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Steady-State Distributions for Models of Bubbles: their Existence and Econometric Implications

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Abstract:

The purpose of this paper is to examine the properties of bubbles in the light of steady state results for threshold auto-regressive (TAR) models recently derived by Knight and Satchell (2011). We assert that this will have implications for econometrics. We study the conditions under which we can obtain a steady state distribution of asset prices using our simple model of bubbles based on our particular definition of a bubble. We derive general results and further extend the analysis by considering the steady state distribution in three cases of a (I) a normally distributed error process, (II) a non normally (exponentially) distributed steady-state process and (III) a switching random walk with a fairly general i.i.d error process We then examine the issues related to unit root testing for the presence of bubbles using standard econometric procedures. We illustrate as an example, the market for art, which shows distinctly bubble-like characteristics. Our results shed light on the ubiquitous finding of no bubbles in the econometric literature.

Keywords: Bubbles, Asset prices, Steady state, Non-linear time series, TAR Models

1. Introduction:

Financial bubbles seem to be a permanent and ongoing event in global financial markets. Economists vary in their definitions of what a bubble is: however, many definitions are very similar to the one we quote below. Kindleberger (1978) defines a bubble as "upward price movement over an extended range that then implodes". He adds (Kindleberger, 1989), "a bubble is "a sharp rise in price of an asset or a range of assets in a continuous process, with the initial rise generating expectations of further rises and attracting new buyers – generally speculators, interested in profits from trading in the asset rather than its use or earning capacity". We shall define a financial bubble as the deviation of prices away from, and above, the fundamental value. The existence of bubbles finds support in financial market experiences and even the most recent financial crisis. Authors like Azariadis and co-authors (1986, 1998), Brunnermeier and Nagel (2004), Garber (2000), Krugman (2000), Poterba and Summers (1988) have documented the presence of bubbles and sunspots to explain financial market crashes.

If bubbles are a characteristic of financial markets, we need to ask if such an occurrence is consistent with the notion of a steady state price or return distribution. We are unaware of any analytical research on this question. In this literature, deviations from fundamentals are modeled either as bubbles or fads which can induce switching behaviour. We can incorporate both these features in our definition of bubbles.

Under the literature for bubbles, one of the earlier models by Blanchard and Watson (1982) proposes a theory of rational bubbles in which agents' (rational) expectations are influenced in part by extrinsic random variables whose properties accord to historical bubble episodes. They consider a price process such that $p_t = \alpha E_t(p_{t+1}) + D_t$ where $\alpha = \frac{1}{1+r_t}$ (less than 1) and D_t is an exogenous stationary dividend process. For algebraic simplicity and tractability, we can assume $r_t = r$. They obtain a solution in which prices equal fundamentals, p* (present discounted value of the dividend stream) by recursive substitution ($p_t = \sum_{i=0}^{\infty} \alpha^i E_t(D_{t+i}) = p^*$). They allow for solutions of the kind $p_t = p^* + c_t$, where c_t can be considered a bubble, under certain assumptions. Since then, there has been a lot of progress in terms dealing with the criticisms of their model and alternative models. Other approaches under rational bubbles include West (1987), Froot and Obstfeld (1991), Santos and Woodford (1997). Evans (1991) constructs rational bubbles that periodically explode and collapse. More recent models make different assumptions about rationality and belief structures including Allen and Gorton (1993), Hong et al. (2005), Lansing (2010), Branch and Evans (2011).

Brunnermeier and Nagel document that, in the late 1990s, hedge funds invested heavily in tech stocks, knowing that they were overvalued. Still, many funds succeeded in timing the market, earning large returns for a while, and selling before the crash. There is also a strand of literature (see Fisher and Kelly, 2000, among others) documenting that, in experimental settings, bubbles are very pervasive. Moreover, recent models of bubbles have become increasingly compatible with standard economic theory. A particularly influential line of research includes Abreu and Brunnermeier (2003), Allen et al. (1993), and Conlon (2004). In these models, asymmetric information deactivates the backward induction mechanism that typically precludes bubbles in other environments.

Our model is also related to commonly used models for "fads". Following Summers (1986) and Camerer (1989), fads can be defined as a deviation between prices and intrinsic or fundamental value that slowly reverts to its mean zero. It can be expressed as $c_t = Ac_{t-1} + e_t$; c_t is given from $p_t = f_t + c_t$, where f_t are the fundamentals. The distinction between fads and bubbles seems one of degree. If $A = 0$, fads disappear. If $A = 1 + r$, we can have the rational expectations bubble like in Blanchard. More studies that test the presence of bubbles and fads include van Norden and Schaller (1997), Roche (2001), Alessandri (2006).

An important stylized fact of asset prices is nonlinearity, which can be explained as a realization of regime switching. This nonlinear behavior can be induced in a number of ways. We attempt here to identify a model for bubbles, using our definition, by explicitly assuming a switching threshold autoregressive process for the price to capture the sudden collapse of the bubble. The key contention here for bubbles is that the trigger for the bubble's collapse is modeled by an exogenous sunspot process. This allows us to use the results derived by Knight and Satchell (2011) to examine conditions under which there exists a steady state distribution of prices. The motivation of this class of models is, in our view, a plausible way to capture the statistical properties observable in the time series of asset prices. Our explanation is closely related to the simple intuitive explanation of the mechanism behind bubble formation suggested by Shiller (2005): "If asset prices start to rise strongly, the success of some investors attracts public attention that fuels the spread of the enthusiasm for the market. New (often, less sophisticated) investors enter the market and bid up prices. This "irrational exuberance" heightens expectations of further price increases, as investors extrapolate recent price action far into the future. The market's meteoric rise is typically justified in the popular culture by some superficially plausible "new era" theory that validates the abandonment of traditional valuation metrics. But the bubble carries the seeds of its own destruction; if prices begin to sag, pessimism can take hold, causing some investors to exit the market. Downward price motion begets expectations of further downward motion, and so on, until the bottom is

eventually reached".

The exact form in which we express our model is nonlinear, which is essential for certain dynamical features. Different studies in the literature have supported this view for instance Leipus et al. (2005) who describe a stationary time series X_t as $X_t=\mu_t+a_tX_{t-1}+\sigma_t\varepsilon_t$ with renewal switching in levels (μ_t), slope (a_t) and/or volatility (σ_t). Such random coefficient AR(1) equations can describe periodically collapsible and restarting bubbles with variance which diverges to infinity exponentially in corresponding random intervals. Bohl (2003) models the existence of periodically collapsing bubbles in stock markets as a momentum threshold autoregressive model (MTAR). Using this nonlinear time series technique, he analyses bubble driven run-ups in stock prices followed by a crash in a cointegration framework with asymmetric adjustment. Phillips et al. (2009) define financial exuberance in the time series context in terms of explosive autoregressive behavior and then introduce some new econometric methodology based on forward recursive regression tests and mildly explosive regression asymptotics to assess the empirical evidence of exuberant behavior in the Nasdaq stock market index. They also note that their approach is compatible with several different explanations of this period of market activity, including the rational bubble literature, herd behavior, and exuberant and rational responses to economic fundamentals. All these propagating mechanisms can lead to explosive characteristics in the data which is what our model will focus on.

We also discuss the issue of testing for the presence of bubbles. Econometric tests based on cointegration techniques often rule out the existence of bubbles. Evans has criticized the use of cointegration techniques for testing the presence of bubbles by demonstrating how the presence of bubbles is often not detected in unit root tests. We add further evidence to illustrate why the null hypothesis of the presence of bubbles does not tend to receive enough statistical support.

Campbell and Shiller (1987) discuss cointegration tests in present value models and show that when a variable is proportional to its present value and these variables are I(1) processes, we can expect a cointegrating relationship to exist. Diba and Grossman (1984, 1988a) proposed the use of standard unit root and cointegration tests for stock prices and observable fundamentals to obtain evidence for the existence of explosive rational bubbles. This approach relies on the argument that if stock prices are not more explosive compared to dividends then rational bubbles do not exist because they generate an explosive component into stock price time series. It has been widely noted that these tests have very low power and erroneously lead to acceptance of the no bubbles hypothesis (see for example Evans, 1991).

More recent work has shown that we cannot apply standard tests in the presence of bubbles in the context of hidden Markov switching processes or in the presence of explosive unit roots. Hall et al. (1999) test for periodically collapsing bubbles

following a hidden Markov switching process by allowing the Augmented Dickey Fuller regression parameters to switch values between different regimes. Homm and Breitung (2012) empirically investigate the ability of a set of different tests to detect bubbles and conclude that standard tests need to be modified to be able to achieve this. Phillips et al. empirically test presence of periodically collapsing bubbles extending the econometric theory of testing under explosive roots developed by Phillips and Magdalinos (2011). Their approach consists of using recursive regression, right-sided unit root tests, and a new method of confidence interval construction. Our analysis differes from the above in that we consider the case in which there exists a steady state distribution under a threshold autoregressive model of bubbles. We demonstrate why tests based on cointegration would tend to not find the presence of bubbles in this case. We conjecture that this particular result has broader applicability.

The direction of our work is different from earlier literature in several respects. First, we intentionally keep the model conceptually as simple as possible. Thus we do not aim at this point to produce a model that can closely explain all of the observable statistical features of complex modern markets, but rather look for the simplest signature model of bubbles, perhaps the next order of approximation to reality after the random walk, which our approach encompasses. One motivation is that, even if not exhaustive, a simple model has a better chance of being capable of econometric estimation without over fitting. There are only two independent parameters in the model plus the choice of an error process, and we investigate the behavior of the model across possible values of these parameters. In the absence of a change in fundamentals, randomness is entirely responsible for igniting the bubble and causing the bubble to collapse. This means that the deterministic part of our dynamics does not suggest any typical time scales for these processes, making them essentially random, and similar to Poisson processes. Indeed, the bubble collapse (or ignition) is hard to predict.

In Section 2 we consider the existence of steady-state distributions and moments of prices for our model of bubbles. Section 3 looks further at steady-state distributions for prices by considering some explicit examples, either by specifying the error process or by reversing the question, and asking what error process will lead to a given distribution. In section 4 we look at the implications of our results on the efficacy of conventional econometric tests for the presence of bubbles. Section 5 looks at an empirical example, the art market, whilst section 6 concludes.

2. Existence of Mean and Variances in the presence of Bubbles

It is useful here to first summarize some of the key results from Knight and Satchell. Consider the following Threshold Autoregressive (TAR) model with an

exogenous trigger:

$$X_t = a + \beta_1 X_{t-1} + \varepsilon_t, \text{ if } Z_{t-1} = 0$$
$$= a + \beta_2 X_{t-1} + \varepsilon_t, \text{ if } Z_{t-1} = 1 \tag{1}$$

and $Z_t \sim$ iid Bernoulli with $P(Z_{t-1} = 1) = \pi$, $P(Z_{t-1} = 0) = 1-\pi$. This model can be written simply as

$$X_t = \alpha + \beta_1 X_{t-1} + (\beta_2 - \beta_1) Z_{t-1} X_{t-1} + \varepsilon_t$$

$$\text{or } X_t = \alpha + (\beta_1 + (\beta_2 - \beta_1) Z_{t-1}) X_{t-1} + \varepsilon_t \tag{2}$$

Transforming so that the random variable in the coefficient has mean zero, (2) is rewritten as

$$X_t = \alpha + (b_0 + b_1 B_{t-1}) X_{t-1} + \varepsilon_t \tag{3}$$

where

$$b_0 = \beta_1 (1-\pi) + \beta_2 \pi$$
$$b_1 = (\beta_2 - \beta_1)$$

and $B_{t-1} = (Z_{t-1} - \pi)$

Back substitution in (3) results in

$$X_t = \alpha \left(1 + \sum_{n=1}^{K-1} \prod_{m=1}^{n} (b_0 + b_1 B_{t-m}) \right)$$
$$+ \prod_{m=1}^{K} (b_0 + b_1 B_{t-m}) X_{t-K}$$
$$+ \sum_{n=1}^{K-1} \prod_{m=1}^{n} (b_0 + b_1 B_{t-m}) \varepsilon_{t-n} + \varepsilon_t$$

Furthermore, following Nicholls and Quinn (1982) and letting

$$S_n(t) = \prod_{m=1}^{n} (b_0 + b_1 B_{t-m})$$

we have

$$\ln|S_n(t)| = \sum_{m=1}^{n} \ln|b_0 + b_1 B_{t-m}|$$

and

$$\frac{1}{n} \ln|S_n(t)| \xrightarrow{as} E\left[\ln|b_0 + b_1 B_{t-m}|\right]$$

This implies that the terms $S_n(t)\varepsilon_{t-n}$ are geometrically bounded as n increases if $E\left[\ln|b_0 + b_1 B_{t-m}|\right] < 0$ and (3) then has the solution

$$X_t = \alpha\left(1 + \sum_{n=1}^{\infty} S_n(t)\right) + \sum_{n=1}^{\infty} S_n(t)\varepsilon_{t-n} + \varepsilon_t \tag{4}$$

Theorem 1 gives the expressions and conditions for existence and stationarity of mean and variance following (4).

Theorem 1

If $(1-\pi)\ ln\ |\beta_1| + \pi\ ln\ |\beta_2| < 0$ then the TAR model given by (1) or (2) has the solution

$$X_t = \alpha\left(1 + \sum_{n=1}^{\infty} S_n(t)\right) + \sum_{n=1}^{\infty} S_n(t)\varepsilon_{t-n} + \varepsilon_t\ .$$

Furthermore its mean is

$$\mu = E(X_t) = \frac{\alpha}{1 - \beta_1(1-\pi) - \beta_2\pi}$$

provided $|\beta_1|(1-\pi)+|\beta_2|\pi<1$. The variance is given by

$$Var(X_t)=\frac{\sigma^2+\mu^2(\beta_2-\beta_1)^2\pi(1-\pi)}{1-\beta_1^2(1-\pi)-\beta_2^2\pi}$$

provided $\beta_1^2(1-\pi)+\beta_2^2\pi<1$.

In the light of these results, our aim is to answer the question here of how we can accommodate the existence of means and variances in a model that includes bubbles. Let us first rewrite a generalized version of our model of bubbles following Knight and Satchell.

Let p_t be the price and $p*$ be the fundamentals. We can then write

$$p_t-p^*=\beta_1(p_{t-1}-p^*)+\varepsilon_t \text{ if } I_{t-1}=0$$

$$p_t-p^*=\beta_2(p_{t-1}-p^*)+\varepsilon_t \text{ if } I_{t-1}=1$$

and I_{t-1} is defined as:

$I_{t-1}=0$ if $Z_t \le c$ with probability $1-\pi$

$I_{t-1}=1$ if $Z_t > c$ with probability π

where Z_t is the forcing variable. Let $\beta_1=0;\beta=\beta_2;X_t=p_t-p*$. Therefore,

$$X_t=\beta I_{t-1}X_{t-1}+\varepsilon_t \tag{5}$$

which is similar now to the framework in Knight and Satchell.

In the Blanchard model, $\beta=1/\pi\alpha \quad (0<\pi<1, 0<\alpha<1)$. This particular value of β corresponds to the bubble as defined very precisely in Blanchard (1979). We do not assume this in our case, so the interpretation changes slightly; essentially we are proposing a system of possibly explosive dynamics for deviations from fair value.

By backward substitution,

$$X_t = \sum_{j=0}^{\infty} \beta^j \prod_{k=1}^{j} I_{t-k} \varepsilon_{t-j} \tag{6}$$

From (6) we can examine conditions for strong stationarity, and the conditions for the existence of moments.

The mean, if it exists, is

$$E(X_t) = 0$$

The variance is given by

$$Var(X_t) = \sigma^2 \sum_{j=0}^{\infty} \beta^{2j} E\left(\prod_{k=1}^{j} I_{t-k} \right)$$

$$= \sigma^2 \sum_{j=0}^{\infty} \beta^{2j} \pi^j$$

$$= \frac{\sigma^2}{1 - \beta^2 \pi} \tag{7}$$

provided $|\beta^2 \pi| < 1$, i.e. $0 < \beta^2 \pi < 1$.

This leads to Theorem 2.

Theorem 2

For the case of the Blanchard model, we see that the steady state distribution, given by (4), always exist; if $\beta = 1/\pi\alpha$, the mean does not exist and hence the variance does not exist. If $\beta \neq 1/\pi\alpha$, that is the generalized model, then the mean exists if $|\beta|\pi < 1$ and is equal to $p*$ and the variance exists if $0 < \beta^2 \pi < 1$.

Comment: Theorem 2 sheds some light on Shiller's notion of excess volatility (1981). If we were in the explosive regime for some period of time, we might expect to see a volatility of prices (or returns) that is incompatible with a dividend discount model, that is the process that determines p*. Again, if we are in a period

of history that is predominantly non explosive, excess volatility should not be present. In the explosive regime, there is no steady state variance but we can compute the conditional variance after being in the explosive regime for k consecutive periods. This is equal to $\frac{\sigma^2\left(\beta^{2k}-1\right)}{(\beta^2-1)}$ which is increasing and tends to infinity for $\beta > 1$.The results are similar if we assume instead of a Bernoulli iid process, the trigger follows a Markov chain (see section 6, Knight and Satchell).

3. Steady State Distributions

We consider the steady state distribution that arises in three cases. These cases differ because of either different assumptions about the error process, or different assumptions about the nature of the steady-state distribution. We consider (I) a normally distributed error process, (II) a non normally (exponentially) distributed steady-state process and (III) a switching random walk with a fairly general i.i.d error process. Before analyzing these cases, we note that the general relationship between the characteristic functions of the dependent variable and the error process, using the obvious notation, will be

$$\varphi_X(s) = \left(\pi\varphi_X(\beta s) + (1-\pi)\right)\varphi_\varepsilon(s)$$

$$\varphi_\varepsilon(s) = \frac{\varphi_X(s)}{\pi\varphi_X(\beta s) + (1-\pi)} \tag{8}$$

We have the following result.

Theorem 3

If the error process has finite kth moments and $\pi|\beta|^k < 1$, then the dependent process has finite kth moments which are given by substituting s=0 in the relationship below.

$$\varphi_X^k(s) = \pi\left(\sum_{j=0}^{k}\frac{k!}{j!(k-j)!}\beta^j\varphi_X^j(\beta s)\varphi_\varepsilon^{k-j}(s)\right) + \left(1-\pi\right)\varphi_\varepsilon^k(s)$$

Proof. We successively differentiate the terms in equation (8) to arrive at the equation in Theorem 3. The inequality $\pi|\beta|^k < 1$ is required to solve the resulting equation. *QED.*

In particular, we see that, assuming the error has a zero mean,

$$\varphi_X^{'}(0) = \frac{\varphi_\varepsilon^{'}(0)}{1-\beta\pi} = 0$$

$$\varphi_x^{''}(0) = \frac{\varphi_\varepsilon^{''}(0)}{1-\beta^2\pi} = \frac{\sigma_\varepsilon^2}{1-\beta^2\pi}$$

$$\varphi_x^{'''}(0) = \frac{\varphi_\varepsilon^{'''}(0)}{1-\beta^3\pi}$$

$$\varphi_X^{''''}(0) = \frac{6\beta^2\pi\varphi_X^{''}(0)\varphi_\varepsilon^{''}(0) + \varphi_\varepsilon^{''''}(0)}{1-\beta^4\pi}$$

$$= \frac{\dfrac{6\beta^2\pi\left(\varphi_\varepsilon^{''}(0)\right)^2}{1-\beta^2\pi} + \varphi_\varepsilon^{''''}(0)}{1-\beta^4\pi}$$

$$= \frac{6\beta^2\pi\left(\varphi_\varepsilon^{''}(0)\right)^2 + \varphi_\varepsilon^{''''}(0)}{\left(1-\beta^2\pi\right)\left(1-\beta^4\pi\right)}$$

These calculations give us general expressions for skewness and kurtosis.

Corollary 3.1. Denoting the skewness and kurtosis of the distributions by S and K respectively,

$$S_x = \frac{(1-\beta^2\pi)^{1.5} S_\varepsilon}{1-\beta^3\pi}$$

$$K_x = \frac{6\beta^2\pi(1-\beta^2\pi) + K_\varepsilon(1-\beta^2\pi)^2}{((1-\beta^4\pi)}$$

We now turn to our three examples:

(I) Using Theorem 2 in Knight and Satchell, we can find the stationary distribution of the deviation of price from its fundamental value in the generalized model. Under the assumption of normality, $\varepsilon_t \sim iid\ N(0,\sigma^2)$, we have the following characteristic function and density,

$$\varphi_X(r) = (1-\pi)\sum_{k=0}^{\infty}\pi^k \exp\left(-\frac{r^2\sigma^2\left(1-\beta^{2k+2}\right)}{2\left(1-\beta^2\right)}\right), |\beta| < \infty$$

$$f(x) = (1-\pi)\sum_{k=0}^{\infty}\pi^k N\left(0,\sigma^2\sum_{n=0}^{k}\beta^{2n}\right), |\beta| < \infty$$

where $N(a, b)$ signifies a Normal pdf with mean a and variance b. By Theorem 2, if the mean and variance exist as in Theorem 1, then, considering the price, its mean and variance are p* and $\frac{\sigma^2}{1-\beta^2\pi}$ respectively.

(II) Instead of looking at additive deviations from fair value which could lead to the possibility of negative prices in the steady state distribution, we consider, following Evans, the following model for multiplicative deviations.

$$X_t = \frac{p_t}{p^*}$$

$$X_t = \beta I_{t-1} X_{t-1} + \varepsilon_t$$

where $\varepsilon_t \sim iid$ and I_{t-1} is exogenous to the innovations ε_t. One approach is to express the price in logarithms or we can work with the positive variables; we shall follow the latter approach. This will require that both our multiplicative deviation and steady state variable will be positive. This can be imposed by assuming that a steady state distribution is that of a non-negative random variable. As an example, we consider the negative exponential distribution. We now want to identify the conditions required so as to ensure that the steady state distribution is exponential. We denote the moment generating function (mgf) of the error term as m_t, and find that

$$\frac{1}{1-\lambda t}=\left(\frac{1-\pi}{1-\lambda\beta_2 t}+\frac{\pi}{1-\lambda\beta_1 t}\right)m_t$$

As $\beta_1=0$, $\beta_2=\beta$

$$\frac{1}{1-\lambda t}=\left(\frac{1-\pi}{1-\lambda\beta t}+\pi\right)m_t$$

$$m_t=\frac{1-\lambda\beta t}{(1-\lambda t)(1-\lambda\beta\pi t)}$$

We assume that $0\le\beta<1$, so $\beta\pi$ will also be less than unity, in conformity with Theorem 2, so that we implicitly assume the existence of an asymptotic mean. Therefore, we can now express the mgf as

$$m_t=\frac{c}{1-\lambda\beta\pi t}+\frac{1-c}{1-\lambda t}$$

for $0<c<1$.

By solving, we find that $c=\frac{\beta(1-\pi)}{1-\beta\pi}$. The distribution of the innovation can be expressed as a weighted sum of two exponential distributions. Denoting the error by ε, we can write the pdf of ε, $f(\varepsilon)$ as

$$f(\varepsilon)=\frac{c\exp\left(-\frac{x}{\lambda\beta\pi}\right)}{\lambda\beta\pi}+\frac{(1-c)\exp\left(-\frac{x}{\lambda}\right)}{\lambda} \tag{9}$$

We now want to express the distribution in terms of just the primitive parameters c, v_1, v_2 where $v_1=\frac{1}{\lambda\beta\pi}$ *and* $v_2=\frac{1}{\lambda}$ while interpreting c as a probability and it follows that $0<v_2<v_1$. Thus,

$$f(\varepsilon)=cv_1\exp(-xv_1)+(1-c)v_2\exp(-xv_2)$$

This implies π can now be written as

$$\pi = \frac{\frac{v_2}{v_1}}{c + (1-c)\frac{v_2}{v_1}}$$

Thus, we can now write the properties of the pdf of the error process and find the expression for the theoretical mean and variance of the error. The variance of the error process is given by

$$= 2\left[\frac{c}{v_1^2} + \frac{(1-c)}{v_2^2}\right] - \mu^2$$

where $\mu = \frac{c}{v_1} + \frac{(1-c)}{v_2}$

As before, we will always have a steady state distribution. The moment conditions are, however, different. By assuming that the steady state is negative exponential, this will imply that all moments exist. Our assumption that $|\beta| < 1$ allows us to interpret the error in (9) as the mixture of two exponentials. This rules out an explosive regime, and is hence less relevant.

(III) The third case is that of a bubble where the process is a random walk ($\beta = 1$). This is interesting because if $\pi = 1$, then we have a conventional random walk in the deviation from fair value in the bubble regime and the steady state distribution does not exist. In the case when this is not so $(i.e\, \pi < 1)$, however, we can identify the steady state distribution for any error distribution.

For $X_t = p_t - p^*$ or $\frac{p_t}{p^*}$, and $\beta = 1$

$$\begin{aligned} X_t &= X_{t-1} + \varepsilon_t \quad \text{if } I_{t-1} = 1 \\ &= \varepsilon_t \qquad\qquad \text{if } I_{t-1} = 0 \end{aligned}$$

Thus, $X_t = I_{t-1}X_{t-1} + \varepsilon_t$

From the above equation, we can compute a relationship for the steady state characteristic function which will exist if $\pi < 1$

$$\varphi_X(s) = \left(\pi\varphi_X(s) + (1-\pi)\right)\varphi_\varepsilon(s)$$

$$\varphi_X(s) = \frac{(1-\pi)\varphi_\varepsilon(s)}{1-\pi\varphi_\varepsilon(s)} \tag{10}$$

We can verify that the expression on the right is indeed a characteristic function given that $\left|\varphi_\varepsilon(s)\right| \le 1$, and $\pi\left|\varphi_\varepsilon(s)\right| < 1$ as $\pi < 1$.

$$\varphi_X(s) = (1-\pi)\varphi_\varepsilon(s)\sum_{j=0}^{\infty}\pi^j\varphi_\varepsilon^j(s)$$

$$= (1-\pi)\sum_{j=0}^{\infty}\pi^j\varphi_\varepsilon^{j+1}(s) \tag{11}$$

Given $\varphi_\varepsilon(s)$ is a characteristic function, $\varphi_\varepsilon^j(s)$ is also a characteristic function by convolution. The uniform convergence of the series in (10) guarantees that the characteristic function of X is indeed a characteristic function.

From the characteristic function in (10), we can also write the moments as follows

The first moment is given by:

$$\mu_X = \varphi_X^{'} = (1-\pi)\sum_{j=0}^{\infty}\pi^j(j+1)\varphi_\varepsilon^j\varphi_\varepsilon^{'} = (1-\pi)\sum_{j=0}^{\infty}\pi^j(j+1)\mu_\varepsilon = 0$$

The second moment is given by:

$$\mu_X^{''} = \varphi_X^{''} = (1-\pi)\sum_{j=0}^{\infty}\pi^j(j+1)j\varphi_\varepsilon^{j-1}\left(\left(\varphi_\varepsilon^{'}\right)^2 + \varphi_\varepsilon^{''}\right) = (1-\pi)\sum_{j=0}^{\infty}\pi^j(j+1)j\mu_\varepsilon^{''}$$

Differentiating $(1-\pi)^{-1} = \sum_{j=0}^{\infty} \pi^j$,

we see that,

$$\mu_X'' = \varphi_X'' = (1-\pi)\sum_{j=0}^{\infty} \pi^j (j+1) j \mu_\varepsilon'' = \frac{2\mu_\varepsilon''}{(1-\pi)^2}$$

and higher moments can be found straightforwardly, although it is simpler to work directly with theorem 3. Using Corollary 3.1, we see that

$$S_X = (1-\pi)^{\frac{1}{2}} S_\varepsilon$$

$$K_X = 6\pi + K_\varepsilon (1-\pi) \tag{12}$$

It is apparent that the skewness of ε and X will have same sign. Note that $\frac{\partial S_X}{\partial \pi} = -\frac{1}{2}(1-\pi)^{-\frac{1}{2}} S_\varepsilon$ so that if the probability of the bubble reoccurring goes up and the error process is positively skewed, then the skewness of the prices process will go down.

It is interesting here to find the kurtosis insofar as Blanchard and Watson themselves refer to the finding of leptokurtosis (excessive fourth moments) as a universal characteristic of practically all financial returns. However, their claim that rational bubbles lead to such time series behavior seems to have not been verified in a formal way. However we can see this from examination of Corollary 3.1. K_X will be greater than 3 if K_ε is greater than $\frac{3-6\pi}{1-\pi}$.

In deriving the properties of steady state distributions, we have arrived at some neglected observable implications of rational bubbles that can be directly compared with empirical findings and how empirical tests for presence of bubbles are tested. We shall do this in the next section.

4. Testing the Existence of Bubbles

Tests based on assumptions of the existence of a stationary cointegrating relationship, as mentioned earlier, very often do not find statistically significant evidence of the presence of bubbles and we provide some simple calculations here as to why that might be the case.

Evans argued that the test approaches put forward by Diba and Grossman are unable to detect periodically collapsing bubbles. The application of standard unit root and cointegration techniques leads, with a high probability, to incorrect conclusions with respect to the presence of bubbles in stock prices. Another related study by Phillips et al. outline the difficulties standard econometric tests encounter in identifying rational asset bubbles. They use recursive tests to locate exploding sub-samples of data and detect periods of exuberance. They construct valid asymptotic confidence intervals for explosive autoregressive processes and tests of explosive characteristics in time series data. This approach can detect the presence of exuberance in the data and date stamp the origination and collapse of periods of exuberance. As is evident, the econometric theory related to explosive unit roots in autoregressive process becomes quite complicated as standard OLS and maximum likelihood methods are shown to be inconsistent if there are common explosive roots (Phillips and Magdalinos).

It is our contention that if the model allows a steady state distribution, econometric tests based on cointegration arguments will not demonstrate that bubbles exist. In each case, misspecification of the model or alternative market fundamentals seems the likely explanation of the findings.

Consider the fundamental relationship

$$p_t = \sum_{i=0}^{\infty} \alpha^i E_t(D_{t+i})$$

where terms are defined as previously. This can be re written in present value terms as follows

$$p_t = \frac{D_t}{r} \varepsilon_t$$

where ε_t is a non negative random process of mean 1. This is implicitly assuming that dividends follow a random walk and that the observed price differs from the true price by a multiplicative error.

So, taking logs of the above present value expression,

$$\ln p_t = \ln D_t - \ln r + \ln \varepsilon_t$$

Hence, if we estimate the above relationship as follows,

$$\ln p_t = c + \beta \ln D_t + v_t$$

and if this is the equilibrium relationship, then we want to test that v_t is stationary.

One way to test this is to conduct a unit root test on the cointegrating term given by

$$v_t = \ln p_t - \ln D_t - c$$

with cointegrating vector $(1,-1,-c)$.

The true model is given by equation (5) which we reproduce below:

$$v_t = \beta I_{t-1} v_{t-1} + \varepsilon_t \tag{13}$$

But we incorrectly assume

$$\Delta v_t = (\rho - 1) v_{t-1} + \eta_t$$

Thus, we want to test whether $\rho = 1$ .We shall assume that the TAR model given by (13) is strictly stationary and has a finite variance. The estimate of ρ is given by $\hat{\rho}$ as follows:

$$\hat{\rho} = p\lim\left(\frac{\sum v_t v_{t-1} / T}{\sum v_{t-1}^2 / T}\right)$$

$$= \frac{Cov(v_t, v_{t-1})}{Var(v_{t-1})}$$

$$= Corr(v_t, v_{t-1})$$

$Corr(v_t, v_{t-1}) = \beta\pi$ following Theorem 2 in Knight and Satchell. Thus,

$$p\lim(\hat{\rho}) = \beta\pi \tag{14}$$

From theorem 2 and our assumptions, we have that $\beta^2\pi < 1$, which implies that $\beta\pi < 1$ or $p\lim(\hat{\rho}) < 1$. This can explain why in many cases we reject the null hypothesis i.e. we fail to find evidence of bubbles.

We can also set up the test as a Dickey Fuller test, in which case we want to test whether $\rho - 1 = 0$.

The t statistic is given as

$$t = \frac{\hat{\rho} - 1}{se(\hat{\rho})}$$

where

$$se(\hat{\rho}) = \sqrt{\frac{\hat{\sigma}^2}{\sum v_{t-1}^2}}$$

$$\hat{\sigma}^2 = \frac{\sum (\Delta v_t - (\hat{\rho} - 1)v_{t-1})^2}{T}$$

$$= \frac{\sum (v_t - \hat{\rho} v_{t-1})^2}{T}$$

$$= \frac{\sum v_t^2}{T} - 2\hat{\rho}\frac{\sum v_t v_{t-1}}{T} + \frac{\sum v_{t-1}^2}{T}$$

$$= \sigma_v^2 - 2\hat{\rho} Cov(v_t, v_{t-1}) + \sigma_v^2 \hat{\rho}^2$$

$$= \sigma_v^2 - 2\hat{\rho}^2\sigma_v^2 + \sigma_v^2\hat{\rho}$$

$$= \sigma_v^2 (1 - \hat{\rho}^2)$$

From (14), we can rewrite the above expression as

$$= \sigma_v^2 (1 - \beta^2\pi^2)$$

Thus,

$$t = \frac{\beta\pi - 1}{\sqrt{\frac{\sigma_v^2(1-\beta^2\pi^2)}{T\sigma_v^2}}}$$

$$= \sqrt{T}\frac{\beta\pi - 1}{\sqrt{(1-\beta^2\pi^2)}}$$

$$= -\sqrt{T}\sqrt{\frac{1-\beta\pi}{1+\beta\pi}}$$

So, $t \to -\infty$ as $T \to \infty$. Again, the misspecified model will lead to a rejection of cointegration when we examine its distribution assuming that the true process includes a bubble.

5. Results for Art data

As an illustration of our results, we take as an example the art market which has seen exceptionally large price changes in recent years. There are a number of methodologies for producing art price indices and the MeiMoses All Art Index and Art Market Research are among the most widely quoted. However both are reliant on data from sales at the main auction houses and the dealer market is largely ignored due to an absence of obtainable data. MeiMoses uses repeat sales performance and auction price records dating back to 1875 and some analysts have raised concerns over a selection bias in the data. The Art Market Research data is available monthly, but only goes back as far as 1976. This uses average returns on a 12 month moving average, according to Campbell (2009,pp121).

An important feature of the data methodology behind the indices is the moving average, which results in a positively auto correlated series. It is important in the analysis on risk and return and on portfolio diversification that the underlying market risk and return levels be calculated and this may require smoothing the data. However, smoothed data according to such calculations may suppress the regime switching that we are looking for in the data.

Campbell gives a good description of the art market which we paraphrase next. In cultural economics, the dealer market is ignored because of an absence of obtainable data. It is clear that dealers make up a good part of the art market.

Dealers typically obtain their stocks from an auction or through direct purchase and their prices are higher (between 20-50 per cent (Anderson (1974)) than auction prices which depend on the original price and anticipated holding period. However the impact on our study of omitting dealer trades is not clear as most dealers are also investors and should, in principle, have their (higher) returns included in the overall art performance index. In this sense, our analysis is conservative.

We present results here from our generalized model of bubbles with the value of art set to zero in tables 1 and 2. We find clear evidence of presence of bubbles in art markets. As an exogenous threshold variable, we use the VIX (monthly data from 1990-2008, giving us 220 observations) and the University of Michigan Consumer Sentiment Index (monthly data from 1978-2008, giving us 364 observations). We could apply this model to other asset classes and consider other candidates for forcing variables, of which interest rates might be of particular interest. We do not claim our model is the true process but purely an illustrative example. Thus, we do not attempt to test the coefficients here because of the complexities that arise as we mentioned before (Phillips and Magdalinos).

Table 1 PARAMETER ESTIMATES USING VIX AS THRESHOLD (MONTHLY, 1990-2008)

	ALL (Price Bracket=Top 100 per cent)	TOP 10 (Price Bracket=Top 10 per cent)	TOP 2 (Price Bracket=Top 10 per cent)
BETA2 (Bubble Regime)	1.008824 (0.015804)	1.013771 (0.014810)	1.011954 (0.013876)
BETA1	0	0	0
Percentage of observations in Bubble Regime	5(value=29.97)	5(value=29.97)	5(value=29.97)

Table 2 PARAMETER ESTIMATES USING MICHIGAN SENTIMENT INDEX AS THRESHOLD (MONTHLY, 1978-2008)

	ALL (Price Bracket=Top 100 per cent)	TOP 10 (Price Bracket=Top 10 per cent)	TOP 2 (Price Bracket=Top 10 per cent)
BETA2 (Bubble Regime)	1.010368 (0.013776)	1.014954 (0.012817)	1.013350 (0.013876)
BETA1	0	0	0
Percentage of observations in Bubble Regime	30(value=83.2)	11(value=69.28)	11(value=69.28)

Tables 1 and 2 above show the results for the values of beta2 for the three indices based on the average price of all sales, the average of the top 10 percent and the average price of the top 2 per cent. We also report the proportion of data points in the bubble regime. In all six cases, we have value of beta greater than 1. In all cases, the steady state means and variances exist. It should be noted that high levels of volatility lead to art market bubbles whilst low levels of sentiment lead to art market bubbles. Switching occurs when VIX is high 5 per cent the time for the market. Periods where crisis seems to have occurred include Sept-Oct 98, Jul-Oct 2002. For sentiment index, the explosive regime occurs 31 per cent of the time for all, and 11 per cent for both top 10 and 2 per cent of the market. The substantial periods of prolonged low sentiment and crisis include the months between 1978-1983, 1990-1993, 2007-2008.

6. Conclusion

In this paper, we have shown that a simple model that includes the Blanchard bubble model as a special case always has a steady state distribution by using a steady state theory for threshold autoregression models. We can also utilize the structure of the threshold autoregression model to give a precise definition of what we mean by a bubble. We show that the properties of the resulting steady state distribution can be analysed both generally and in a number of relevant cases. Given the existence of a steady state distribution, we also show that standard econometric methods of testing for bubbles based on cointegration will tend to reject the null of cointegration between price and dividend. In our

definition of a bubble, which implies an explosive regime in a regime switching model for the deviation from fair value, and assuming the existence of a steady state distribution, cointegration will be an inappropriate procedure. A more natural one would involve testing whether all regimes are stationary. Such a test awaits further research.

Viewing history would allow us to count a number of extinct, or at least dormant, financial/asset markets, consistent with our interpretation of price limiting behaviour. It is probably difficult to persuade some of the profession that prices do not follow a random walk but it is worth noting that our results allow prices to follow a random walk most of the time, we only need a little bit of mean-reversion to allow for a well-defined steady-state, something that the limiting behaviour of random walks will not deliver.

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