can design a MDM to allocate the object (almost) efficiently and can extract (almost) all surplus. More specifically, for any given $\varepsilon > 0$, there is $\delta^*(\varepsilon)$ such that for any chosen $\delta \in (0, \delta^*(\varepsilon))$ and any $\eta > 0$, the reserve type is no greater than $\eta$ (i.e., the item is sold if at least one buyer's valuation is greater than $\eta$) and no buyer type obtains (an ex post) surplus greater than $\delta$. (Of course, the types that do not buy get zero surplus. However, the seller makes zero revenue from them as well and so an important point of the result is that while extracting almost all surplus from the types that buy, the mass of non-buying types can be made to be arbitrarily small.) Since the set of types who are excluded are at most $[0, \eta]$ and the ex post surplus of the types who buy is at most $\delta$, and since both $\delta$ and $\eta$ can be arbitrarily small, the result follows.

**Proposition 4** *For any preference parameter $\varepsilon > 0$, there exists $\delta^*(\varepsilon) > 0$ such that for any $\delta < \delta^*(\varepsilon)$, and $\eta > 0$, there is a MDM such that in any equilibrium of the game induced by the MDM, the item is sold if at least one buyer has valuation greater than $\eta$ and no type obtains an ex post surplus greater than $\delta$.*

**Proof:** The results in the previous section show that for any $\varepsilon > 0$, there is $\delta^*(\varepsilon) > 0$ such that whenever $\delta < \delta^*(\varepsilon)$, an equilibrium exists, and all equilibria can be characterized as in Proposition 2. Further, as noted in section 3.1, for any $\eta \in (0, 1)$, there exists an integer $T$ such by choosing $n = T$, the price sequence (which consists of $n$ prices) of the MDM covers at least a fraction $(1 - \eta)$ of types so that the item is not sold to at most types in $[0, \eta]$. Thus, it only remains to show that no type that buys gets an ex post surplus greater than $\delta$.

Now, since types in $[v_k, v_{k-1})$ buy at price $p_k$, the ex post surplus of any type buying at $p_k$ is at most $v_{k-1} - p_k$, which is bounded above by $\delta$ as follows: From the necessary

conditions for equilibrium presented in Proposition 2, we have

$$\begin{aligned} v_{k-1} - p_k &= p_{k-1} - p_k + \Delta_{k-1}\frac{(1-\varepsilon)H_{k-1}}{1-(1-\varepsilon)H_{k-1}} \\ &< \Delta_{k-1} + \Delta_{k-1}\frac{(1-\varepsilon)}{\varepsilon} \\ &= \frac{\Delta_{k-1}}{\varepsilon} = \delta\left(\frac{1-\delta}{1-\delta+\delta\varepsilon}\right)^{k-1} < \delta \end{aligned}$$

where the second step follows from the fact that $p_{k-1} - p_k = \Delta_{k-1}$ and the fact that $\frac{(1-\varepsilon)H_{k-1}(v_{k-1})}{1-(1-\varepsilon)H_{k-1}(v_{k-1})} < \frac{(1-\varepsilon)}{\varepsilon}$ since $H_{k-1}(v_{k-1}) < 1$. The final inequality follows from the fact that $\left(\frac{1-\delta}{1-\delta+\delta\varepsilon}\right)^{k-1} < 1$ for any $\varepsilon > 0$. This completes the proof.||

The basic intuition for the result is that for any $v$ and any price $p$ where $p < v$, the payoff from buying at $p$ is $v - p$ whereas the payoff from waiting one more period is $v - p + \Delta_p$ times the probability that the current buyer obtains the item in the next period. (Here $p - \Delta_p$ is the next price). With epsilon contamination preferences, the buyer attaches at least probability $\varepsilon$ that the item gets sold before he has the chance to obtain it next period. Thus the loss from waiting is at least $(v - p)(\varepsilon)$ whereas the gain from waiting is of the order $\Delta_p$. For any given $\varepsilon$, by making $\Delta_p$ successively small, the gain from waiting can be made arbitrarily small. However, in order to extract at least $\delta$ amount of surplus, we need buyer types to buy at (sufficiently high) prices such that the ex post surplus, $v - p$, is at most $\delta$. In other words, the loss from waiting, which is small when $\delta$ is small still needs to be larger than the gain from waiting and the price sequence is *constructed* in such a way that this is achieved.

Note that this cannot happen in the standard (i.e., the unique prior) model. There, for any type $v$, as long as the seller is selling to types below $v$ with positive probability, the surplus of type $v$ cannot be made to be arbitrarily small.[25] Roughly speaking, in the

[25] Of course in the standard setting, the price sequence considered above will not work since when $\varepsilon = 0$, $p_k = 1 - \delta$ for all $k$. What we mean is that there is no price sequence that can extract almost full surplus in the ambiguity-neutral setting. Perhaps this is obvious from the standard mechanism design exercise; there, the optimal revenue maximizing mechanism does not allocate the object if no bidder's type is greater than $v_*$ where $v_* - \dfrac{1-F(v_*)}{f(v_*)} = 0$ and types $v \geq v_*$ obtain expected surplus equal to $\int_{v_*}^{v} F(y)dy$. Given that the probability of obtaining the item is bounded, ex post surplus cannot be made arbitrarily small.

absence of ambiguity, given that $F$ is smooth, the expected gain and expected loss from waiting shrink at the same rate as the price gap becomes smaller.

## 5 A Numerical Example

Suppose $F$ is the uniform distribution on the unit interval. We know that for any $k < n$ the equation for $v_k$ is

$$v_k = p_k + \Delta_k \frac{(1-\varepsilon)H_k}{1-(1-\varepsilon)H_k}$$

where

$$H_k = \begin{cases} (v_1 + v_2)/(1 + v_1) & \text{for } k = 1, \text{ and} \\ (v_k + v_{k+1})/(v_{k-1} + v_k) & \text{for } 2 \leqslant k \leqslant (n-1) \end{cases}$$

Given $v_n = p_n$, the equations can be solved for any given $n$. It can be directly verified (as well as already noted in Proposition 3) that there is a unique positive solution for any $v_k$.

The following table shows a few steps for $\delta = 0.05$, and $\varepsilon = 0.2$. We stop as soon as we cross 0.9 (i.e. in this exercise we extract a rent of at least 0.95 from the top 10% types). In this case $n = 7$, and the prices $p_k$ and cutoffs $v_k$ are as shown. The right hand column shows the maximum rent obtained by any type. The rent obtained by any type $v \in [v_{(k+1)}, v_k)$ is given by $v - p_{(k+1)} \leqslant v_k - p_{(k+1)}$, which is the maximum rent.

| Price | $V_k$ | Maximum Rent |
|---|---|---|
| 0.9500 | 0.9874 | 0.0500 |
| 0.9401 | 0.9773 | 0.0473 |
| 0.9303 | 0.9671 | 0.0470 |
| 0.9206 | 0.9570 | 0.0465 |
| 0.9110 | 0.9468 | 0.0460 |
| 0.9015 | 0.9342 | 0.0453 |
| 0.8922 | 0.8922 | 0.0421 |

It is interesting to compare this with the outcome of the static optimal mechanism. Bose et al. (2006) show that in their mechanism, the reserve type, $v_*$ is given by the equation

$$v_* - (1-\varepsilon)\frac{1-F(v_*)}{f(v_*)} = 0$$

and the (expected) surplus of type $v$ is equal to $(1-\varepsilon)\int_{v_*}^{v} F(y)dy$.[26] When $F$ is the uniform distribution, the surplus is approximately equal to 0.32 for $v = 1$. In contrast, in the MDM, type $v = 1$ gets a surplus of exactly $\delta$ which in this numerical example is 0.05 (and, in general, can be made arbitrarily small).

Returning to the example, note that continuing in this fashion (i.e. by increasing $n$ beyond 7), it is possible to extract a rent of at least 0.95 from any fraction of types less than 1 (i.e. any type is left with a rent of at most 0.05). The figure below shows the price steps for $n = 1000$ for different values of $\delta$ (given $\varepsilon = 0.2$). The rate of change of prices is given by

$$\frac{p_k - p_{(k+1)}}{p_k} = \frac{\delta\varepsilon}{(1-\delta+\delta\varepsilon)}$$

This is increasing in $\delta$. So with lower $\delta$, prices fall "more slowly."

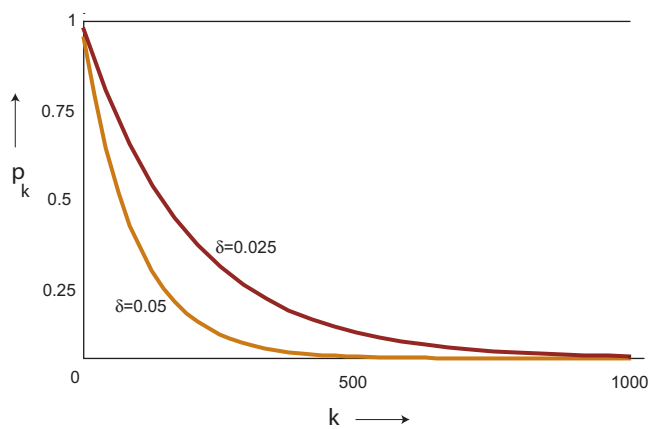


Figure 2: With $\varepsilon = 0.2$ and $\delta = 0.05$, 500 steps is enough to for price to get very close to zero - extracting a rent of at least $1-\delta = 0.95$ from almost all types. With $\delta = 0.025$ the rent extracted from each type is at least $0.975$, but extracting this from almost all types requires about 1000 price steps.

[26] They show that the optimal (static) direct revelation mechanism is a full insurance mechanism, and a type's surplus, when reporting its type truthfully is a function of its own report only and do not vary with the report of the other buyer. Hence, under truthtelling, expected surplus is also ex post surplus. For details see Bose et al. (2006).

# 6 Discussion

Maxmin preferences, and in particular the epsilon contamination formulation, have been used in both economics and statistics literatures. Our results show the effects that a dynamic mechanism can have in a setting which is IPV in all aspects other than the fact that the buyers are ambiguity averse. The preferences are modeled using the epsilon contamination specification. In this section we discuss our modeling choices and certain aspects of our results. Some of these issues were briefly mentioned in earlier sections.

The driving force behind the result is that the seller has the ability to make repeated offers to the buyers. The game is constructed in such a way that at each stage the buyer can accept, and get a sure payoff, or wait, and get an ambiguous payoff. Since the results depend on the maxmin expected payoffs (from waiting) that the buyers calculate at each stage using some updating rule (from an updated set of distributions), let us comment on the updating rule first. In particular, as we mentioned before (section 3.2), the two most commonly used updating rules are the full Bayesian and the generalized maximum likelihood rules and we first argue that our results hold under both of these rules.

## 6.1 Full Bayesian and Generalized Maximum Likelihood Updating Rules

In the full Bayesian updating rule the decision maker uses Bayes rule to update all the distributions (except those under which the observed event is impossible to have occurred) and the payoff is equal to the minimum expected utility calculated by considering this entire set of updated distributions. Since it is clear that with epsilon contamination preference where the decision maker puts epsilon weight on the worst possible distribution from the set of all distributions, it is clear that our results hold under the full Bayesian updating rule, we focus our discussion on the generalized maximum likelihood rule. Under this rule only those distributions are retained (and updated) that gives the maximum likelihood to the event known to have occurred. In our case, in any period $k$, faced with an offer $p_k$, a buyer puts a weight $(1-\varepsilon)$ on his chances of getting the item in future under the distribution $F$, and puts a weight $\varepsilon$ on the worst distribution in terms of his getting the item if he passes, conditional on the event that

he knows has happened. But, in period $k$, the fact that the seller makes an offer to a buyer means that the buyer knows that the other buyer has not bought the item in any previous period. In other words, the buyer knows that the other buyer's type must be in $[0, v_{k-1})$. Therefore, when calculating the minimum expected payoff from waiting, instead of taking the expectation with respect to *all* updated distributions, the buyer considers only those that are "most favorable" in terms of the event $[0, v_{k-1})$ which is known to have occurred. However, within this latter set, the worst distribution is still the one the puts the entire weight on the event that the current buyer will not obtain the item if he waits. Therefore the minimum expected payoff is the same as under the full Bayesian rule. Put differently, in our setting, the maximum likelihood rule has no "real" additional bite over the full Bayesian rule.

Note that in general, even though we have discussed only two updating rule (prevalent in the literature), as long as the updating rule does not throw away these worst distributions, the results in the paper should go through. We are not aware of any general argument that would require removal of these worst distributions as the game progresses. To be specific, consider an event $[v_k, v_{k-1})$ and note that the set of distributions that the buyers consider initially has amongst it (at least) one distribution $\widetilde{F}$ which puts an epsilon weight on this event. Now, suppose the buyer is told that the event $[v_{k-1}, 1]$ has not occurred. There is no obvious reason to suggest that this extra information should make $\widetilde{F}$ irrelevant.

### 6.2 Randomization

We have formally defined a buyer's decision making problem in any period as a choice between accepting or rejecting the seller's offer. This obviously means that we have restricted the buyers' action sets in any period (and hence strategies over the entire game) to be pure. A question naturally arises therefore as to whether the results continue to hold if we removed this restriction and allowed buyers to randomize over the pure actions if they so desired. Now, with non-EU preferences, this is a particularly delicate issue since there might be a strict preference for randomization (see Crawford (1990) for the seminal contribution). In fact the (crucial) assumption on uncertainty aversion in Gilboa and Schmeidler (1989)[27] shows that unlike EU preferences, a player with

[27] Assumption A5, page 144.

maxmin preferences may be indifferent between two pure strategies, yet strictly prefer a mixture of the two to either pure strategy. However, we now argue that allowing for randomization does not create a problem in our model.

To see this, note that in any period a buyer has only two pure actions (accept or reject), and, crucially, the action "accept" gives a *certain* (i.e. non-random) payoff. Let $A$ and $R$ denote accept and reject respectively. Consider the problem of a buyer in period $k$ who receives an offer to buy at price $p_k$ from the seller. The payoff $E(A)$ from $A$ is simply $v - p_k$. Let $E_{\min}(R)$ denote the maxmin payoff from rejecting offer $p_k$ (and choosing the optimal action in the future). Finally, let $E_{\min}(\alpha)$ denote the maxmin payoff from any randomized strategy, where $\alpha \in [0, 1]$ is the probability of accepting. Since $E(A)$ is a certain payoff, it follows that $E_{\min}(\alpha) = \alpha E(A) + (1 - \alpha)E_{\min}(R) \leqslant \max[E(A), E_{\min}(R)]$, where the inequality is strict whenever $E(A) \neq E_{\min}(R)$. Hence, whenever a pure action $A$ or $R$ is strictly preferred over the other, the preferred pure action continues to be optimal even if we allowed for randomizations. When $E(A) = E_{\min}(R)$, a randomized strategy gives the same payoff and therefore our assumption that a buyer accepts when indifferent between accepting and rejecting any price $p_k$ is without loss of generality.

### 6.3 Dynamic Consistency

Preferences satisfy dynamic consistency if an optimal plan based on prior preferences (ex ante plan) coincides with the sequentially optimal plan in a decision tree. This is unproblematic in the expected utility paradigm,[28] but does not arise naturally under ambiguity. In this section we discuss the ramifications of this for our model. We explain below how this relates to our work and why this issue is not central to deriving our formal results. The next part of this section provides a discussion of the general issues relating to the problem of dynamic consistency under ambiguity. Readers not interested in this specific issue can skip the later part of this section without any loss of continuity.

[28] In the expected utility paradigm, assuming that preferences satisfy dynamic consistency is not a problem if one assumes that the updating follows Bayes rule. It is well known (see Epstein and Schneider (2003)) that if the conditional preferences at every time-event pair satisfy expected utility theory, they satisfy dynamic consistency if and only if the updating is done using Bayes Rule.

### 6.3.1 Sophisticated dynamic choice: consistent plans

Siniscalchi (2006) separates the issue of dynamic consistency from the idea of sophisticated behavior by axiomatizing preferences over decision trees rather than over acts. Dynamic consistency is a property of preferences over acts at different nodes in a tree. Assuming properties of preferences over trees allows abstraction from the from the issue of dynamic consistency. However, this allows a characterization of consistent planning, which formalizes the idea of "sophisticated behavior": a sophisticated decision maker correctly anticipates his future preferences. Siniscalchi provides a coherent theory of dynamic choice without the need to appeal to dynamic consistency.

This idea is very much the spirit of the model here, and therefore the issue of dynamic consistency does not affect our formal results directly. Our model assumes sophisticated behavior by agents in the above sense, and therefore in all our formal derivations we use the idea that agents are forward looking and form consistent plans. We do not at any point invoke ex ante planning, and therefore the question of whether such plans coincide with sequential choice has no direct bearing on our formal results. However, the issue affects the interpretation of the results, in particular the role of dynamic mechanisms in contrast to static ones. The question of ensuring dynamic consistency is discussed further below.

### 6.3.2 General issues

We now proceed to explain why violations of dynamic consistency arises naturally under ambiguity, and discuss implications of enforcing consistency by restricting priors.

It is well known that without further restrictions on the set of priors, the full Bayesian updating rule (as well as the maximum likelihood updating rule) gives rise to dynamically inconsistent preferences over acts when conditional preferences at every time-event pair satisfy the maxmin utility theory. Several papers (e.g. Epstein and Schneider 2003, Hanany and Klibanoff 2006, Maccheroni et al. 2006, Klibanoff et al. 2006) propose theories of dynamic behavior by imposing dynamic consistency as an axiom that preferences must satisfy.

While it is often considered desirable, especially for practical purposes, to have dy-

namic consistency, it has also been pointed out that in certain situations it makes more intuitive sense to allow for preferences that violate dynamic consistency.[29] Here we briefly sketch the argument depicting one such situation. The following example and the associated discussion is borrowed from Epstein and Schneider (2003). The reader should consult the original article for a fuller exposition.[30]

Consider the Ellsberg urn experiment in which there are 30 balls that are red and 60 that are either blue or green. A ball is drawn at random from this urn and we are interested in the decision-maker's preference over acts whose payoffs are dependent on the color of the ball drawn. A natural state space is $\Omega = \{R, B, G\}$ and acts are, as usual, mapping from states to real numbers. So for example, the act $(1, 0, 0)$ pays 1 if the ball drawn is red and zero otherwise. To introduce dynamics in the simplest possible way suppose a ball is drawn in $t = 0$, some information is revealed at $t = 1$ and finally the color is revealed (hence all uncertainty is resolved) and payoffs given at $t = 2$. Specifically, suppose the information revealed in $t = 1$ can be expressed by the following filtration of the state space: $\mathcal{F}_1 = \{\{R, B\}, \{G\}\}$.

Consider first the atemporal ranking of two acts $(1, 0, 1)$ and $(0, 1, 1)$. Typical choice shows the time 0 ranking to be

$$(0, 1, 1) \succ_0 (1, 0, 1) \tag{6.1}$$

which is intuitive given the decision-maker's ambiguity about the exact number of blue versus green balls (there is no ambiguity about the total number of blue *and* green balls). For maxmin utility, this ranking is supported by the set of priors $\mathcal{P}$ given by

$$\mathcal{P} = \left\{ p = \left( \frac{1}{3}, p_B, \frac{2}{3} - p_B : \frac{1}{6} \leq p_B \leq \frac{1}{2} \right) \right\}$$

where ambiguity about the number of blue versus green balls is reflected in the range of $p_B$. However, to see the problem that arises in the dynamic model, note that with maxmin utility and using the full Bayesian updating rule (i.e., at time $t = 1$, the updated priors are obtained from using Bayes rule on all the priors in the set $\mathcal{P}$), one gets the $t = 1$ period ranking as

$$(1, 0, 1) \succ_{1,\{R,B\}} (0, 1, 1) \text{ and } (1, 0, 1) \sim_{1,\{G\}} (0, 1, 1) \tag{6.2}$$

[29] Further, as noted above, even for practical purposes it is possible to model rational agents who carry our consistent plans if, the conditional preferences are defined over *trees* (rather than *acts*) and the axiom of *sophistication* is imposed.

[30] See also Hanany and Klibanoff (2006).

In other words, in period $t = 1$, the decision-maker strictly prefers $(1, 0, 1)$ to $(0, 1, 1)$ if event $\{R, B\}$ obtains and is indifferent over the two acts if the complement of event $\{R, B\}$ obtains. Dynamic consistency then requires that in period $t = 0$, act $(1, 0, 1)$ should be strictly preferred to act $(0, 1, 1)$. But this contradicts (6.1), and therefore dynamic consistency is violated.

As Epstein and Schneider go on to explain in their paper, to ensure dynamic consistency one needs a restriction on the set of allowable priors; in particular one needs the set of priors to be what they call rectangular.[31] For the example above, $\mathcal{P}$ is not $\mathcal{F}_t-$ rectangular and they show the smallest rectangular set of priors (containing $P$), is given by:

$$\mathcal{P}' = \left\{ p = \left( \frac{1}{3} \frac{\frac{1}{3} + p'_B}{\frac{1}{3} + p_B}, p_B \frac{\frac{1}{3} + p'_B}{\frac{1}{3} + p_B}, \frac{2}{3} - p'_B \right) : \frac{1}{6} \leq p_B, p'_B \leq \frac{1}{2} \right\}$$

With the set of priors $\mathcal{P}'$, the period $t = 0$ ranking is dynamically consistent with the period $t = 1$ ranking as in (6.2). Of course, as Epstein and Schneider point out, this comes at the cost of reversing the ranking in (6.1). The lesson from this is that in some settings where there are intuitive choices for *different* periods, ambiguity may result in dynamic consistency being problematic.[32]

In fact, we feel that imposing dynamic consistency, in some sense, goes against the spirit of ambiguity aversion. To see what we mean, note that the essence of ambiguity aversion is relaxation of the sure thing principle. Under the sure thing principle, if the decision maker's conditional preferences are such that act $h$ is preferred to $g$ irrespective of whether event $A$ happens or $A^{\sim}$ (the complement of event $A$) happens, the unconditional preference should be such that act $h$ is preferred to act $g$. Dynamic consistency requires that if at period $t = 1$ act $h$ is preferred to $g$ if either event $A$ or $A^{\sim}$ is known to have happened, at the period $t = 0$ act $h$ must be preferred to act $g$. Hence when preferences exhibit ambiguity aversion (and involve violation of sure thing principle *within* a period), imposing this sort of restriction over conditional preferences *across periods* may not be the right thing to do.

[31] The basic idea is to construct the set of priors via backward induction by considering one-step-ahead conditionals. Clearly, this ensures dynamic consistency (under full Bayesian updating rule) by construction.

[32] Another way to see the problem with $\mathcal{P}'$. is as follows. $\mathcal{P}'$, gives a *range* of values for the odds of drawing a red ball, which differ from $1/3$ except when $p_B = p'_B$. However, at $t = 0$, it is *known* that there are *exactly* 30 red balls in the urn containing 90 balls.

We should add, lest we be misunderstood, that we do not mean to suggest that ambiguity averse decision-makers should never satisfy dynamic consistency. In fact, Epstein and Schneider give another example in their paper where no intuitive problem arises in making the set of priors rectangular (and hence ensuring dynamically consistent preferences). Also, we obviously do not mean that there is anything *formally* wrong in having ambiguity aversion with dynamic consistency; they are logically distinct features and in fact the papers we cite do just that - they construct formal models where decision-makers have ambiguity averse but dynamically consistent preference. What we mean, and here we are probably just echoing Epstein and Schneider, is that imposing dynamic consistency as a condition–that is insisting that it should *always* hold–leaves out interesting and intuitive behavior in settings with ambiguity. Our objective in this paper is to study a dynamic situation with ambiguity averse preferences where we do not have dynamic consistency (but as mentioned before, the buyers do carry out consistent plans since they are sophisticated and know their future behavior). The main result we get *is* surprising but that only goes on to show that much more work is needed before we can fully understand ambiguity averse (and in general many non-expected utility) preferences in dynamic settings.[33]

### 6.4 Restricting the contaminating set of distributions

Finally, it has often been pointed out, especially in the statistics literature, that the version of epsilon contamination that we use is too general: namely putting epsilon weight on the possibility that the true distribution might be *any* distribution may actually be allowing for too much. In particular, a "reasonable" modification of the model might involve restricting the set of contaminating distributions so that each element of the set satisfies certain properties. Here we sketch an informal argument involving an example to suggest why we think that in many situations our results will go through even if one made some of these modifications. So, suppose the contaminating set of distributions on $[0, 1]$ is restricted to contain only those that satisfy differentiability and

[33] As Epstein and Schneider show, the set of rectangular priors can be constructed by backward induction from one-step-ahead conditionals when there is an exogenously given filtration. However, in a game the equilibrium strategies determine how information is revealed - in other words the filtration is endogenous. It might be an interesting question to explore whether, or how, rectangularity can be extended to these settings.

monotone hazard rate.[34] More specifically, consider the example where the set of contaminating distributions are of the form $L_n(v) = v^n$. Each distribution in this family is differentiable and satisfies monotone hazard rate.[35] Now, the conditional probability of the event $[v_k, v_{k-1})$, given the event $[0, v_{k-1})$ is given by $1 - \left(\frac{v_k}{v_{k-1}}\right)^n$, and note that $\inf_n \left(\frac{v_k}{v_{k-1}}\right)^n = 0$. Hence, in this case, it is as if, for all practical purposes, there is a contaminating distribution that puts the entire mass on $[v_k, v_{k-1})$ which is what is done in the formal model of the paper.

## 7 Conclusion

Evidence (experimental and otherwise) suggests that it is important for economic models to explore the consequence of non-expected utility preferences. The fairly large (and growing) literature in this area has given us many valuable insights.

In this paper, we consider a private values auction model with ambiguity and buyers with ambiguity averse preferences. In the standard setting with a unique prior, the optimal mechanism leaves all but the lowest participating type with information rent. Previous work shows that even under ambiguity aversion, the optimal static mechanism leaves buyer types with rent. In contrast, we show that in the latter environment, dynamic mechanisms have more power, and using the epsilon contamination specification to model ambiguity aversion, we construct a very simple dynamic mechanism that extracts almost all surplus.

We view the contribution of our work as providing an example of the non-standard effects that ambiguity aversion can have on mechanism design. Our formal model uses the epsilon contamination specification and clearly our result of full surplus extraction is related to this setting. Nevertheless, the idea that in auction like settings, dynamic mechanisms can extract greater surplus than static ones by exploiting ambiguity aversion is a more general one. By showing that the equivalence between static

[34] The interpretation is that as before the decision maker puts a weight of $\varepsilon$ on the true distribution not being $F$. Now, however, he has confidence that the true distribution, even if not $F$, has certain properties similar to $F$.

[35] Of course this is only a particular family of distributions having the property of being differentiable and having monotone hazard rate. However, since the minimum cannot increase if sets are made bigger, considering only a particular family like this is enough for the purpose of illustrating our point.

and dynamic mechanisms (standard under the unique prior model) need not extend to a dynamic setting, our results strike a cautionary note when working in the non-unique prior environment. This is further highlighted by the contrast between our results and those in Bose et al. (2006) who study optimal static auctions under ambiguity, and leads us to conclude that a straightforward application of the revelation principle has its limitations when preferences are no longer characterized by subjective expected utility. Understanding the proper scope of the revelation principle with such "non-probabilistically sophisticated" preferences is an interesting question that we hope to address in future research.

# 8 Appendix: Proofs

## A.1 Some Conditional Probabilities

This section derives some conditional probabilities that are used repeatedly in the analysis.

Let $H_k^i$ denote the probability under the distribution $F$ (i.e., if there were no ambiguity) that $i$ wins the item at $p_{k+1}$ given that he refuses the current offer of $p_k$.

This can be calculated in two parts.

First, let $\phi_k^i$ denote the probability under the distribution $F$ that $i$ wins the item at $p_{k+1}$ conditional on the item not being sold at $p_k$.

Second, let $\pi_k^i$ denote the probability (again, this is the probability under $F$) that if $i$ refuses the current offer $p_k$ the object remains unsold till the next price $p_{k+1}$.

Then we have $H_k^i = \pi_k^i \phi_k^i$.

**Calculating $\boldsymbol{\phi_k^i}$:** $\phi_k^i$ can be derived is as follows. If buyer $i$ is asked first in period $k+1$ (which happens with probability $1/2$), he wins for sure. If $j$ is asked first (probability $1/2$), $i$ wins only if $j$ passes. Given that the object is unsold at $p_k$, we know that the type of $j$ is lower than $v_k^j$. Therefore the probability that $j$ will refuse $p_{k+1}$ given that he has refused $p_k$ is given by $Prob(v^j < v_{k+1}^j | v^j < v_k^j) = \dfrac{F(v_{k+1}^j)}{F(v_k^j)}$. Therefore

$$\phi_k^i = \frac{1}{2} + \frac{1}{2}\frac{F(v_{k+1}^j)}{F(v_k^j)} \tag{A.1}$$

**Calculating $\boldsymbol{\pi_k^i}$:** Next, $\pi_k^i$ can be derived as follows.

First, we need to work out the probability that a buyer is being asked first given that he is asked whether he wants to buy at $p_k$. The conditioning on being asked is important since the fact that a buyer is asked whether he wants to buy at $p_k$ conveys information about whether he is first or second. Let $q^i \in \{1, 2\}$ denote the position (1st or 2nd) of buyer $i$ in any period. Further, let $A^i$ denote the event that "buyer $i$ is asked whether

he wants to buy at $p_k$." We want to determine $Prob(q^i = 1|A^i)$.

$$\begin{aligned} Prob(q^i = 1|A^i) &= \frac{Prob(q^i = 1)Prob(A^i|q^i = 1)}{Prob(q^i = 1)Prob(A^i|q^i = 1) + Prob(q^i = 2)Prob(A^i|q^i = 2)} \\ &= \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{2}\frac{F(v_k^j)}{F(v_{k-1}^j)}} \\ &= \frac{F(v_{k-1}^j)}{F(v_{k-1}^j) + F(v_k^j)} \end{aligned}$$

where $v_0^j \equiv 1$.

Similarly,

$$Prob(q^i = 2|A^i) \;=\; 1 - Prob(q^i = 1|A^i) \;=\; \frac{F(v_k^j)}{F(v_{k-1}^j) + F(v_k^j)}$$

where $v_0^j \equiv 1$ as before.

We are now ready to derive $\pi_k^i$. Note that given $i$ refuses $p_k$, the probability of the object being unsold if $i$ is second ($q^i = 2$) is 1, and the probability of the object being unsold if $i$ is first ($q^i = 1$) is $\dfrac{F(v_k^j)}{F(v_{k-1}^j)}$. Therefore

$$\begin{aligned} \pi_k^i &= Prob(q^i = 1|A^i)\frac{F(v_k^j)}{F(v_{k-1}^j)} + Prob(q^i = 2|A^i)(1) \\ &= \frac{2F(v_k^j)}{F(v_{k-1}^j) + F(v_k^j)} \end{aligned} \tag{A.2}$$

where $v_0^j \equiv 1$.

Finally, using equations (A.1) and (A.2), we get

$$H_k^i = \pi_k^i\, \phi_k^i \;=\; \frac{F(v_k^j) + F(v_{k+1}^j)}{F(v_k^j) + F(v_{k-1}^j)}$$

where $v_0^j \equiv 1$.

## A.2 Proof of Proposition 1

In this section we prove Proposition 1. The basic outline of our argument is as follows. In Lemma 3 we show that in any equilibrium, for both buyers, the cutoff type for price $p_n$ is in fact $p_n$ and that there are types of positive measure who plan to buy at price $p_n$. Next, we show in Lemma 4 that for both buyers, there are types of positive measure that plan to buy at $p_1$. Lemma 5 is crucial, it shows that whenever $\delta$ is sufficiently small, given that a positive measure of types of both buyers buy at prices $p_1$ and $p_n$, there must be a positive measure of types of both buyers who buy at price $p_{n-1}$. Proposition 1 now follows from a recursive argument: provided types of positive measure plan to buy at prices $p_1$ and $p_{k+1}, ..., p_n$, there must be types of positive measure who plan to buy at $p_k$ as well.

We remind the reader that the term $v_k^i$, is used to denote the lowest type of buyer $i$ who plans to buy at price $p_k$. Also, to avoid confusion with respect to superscripts versus exponents, in the rest of this Appendix, we refer to the two buyers as $i$ and $j$ instead of 1 and 2.

**Lemma 3** *In any equilibrium, $v_n^i = v_n^j = p_n$. Further, a positive measure of types of both buyers plan to buy at price $p_n$ but not at any earlier price.*

**Proof:** Consider a type $v \in (p_n, p_{n-1})$ of either buyer. Buying at any price greater than $p_{n-1}$ is dominated by not buying at all but not buying at price $p_n$ gives a zero surplus whereas buying at price $p_n$ gives a surplus $v - p_n > 0$. Hence types of positive measure $(p_n, p_{n-1})$ must plan to buy at $p_n$ but not at any earlier price. Furthermore, the lowest type (the type that is indifferent between buying at price $p_n$ and not buying at all) that buys at $p_n$ is $p_n$, so that $v_n^i = v_n^j = p_n$. ||

In what follows, we use the word "probability" to mean probability with respect to the distribution $F$. (This allows us to avoid writing the phrase "with respect to the distribution $F$" repeatedly.)

**Lemma 4** *In equilibrium a positive measure of types of each buyer plan to buy at $p_1$.*

**Proof:** Consider any buyer, say, buyer $j$ and suppose on the contrary that no type of buyer $j$ plans to buy at price $p_1$. More generally, for $1 \leqslant k < n$ suppose buyer $j$ does

not plan to buy at prices $p_1, \ldots, p_k$ so that $p_{k+1}$ is the first price at which buyer $j$ buys. (Formally, this is denoted as $v_1^j = \ldots = v_k^j = 1$ and $v_{k+1}^j < 1$).

Since $j$ does not plan to buy at prices $p_1, \ldots, p_k$, it is clear that $i$ should not plan to buy at prices $p_1, \ldots, p_{k-1}$. Recall (from appendix **A**.1) that $H_k^i$ is the probability that buyer $i$ wins at $p_{k+1}$ given that he refuses the current offer of $p_k$. Note that if $i$ refuses $p_k$, the probability that the game reaches $p_{k+1}$ is 1. Thus $\pi_k^i = 1$ Therefore $H_k^i = \pi_k^i \phi_k^i = \phi_k^i = 1/2 + (1/2)F(v_{k+1}^j)$.[36] Define the following function.

$$G_k^i(v) \equiv v - p_k - (1-\varepsilon)(v - p_{k+1})H_k^i$$

$G_k^i(v)$ can be rewritten as $(v - p_k)(1 - (1-\varepsilon)H_k^i) - (1-\varepsilon)\Delta_k H_k^i$. Note that

$$\begin{aligned}
G_k^i(1) &= (1-p_k)(1-(1-\varepsilon)\phi_k^i) - (1-\varepsilon)\Delta_k H_k^i \\
&> \delta\varepsilon - (1-\varepsilon)\Delta_k \\
&\geqslant \delta\varepsilon - (1-\varepsilon)\Delta_1 \\
&= \delta\varepsilon\left(1 - \frac{(1-\varepsilon)(1-\delta)}{1-\delta+\delta\varepsilon}\right) \\
&= \frac{\delta\varepsilon^2}{1-\delta+\delta\varepsilon} \\
&> 0
\end{aligned}$$

where the second step follows from the fact that $(1-p_k) \geqslant (1-p_1) = \delta$, and the fact that $H_k^i < 1$, and the third step uses $\Delta_1 \geqslant \Delta_k$.

Since $G_k^i(v)$ is continuous, increasing in $v$, and negative at $v = p_k$, there exists $v_k^i$ such that $G_k^i(v) > 0$ for $v > v_k^i$ and $G_k^i(v_k^i) = 0$. Since we know that $i$ does not plan to buy at any earlier price than $p_k$, it must be that types $[v_k^i, 1]$ of buyer $i$ plan to buy at $p_k$.

Now, let $H_k^j$ be the probability that $j$ wins at $p_{k+1}$ if $j$ refuses the current offer of $p_k$ in period $k$. Since we have just shown that some types of $i$ plan to buy at price $p_k$, irrespective of what $i$ plans to do at price $p_{k+1}$, buyer $j$ knows in period $k$ that he cannot expect to obtain the item for sure in period $k+1$. Now, for buyer $j$, let

$$G_k^j(v) \equiv v - p_k - (1-\varepsilon)(v - p_{k+1})H_k^j$$

We have

[36] Note that this is the same formula as in equation (A.1), since here $F(v_k^j) = F(v_{k-1}^j) = 1$.

$$
\begin{aligned}
G_k^j(1) &= 1 - p_k - (1-\varepsilon)(1-p_{k+1})H_k^j \\
&= (1-p_k)(1-(1-\varepsilon)H_k^j) - (1-\varepsilon)\Delta_k H_k^j \\
&> \delta\varepsilon - (1-\varepsilon)\Delta_k \\
&\geq \delta\varepsilon - (1-\varepsilon)\Delta_1 \\
&= \frac{\delta\varepsilon^2}{1-\delta+\delta\varepsilon} \\
&> 0
\end{aligned}
$$

where the first inequality follows since $H_k^j < 1$, and $(1-p_k) \geqslant (1-p_1) = \delta$ and the second one follows since $\Delta_k \leq \Delta_1$. Since $G_k^j(v)$ is increasing and continuous, there are types of $j$ of positive measure near 1 who would deviate and buy at $p_k$. Contradiction.‖

From above, we know that each buyer has types who buy at both $p_1$ and $p_n$. To complete the proof we need to show that whenever $\delta$ is sufficiently small, this is true at other prices between $p_1$ and $p_n$ as well.

To prove this, we start by showing that both buyers must have types who plan to buy at $p_{n-1}$. Then we show that if there are buyer types who plan to buy at prices $p_{n-k}$ to $p_n$ for $k \geq 2$, then there must also be types who plan to buy at price $p_{n-k-1}$. This completes the proof.

Let us now show that both buyers have types who plan to buy at price $p_{n-1}$.

Suppose this is not true. In particular, suppose buyer $j$ does not plan to buy at prices $\{p_{n-\ell+1}, \ldots, p_{n-1}\}$ where $2 \leqslant \ell \leqslant n-1$, but plans to buy at $p_{n-\ell}$ (and of course at $p_n$).

Since $j$ does not plan to buy at prices $\{p_{n-\ell+1}, \ldots, p_{n-1}\}$, the best response by $i$ involves not planning to buy at prices $\{p_{n-\ell+1}, \ldots, p_{n-2}\}$ whenever $\ell > 2$. Note further that in that case, there must be types of $i$ who plan to buy at $p_{n-\ell}$. (Otherwise types of $j$ buying at $p_{n-\ell}$ can profitably deviate to, say, $p_{n-2}$. This contradicts the assumption that $j$ buys at $p_{n-\ell}$). Armed with these facts, let us now show the result.

**Lemma 5** *There is $\overline{\delta} > 0$ such that for $\delta < \overline{\delta}$ there are types (of positive measure) of $j$ who buy at $p_{n-1}$.*

**Proof:** In the proposed equilibrium, types $v \geqslant v_{n-\ell}^j$ of $j$ buy at prices $p \geqslant p_{n-\ell}$, with type $v_{n-\ell}^j$ and some types just above buying at price $p_{n-\ell}$. But since $j$ does not buy at

prices $\{p_{n-\ell+1}, \ldots, p_{n-1}\}$, types just below $v^j_{n-\ell}$ must buy at $p_n$ and not before. Therefore, in the proposed equilibrium, it must be that $v^j_{n-\ell}$ is indifferent between buying at $p_{n-\ell}$ or $p_n$. So we have, for buyer $j$,

$$v^j_{n-\ell} - p_{n-\ell} = (1-\varepsilon)(v^j_{n-\ell} - p_n)H^j_{n-\ell} \tag{A.3}$$

where $H^j_{n-\ell} = \pi^j_{n-\ell}\ \widehat{\pi}^j_{n-1}\ \phi^j_{n-1}$, where $\pi^j_{n-\ell}$ is the probability that the object is unsold at $p_{n-\ell}$ given that $j$ refuses the current offer of $p_{n-\ell}$, $\widehat{\pi}^j_{n-1}$ is the probability that the object will remain unsold at $p_{n-1}$, and $\phi^j_{n-1}$ is the probability that $j$ wins at price $p_n$.

We know some types of $i$ buy at price $p_{n-\ell}$, without loss of generality, for $t \geq 1$, let $p_{n-\ell-t}$ be the price before $p_{n-\ell}$ at which some types of $i$ buy in equilibrium. We have

$$\pi^j_{n-\ell} = \frac{2F(v^i_{n-\ell})}{F(v^i_{n-\ell}) + F(v^i_{n-\ell-t})}$$

and

$$\widehat{\pi}^j_{n-1} = \frac{F(v^i_{n-1})}{F(v^i_{n-\ell})}$$

Note that if there are no types of $i$ who buy at $p_{n-1}$, then $F(v^i_{n-1}) = F(v^i_{n-\ell})$, and $\widehat{\pi}^j_{n-1} = 1$. Otherwise $\widehat{\pi}^j_{n-1}$ is less than 1.

Finally

$$\phi^j_{n-1} = \frac{1}{2} + \frac{1}{2}\frac{F(v^i_n)}{F(v^i_{n-1})} \tag{A.4}$$

where, again, if there are no types of $i$ who buy at $p_{n-1}$, then we have $F(v^i_{n-1}) = F(v^i_{n-\ell})$.

From the above, we have

$$H^j_{n-\ell} = \frac{F(v^i_{n-1}) + F(v^i_n)}{F(v^i_{n-\ell}) + F(v^i_{n-\ell-t})}$$

Now, we can rewrite equation (A.3) above as

$$v^j_{n-\ell} - p_{n-\ell} = \frac{(1-\varepsilon)(p_{n-\ell} - p_n)H^j_{n-\ell}}{1-(1-\varepsilon)H^j_{n-\ell}} \tag{A.5}$$

Let

$$G^j_{n-1}(v) \equiv v - p_{n-1} - (1-\varepsilon)(v - p_n)\ H^j_{n-1}$$

where $H^j_{n-1} = \pi^j_{n-1}\,\phi^j_{n-1}$, where $\phi^j_{n-1}$ is as above (given by equation (A.4)), and $\pi^j_{n-1}$ is the probability that the object is period $n-1$ given that $j$ refuses the current offer of $p_{n-1}$. Note that $\pi^j_{n-1} = 1$ if no types of $i$ buy at price $p_{n-1}$, otherwise it is equal to $\frac{2F(v^i_{n-1})}{F(v^i_{n-1})+F(v^i_n)}$. In either case, since $\phi^j_{n-1} < 1$, we have $H^j_{n-1} < 1$ as well. To establish that contrary to what has been supposed, there are types of $j$ who, not having bought before, will in fact want to buy at price $p_{n-1}$, it is useful to break up the analysis into several cases.

**Case 1:** $\ell$ and $t$ are fixed positive integers.

Intuitively, this is the case where both $i$ and $j$ follow strategies where they do not buy for some finite number of prices. Note that in this case, as $\delta \to 0$, the real length of the interval over which they don't buy converge to zero. More specifically, $\delta(\ell + t - 1) \to 0$, as $\delta \to 0$. The fact that $v^i_{n-\ell-t} - v^i_{n-1} < \delta(\ell + t - 1)$, (shown below in Lemma (6)), is used for proving this case.

Now, since there are no types of $j$ who buy at $p_{n-1}$, it must be that $G^j_{n-1}(v)$ is not strictly positive for any $v \in [p_{n-1}, v^j_{n-\ell}]$. Consider the value of $G^j_{n-1}(\cdot)$ at $v^j_{n-\ell}$. We have

$$\begin{aligned}
G^j_{n-1}(v^j_{n-\ell}) &= v^j_{n-\ell} - p_{n-1} - (1-\varepsilon)(v^j_{n-\ell} - p_n)\; H^j_{n-1} \\
&= (v^j_{n-\ell} - p_{n-\ell}) + (p_{n-\ell} - p_n) - \Delta_{n-1} \\
&\qquad - (1-\varepsilon)\left[(v^j_{n-\ell} - p_{n-\ell}) + (p_{n-\ell} - p_n)\right] H^j_{n-1} \\
&= (p_{n-\ell} - p_n)\left(\frac{1-(1-\varepsilon)\; H^j_{n-1}}{1-(1-\varepsilon)\; H^j_{n-\ell}}\right) - \Delta_{n-1} \\
&> \left[2\left(\frac{1-(1-\varepsilon)\; H^j_{n-1}}{1-(1-\varepsilon)\; H^j_{n-\ell}}\right) - 1\right]\Delta_{n-1}
\end{aligned}$$

where the second step follows from equation (A.5), and the third step follows from the fact that $p_{n-\ell} - p_n \geqslant p_{n-2} - p_n = \Delta_{n-2} + \Delta_{n-1} > 2\Delta_{n-1}$.

Now, since

$$H^j_{n-\ell} = \frac{F(v^i_{n-1}) + F(v^i_n)}{F(v^i_{n-\ell}) + F(v^i_{n-\ell-t})}$$

and

$$H^j_{n-1} = \begin{cases} \dfrac{F(v^i_{n-1}) + F(v^i_n)}{F(v^i_{n-1}) + F(v^i_{n-\ell})} & \text{if some types of } i \text{ buy at } p_{n-1} \\ \\ \dfrac{F(v^i_{n-\ell}) + F(v^i_n)}{2F(v^i_{n-\ell})} & \text{otherwise} \end{cases}$$

we have,

$$\frac{H^j_{n-1}}{H^j_{n-\ell}} = \begin{cases} \dfrac{F(v^i_{n-\ell}) + F(v^i_{n-\ell-t})}{F(v^i_{n-\ell}) + F(v^i_{n-1})} & \text{if some types of } i \text{ buy at } p_{n-1} \\ \\ \dfrac{F(v^i_{n-\ell}) + F(v^i_{n-\ell-t})}{2F(v^i_{n-\ell})} & \text{otherwise} \end{cases}$$

From Lemma 6, $v^i_{n-\ell-t} - v^i_{n-1} < \delta(\ell + t - 1)$. Therefore, as $\delta \to 0$, the ratio $\frac{H^j_{n-1}}{H^j_{n-\ell}}$ converges to 1. Hence for sufficiently small $\delta$, the term $\frac{1-(1-\varepsilon)\ H^j_{n-1}}{1-(1-\varepsilon)\ H^j_{n-\ell}}$ is greater than $\frac{1}{2}$ and we have $G^j_{n-1}(v^j_{n-\ell}) > 0$.

<u>Case 2: $t$ is arbitrary and $\ell$ varies with $n$.</u>

This is the case when the gap $p_{n-\ell} - p_{n-1}$ does not vanish as $\delta \to 0$.

Consider again $G^j_{n-1}(v^j_{n-\ell})$.

From equation (A.5),

$$v^j_{n-\ell} - p_{n-\ell} = \frac{(1-\varepsilon)(p_{n-\ell} - p_n)H^j_{n-\ell}}{1-(1-\varepsilon)H^j_{n-\ell}}$$

As $\delta \to 0$, since $(p_{n-\ell} - p_n)$ does not vanish, and since for any given $\eta > 0$, $H^j_{n-\ell}$ is bounded away from zero, $v^j_{n-\ell} - p_{n-\ell}$ does not vanish. Therefore, $v^j_{n-\ell} - p_{n-1}$ does not vanish. However, $p_n - p_{n-1} \to 0$, and $(1-\varepsilon)H^j_{n-1} < 1$. Therefore for $\delta$ small enough, $G^j_{n-1}(v^j_{n-\ell}) > 0$.

In the two cases above, we have shown that $G^j_{n-1}(v^j_{n-\ell}) > 0$. But since $G^j_k(\cdot)$ is strictly increasing, continuous, and negative at $p_{n-1}$, $G^j_{n-1}(v^j_{n-\ell}) > 0$ implies that there is

$v^j_{n-1} \in (p_{n-1}, v^j_{n-\ell})$ such that $G^j_{n-1}(v) > 0$ for $v \in (v^j_{n-1}, v^j_{n-\ell})$. Since types below $v^j_{n-\ell}$ do not buy at any price greater than or equal to $p_{n-\ell}$, these types (of positive measure) strictly prefer to stop at $p_{n-1}$ rather than wait till $p_n$. This contradicts the supposition that there are no types of $j$ who buy at $p_{n-1}$.

We need to consider a third possibility in order to complete the Lemma.

Case 3: $\ell$ is a fixed integer and $t$ varies with $n$.

This is the case when as $\delta \to 0$, $\delta(\ell + t - 1)$ does not go to zero because $t$ (and $n$) become arbitrarily large as $\delta$ becomes small. However, this is analogous to the cases we have analyzed before with $i$ and $j$ roles being switched. We know that in equilibrium, both buyers have types who plan to buy at price $p_{n-\ell}$. If $i$ plans to buy at prices $p_{n-\ell-t}$ and $p_{n-\ell}$, but does not plan to buy at prices $\{p_{n-\ell-t+1}, ..., p_{n-\ell-1}\}$ then the best response of $j$ should include not to plan to buy at prices $\{p_{n-\ell-t+1}, ..., p_{n-\ell-2}\}$.If $p_{n-\ell-t} - p_{n-\ell-1}$ does not go to zero, then we can use the arguments of case 2 above to argue that contrary to what is being supposed, for small $\delta$, buyer $i$ will in fact have some types of positive measure who will want to buy at $p_{n-\ell-1}$ rather than waiting till $p_{n-\ell}$.

This completes the proof of the lemma.‖

To continue now with the proof of the Proposition, suppose both buyers have a positive measure of types buying at prices $p_{n-k}$ to $p_n$, where $1 \leqslant k \leqslant n-2$. By exactly the same argument as above we can establish that both buyers must also buy at $p_{k-1}$. This, combined with the previous steps complete the proof of proposition 1.‖

Finally we show in Lemma 6 below the fact we have used in case 1, namely that $v^i_{n-\ell-t} - v^i_{n-1} < \delta(\ell + t - 1)$.

**Lemma 6** $v^i_{n-\ell-t} - v^i_{n-1} < \delta(\ell + t - 1)$.

**Proof:** $v^i_{n-\ell-t}$ is given by $G^i_{n-\ell-t}(v) = 0$, i.e.

$$\begin{aligned} v^i_{n-\ell-t} - p_{n-\ell-t} &= (1-\varepsilon)(v^i_{n-\ell-t} - p_{n-\ell})H^i_{n-\ell-t} \\ &= (1-\varepsilon)(v^i_{n-\ell-t} - p_{n-\ell-t} + \Delta_{n-\ell-t} + \ldots + \Delta_{n-\ell-1})H^i_{n-\ell-t} \end{aligned}$$

Solving,

$$\begin{aligned} v^i_{n-\ell-t} - p_{n-\ell-t} &= (\Delta_{n-\ell-t} + \ldots + \Delta_{n-\ell-1}) \frac{(1-\varepsilon)H^i_{n-\ell-1}}{1-(1-\varepsilon)H^i_{n-\ell-1}} \\ &< (\Delta_{n-\ell-t} + \ldots + \Delta_{n-\ell-1}) \frac{(1-\varepsilon)}{\varepsilon} \end{aligned} \tag{A.6}$$

Let $\alpha \equiv \frac{1-\delta}{1-\delta+\delta\varepsilon}$. From equation (3.2), we have $\Delta_k = \delta\epsilon\alpha^k$. Therefore

$$\begin{aligned} v^i_{n-\ell-t} - p_{n-1} &= v^i_{n-\ell-t} - p_{n-\ell-t} + p_{n-\ell-t} - p_{n-1} \\ &= v^i_{n-\ell-t} - p_{n-\ell-t} + \Delta_{n-\ell-t} + \ldots + \Delta_{n-2} \\ &< (\Delta_{n-\ell-t} + \ldots + \Delta_{n-\ell-1}) \frac{(1-\varepsilon)}{\varepsilon} + \Delta_{n-\ell-1} + \Delta_{n-\ell} + \ldots + \Delta_{n-2} \\ &= (\Delta_{n-\ell-t} + \ldots + \Delta_{n-\ell-2}) \frac{(1-\varepsilon)}{\varepsilon} + \frac{\Delta_{n-\ell-1}}{\varepsilon} + \Delta_{n-\ell} + \ldots + \Delta_{n-2} \\ &= \delta(1-\varepsilon)\left[\alpha^{n-\ell-t} + \ldots + \alpha^{n-\ell-2}\right]\delta + \delta\varepsilon\left[\alpha^{n-\ell} + \ldots + \alpha^{n-2}\right] \\ &< \delta(1-\varepsilon)(t-1) + \delta + \delta\varepsilon(\ell-1) \\ &< \delta(\ell+t-1) \end{aligned}$$

where the third step follows from the inequality (A.6) above, and the fifth and the last step follow, respectively, from the facts that $\alpha < 1$ and $\varepsilon < 1$.

Finally, since $v^i_{n-1} > p_{n-1}$, $v^i_{n-\ell-1} - v^i_{n-1} < v^i_{n-\ell-1} - p_{n-1} < \delta(\ell+t-1)$. This completes the proof.‖

## **A**.3 Proof of Proposition 3

To show that a symmetric equilibrium exists, we need to show first that the set of cut-off vectors $\equiv \{v_0, v_1, \ldots, v_n\}$ (denoted by $E$) is convex and compact, set up a best response mapping from $E$ to $E$, and establish that the mapping is continuous. Once this is done, Brouwer's fixed point theorem tells us that the mapping has a fixed point, which is therefore a symmetric equilibrium.

As the formal proof shows, this can be done easily except that a trick is required to establish a continuous mapping from $E$ to $E$. Below we first explain informally what the problem is and how the "trick" that solves the problem works. The formal proof is then presented in section **A**.3.2.

### A.3.1 Informal Exposition

To clarify the idea of the proof, consider an example with $n = 3$. Since we already know $v_0 \equiv 1$ and $v_3 = p_3$, let us first describe the set of the rest of the cut-offs. The figure below illustrates the set $D$ of cut-offs $(v_1, v_2)$. The shaded area in figure 3 below shows the set $D$.

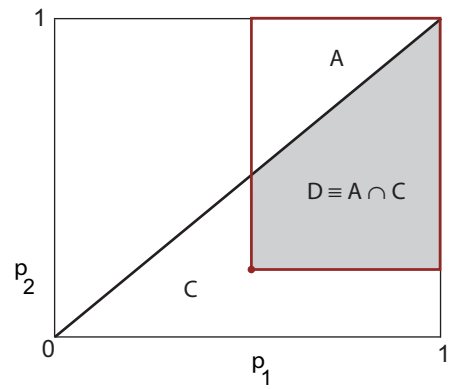


Figure 3: Any $(v_1, v_2)$ is in the unit square and must satisfy $v_1 \geqslant p_1$ and $v_2 \geqslant p_2$ and $v_1 \geqslant v_2$. The upper right hand rectangle $A$ is the set of numbers $(x_1, x_2)$ in the unit square satisfying $x_1 \geqslant p_1$ and $x_2 \geqslant p_2$. The set of cut-offs $D$ is the intersection of this set and the lower triangle $C$ which is the set of numbers $(x_1, x_2)$ in the unit square with $x_1 \geqslant x_2$.

Now, for this example, let $E \equiv \left\{\mathbf{v} \in [0,1]^4 \ \ |v_0 = 1, \{v_1, v_2\} \in D, v_3 = p_3\right\}$. This is the set of cut-off vectors $\equiv (v_0, v_1, v_2, v_3)$. Note that this set is compact and convex. In the proof the set $E$ is constructed in a similar way for arbitrary $n$ : first we define the set $D \subset [0,1]^{n-1}$ and then $E$ is constructed by adding the numbers 1 and $p_n$ at the beginning and the end respectively to get a set in $[0,1]^{n+1}$.

Next we need to establish a best response mapping from $E$ to $E$.

We say that any cut-off vector $\mathbf{v}$ is in the interior of $E$ if $(v_1, ...v_{n-1})$ is in the interior of $D$. (For the example, $\mathbf{v}$ is in the interior if $(v_1, v_2)$ is in the interior of the region $D$ shown above.) Now for any such $\mathbf{v}$, we can show that the best response to $\mathbf{v}$ is continuous; moreover, when $\delta$ is small (i.e. $n$ is large), the best response is in the interior of $E$ as well. The intuition for this result is similar to that for Proposition 2. Suppose a positive measure of types of $j$ buy at each price (i.e. the cutoff vector of $j$ is in the interior of $E$). Now at any price $p_k$, suppose the seller asks buyer $i$ whether he

wants to buy. By accepting, type $v$ of $i$ gets a certain payoff of $v - p_k$, while waiting till $p_{k+1}$ involves facing ambiguity, and therefore the payoff is $(1-\varepsilon)(v - p_{k+1})H_k^i$ where $H_k^i$ is the probability of being offered $p_{k+1}$ conditional on rejecting $p_k$. It follows from Proposition 2 that when the difference between prices is small (i.e. $\delta$ small), there are types of $i$ who find it optimal to buy at $p_k$ rather than wait.

Therefore, for small $\delta$, the interior of $E$ maps to the interior of $E$ in a continuous manner. However, the same is not true for boundary points. In the example above, suppose that the strategy of a buyer (say $j$) is that "all types buy only at $p_3$." Since no type of $j$ buys either at $p_1$ or at $p_2$, we have $v_0 = v_1 = v_2 = 1$. Such a cut-off vector is clearly not in the interior of $E$ (in figure 3 the corresponding $(v_1, v_2)$ is the point $(1,1)$). For such "border points" the best response by $i$ clearly does not involve buying at $p_1$.

But as Proposition 2 shows (and as we have argued in the preceding paragraphs), when $\delta$ is small, for any $\mathbf{v}$ in the interior of $E$, the best response is in the interior of $E$ as well. This therefore creates a discontinuity in the best response when $\delta$ is small.

To be specific, consider a sequence of vectors $\mathbf{v}$ that converge to some point in the boundary (e.g. the point $(1,1)$ in the example). For every point along the sequence, the best response involves a measure of types buying at all prices, where the measure is uniformly bounded away from zero.[37] But at a boundary point the best response does involve not buying at some prices so the measure of types buying at some prices drops discontinuously to zero.[38]

Given this discontinuity, Brouwer's theorem cannot be used to show existence. However, there is an easy solution to this problem.

Now from Proposition 2, for any $\mathbf{v}$ in the interior of $E$, the $k$-th element of the best response cut-off vector (denoted by $y_k(\mathbf{v})$) is given by

$$y_k(\mathbf{v}) = p_k + \Delta_k \frac{(1-\varepsilon)H_k(\mathbf{v})}{1-(1-\varepsilon)H_k(\mathbf{v})}$$

Consider any cut-off vector $\widetilde{\mathbf{v}}$ on the border of $E$ at which the best response function is discontinuous (say the best response is to not buy at price $p_k$). We then replace this with a "pseudo best response" function which is a continuous extension of the best

[37] As long as *some* types of the other buyer buys at some price, while calculating the expected benefit and expected cost from waiting, a buyer puts at least $\varepsilon$ weight on the worst outcome.

[38] See Nishimura and Ozaki (2004) for a different kind of fix to the similar problem that the Choquet capacity resulting from epsilon contamination preferences is discontinuous.

response function from the interior of $E$ to the relevant border point of $E$. Specifically, let $\widehat{y}_k(\widetilde{\mathbf{v}})$ be such that

$$\widehat{y}_k(\widetilde{\mathbf{v}}) = p_k + \Delta_k \frac{(1-\varepsilon)}{\varepsilon}$$

Clearly, $\lim_{\mathbf{v}\to\widetilde{\mathbf{v}}} y_k(\mathbf{v}) = \widehat{y}_k(\widetilde{\mathbf{v}})$. Thus replacing $y$ by $\widehat{y}$ at the discontinuity points preserves continuity of the best response mapping.

With this in mind, suppose we propose the following "pseudo best response" function $\Psi : E \to E$. For any strategy with a cut-off vector in the interior in $E$, the function $\Psi$ coincides with the actual best response function. However, for any point on the border of $E$ at which the actual best response function is discontinuous, the function coincides with the pseudo best response function.

The function $\Psi$ is constructed to be continuous, and therefore by Brouwer's theorem it has a fixed point. Since the best response to any $\mathbf{v}$ in the interior of $E$ is in the interior of $E$ and away from the boundary, $\Psi$ maps any boundary point to the interior as well. Since $\Psi(\mathbf{v})$ is in the interior of $E$ for all $\mathbf{v} \in E$, the fixed point must be in the interior of $E$. But $\Psi$ is the true best response function for any $\mathbf{v}$ in the interior of $E$, and therefore the fixed point result establishes the existence of a symmetric equilibrium.

### A.3.2 Formal Proof

Define $A^k = [p_k, 1]$ for $k = 1, 2, \ldots, n-1$. Let $A$ be the cartesian product of $A^k$. A vector $\mathbf{x} \in A$ is of the form: $\mathbf{x} = \{x_1, \ldots, x_{n-1}\}$, such that $x_k \in [p_k, 1]$. Note that $A$ is closed and bounded and hence compact; it is also convex.

Let $B$ be the cartesian product of $[0, 1]$ taken $n-1$ times. Let $C$ be the subset of $B$ such that

$$C = \left\{ \mathbf{x} \in [\mathbf{0}, \mathbf{1}]^{n-1} \ \ | x_1 \geq x_2 \geq \ldots \geq x_{n-1} \right\}$$

$C$ is closed and bounded and hence compact. It is also convex.[39] Let $D$ be the intersection of $C$ and $A$; that is $D \equiv C \cap A$. Since $C$ and $A$ are both finite dimensional compact convex sets, $D$ is also compact and convex. Finally, we can define the set of cut-off vectors $E$:

$$E = \left\{ \mathbf{v} \in [\mathbf{0}, \mathbf{1}]^{n+1} \ \ | v_0 = 1, \{v_1, \ldots, v_{n-1}\} \in D, v_n = p_n \right\}$$

[39] If $x_k \geq x_{k+1}$ and $y_k \geq y_{k+1}$, then clearly $\lambda x_k + (1-\lambda) y_k \geq \lambda x_{k-1} + (1-\lambda) y_{k-1}$ for $\lambda \in (0, 1)$.

That is $E$ is the set of cut-off vectors $D$ with each vector augmented by an initial and final element, which are fixed at 1 and $p_n$, respectively.

Throughout the proof we assume that $\delta$ is small enough so that all previous results hold.

The following definitions are used throughout the proof.

Any vector $(v_1, \ldots, v_{n-1})$ is said to be in the interior of $D$ if $v_k > v_{k+1}$ for all $k \in \{1, \ldots, n-2\}$, and any vector in $D$ not in the interior of $D$ is said to be in the border of $D$. Any vector  is said to be in the **interior (border) of** $E$ if $(v_1, \ldots, v_{n-1})$ is in the interior (border) of $D$.

Next, similar to the term $H_k^i$ in lemma 1 (as well as in appendix **A**.1), let $H_k(\mathbf{v})$ denote the probability that the a buyer can buy at $p_{k+1}$ conditional on passing at $p_k$. As before, this is given by $H_k(\mathbf{v}) = \dfrac{F(v_k) + F(v_{k+1})}{F(v_k) + F(v_{k-1})}$.

Now, consider any $\in E$ with at least three elements coinciding, i.e. $v_{k-1} = v_k = v_{k+1}$ for some $k$ (figure 1 in section 3.4 provides an example). This implies that the buyer associated with  does not bid at $p_k$ and $p_{k+1}$, and therefore $H_k(\mathbf{v}) = 1$. This in turn implies that the best response by $i$ involves not buying at $p_k$.

Let $E_B$ denote the set of such vectors, i.e.

$$E_B \equiv \{\mathbf{v} \in E | v_{k-1} = v_k = v_{k+1} \text{for some } k \in \{1, \ldots, n-1\}\}$$

Let $E_I \equiv E \backslash E_B$.

Note that $E_B$ is a subset of the border of $E$. In the example in the previous section with $n = 3$, $E_B$ is the singleton $(1, 1, 1, p_3)$, and the corresponding point in the set $D$ in figure 3 is the corner point $(1, 1)$. $E_I$ then contains the interior of $E$ as well as those border vectors that have the property that for any $k$, at most two successive components are equal.

Let $\mathbf{y}(\mathbf{v}) \equiv \{y_0(\mathbf{v}), \ldots, y_n(\mathbf{v})\}$ denote the best response to any $\mathbf{v} \in E$.

**STEP 1:** First, consider the set of vectors in $E_I$.

We know that $y_0(\mathbf{v}) = 1$ and $y_n(\mathbf{v}) = p_n$. From Proposition 2, for any $\mathbf{v}$ in the interior of $E$, $y_k(\mathbf{v})$ is well defined, unique, and continuous in $\mathbf{v}$ for any $k \in \{1, \ldots, n-1\}$.

From Proposition 2, we also know that the best response is given by

$$y_k(\mathbf{v}) = p_k + \Delta_k \frac{(1-\varepsilon)H_k(\mathbf{v})}{1-(1-\varepsilon)H_k(\mathbf{v})} \tag{A.7}$$

Finally, consider vectors $\widetilde{\mathbf{v}} \in E_I$ that do not belong to the interior of $E$. As noted before, these are vectors in the border of $E$ such that if $v_{k-1} = v_k$ for any $k \in \{1, \ldots, n\}$, then $v_{k-2} > v_{k-1}$ whenever $k-2 \geqslant 0$, and $v_k > v_{k+1}$ whenever $k+1 \leqslant n$. In other words, these are vectors such that at any $k$ at most two successive components coincide.

Suppose $v_{k-1} = v_k$. Then, as shown in appendix **A**.1, $\phi_{k-1} = 1$, and $H_{k-1} = \pi_{k-1}$ where $\pi_{k-1} = \frac{2F(v_{k-1})}{F(v_{k-2})+F(v_{k-1})} < 1$. Further, $\pi_k = 1$, and $H_k = \phi_k$ where $\phi_k = 1/2 + 1/2\frac{F(v_k)}{F(v_{k+1})} < 1$.

Now, let $\widetilde{\mathbf{v}}$ be any vector in $E_I$ not in the interior of $E$. It follows from Proposition 2 that

$$\begin{aligned} y_{k-1}(\widetilde{\mathbf{v}}) &= p_k + \Delta_k \frac{(1-\varepsilon)\phi_{k-1}}{1-(1-\varepsilon)\phi_{k-1}} \\ y_k(\widetilde{\mathbf{v}}) &= p_k + \Delta_k \frac{(1-\varepsilon)\pi_k}{1-(1-\varepsilon)\pi_k} \end{aligned}$$

But for any such $\widetilde{\mathbf{v}}$ it is also true that $\lim_{\mathbf{v}\to\widetilde{\mathbf{v}}} H_{k-1} = \phi_{k-1}$ and $\lim_{\mathbf{v}\to\widetilde{\mathbf{v}}} H_k = \pi_k$. It follows that $\lim_{\mathbf{v}\to\widetilde{\mathbf{v}}} y_{k-1}(\mathbf{v}) = y_{k-1}(\widetilde{\mathbf{v}})$ and $\lim_{\mathbf{v}\to\widetilde{\mathbf{v}}} y_k(\mathbf{v}) = y_k(\widetilde{\mathbf{v}})$.

This proves that $\mathbf{y}(\mathbf{v})$ is continuous at any $\mathbf{v} \in E_I$.

**STEP 2:** Next consider $\mathbf{v} \in E_B$. For any such vector, there is some $k$ for which $H_k = 1$.

Note that for any $\in E_I$, $H_k < 1$. The argument in step 1 shows that for any $H_k < 1$, $y_k < y_{k-1}$. This also implies that $\lim_{H_k \to 1} y_k < y_{k-1}$ But if $H_k = 1$, $y_k = y_{k-1}$. Thus the best response mapping is discontinuous at any $\mathbf{v} \in E_B$.

To solve the problem we proceed as follows. For any $\in E_B$ we let $\widehat{y}_k(\mathbf{v})$ be a "pseudo best response" where

$$\widehat{y}_k(\mathbf{v}) = p_k + \Delta_k \frac{(1-\varepsilon)}{\varepsilon}$$

Consider any $\widetilde{\mathbf{v}} \in E_B$. From equation (A.7), clearly $\lim_{\mathbf{v}\to\widetilde{\mathbf{v}}} y_k(\mathbf{v}) = \widehat{y}_k(\widetilde{\mathbf{v}})$. Thus replacing $y$ by $\widehat{y}$ on $E_B$ preserves continuity of the best response mapping.

With this specification, the calculations in Proposition 2 can be retraced and it can be easily seen that all conclusions are exactly the same (we are simply putting $H_k = 1$

but preserving the factor $(1-\varepsilon)$, and none of the results require $H_k < 1$). In particular, note that $\widehat{y}_k < \widehat{y}_{k-1}$ for all $k \in \{1, \ldots, n\}$, and therefore the pseudo best response vector belongs in the interior of $E$.

**STEP 3:** Finally, define the mapping $\Psi : E \to E$ such that

$$\begin{aligned}
\Psi_0(\mathbf{v}) &= 1 \\
\Psi_k(\mathbf{v}) &= \begin{cases} y_k(\mathbf{v}) & \text{if } \mathbf{v} \in E_I \\ \widehat{y}_k(\mathbf{v}) & \text{if } \mathbf{v} \in E_B \end{cases} \\
\Psi_n(\mathbf{v}) &= p_n
\end{aligned}$$

Since $\Psi$ maps $E$ continuously to itself, by Brouwer's fixed point theorem, there exists a fixed point of $\Psi$, i.e. there exists $\mathbf{v}^*$ such that $\Psi(\mathbf{v}^*) = \mathbf{v}^*$.

We know from Proposition 2 that for any $\mathbf{v} \in E_I$, $\Psi(\mathbf{v})$ belongs to the interior of $E$. As noted at the end of step 2, the same is true for vectors in $E_B$. Thus the range of $\Psi$ is a subset of the interior of $E$. Therefore any fixed point must be in the interior of $E$. It follows that any fixed point must be a true mutual best response, and therefore a symmetric equilibrium.

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