BIROn - Birkbeck Institutional Research Online

Schröder, David (2020) Real options, ambiguity, and dynamic consistency - a technical note. International Journal of Production Economics 229 (107772), ISSN 09255273.

Real Options, Ambiguity, and Dynamic Consistency – A Technical Note^{†}

David Schröder
Birkbeck College (University of London)*

Abstract

Recent research on real options does not only consider optimal investment decisions under risk, but also under ambiguity. However, most models that allow for ambiguity are generally not dynamically consistent. Examples are, among others, the α-MEU model, the imprecision aversion model, or the NMEU model. Dynamic consistency is however required to solve optimal stopping real options problems analytically or in closed-form. This paper highlights the resulting difficulties, which are often overlooked, exemplarily for the NMEU model.

JEL Classification: D81, G11

Keywords: Ambiguity; dynamic consistency; real options; NMEU preferences; rectangularity; optimal stopping

1   Introduction

In the last decades, the real options literature has seen a tremendous development. Following the seminal works by McDonald and Siegel (1986) and Dixit and Pindyck (1994), the standard real options model has been applied to many areas in Economics and Finance, and has improved our understanding of the firms' investment decisions.

Real options timing and investment problems are usually modelled as optimal stopping problems in continuous time with an infinite time horizon. As Dixit and Pindyck (1994) show, the recursive structure of optimal stopping problems allows to transform them into non-stochastic Hamilton-Jacobi-Bellman partial differential equations (HJB PDE), which can often be solved analytically. While the real option approach traditionally focusses on assessing optimal investments under risk, the work by Nishimura and Ozaki (2007) is the first to explore investment under another type of uncertainty, called ambiguity. Different from risk, which refers to a situation with an uncertain outcome where the probabilities for each of the outcomes are known, ambiguity denotes the absence of accurate information on probabilities.

The contribution of Nishimura and Ozaki (2007) is to show that the combination of the maximin expected utility (MEU) model by Gilboa and Schmeidler (1989) together with updating rule for probabilities of strong rectangularity (Chen and Epstein, 2002; Epstein and Schneider, 2003) allows preserving the recursive structure of optimal stopping problems under ambiguity.[1] The optimal investment rule is then derived by transforming the optimal stopping problem into a HJB PDE, similar to Dixit and Pindyck (1994).

However, most models that allow for ambiguity are not dynamically consistent. Examples include the $\alpha-$MEU model by Ghiradato et al. (2004) as employed by Peijnenburg (2018), or the NMEU model, as recently used in Gao et al. (2018). Dynamic consistency means that an ex-ante optimal intertemporal (i.e., dynamic) decision plan is still optimal ex-post, as time goes by. Put differently, a decision maker will never revise the ex-ante optimal plan at later points of time.[2] If preferences are not dynamically consistent, however, the recursive structure of optimal stopping problems is generally lost. This paper highlights the resulting difficulties, which are often overlooked, exemplarily for the NMEU model.

In a recent paper, Gao et al. (2018) examine the effect of ambiguity aversion on price negotiations between buyers and sellers of goods and services. They extend the bilateral negotiation results of Moon et al. (2011) to the case of ambiguity, and analyze how ambiguity and network structures might affect buyer-seller interactions. To derive the optimal negotiation strategy of buyers and sellers, the paper relies on the real options approach. They show that higher ambiguity aversion raises the threshold for commitment for the seller, and that it has equivocal effects for the buyer's negotiation prospects.

Gao et al. (2018) model preferences under ambiguity using the multiple-priors expected utility

---

[1]The notion of rectangularity is introduced by Chen and Epstein (2002) and Epstein and Schneider (2003). Nishimura and Ozaki (2007) refine the concept by introducing the notion of strong rectangularity.

[2]See Hill (2020) for a detailed discission on dynamic consistency.

with non-extreme outcomes (NMEU). This preference model consists of taking a convex combination of the minimum expected utility with respect to the set of priors (worst case scenario, similar to Gilboa and Schmeidler (1989)), and the standard expected utility with respect to a reference probability measure (objective scenario):

$$NMEU\,[x] = \rho \inf_{p \in \mathcal{P}} E^p\,[x] + (1 - \rho)E\,[x] \tag{1}$$

where $\mathcal{P}$ is a set of probability distributions and ρ is a preference parameter denoting the degree of ambiguity aversion. If $\rho = 0$, the model coincides with expected utility; if $\rho = 1$ the model is equivalent to the maxmin expected utility by Gilboa and Schmeidler (1989).[3]

Following Nishimura and Ozaki (2007), Gao et al. (2018) express the seller's and buyer's negotiating problem as an optimal stopping problem in continuous time, and transform it into an HJB PDE. However, since NMEU preferences are generally not dynamically consistent, the optimal stopping problems lack their recursive structure. As a result, the real options problem cannot be solved analytically unless $\rho = 0$ or $\rho = 1$.

The next section of this note provides a formal argument why NMEU preference are generally not dynamically consistent. Finally, section 3, discusses the implications of dynamic inconsistency on optimal stopping problems, as used in the real options literature.

2 Dynamic inconsistency of NMEU preferences

Dynamic consistency of the expected utility model is due to the law of iterated expectations (Epstein and Schneider, 2003):

$$E_s[x_u] = E_s[E_t[x_u]] \qquad \forall s < t < u. \tag{2}$$

Put differently, the expected value of a random variable x at time u should be the same regardless whether the expectation is taken as of time s, or whether the expectation is first taken as of time t, and then that expectation is taken at an earlier time s. For NMEU preferences to be dynamically consistent, and equivalent condition would need to hold:

$$NMEU_s[x_u] = NMEU_s[NMEU_t[x_u]] \qquad \forall s < t < u. \tag{3}$$

This is generally not the case. Using (1), the term on the left hand side can be expanded into

$$NMEU_s\,[x_u] = \rho \inf_{p \in \mathcal{P}} E_s^p\,[x_u] + (1 - \rho)E_s\,[x_u],$$

while the term on the right is given by

---

[3]Gao et al. (2018) do not provide an axiomatic foundation of their ambiguity preference model. However, the model shares some similarity to the utility representation by Kopylov (2016). The NMEU can also be viewed as a special case of the imprecision aversion model by Gajdos et al. (2008). Gao et al. (2018) introduce the NMEU model in continuous time, but for ease of exposition this paper uses a discrete-time framework.

$$
\begin{aligned}
NMEU_s\left[NMEU_t\left[x_u\right]\right] &= \rho \inf_{p\in\mathcal{P}} E_s^p\left[NMEU_t\left[x_u\right]\right] + (1-\rho)E_s\left[NMEU_t\left[x_u\right]\right] \\
&= \rho \inf_{p\in\mathcal{P}} E_s^p\left[\rho \inf_{p\in\mathcal{P}'} E_t^p\left[x_u\right] + (1-\rho)E_t\left[x_u\right]\right] + \\
&\quad (1-\rho)E_s\left[\rho \inf_{p\in\mathcal{P}'} E_t^p\left[x_u\right] + (1-\rho)E_t\left[x_u\right]\right] \\
&= \rho^2 \inf_{p\in\mathcal{P}} E_s^p\left[x_u\right] + \\
&\quad \rho \inf_{p\in\mathcal{P}} E_s^p\left[(1-\rho)E_t\left[x_u\right]\right] + (1-\rho)E_s\left[\rho \inf_{p\in\mathcal{P}'} E_t^p\left[x_u\right]\right] + \\
&\quad (1-\rho)^2 E_s\left[x_u\right]
\end{aligned}
\tag{4}
$$

where $\mathcal{P}'$ is the set of updated priors at time t under strong rectangularity.[4] In expression (4), the first term is simplified using the law of iterated expectations for maximin preferences under strong rectangularity (Nishimura and Ozaki, 2007). The last term of (4) is simplified using the standard law of iterated expectations, see equation (2). However, unless $\rho = 0$ or $\rho = 1$ (or if $\mathcal{P}$ is singleton and coincides with the objective probability measure, i.e., there is no ambiguity), the second and the third terms of (4) cannot be simplified, such that $NMEU_s[x_u] \neq NMEU_s[NMEU_t[x_u]]$. Intuitively, a decision maker evaluates x at each point of time as the weighted average of the worst case scenario and the objective scenario. Hence, at a future time $t > s$, the remaining possibilities are evaluated under the worst case scenario $\inf_{p\in\mathcal{P}'} E_t^p[x_u]$ and the objective scenario $E_t[x_u]$ to obtain $NMEU_t[x_u]$. Yet, not all of these scenarios are being considered when evaluating the same random variable x at an earlier time $s < t$. In this case, the decision maker only combines the overall worst case scenario $\inf_{p\in\mathcal{P}} E_s^p[x_u]$ and the overall objective scenario $E_s[x_u]$. In contrast, the worst case scenario from the perspective of time s of the objective scenario from the perspective of time t (i.e., the second term of expression (4)) is not reflected when calculating the NMEU value at time s. Similarly, the third term of (4) is not included in $NMEU_s[x_u]$.

While NMEU preferences allow for an intratemporal weighting of the worst case scenario and the objective scenario, they do not allow for an intertemporal weighting of these two probability measures. As a consequence, $NMEU_s[x_u]$ does not generally include the same information as $NMEU_s[NMEU_t[x_u]]$, such that both expectations differ.

3 Implications and concluding remarks

The dynamic inconsistency of NMEU preferences implies that optimal stopping problems under NMEU preferences are generally not recursive, as this would require expression (3) to hold (Nishimura and Ozaki, 2007). As a result, optimal stopping problems cannot be transformed

---

[4]The model by Nishimura and Ozaki (2007) is in continuous time. However, the concept of strong rectangularity is can also be applied in a discrete-time framework (Chen and Epstein, 2002; Epstein and Schneider, 2003).

into non-stochastic HJB equations, such that no analytical solution exists. This problem arises despite the set of priors being recursive using the strong rectangularity assumption.[5]

Ambiguity preferences are often difficult to reconcile with dynamic consistency (Epstein and Schneider, 2003; Klibanoff et al., 2009). Most ambiguity preference models do not exhibit this feature. In the literature, this problem is often overlooked. In particular, any preference model that uses a convex combination of several dynamically consistent preferences models is generally not dynamically consistent. This does not only apply to the NMEU model above, but also to the the $\alpha-$MEU model by Ghiradato et al. (2004) as shown in Schröder (2011) and Beissner et al. (2019), the neo-additive capacity model by Chateauneuf et al. (2007), or the imprecision aversion model by Gajdos et al. (2008) as discussed in Riedel et al. (2018).

When modelling real options under ambiguity in continuous time, the best approach therefore remains using the maximin expected utility (MEU) model by Gilboa and Schmeidler (1989), as shown in Nishimura and Ozaki (2007). For discrete-time applications, the ambiguity model proposed by Klibanoff et al. (2009) might be a suitable alternative.

---

[5]For Gao et al. (2018) this means that their results remain valid if $\rho = 0$ (no ambiguity aversion), $\rho = 1$ (extreme ambiguity aversion), or if $\mathcal{P}$ is singleton (i.e., there is no ambiguity).

References

Beissner, Patrick, Qian Lin and Frank Riedel (2019), 'Dynamically consistent alpha-maxmin expected utility', *Mathematical Finance* . forthcoming.

Chateauneuf, Alain, Jürgen Eichberger and Simon Grant (2007), 'Choice under uncertainty with the best and worst in mind: Neo-additive capacities', *Journal of Economic Theory* **137**, 538–567.

Chen, Zengjing and Larry Epstein (2002), 'Ambiguity, risk, and asset returns in continuous time', *Econometrica* **70**(4), 1403–1443.

Dixit, Avinash K. and Robert S. Pindyck (1994), *Investment under Uncertainty*, Princeton University Press, Princeton.

Epstein, Larry G. and Martin Schneider (2003), 'Recursive multiple-priors', *Journal of Economic Theory* **113**(1), 1–31.

Gajdos, T., T. Hayashi, Jean-Marc Tallon and J.-C. Vergnaud (2008), 'Attitude toward imprecise information', *Journal of Economic Theory* **140**(1), 27–65.

Gao, Yongling, Tarik Driouchi and David Bennett (2018), 'Ambiguity aversion in buyer-seller relationships: A contingent-claim and social network explananation', *International Journal of Production Economics* **200**, 50–67.

Ghiradato, Paolo, Fabio Maccheroni and Massimo Marinacci (2004), 'Differentiating ambiguity and ambiguity attitude', *Journal of Economic Theory* **118**, 133–173.

Gilboa, Itzhak and David Schmeidler (1989), 'Maxmin expected utility with non-unique prior', *Journal of Mathematical Economics* **18**, 141–153.

Hill, Brian (2020), 'Dynamic consistency and ambiguity: A reappraisal', *Games and Ecoomic Behaviour* **120**, 289–310.

Klibanoff, Peter, Massimo Marinacci and Sujoy Mukerji (2009), 'Recursive smooth ambiguity preferences', *Journal of Economic Theory* **144**(3), 930–976.

Kopylov, Igor (2016), 'Subjective probability, confidence and baysian updating', *Economic Theory* **62**(4), 635–658.

McDonald, Robert and Daniel Siegel (1986), 'The value of waiting to invest', *Quarterly Journal of Economics* **101**(4), 707–728.

Moon, Yongma, Tao Yao and Sungsoon Park (2011), 'Price negotiation under uncertainty', *International Journal of Production Economics* **134**, 413–423.

Nishimura, Kiyohiko G. and Hiroyuki Ozaki (2007), 'Irreversible investment and knightian uncertainty', *Journal of Economic Theory* **136**(1), 668–694.

Peijnenburg, Kim (2018), 'Life-cycle asset allocation with ambiguity aversion and learning', *Journal of Financial and Quantitative Analysis* **53**(5), 1963–1994.

Riedel, Frank, Jean-Marc Tallon and Vassili Vergopoulos (2018), 'Dynamically consistent preferences under imprecise probabilistic information', *Journal of Mathematical Economics* **79**, 117–124.

Schröder, David (2011), 'Investment under ambiguity with the best and worst in mind', *Mathematics and Financial Economics* **4**(2), 107–133.