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Journal Article

Version: Pre-print

Citation:

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Minimal Factorizations of Permutations Into Star Transpositions

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Abstract

We give a compact expression for the number of factorizations of any permutation into a minimal number of transpositions of the form $(1\,i)$. Our result generalizes earlier work of Pak in which substantial restrictions were placed on the permutation being factored.

Key words: factorizations, permutations, star transpositions, symmetric group

1 Introduction

It is well known that the symmetric group $\mathfrak{S}_n$ is generated by various sets of transpositions, and it is natural to ask for the number of decompositions of a permutation into a minimal number of factors from such a set. For instance, a famous paper of Dénes [1] addresses this question when the generating set is taken to consist of all transpositions. Stanley [9] has also considered the problem for the set of Coxeter generators $\{(i\ i+1) : 1 \le i < n\}$.

More recently, Pak [8] considered minimal decompositions of permutations relative to the generating set $S = \{(1\,i) : 2 \le i \le n\}$. The elements of S are called *star transpositions* because the labelled graph on vertex set $[n] =$

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$\{1, \ldots, n\}$ obtained from them by interpreting $(a\,b)$ as an edge between vertices a and b is star-shaped. Pak proves that any permutation $\pi \in \mathfrak{S}_n$ that fixes 1 and has m cycles of length $k \geq 2$ admits exactly

$$\frac{k^m(mk+m)!}{n!} \tag{1}$$

decompositions into the minimal number $n + m - 1$ of star transpositions. He leaves open the problem of extending (1) to more general target permutations π, and it is the purpose of this paper to answer this question.

Our result is best expressed in terms of *minimal transitive star factorizations*, which we now define. A *star factorization* of $\pi \in \mathfrak{S}_n$ of *length* r is an ordered list $f = (\tau_1, \ldots, \tau_r)$ of star transpositions τ_i such that $\tau_1 \cdots \tau_r = \pi$.[1] We say f is *minimal* if π admits no star factorization of length less than r, and *transitive* if the group generated by its factors acts transitively on $[n]$.

Observe that a permutation $\pi = (1\ b_2 \cdots b_{\ell_1})(a_1^2 \cdots a_{\ell_2}^2) \cdots (a_1^m \cdots a_{\ell_m}^m) \in \mathfrak{S}_n$ with m cycles admits the transitive star factorization

$$\pi = \underbrace{(1\ b_{\ell_1})(1\ b_{\ell_1-1}) \cdots (1\ b_2)}_{=(1\ b_2 \cdots b_{\ell_1})} \underbrace{(1\ a_1^2)(1\ a_{\ell_2}^2)(1\ a_{\ell_2-1}^2) \cdots (1\ a_1^2)}_{=(a_1^2 \cdots a_{\ell_2}^2)} \cdots$$
$$\underbrace{(1\ a_1^m)(1\ a_{\ell_m}^m)(1\ a_{\ell_m-1}^m) \cdots (1\ a_1^m)}_{=(a_1^m \cdots a_{\ell_m}^m)}$$

of length $\ell_1 - 1 + \sum_{i=2}^m (\ell_i + 1) = n + m - 2$. Moreover, it is well known [4, Proposition 2.1] that *any* transitive star factorization of π requires at least this many factors.[2] Thus a transitive star factorization of π of length exactly $n + m - 2$ is said to be *minimal transitive*.

Our main result is the following:

Theorem 1 *Let $\pi \in \mathfrak{S}_n$ be any permutation with cycles of lengths $\ell_1, \ldots, \ell_m$. Then there are precisely*

$$\frac{(n+m-2)!}{n!} \ell_1 \cdots \ell_m$$

minimal transitive star factorizations of π.

Notice that Pak's formula (1) is recovered from Theorem 1 by setting $\ell_1 = 1$ and $\ell_2 = \cdots = \ell_{m+1} = k$ and observing that a star factorization of a permutation with no fixed points other than (possibly) 1 must be transitive, since $\pi(a) \neq a$ means any star factorization of π involves the factor $(1\,a)$.

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[1] We multiply permutations in the usual order, so $\rho\sigma(j) = \rho(\sigma(j))$.
[2] In fact, this holds true when arbitrary transposition factors are allowed.

Given the special role played by the symbol 1 in star factorizations, the lack of bias towards this symbol in the enumerative formula of Theorem 1 is quite surprising. Indeed, this symmetry is a very compelling aspect of the theorem, and it is not yet understood.

Consider now a permutation π having fixed points $i_1, \ldots, i_k$ and possibly 1. A minimal (though not transitive) star factorization of π should certainly *not* contain any of the factors $(1\,i_1), \ldots, (1\,i_k)$. Indeed, π naturally induces a permutation π' on $[n] \backslash \{i_1, \ldots, i_k\}$ having no fixed points other than (possibly) 1, and minimal star factorizations of π are simply minimal transitive star factorizations of π'. Since π' has $m - k$ cycles when π has m cycles, we obtain the following result by setting $n = n - k$ and $m = m - k$ in Theorem 1.

Corollary 2 *Let $\pi \in \mathfrak{S}_n$ be any permutation with cycles of lengths $\ell_1, \ldots, \ell_m$ including exactly k fixed points not equal to 1. Then there are*

$$\frac{(n + m - 2(k+1))!}{(n-k)!} \ell_1 \cdots \ell_m$$

minimal star factorizations of π.

We prove Theorem 1 in two stages. In Section 2, we begin by giving a complete characterization of minimal transitive star factorizations (Lemma 5). We then use this characterization in Section 3 to build a correspondence between star factorizations and certain restricted words, finally using the cycle lemma to count these words and hence prove Theorem 1.

This path to Theorem 1 is deliberately similar to that followed in [8]. However, in Section 4, we briefly describe an elegant graphical approach to this problem that employs the well-known connection between factorizations of permutations and embeddings of graphs on surfaces (*i.e.* maps). Finally, Section 5 contains some brief comments on recent extensions of Theorem 1 and its curious symmetry.

2   Characterizing Star Factorizations

Throughout this section we have in mind a fixed permutation $\pi \in \mathfrak{S}_n$ and a minimal transitive star factorization $f = (\tau_1, \ldots, \tau_r)$ of π.

Our arguments are best understood with a concrete example at hand. For this purpose, we will often refer to the factorization

$$\overset{\tau_1}{(1\,9)}\overset{\tau_2}{(1\,11)}\overset{\tau_3}{(1\,9)}\overset{\tau_4}{(1\,2)}\overset{\tau_5}{(1\,10)}\overset{\tau_6}{(1\,5)}\overset{\tau_7}{(1\,3)}\overset{\tau_8}{(1\,3)}\overset{\tau_9}{(1\,4)}\overset{\tau_{10}}{(1\,7)}\overset{\tau_{11}}{(1\,6)}\overset{\tau_{12}}{(1\,6)}\overset{\tau_{13}}{(1\,10)}\overset{\tau_{14}}{(1\,8)} \qquad (2)$$

of
$$\pi = (1\ 8\ 2)(3)(4\ 5\ 10\ 7)(6)(9\ 11) \in \mathfrak{S}_{11}. \tag{3}$$

Let us say that a transposition $(1\,i)$ *meets* a cycle σ (and vice versa) if σ contains the symbol i. We say a factor τ_i is to the *left* of τ_j if $i < j$, and to the *right* if $j > i$. So, in the example above, $(1\,5)$ meets $(4\ 5\ 10\ 7)$, and $(1\,2)$ is to the left of $(1\,7)$. Clearly every star transposition meets exactly one cycle of π.

Our goal here is to characterize minimal transitive star factorizations of π.

Lemma 3 *Let σ be a cycle of π.*

(1) *If $\sigma = (a_1\,a_2\,\ldots\,a_\ell)$, where $a_i \neq 1$ for all i, then some transposition $(1\,a_j)$ appears exactly twice in f, while all transpositions $(1\,a_i)$ with $i \neq j$ appear exactly once.*

(2) *If $\sigma = (1\,b_2\,\cdots\,b_\ell)$, then each transposition $(1\,b_i)$ appears only once in f.*

Moreover, in the first case, if $(1\,a_1)$ appears twice, then the factors of f meeting σ appear in left-to-right order $(1\,a_1),(1\,a_\ell),\ldots,(1\,a_2),(1\,a_1)$. In the second case, the factors meeting σ appear in left-to-right order $(1\,b_\ell),(1\,b_{\ell-1}),\ldots,(1\,b_2)$.

PROOF. Suppose $\sigma = (a_1\,a_2\,\ldots\,a_\ell)$ with $a_i \neq 1$. It is clear that for f to be transitive every transposition $(1\,a_i)$ must appear at least once as a factor. Let $(1\,a_j)$ be the leftmost (last in order of multiplication) factor of f that meets σ. If $(1\,a_j)$ appeared only this once, then we would have $\pi = \pi_1\,(1\,a_j)\,\pi_0$, where π_0 fixes a_j and π_1 fixes all a_i. In particular, $\sigma(a_j) = \pi(a_j) = \pi_1(1) \neq a_i$ for any i, a contradiction.

On the other hand, if $\sigma = (1\,b_2\,\cdots\,b_\ell)$ then again transitivity requires that f contain factors $(1\,b_2),\ldots,(1\,b_\ell)$. So if the cycles of π are $\sigma_1,\ldots,\sigma_m$, where σ_1 contains symbol 1, then σ_i meets at $\ell_i + 1$ factors of f for $i \neq 1$, while σ_1 meets at least $\ell_1 - 1$ factors. That is, f has at least $(\ell_1 - 1) + \sum_{i=2}^{m}(\ell_i + 1) = n + m - 2$ factors. But since f is minimal transitive, it has exactly this many factors. Hence all factors are accounted for and parts (1) and (2) of the lemma follow. It remains to determine the relative ordering of the factors meeting σ.

We return to the case $\sigma = (a_1\,a_2\,\ldots\,a_\ell)$ with $a_i \neq 1$, and assume without loss of generality that $(1\,a_1)$ appears twice in f. The proof given above identified $(1\,a_1)$ is the leftmost factor of f meeting σ. However, reading the factors of f in reverse order yields a factorization f' of π^{-1}, and the same logic now identifies $(1\,a_1)$ as the leftmost factor of f' meeting σ. Thus $(1\,a_1)$ appears in f in the leftmost and rightmost positions amongst all factors meeting σ. Finally, note that for $1 \leq i < \ell$ the factor $(1\,a_{i+1})$ is to the left of the rightmost occurrence of $(1\,a_i)$ in f, as otherwise we would have $\pi = \pi_1\,(1\,a_i)\,\pi_0$, where π_0 fixes a_i and π_1

fixes a_{i+1}, and this gives the contradiction $\sigma(a_i) = \pi(a_i) = \pi_1(1) \neq a_{i+1}$. It follows that the factors meeting σ appear in order $(1\,a_1), (1\,a_2), \dots, (1\,a_\ell), (1\,a_1)$.

If instead $\sigma = (1\,b_2 \cdots b_\ell)$, then the same logic just applied shows that for $2 \leq i < \ell$, the factor $(1\,b_{i+1})$ appears to the left of $(1\,b_i)$ in f. Thus the factors meeting σ appear in order $(1\,b_2), (1\,b_3), \dots, (1\,b_\ell)$, as claimed. $\square$

The next lemma asserts that the factors of a minimal transitive star factorization are nested in a well defined manner. This "non-crossing" property makes it unsurprising that such factorizations can be encoded as trees. (See Section 4 for details.) Note that one immediate consequence of the lemma is that all the factors meeting one of the cycles of π must appear consecutively.

Lemma 4 *Let σ and $\hat{\sigma}$ be distinct cycles of π. Suppose there exist $s < v < t$ such that factors τ_s and τ_t of f meet σ while τ_v meets $\hat{\sigma}$. Then $\hat{\sigma}$ does not contain the symbol 1, and all τ_j that meet $\hat{\sigma}$ have $s < j < t$.*

PROOF. Without loss of generality we may assume that indices s and t are "extremal", in the sense that if $\tau_{s'}$ and $\tau_{t'}$ meet the same cycle of π, with $s < s' < v < t' < t$, then this common cycle is $\hat{\sigma}$. (If not, simply restart by letting σ be the common cycle and replacing s and t with s' and t'.)

Let $\tau_s = (1\,b)$ and $\tau_t = (1\,a)$. Since τ_s and τ_t are assumed to meet the same cycle of π, Lemma 3 implies τ_s is the leftmost copy of $(1\,b)$ in f, τ_t is the rightmost copy of $(1\,a)$ in f, and $\pi(a) = b$. It follows from these criteria that the permutation $\tau_{s+1} \cdots \tau_{t-1}$ fixes 1, and therefore

$$i := \max\{k \ : \ k \leq v \text{ and } \tau_k \cdots \tau_{t-1} \text{ fixes } 1\}$$

is well defined. Say $\tau_i = (1\,c)$. Notice that this factor must occur twice amongst those of $\gamma = \tau_i \cdots \tau_{t-1}$, as otherwise $\gamma(c) = 1$ and hence γ does not fix 1, contrary to the definition of i.

Suppose $\tau_j = \tau_i = (1\,c)$ for some $j > i$. Then Lemma 3 implies c cannot appear in any factor between τ_i and τ_j, so the permutation $\tau_i \cdots \tau_j$ fixes 1. But since $\tau_i \cdots \tau_{t-1}$ fixes 1, it follows that $\tau_{j+1} \cdots \tau_{t-1}$ also fixes 1. Thus the maximality of i forces $j \geq v$. However, if $j > v$ then we have two identical factors τ_i and τ_j, with $s < i < v < j < t$, that meet the same cycle of π, and by hypothesis this common cycle must be $\hat{\sigma}$. In this case, Lemma 3 rules out the possibility of $\hat{\sigma}$ containing symbol 1 (because no transposition meeting the cycle containing 1 can appear twice in f), and also implies any other factor of f that meets $\hat{\sigma}$ lies between τ_i and τ_j, as desired. The remaining case is $j = v$, in which τ_v occurs twice between τ_s and τ_t. Again the result follows from Lemma 3. $\square$

The statements of Lemmas 3 and 4 are crafted with the implicit assumption that $f = (\tau_1, \ldots, \tau_r)$ is a minimal transitive star factorization of π. We now show that this can, in fact, be deduced from the conditions on f established by the lemmas. That is to say, if f is a star factorization whose factors are related to the permutation π in the manner described by the lemmas above, then f is necessarily a minimal transitive star factorization of π.

Lemma 5 *The conditions on f guaranteed by Lemmas 3 and 4 characterize minimal transitive star factorizations of π.*

PROOF. Let $f' = (\tau_1', \ldots, \tau_r')$ be a star factorization that satisfies the conditions described by Lemmas 3 and 4. For brevity we shall refer to these conditions as C1 and C2, respectively. Suppose the cycles of π are $\sigma_1, \ldots, \sigma_m$, with σ_1 containing symbol 1. We wish to show $\pi' = \pi$, where $\pi' := \tau_1' \cdots \tau_r'$. (Note that the transitivity and minimality of f' are then immediately implied by C1.)

If π has only one cycle, say $\pi = (1\, b_2\, \cdots\, b_n)$, then C1 implies $\pi' = \tau_1' \cdots \tau_r' = (1\, b_n)(1\, b_{n-1}) \cdots (1\, b_2) = \sigma_1$. Hence $\pi = \pi'$ in this case. Otherwise, by C2 there exists some cycle $\sigma_j = (a_1\, \cdots\, a_k) \neq \sigma_1$ of π such that the factors τ_i' that meet σ_j occur contiguously in f'. By C1 this means that for some s we have

$$\tau_s' \tau_{s+1}' \cdots \tau_{s+k}' = (1\, a_1)(1\, a_k) \cdots (1\, a_1) = (a_1\, a_2\, \cdots\, a_k),$$

and no factors of f' other than $\tau_s', \ldots, \tau_{s+k}'$ meet σ_j. Thus π' agrees with π on $S := \{a_1, \ldots, a_k\}$, and $f'' = (\tau_1', \ldots, \tau_{s-1}', \tau_{s+k+1}', \ldots, \tau_r')$ is a star factorization of $\pi'' := \pi'|_{[n] \setminus S}$. But f'' satisfies C1 and C2 relative to the permutation $\pi|_{[n] \setminus S}$, so we can iterate this argument to see that π' agrees with π on all of $[n]$. $\square$

3   Counting Star Factorizations

Let $\pi \in \mathfrak{S}_n$ be a permutation with cycles $\sigma_1, \ldots, \sigma_m$, listed in increasing order of least element (in particular, σ_1 contains symbol 1). Set $r := n + m - 2$, and let $f = (\tau_1, \ldots, \tau_r)$ be a minimal transitive star factorization of π. Define the word $w = w_1 \cdots w_r \in [m]^r$ by setting $w_i = j$ if τ_i meets σ_j. Moreover, for $2 \leq j \leq m$, define k_j by the condition that the rightmost factor of f meeting σ_j is $(1\, k_j)$

Example 6 *Consider f and π as defined in (2) and (3). Under each factor τ_i, we write the unique value of j such that τ_i meets σ_j, and we distinguish the*

rightmost occurrence of each symbol $j \geq 2$:

$$(1\,9)(1\,11)(1\,9)(1\,2)(1\,10)(1\,5)(1\,3)(1\,3)(1\,4)(1\,7)(1\,6)(1\,6)(1\,10)(1\,8)$$

5	5	**5**	1	3	3	2	**2**	3	3	4	**4**	**3**	1

This yields the word

$$w = 5\,5\,5\,1\,3\,3\,2\,2\,3\,3\,4\,4\,3\,1,$$

while the transpositions in the distinguished positions give the values of k_j, in this case $(k_2, k_3, k_4, k_5) = (3, 10, 6, 9)$. $\square$

Let $\mathcal{W}_\pi \subset [m]^r$ be the set of words such that

- 1 appears $\ell_1 - 1$ times,
- j appears $\ell_j + 1$ times for $2 \leq j \leq m$, and
- there are no occurrences of the subwords $abab$ or $a1a$ for distinct $a, b \neq 1$.

If $\mathcal{O}_1, \ldots, \mathcal{O}_m$ are the orbits of π, listed in increasing order of least element, then the correspondence described above is clearly one-one between tuples $(w, k_2, \ldots, k_m) \in \mathcal{W}_\pi \times \mathcal{O}_2 \times \cdots \times \mathcal{O}_m$ and star factorizations of π satisfying the conditions of Lemmas 3 and 4. Lemma 5 then establishes that this is, in fact, a bijection between such tuples and the set $\mathcal{F}_\pi$ of all minimal transitive star factorizations of π. Thus we have

$$|\mathcal{F}_\pi| = |\mathcal{W}_\pi| \cdot |\mathcal{O}_2| \cdots |\mathcal{O}_m|. \tag{4}$$

We are now in a position to prove Theorem 1.

Proof of Theorem 1: Assume the notation above, and for convenience let $\ell_j = |\mathcal{O}_j|$, for $j = 1, \ldots, m$. Consider the set of sequences $(d_0, d_1, \ldots, d_r)$ whose entries d_i are either 1 or $-\ell_j^{(j)}$ for some $j \geq 2$, where the exponent (j) is considered to be a decoration. Of these, let $\mathcal{D}_\pi$ be the subset satisfying the following properties:

- $-\ell_j^{(j)}$ appears exactly once, for $2 \leq j \leq m$, and
- all partial sums (ignoring decorations) are positive.

Thus $\mathcal{D}_\pi$ describes a type of decorated Dyck sequence. For example, with π defined by (3), we have

$$(1, 1, 1, -2^{(5)}, 1, 1, 1, 1, -1^{(2)}, 1, 1, 1, -1^{(4)}, -4^{(3)}, 1) \in \mathcal{D}_\pi. \tag{5}$$

Define $\Psi_\pi : \mathcal{W}_\pi \longrightarrow \mathcal{D}_\pi$ by the following rule. The image $(d_0, d_1, \ldots, d_r)$ of the word $w_1 \cdots w_r \in \mathcal{W}_\pi$ is given by

(1) $d_0 = 1,$

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(2) if $w_i = 1$ then $d_i = 1$,

(3) if $w_i = j$, for $j \geq 2$, then

   (a) $d_i = -\ell_j^{(j)}$ when w_i is the rightmost occurrence of j in w

   (b) $d_i = 1$ otherwise.

For instance, sequence (5) is the image under Ψ_π of the word w of Example 6.

Clearly Ψ_π is well defined. It is also easily seen to be bijective. Indeed, suppose $\mathbf{d} = (d_0, d_1, \ldots, d_r) \in \mathcal{D}_\pi$, and let $d_i = -\ell_j^{(j)}$ be the first negative entry of $\mathbf{d}$. Note that the value of j is known via the decoration. Remove d_i and the previous ℓ_j 1s from $\mathbf{d}$ to obtain a new sequence $\mathbf{d}'$. Inductively, $\mathbf{d}' = \Psi_{\pi'}(w')$ for a unique word $w' \in \mathcal{W}_{\pi'}$, where π' is the permutation obtained from π by removing cycle σ_j. Adding $\ell_j + 1$ copies of j after $w'_{i-\ell_j-1}$ gives the unique desired preimage $w = \Psi_\pi^{-1}(\{\mathbf{d}\})$.

Thus we have $|\mathcal{W}_\pi| = |\mathcal{D}_\pi|$. So we now turn to enumerating $\mathcal{D}_\pi$. Our main tool is the cycle lemma of Dvoretzky and Motzkin [2], one version of which states that any sequence with integral entries ≤ 1 and total sum $s \geq 0$ has exactly s cyclic rotations with all partial sums positive.

Any sequence in $\mathcal{D}_\pi$ has terms $-\ell_2^{(2)}, \ldots, -\ell_m^{(m)}$ along with $r + 1 - (m - 1) = n$ entries equal to 1. Note that there are $(n + m - 1)!/n!$ sequences with exactly these terms. The sequences $(d_0, d_1, \ldots, d_r) \in \mathcal{D}_\pi$ we wish to count are characterized by having total sum (ignoring decorations)

$$\sum_{i=0}^{r} d_i = n \cdot 1 - (\ell_2 + \cdots + \ell_m) = n - (n - \ell_1) = \ell_1$$

with all partial sums positive. Since a sequence of length $n + m - 1$ admits $n + m - 1$ cyclic rotations, the cycle lemma implies that

$$|\mathcal{D}_\pi| = \frac{(n + m - 1)!}{n!} \cdot \frac{\ell_1}{n + m - 1}.$$

Theorem 1 now follows from identity (4), since $|\mathcal{W}_\pi| = |\mathcal{D}_\pi|$ and $|\mathcal{O}_j| = \ell_j$. $\square$

4   A Graphical Approach

Transitive factorizations in the symmetric group are well known to be in correspondence with certain classes of labelled maps, and our characterization of star factorizations (Lemmas 3, 4, and 5) can be derived elegantly through this connection. We now briefly describe how this is done, using a version of the factorization-map correspondence introduced in [6]. Indeed, it was by this method that Theorem 1 was originally discovered. We elected to frame
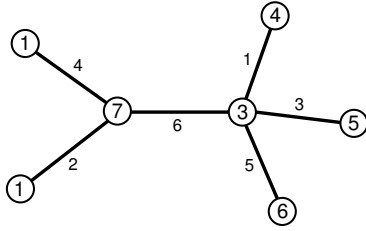
Fig. 1. The planar map corresponding to $(1\,2\,3\,4\,5\,6\,7) = (2\,5)(3\,6)(2\,7)(3\,5)(1\,7)(3\,4)$.
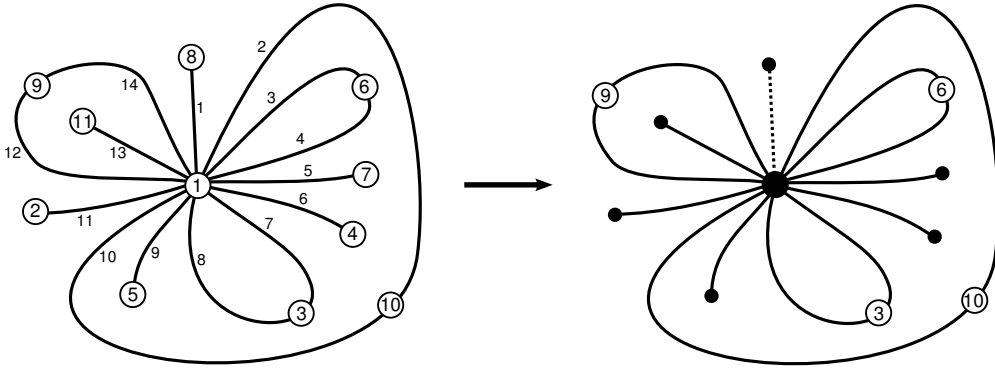

Fig. 2. The planar map corresponding to a star factorization, and its reduced form.

our proof in Pak's techniques to demonstrate how they generalize and to keep this paper self contained. We note that an alternative formulation of the factorization-map correspondence, developed with great effect in [7], can be applied here with equal ease.

Let $f = (\tau_1, \ldots, \tau_r)$ be a transitive factorization of $\pi \in \mathfrak{S}_n$, where the factors τ_i are arbitrary transpositions. Then f naturally induces a graph G_f on n labelled vertices and r labelled edges, as follows: the vertex set of G_f is $[n]$, and there is an edge with label i between vertices a and b whenever $\tau_i = (a\,b)$. The transitivity of f ensures G_f is connected, so G_f admits a 2-cell embedding in an orientable surface of minimal genus. A unique such map M_f is determined by insisting that the edge labels encountered on clockwise traversals of small circles around the vertices are cyclically increasing. Faces of M_f correspond with the cycles of π, and the Euler-Poincaré formula thus implies M_f *is planar precisely when f is minimal transitive.*

Example 7 *The factorization $(1\,2\,3\,4\,5\,6\,7) = (2\,5)(3\,6)(2\,7)(3\,5)(1\,7)(3\,4)$ is minimal transitive. Its corresponding planar map is shown in Figure 1.* $\square$

The maps corresponding to minimal transitive star factorizations are particularly simple. This is illustrated in Figure 2, where the planar map associated with our primary example factorization (2) is drawn.

Since such a map must be planar with edge labels increasing clockwise around the central vertex 1, no edge $\{1, a\}$ can appear more than twice. When two
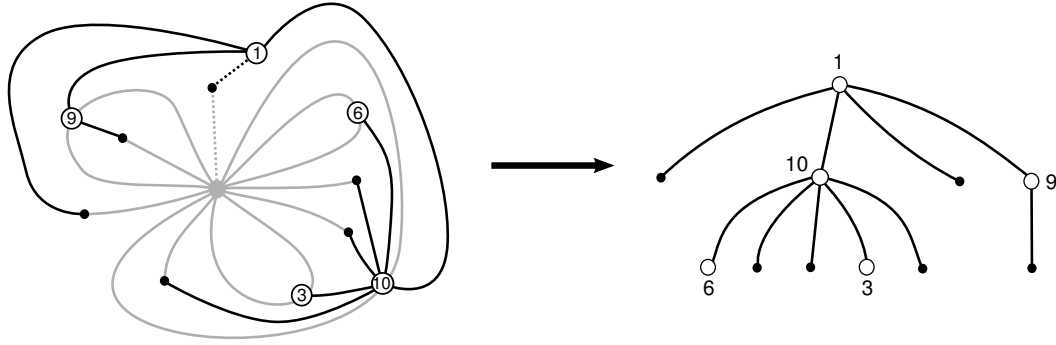
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Fig. 3. From maps to trees.

copies of $\{1, a\}$ are present they enclose a face of the map. It is this face that is associated with the cycle of the target permutation containing symbol a, and a vertex b of degree one lies within it precisely when b belongs to this same cycle. Translated from the language of maps to that of factorizations, these observations are equivalent to Lemmas 3 and 4.

The canonical labelling of edges around the central vertex makes all but one of the edge labels superfluous. Moreover, the labels of all vertices of degree 1 may be deduced from the target permutation and the labels of the other vertices. Thus all maps corresponding to minimal transitive star factorizations may be reduced in the manner demonstrated on the right of Figure 2.

From this reduced form, create a tree as follows. Begin by placing a vertex with label 1 in the outer face, so that every labelled vertex is naturally associated with one face of the map. Then draw an edge between each labelled vertex and all (non-central) vertices that are incident with its associated face. See Figure 3 for an example.

This transition from factorization to map to tree is reversible. A minimal transitive factorization of a permutation $\pi \in \mathfrak{S}_n$ with orbits $\mathcal{O}_1, \ldots, \mathcal{O}_m$ (listed, as usual, in increasing order of least element) corresponds with a tree on m labelled white vertices and $n - m$ black vertices in which

(1) the root is white with label 1,
(2) the non-root white vertices are labelled $\{a_2, \ldots, a_m\}$, where $a_j \in \mathcal{O}_j$,
(3) the white vertex with label a_j has $|\mathcal{O}_j| - 1$ black children, for $j = 1, \ldots, m$.

Such trees can be encoded using Dyck-type sequences, as follows: traverse the boundary, beginning at the root and proceeding clockwise, writing 1 whenever a vertex is encountered for the first time, and $-i^{(j)}$ when a white vertex with label $j \geq 2$ and $i - 1$ black children is encountered for the last time. For instance, the tree in Figure 3 yields the following sequence (compare with (5)):

$$(1, 1, 1, -2^{(9)}, 1, 1, 1, 1, -1^{(3)}, 1, 1, 1, -1^{(6)}, -4^{(10)}, 1).$$

These sequences are counted as in Section 3 to yield Theorem 1.

5 Further Questions

Notice that Theorem 1 asserts that the number of minimal transitive star factorizations of a permutation π depends only on the conjugacy class of π (that is, the length of its cycles). This is certainly not obvious from the formulation of the problem, since one would certainly expect that the length of the cycle of π containing symbol 1 would play a special role.

Moreover, while this article was being refereed, Goulden and Jackson [5] extended Theorem 1 to compute the number of transitive star factorizations of any permutation into an *arbitrary* number of factors (that is, minimality is not assumed). Interestingly, they witness the same symmetry in their results: the number of transitive star factorizations of π of length r is dependent only on the conjugacy class of π.

Finding a simple combinatorial explanation for this curious symmetry remains an interesting open problem. Further open questions regarding star factorizations and their role in the general interplay between factorizations and geometry are discussed in [5].

Acknowledgements

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