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Regulating Altruistic Agents

Anthony Heyes
Royal Holloway, University of London

Sandeep Kapur
Birkbeck, University of London

May 2010
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Anthony Heyes
Royal Holloway
University of London

Sandeep Kapur†
Birkbeck College
University of London

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Abstract

Altruism or ‘regard for others’ can encourage self-restraint among generators of negative externalities, thereby mitigating the externality problem. We explore how introducing impure altruism into standard regulatory settings alters regulatory prescriptions. We show that the optimal calibration of both quantitative controls and externality taxes are affected. It also leads to surprising results on the comparative performance of instruments. Under quantity-based regulation welfare is increasing in the propensity for altruism in the population; under price-based regulation the relationship is non-monotonic. Price-based regulation is preferred when the population is either predominantly altruistic or predominantly selfish, quantity-based regulation for cases in between.

JEL Codes: D62, D64, H42

*We thank Karine Nyborg, Arun Malik, John List, Colin Rowat, Ron Smith, Tim Swanson, and seminar participants at Birmingham, UCL, Toulouse and Heidelberg universities for very valuable comments. All errors are ours.

†Department of Economics, Malet St, London WC1E 7HX, United Kingdom.
email: s.kapur@bbk.ac.uk
1 Introduction

People often face situations in which their private activities – such as driving the family car or generating household waste – impose negative externalities. In many cases those activities are subject to regulatory control or influence, in the form of quantitative limits or taxation. But even in the absence of regulation, it is plausible that ‘regard for others’ or the desire to ‘do the right thing’ – the two will be analytically equivalent in the framework that we develop – might induce people to moderate, to varying extents, their behavior voluntarily.

Many British motorists report moderating their driving habits in order to reduce their impact on others (see, for example, Daily Telegraph (2007)). The desire for moderation is present despite there being a corrective tax levied on petrol in the United Kingdom that is – according to estimates by Parry and Small (2005) – about twice the level needed to internalize the air quality and accident risk externalities associated with motoring.

Production of domestic waste is another area where surveys suggest that households are sensitive to the external impact of their actions. According to Berglund (2006), 73% of people report that part of the motive for their recycling of waste is that they “want to contribute to a better local environment”. Waste reduction provides an example where there are alternative regulatory approaches in use. In the English county of Surrey household waste is subject to ‘quantity caps’: the local council issues a ‘wheelie bin’ of particular dimension to each address and will only collect rubbish up to that volume – requiring that the lid be closed to avoid ambiguity about the allowance. In neighboring Buckinghamshire a price instrument is used – households can put out as much waste as they wish but it has to be in designated refuse sacks, which can be bought from the council.

In this paper we introduce altruistic or other-regarding tendencies into a regulated population. In particular we contemplate impure altruism, and the associated notions of ‘warm glow’ and ‘cold prickle’ introduced by Andreoni (1991). The suggestion here is that agents derive a warm glow benefit from contributing to a clean environment or, conversely, a cold prickle disbenefit from damaging it. In our model the behavior of agents is moderated, to varying degrees, by the ‘guilt’ or otherwise negative feeling that they derive from imposing unpleasantness on others in their
community.\footnote{Alternatively, we could interpret the altruism parameter that we introduce in our model as capturing ‘principles’ or morals. These may derive from religious conviction, from cultural norms reinforced by education, etc. Christians are told to “Do unto others as you would have them do to you” (Luke ch. 6, v. 31). Taoists are taught to “Regard your neighbour’s gain as your gain, and your neighbour’s loss as your loss” (teachings of Tai Shang Kan Yin P’ien).} We explore how such other-regarding tendencies impact policy prescription: specifically, (a) the way in which any particular regulatory instrument is calibrated, and, (b) the comparative merits of alternative instruments. We develop a model in the context of a depletable or local externality, but the model could readily be extended to the case of a public bad.

There is a long tradition of instrument comparison in environmental economics. Baumol and Oates (1971) made an early case for the use of taxation in the control of externalities, while Weitzman (1974) pointed out that, when benefits and abatement costs are uncertain, the choice between price and quantity instruments depends upon the comparative slopes of the marginal benefit and marginal cost curves. In other well-known early contributions the comparative merits of price versus quantity instruments were explored in different frameworks in Adar and Griffin (1976) and Roberts and Spence (1976).\footnote{More recently the basic model has been extended in various directions. Williams (2002) differentiates between tradable and non-tradable quantity instruments; Kaplow and Shavell (2002) allow for non-linear taxes; Stavins (1996) considers the effect of correlation between cost and benefit uncertainties; Montero (2002) allows for less-than-full compliance.} Economists usually express a preference for price-based instruments over quantitative restrictions. This is largely because tax-based restrictions are able to deliver efficiency in contexts where different regulated parties face different costs of compliance – they provide for enough flexibility that an agent with very high compliance costs can emit more than his counterpart with lower ones.

We present a model in which individuals’ choices impose external costs. The model is populated by agents exhibiting different degrees of impure altruism – that is, individuals suffer different degrees of the ‘cold prickle’ sensation from imposing externalities on others. We introduce, in other words, motivational heterogeneity. The model has otherwise standard features, including heterogeneity in compliance costs. We investigate how introducing altruism leads us to update our views regarding optimal specification of a particular instrument and optimal instrument choice. The scenario absent altruism (the conventional textbook case) remains nested here as a special case, and the benchmark against which our results are compared.

We find that the choice between regulatory instruments depends on the nature of
the heterogeneity in the regulated population. Price-based instruments are good at handling heterogeneity in compliance costs (the only sort of variation that features in many existing models) but less-suited to handling motivational heterogeneity. When there is no motivational heterogeneity the price instrument is unambiguously superior in our model. Once motivational heterogeneity is introduced the ranking of instruments becomes more complex, and quantity-based instruments may dominate. While we develop our results in a relatively general setting, the complexity is best illustrated through a simple example. Imagine that there are just two types of individuals: some who are entirely selfish and others who are altruistic to the same extent. Our example compares the performance of price- and quantity-based instruments across populations that differ only in the proportions of these two types. We find that under optimal quantity-based regulation welfare is increasing in the proportion of the population that is altruistic. Under optimal price-based regulation the relationship turns out to be non-monotonic. In particular, price-based regulation is superior when the regulated population is either predominantly altruistic or predominantly selfish, and quantity-based regulation may dominate for intermediate cases.

2 The Model

An agent chooses a level \( x \in \mathbb{R}_+ \) of some anti-social activity that imposes external costs \( cx \) on some other individual. For concreteness we regard \( x \) as ‘emissions’ but we encourage the reader to think of \( x \) as the level of any externality-generating behavior, rather than merely the headline categories of pollution. For modeling purposes we assume that this negative externality is depletable – the experience of the externality by one agent reduces the amount felt by others. This is natural in settings where the externality has a neighborhood dimension, for example, or in contexts in which agents are imposing externalities by their exploitation of a common pool resource or commons.\(^3\) For simplicity we assume that the external cost is linear in \( x \), with constant marginal cost \( c > 0 \).

\(^3\)Alternatively, we could have assumed that the externality is non-depletable, that is, in the nature of a ‘public bad’. See Mas-Colell et al. (1995: 364) for an elaboration of the difference in two approaches. The qualitative results of our model would be preserved in that alternative setting; our arguments require only that we have a private actor who benefits from the activity and imposes an external cost on others.
Agents differ in two important ways. One, they differ in the benefit they derive from the activity. Two, they differ in the extent to which they are sensitive to the impacts of their activity upon others. It is the latter variation that is the novelty in our framework. To capture these effects, we endow our agents with utility functions of the form

\[ \theta_i b(x_i) - \alpha_i c x_i. \] (1)

Agent \( i \) derives private benefit \( \theta_i b(x_i) \) from the activity level \( x_i \), where \( \theta_i > 0 \) captures the heterogeneity in the benefit that different agents derive from the activity. We assume that private benefit increases concavely in \( x \) – that is, \( b'(x) \geq 0 \) and \( b''(x) < 0 \) – which corresponds to diminishing marginal utility to engagement in the activity.

The second term captures the ‘cold prickle’ that the agent suffers on imposing the externality on some neighbor. The term \( \alpha_i \in [0,1] \) measures the intensity of the cold prickle and can be viewed as the degree of altruism. It captures the extent to which altruism induces behavior as if the externality were internalized.\(^4\) We do not assume that all agents are altruistic to the same degree, though the model allows for that possibility. The standard ‘textbook’ case with selfish agents is nested here as the special case in which \( \alpha_i = 0 \) for all \( i \).\(^5\)

An agent, then, is characterized by the pair \((\theta, \alpha)\): we drop subscripts where it causes no ambiguity. These characteristics are assumed to be private information though their distribution in the population is known, and given by distribution functions \( F(\theta) \) and \( G(\alpha) \). We assume that the two distributions are independent and, importantly, we leave the form of \( G \) entirely general.\(^6\)

In the absence of regulation an agent chooses \( x \) to maximize (1). Under the usual

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\(^4\)As noted above, \( \alpha \) could also capture ‘principles’ – views about the ‘right way to behave’ – rather than altruism. For instance, perfect conformance with Taoist principles would imply acting as if \( \alpha = 1 \), although we could expect adherents to vary in the extent of their resolution (that is, act in a way consistent with \( \alpha \) less than 1). The case with \( \alpha = 1 \), if universalized, maximizes social welfare and corresponds to the Kantian ideal. Bilodeau and Gravel (2004) examine the implications of agents exhibiting a Kantian ethic in a game of voluntary public good contributions.

\(^5\)For simplicity we have assumed that, apart from cold prickle, there are no other direct costs of the activity for the agent. If such costs exist, these can incorporated in \( \theta_i b(x_i) \) – then to be regarded as private benefit net of private costs – with only minor modifications to the arguments that follow.

\(^6\)We also assume that \( G(\alpha) \) is exogenously fixed. Frey and Oberholzer-Gee (1997) and Gneezy and Rustichini (2000b) provide evidence that a policy instrument in use can itself change attitudes towards misbehavior. We later discuss the implications of those findings.
conditions the solution is interior and implicitly defined by the first-order condition

$$\theta b'(x) - \alpha c = 0.$$  

(2)

We denote this choice as $x_\emptyset(\theta, \alpha)$. The subscript $\emptyset$ will be used to refer to the unregulated setting, to distinguish it from choices and outcomes under quantity-based and price-based regulation.

2.1 Altruism and welfare

There has been some debate in the literature as to how to treat warm-glow benefits/cold-prickle costs in the evaluation of social welfare. We follow Diamond and Hausman (1994), Andreoni (2004), and Diamond (2006) in assuming that warm glow benefits should be ignored in the welfare evaluation. Bernheim and Rangel (2005: 59) offer the following summary and assessment: “Diamond (2006) argues that measures of social welfare should exclude the apparent benefits from the warm glow. He advocates using the warm glow model for positive purposes (that is, to describe behavior), but favors the standard model for evaluating welfare. Andreoni (2004) expresses a similar view.”

If so, the evaluation of the welfare associated with an agent’s chosen activity level includes only the benefits that he derives from the activity and the negative externality that he imposes on others. Formally, the ‘welfare contribution’ of the $i$-th agent’s choice is

$$w(\theta_i, x_i) = \theta_i b(x_i) - c x_i.$$  

(3)

The level of activity that maximizes an agent’s welfare contribution – call this socially-optimal level of activity $\hat{x}$ – then satisfies

$$\theta_i b'(\hat{x}) - c = 0.$$  

Given that $b(x)$ is concave, the socially-optimal level $\hat{x}(\theta_i)$ is increasing in $\theta_i$. Given that we ignore warm-glow benefits in the evaluation of welfare, the socially-optimal choice is independent of $\alpha_i$.

Of course the actual choice of unregulated agents does vary with $\alpha$. Agents with $\alpha < 1$ choose activity levels that exceed the first-best level: $x_\emptyset(\theta, \alpha) \geq \hat{x}(\theta)$. The
chosen level is decreasing in $\alpha$, and a perfectly altruistic agent’s choice coincides with the socially-optimal level: $x_\emptyset(\theta, 1) = \hat{x}(\theta)$.

An unregulated agent’s choice has associated welfare contribution

$$w_\emptyset(\theta, \alpha) = \theta b(x_\emptyset(\theta, \alpha)) - c x_\emptyset(\theta, \alpha).$$

It is straightforward to establish that $w_\emptyset(\theta, \alpha)$ is increasing in $\alpha$, with $\alpha = 1$ achieving the first-best.

Aggregate welfare depends on the distribution of $\theta$ and $\alpha$ in the population. Given distribution functions $F(\theta)$ and $G(\alpha)$, aggregate welfare in the absence of regulation is

$$W_\emptyset = \int_\theta \int_\alpha w_\emptyset(\theta, \alpha)dG(\alpha)dF(\theta).$$

3 Quantity Regulation

Quantity regulation entails the regulator imposing upon each agent a maximum admissible level of the activity. If, for example, the externality follows from the over-production of household waste the local government could set a cap on the volume of waste it is willing to collect from any given address. The cap is fully and costlessly enforced.

Recall that the welfare-optimizing level of activity $\hat{x}(\theta_i)$ varies with $\theta_i$. If the regulator were able to observe agent-specific characteristics then he could implement the first-best outcome by imposing agent-specific regulatory caps equal to $\hat{x}(\theta_i)$. Given that $\theta_i$ is private information, though, the regulator can do no better than set a uniform regulatory cap, call it $\overline{x}$.

The existence of altruistic agents – those with strictly positive $\alpha$’s – creates the possibility that a uniform cap may not bind on all agents. Not everyone lives ‘at the limits of the law’. Some, guided to restraint by their altruistic concern for damage imposed upon others, will choose a level of activity strictly less than the cap.\footnote{We are familiar with two other papers in which (for quite different reasons) caps may not bind on all regulated parties. Brozovic, Sunding and Zilberman (2004) assume an interior solution to the firms unregulated pollution problem. This contrasts with the (more conventional) assumption that costs are everywhere decreasing in emissions, such that all firms produce at a point where any quantitative emissions cap is binding. Bandyopadhyay and Horowitz (2006) point to a ‘safety
concretely, the cap is binding on agents for whom \(x_0(\theta, \alpha) \geq \bar{x}\), which includes those who derive high benefit from the activity (large \(\theta\)) or those who not very altruistic (small \(\alpha\)). Thus, under a regime of quantity caps the agent’s choice is

\[x_Q(\theta, \alpha, \bar{x}) = \min [x_0(\theta, \alpha), \bar{x}].\]

With quantity capped at \(\bar{x}\) an agent’s welfare contribution is

\[w_Q(\theta, \alpha, \bar{x}) = \theta b(x_Q(\theta, \alpha, \bar{x})) - cx_Q(\theta, \alpha, \bar{x}),\]

and aggregate welfare, as function of the chosen quantity cap, is

\[W_Q(\bar{x}) = \int_{\theta} \int_{\alpha} w_Q(\theta, \alpha, \bar{x})dG(\alpha)dF(\theta).\]

For particular values of \(\theta\) and \(\bar{x}\), define the critical value \(\alpha(\theta, \bar{x}) \in [0, 1]\) such that the given regulatory cap binds for agents with lower \(\alpha\). Then

\[W_Q(\bar{x}) = \int_{\theta} \left[ \int_{0}^{\alpha(\theta, \bar{x})} w(\theta, \bar{x})dG(\alpha) + \int_{\alpha(\theta, \bar{x})}^{1} w_0(\theta, \alpha)dG(\alpha) \right] dF(\theta).\]

The first integral captures the welfare contribution of agents for whom the regulatory cap binds, the second of those whose behavior is restrained by altruism. The regulator chooses \(\bar{x}\) to maximize aggregate welfare. Assuming differentiability, and using the fact that \(w_0(\theta, \alpha(\theta, \bar{x})) = w(\theta, \bar{x})\), we have

\[W_Q'(\bar{x}) = \int_{\theta} \left[ \int_{0}^{\alpha(\theta, \bar{x})} w'(\theta, \bar{x})dG(\alpha) \right] dF(\theta),\]

where \(w'(\theta, \bar{x}) = \theta b'(\bar{x}) - c\). Assuming an interior solution, the optimal regulatory cap \(\bar{x}^*\) is given by the first-order condition \(W_Q'(\bar{x}^*) = 0\).\(^8\)

How does the presence of altruistic agents affect the optimal regulatory cap? We compare the general case analyzed here with the ‘traditional’ economic analysis that margin’ effect whereby a firm would stay inside a quantitative limit if its realized pollution discharge had a stochastic element (it could only control its pollution level noisily) and penalties for violation were sufficiently large. They use this as an alternative to the altruistic ‘explanation’ for why firms appear to overcomply.

\(^8\)The second-order condition is satisfied as \(W(\bar{x})\) is concave in \(\bar{x}\). To see why, note that \(w_Q(\theta, \alpha, \bar{x})\) is weakly concave in \(\bar{x}\). Integration preserves concavity in \(\bar{x}\).
ignores any self-restraint due to altruism – the special case in which \( \alpha_i = 0 \) for all \( i \) – and find that the presence of altruistic agents allows for less stringent regulation.

**Proposition 1** The presence of altruistic agents raises the optimal regulatory cap.

**Proof:** Let \( \bar{x}'(G) \) be the optimal regulatory cap for any non-generate distribution \( G(\alpha) \). The first-order condition for optimality requires

\[
\int_{\theta} \int_{0}^{\alpha(\theta, \bar{x}'(G))} w'(\theta, \bar{x}'(G))dG(\alpha)dF(\theta) = 0. \tag{5}
\]

The traditional case, with selfish agents, amounts to one in which \( G(\alpha) \) is degenerate at \( \alpha = 0 \), or that \( G(0) = 1 \). For this case any regulatory cap binds on all agents, and the optimal cap – call it \( \tilde{x} \) – must be such that

\[
W'(\tilde{x}) = \int_{\theta} w'(\theta, \tilde{x})dF(\theta) = 0. \tag{6}
\]

As \( w'(\theta, x) \) does not vary with \( \alpha \), we can rewrite the above as

\[
\int_{\theta} \int_{0}^{\alpha(\theta, \bar{x}'(G))} w'(\theta, \bar{x})dG(\alpha)dF(\theta) + \int_{\theta} \int_{\alpha(\theta, \bar{x}'(G))}^{1} w'(\theta, \bar{x})dG(\alpha)dF(\theta) = 0. \tag{7}
\]

To demonstrate that \( \bar{x}'(G) \geq \tilde{x} \), assume, to the contrary, that \( \tilde{x} > \bar{x}'(G) \). Noting that \( w'(\theta, x) = \theta b'(x) - c \), due to the concavity of \( b(x) \) we must have \( w'(\theta, \tilde{x}) < w'(\theta, \bar{x}'(G)) \). Then, comparing (5) and (7), it follows that the first term in (7) must be negative; if so the second term in (7) must necessarily be positive. Thus our assumption requires

\[
\int_{\theta} \int_{\alpha(\bar{x}'(G), \theta)}^{1} w'(\theta, \tilde{x})dG(\alpha)dF(\theta) > 0. \tag{8}
\]

Note that the marginal welfare contribution is negative for \( \alpha > \alpha(\tilde{x}, \theta) \),

\[
w'(\tilde{x}, \theta) = \theta b'(\tilde{x}) - c < \theta b'(\tilde{x}) - \alpha c < 0, \tag{9}
\]

and so also for \( \alpha > \alpha(\bar{x}'(G), \theta) > \alpha(\tilde{x}, \theta) \). If so, (8) cannot hold. We have a contradiction. \( \blacksquare \)

So in this setting altruistic tendencies can be regarded as a substitute for regula-
tion – more of the former implies there should be less of the latter. In the absence of altruism, the optimal quota trades off allowing some agents (those with low $\theta$) to do an inefficiently high level of the activity with constraining others (with high $\theta$) to do too little. The introduction of altruism means that the costs associated with allowing low benefit agents to do too much are now decreased, because some agents will exercise voluntary self-restraint. This leads to an increase in the optimal quota.

4 Price Regulation

An alternative to setting quantitative limits on the activity is to impose a tax or charge on it. If the externality derives from the over-production of household waste then the local government could charge by volume or weight for collection. We assume throughout that the tax is fully and costlessly collected. If the activity is taxed at rate $t$, the agent characterized by $(\theta, \alpha)$ will choose $x$ to maximize

$$\theta b(x) - tx - \alpha cx.$$ 

The first-order condition associated with an interior maximum is

$$\theta b'(x) - t - \alpha c = 0.$$ 

From the concavity of $b(x)$, we know that an agent’s optimal choice, $x_P(\theta, \alpha, t)$, is increasing in $\theta$ and decreasing in $t$ or $\alpha$: other things equal, agents will do more of the activity if they find cutting back expensive, and will do less if they are more altruistic or if the tax rate is high.

In this regime, the agent’s welfare contribution is

$$w_P(\theta, \alpha, t) = \theta b(x_P(\theta, \alpha, t)) - cx_P(\theta, \alpha, t),$$

so that aggregate welfare

$$W_P(t) = \int_\theta \int_\alpha w_P(\theta, \alpha, t)dG(\alpha)dF(\theta)$$

(10)

varies with chosen tax rate $t$. The first-order condition\(^9\) for the optimal tax rate $t^*$ is

\(^9\)To see that the second-order condition holds note that $w_P(\theta, \alpha, t)$ is concave in $t$ as the deriva-
\( W_p'(t^*) = 0 \), where

\[
W_p'(t) = \int_\theta \int_\alpha \frac{\partial}{\partial t} [w_p(\theta, \alpha, t)] dG(\alpha) dF(\theta).
\]

We have

\[
\frac{\partial}{\partial t} [w_p(\theta, \alpha, t)] = \left[ \theta b'(x_p) - c \right] \frac{dx_p}{dt} + \left[ t - (1 - \alpha)c \right] \frac{1}{\theta b''(x_p)}.
\]

The last equality obtains from manipulation of the first-order condition for the agent’s maximum, and makes transparent the role of taxation. The tax can be thought of as correcting the insufficiency of altruism in restraining the anti-social activity. Given heterogeneity in altruistic concerns, the ideal corrective tax rate must vary too: the optimal tax to face agent \( i \) would be \( t_i = (1 - \alpha_i)c \). If \( \alpha_i = 1 \), no taxation is necessary – in making decisions the agent is already acting as if the externality were incident upon himself. If \( \alpha_i = 0 \), the optimal tax rate is \( c \), the standard Pigovian prescription. In general, for \( \alpha \)’s that are positive but less than 1, the optimal tax rate is positive but less than \( c \).

We note that if \( \alpha_i = k \) for all \( i \) – then the regulator could implement the first best by setting a tax equal to \((1 - k)c\). By appropriate manipulation of the tax rate the same first best level of welfare could be achieved regardless of the value of \( k \).

With unobservable heterogeneity in \( \alpha \), however, optimal tax policy must pick a uniform tax rate that minimizes the welfare impact of distortions across agents. At the optimum an interval of the most altruistic agents in the population will face a tax that is ‘too high’, an interval of the least so face a tax that is too low. It is clear that at the optimum, the tax rate will not exceed \( c \), which leads us to:

**Proposition 2** The presence of altruistic agents lowers the optimal tax rate.

**Proof.** For the traditional textbook case without altruism, \( \alpha_i = 0 \) for all agents, so the optimal tax rate is simply \( c \). We show that for the general case with altruism, \( \frac{\partial w_p}{\partial t} = \left[ \theta b'(x_p) - c \right] \frac{\partial x_p}{\partial t} + \left[ t - c(1 - \alpha) \right] \frac{\partial x_p}{\partial t} \) is positive for small \( t \) (i.e., \( t - c(1 - \alpha) < 0 \)) and negative for large \( t \). The integral, a convex combination of these concave functions, inherits the concavity.
the optimal tax rate \( t^* < c \). The optimal tax rate satisfies

\[
W'_P(t^*) = \int_\alpha \int_\theta \left[ t^* - (1 - \alpha)\frac{c}{\theta} \right] dG(\alpha)dF(\theta) = 0.
\]

For this to hold for a non-degenerate distribution \( G(\alpha) \), we must have \( t^* - (1 - \alpha)c \) strictly positive for some \( \alpha_i \) and negative for other \( \alpha_i \) in the support of distribution \( G \). [Recall that \( b'' < 0 \) due to the concavity of \( b \).] Let the support of \( \alpha_i \) be \([\alpha_{\min}, \alpha_{\max}]\). If \( t^* - c + \alpha_{\min}c < 0 \), we must have \( t^* - c < 0 \) as long as \( \alpha_{\min} \geq 0 \) (which is true by assumption).

5 Instrument Choice

How do the two forms of regulation compare in terms of welfare outcomes? How does the insertion of altruism into a model of externality with otherwise standard features lead us to reconsider the comparative merits of quantity- versus price-based policy?

We compare optimally-calibrated quantity- and price-based regulation in the presence of altruism. Let \( W^*_Q \) denote aggregate welfare under quantity regulation when the regulatory cap is chosen optimally. That is, \( W^*_Q = W_Q(x^*) \). Similarly, let \( W^*_P = W_P(t^*) \) denote the optimized value of welfare under price regulation when tax rate is chosen optimally. It is clear from the analysis of previous section that these optimized values vary with the distributions \( G(\alpha) \) and \( F(\theta) \). Our focus in this paper is on how \( G(\alpha) \), the distribution of altruism in the population, affects the relative performance of price and quantity regulation.

To compare price and quantity regulation, we begin by examining how welfare under these regimes varies with an increase in the degree of altruism in the population (in the sense of first-order stochastic dominance\(^{10} \), in effect, more mass on higher values of \( \alpha \)) and by an increase in the heterogeneity of the altruism parameter (in the sense of mean-preserving spreads, or more mass on the tails of the distribution).

We have the following propositions.

\(^{10}\)For any two distributions \( G_1(\alpha) \) and \( G_2(\alpha) \), we say that \( G_1(\alpha) \) first-order stochastically dominates \( G_2(\alpha) \) if \( G_1(\alpha) \leq G_2(\alpha) \) for all \( \alpha \).
Proposition 3  Given distribution $G(\alpha)$, let $W^*_Q(G)$ be the optimized level of welfare under quantity regulation. An increase in the degree of altruism in the sense of first-order stochastic dominance increases $W^*_Q(G)$. In contrast, a mean-preserving spread of $G(\alpha)$ may increase or decrease the value of $W^*_Q(G)$.

Proof. To show that welfare under quantity regulation is increasing in altruism, we begin by considering a fixed regulatory cap. Given $\bar{x}$, there exists $\underline{\alpha}(\theta, \bar{x})$ such that the cap binds for $\alpha < \underline{\alpha}(\theta, \bar{x})$. Aggregate welfare is
\[
W_Q(\bar{x}; G) = \int_0^{\underline{\alpha}(\theta, \bar{x})} w(\theta, \bar{x}) dG(\alpha) + \int_{\underline{\alpha}(\theta, \bar{x})}^{1} w_\emptyset(\theta, \alpha) dG(\alpha) dF(\theta).
\]
We claim that $w_\emptyset(\theta, \alpha) > w(\theta, \bar{x})$ for $\alpha > \underline{\alpha}(\theta, \bar{x})$. To see why, note that from the optimality of $x_\emptyset$, we have $\theta b'(x_\emptyset) - \alpha c = 0$. If so, $\theta b'(x_\emptyset) - c < 0$ for $\alpha < 1$, or that $w(\theta, x) = \theta b(x) - cx$ is decreasing in the neighborhood of $x_\emptyset$. Given that $x_\emptyset(\theta, \alpha) < \bar{x}$ for $\alpha > \underline{\alpha}(\theta, \bar{x})$, the claim follows. If $G_1(\alpha)$ first-order stochastically dominates $G_2(\alpha)$, it attaches more weight to relatively high values of $\alpha$. Given that $w_\emptyset(\theta, \alpha) \geq w(\theta, \bar{x})$, we have $W_Q(\bar{x}; G_1) \geq W_Q(\bar{x}; G_2)$. Given this inequality holds for arbitrary $\bar{x}$, the optima can be ranked: $W^*_Q(G_1) \geq W^*_Q(G_2)$.

The ambiguity of mean-preserving spreads on $W^*_Q$ follows from the fact that, as a function of $\alpha$, individual welfare contribution $w_Q(\theta, \alpha, \bar{x})$ has both convex and concave intervals. If so, the impact of a mean-preserving spread on aggregate welfare $W^*_Q$ is sensitive to the choice of distribution $G(\alpha)$.

Proof. To show that welfare under quantity regulation could fall or rise as the population becomes more altruistic, compare welfare under three distributions. Let the first be one in which all agents are selfish ($\alpha_i = 0$ for all $i$); in the second half the population is selfish and half is perfectly altruistic ($\alpha = 1$); in the third all agents are perfectly altruistic. By construction, these capture increasing altruism in the sense

\[\text{11} \text{This ambiguity is easily established by numerical example, available from the authors.}\]
of first-order stochastic dominance. Note that optimal price regulation achieves the first-best outcome in the first case (by setting tax equal to marginal external damage \( c \)) and in the third case (setting tax equal to zero), while no uniform rate of tax can achieve that in the second population.

The second part follows from the fact that \( w_P(\theta, \alpha, t) \) is concave in \( \alpha \). To see why, note that the derivative

\[
\frac{\partial w_P}{\partial \alpha} = \left[ \theta b'(x_P) - c \right] \frac{\partial x_P}{\partial \alpha} = \left[ t - c(1 - \alpha) \right] \frac{\partial x_P}{\partial \alpha}
\]

is positive for small \( \alpha \) (i.e., \( t - c(1 - \alpha) < 0 \)) and negative for large \( \alpha \). A mean preserving spread of the distribution \( G(\alpha) \) lowers \( W_P(t) \) for any fixed tax rate \( t \). As a mean preserving spread of \( G(\alpha) \) causes the entire function \( W_P(t) \) to move down, the maximized value of this function – that is, \( W_P^* \) – must necessarily fall. ■

To sum up, an increase in altruistic tendency weakly increases \( W_Q^* \) but has an ambiguous effect on \( W_P^* \). On the other hand, greater dispersion in the degree of altruism lowers \( W_P^* \) but has an ambiguous impact on \( W_Q^* \). These properties are useful in comparing the relative merits of the two regulatory instrument under alternative distributions of altruism.

Before comparing policies in the presence of altruistic agents, we note the benchmark result that obtains in our model in the absence of altruism. If \( \alpha_i = 0 \) for all \( i \), price-regulation achieves the first-best outcome in our setting. Given our assumption that the cost of the externality is linear, the marginal cost of the externality is invariant to choices, and the externality can entirely be corrected by setting the tax at the Pigovian level. Any heterogeneity in compliance costs \( \theta_i \) ensures that no quantity cap can match that, and must deliver a strictly inferior outcome. Thus, in our setting price-based regulation (weakly) dominates quantity-based regulation in the absence of altruism. In so biasing policy preference in favor of price instruments, we can highlight how the introduction of altruism qualifies that preference.

**Proposition 5** In the presence of altruism, quantity-based regulation dominates price-based regulation when there is no or little heterogeneity in \( \theta \), while price-based regulation dominates when there is no or little heterogeneity in \( \alpha \).

**Proof.** To establish the first part, let the distribution \( F(\theta) \) collapse around its
mean $\bar{\theta}$ (that is, let $\theta_i = \bar{\theta}$ for all $i$). Then the optimal cap, implicitly defined by $\bar{\theta}b'(\bar{x}^*) - c = 0$, achieves the first-best outcome, while a tax regime does not as long as there is some heterogeneity in $\alpha$. Next, consider perturbations of $F(\theta)$ that allow some heterogeneity around $\bar{\theta}$. Given the strict ranking above, by continuity the outcome under quantity regulation dominates as long as heterogeneity in $\theta$ is sufficiently small relative to heterogeneity in $\alpha$.

To establish the second part, let the distribution $G(\alpha)$ collapse around some value $\bar{\alpha}$ (that is, let $\alpha_i = \bar{\alpha}$ for all $i$). Then the optimal tax, defined by $t^* = (1 - \bar{\pi})c$, achieves the first-best outcome, while the quantity-based regime does not as long as there is some heterogeneity in $\theta$. As in the previous case, by continuity the outcome under price regulation dominates the outcome under quantity regulation as long as heterogeneity in $\alpha$ is sufficiently small. 

The proof highlights the comparative efficiency of the two regulatory instruments by focusing on extreme cases. Removing variability in compliance costs but leaving some variability in altruism implies that quantity regulation can implement the first best, while price regulation cannot. Conversely removing variability in altruism but leaving some variability in compliance costs implies that price regulation can implement the first best, while quantity regulation cannot. In fact the implicit preference for price-based regulation in our benchmark with purely selfish agents follows not from the absence of altruistic tendencies but rather from their homogeneity: that all the $\alpha_i$ were of common value, rather than the fact that the common value was zero.

In sum, other things equal, price regulation is better at handling technological heterogeneity (variation in $\theta$) whilst quantity-based regulation is better at handling ‘motivational’ heterogeneity (variation in $\alpha$).

5.1 An Example

More generally, once we allow for significant heterogeneity in both $\alpha$ and $\theta$, there is no natural ordering of one type of instrument over the other. Welfare under quantity regulation may be higher or lower than that under price regulation for arbitrary distributions $F(\theta)$ and $G(\alpha)$. In settings that display both forms of heterogeneity, the relative performance of the two regimes is sensitive to the specifics of underlying

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distributions.

However some qualitative conclusions emerge in simple settings. Assume that the distribution \( F(\theta) \) is non-degenerate and invariant in the analysis that follows. Consider a population with a two-point distribution for the altruism parameter, with a fraction \( \pi \) of the population being highly altruistic, with \( \alpha = \alpha_H > 0 \), and the rest being selfish, with \( \alpha = 0 \). Our aim is to compare the relative performance of the two regulatory instruments as \( \pi \), the proportion of altruistic agents in the population, varies over the interval \([0, 1]\). With slight abuse of notation we now write \( W_Q^* \) and \( W_P^* \) as functions of \( \pi \), namely \( W_Q^*(\pi) \) and \( W_P^*(\pi) \).

The general propositions above shed light on the behavior of these functions. One, it follows from Proposition 4 that price-based regulation achieves the first-best outcome when \( \pi = 0 \) (by setting the tax optimally at \( c \)) or when \( \pi = 1 \) (by setting the tax optimally at \((1 - \alpha_H)c\)). The level of welfare delivered is the same in either case, or \( W_P^*(0) = W_P^*(1) = W^{FB} \) (where the superscript ‘FB’ denotes the ‘first-best’). Two, values of \( \pi \) strictly within the unit interval imply dispersion in the altruism parameter, and must deliver strictly lower welfare: \( W_P^*(\pi) < W^{FB} \) for \( 0 < \pi < 1 \). So the graph of optimized welfare delivered by the tax regime is non-monotonic, in fact \( U \)-shaped, in \( \pi \).

What of \( W_Q^* \)? Given the assumed heterogeneity in \( \theta \), we have \( W_Q^*(0) < W_P^*(0) \) and \( W_Q^*(1) < W_P^*(1) \): at each extreme quantity-based regulation cannot quite match price-based regulation, which delivers the first-best outcome. Further, an increase in \( \pi \) amounts to an increase in altruism in the sense of first-order stochastic dominance, so by Proposition 3, \( W_Q^*(\pi) \) is weakly increasing in \( \pi \).

Figure 1 plots \( W_Q^*(\pi) \) and \( W_P^*(\pi) \) as \( \pi \) varies in the unit interval. It is plotted for the most interesting case, namely that in which \( W_Q^* \) and \( W_P^* \) cross. We cannot rule out, but ignore here, the possibility that for some distributions \( F(\theta) \) and \( G(\alpha) \), \( W_Q^* \) may lie everywhere below \( W_P^* \).

In terms of Figure 1 we note that there are intervals of \( \pi \) close to zero and close to one where \( W_P^* > W_Q^* \). The tax instrument performs best when altruistic tendencies are relatively homogeneous in the population, which happens in the vicinity of \( \pi = 0 \) and \( \pi = 1 \). With unobservable heterogeneity in the \( \alpha \)'s the regulator is obliged to set a tax based on average responses – one that is too high than would ideally be set for an altruistic agent and too low than would ideally be set for a selfish one.
When there is greater dispersion in the level of altruism – values of $\pi$ close to the middle of the unit interval amount to greater dispersion – the losses due to those departures are their greatest.

More generally, what policy conclusions can we draw in such settings?

**Remark 1** Price-based regulation is better than quantity-based regulation when the proportion of altruistic agents in the population is either sufficiently high or sufficiently low. Quantity-based regulation may be better for a range of intermediate values.

We determined these intervals numerically for a calibrated example. We considered the case where $b(x)$ is logarithmic, $c = 1$, and $\theta$ is distributed uniformly over the interval $[10, 30]$. If $\alpha_H = 0.75$, we found that tax-based regulation is superior when $\pi$ is below 0.31 or above 0.98, while quantity caps are superior in the intermediate range.

### 5.2 Mixed instruments

Given that price and quantity regulation cope differently with the two categories of heterogeneity, it seems plausible that a richer class of hybrid regulatory regimes might do better. Consider a regulatory regime that combines price and quantity regulation, say by allowing agents to carry out the activity to some specified level $\bar{x}$
at no cost, and charging a tax rate $t$ for any excess over that. The regimes considered above can be viewed as special cases within this class, with price regulation being the special case where $\overline{x} = 0$ (so that all positive levels of activity were taxed at rate $t$) and quantity caps being the special case where the tax rate above $\overline{x}$ was prohibitive (so that no agent would choose to exceed $\overline{x}$).

How does this richer regime affect choices made by altruistic agents? The agent’s choice problem now is to choose $x$ to maximize

$$\theta_i b(x_i) - \alpha_i c x_i - t \max\{x - \overline{x}, 0\}$$

For given $(\theta, \overline{x}, t)$, we can now identify two critical values that partition the set of agents according to their altruism: as before there exists $\underline{\alpha}(\theta, \overline{x}) \in [0, 1]$ such that individuals with higher $\alpha$ are restrained below $\overline{x}$ by their altruism and choose $x_\theta(\theta, \alpha)$. Further, there exists $\alpha_c(\theta, \overline{x}, t) \leq \underline{\alpha}(\theta, \overline{x})$ such that individuals with $\alpha < \alpha_c$ choose $x_P(\theta, \alpha, t) > \overline{x}$ and pay tax accordingly. Individuals whose altruism parameter lies within these two thresholds choose precisely $\overline{x}$.

Aggregate welfare $W(t, \overline{x})$, as a function of the policy parameters $t$ and $\overline{x}$, is

$$\int_\theta \left[ \int_0^{\alpha_c} w_P(\theta, \alpha, t) dG(\alpha) + \int_{\alpha_c}^\alpha w(\theta, \overline{x}) dG(\alpha) + \int_\alpha^1 w_0(\theta, \alpha) dG(\alpha) \right] dF(\theta).$$

Optimal calibration of policy instruments, $t \geq 0$ and $\overline{x} \geq 0$ would satisfy the first-order conditions

$$\frac{\partial W}{\partial t}(\overline{x}^*, t^*) = \int_\theta \left[ \int_0^{\alpha_c} w_P'(\theta, \alpha, t) dG(\alpha) \right] dF(\theta) = 0,$$

and

$$\frac{\partial W}{\partial \overline{x}}(\overline{x}^*, t^*) = \int_\theta \left[ \int_{\alpha_c}^\alpha w_Q'(\theta, \overline{x}) dG(\alpha) \right] dF(\theta) = 0,$$

for an interior solution, and with $t^* = 0$ if $\frac{\partial W}{\partial t} < 0$, and $\overline{x}^* = 0$ if $\frac{\partial W}{\partial \overline{x}} < 0$.

A regime that combines the use of price and quantity instruments can only improve the outcome over regimes that restrict the value of any one of these instruments

\footnote{Of course, this two-step regulatory regime can be generalized even further, by considering general non-linear tax schedules $T(x)$ that specify the total tax to be paid for any given level of activity $x$. We do not pursue this possibility here, and note that complex instruments are not often observed in practice.}
two instruments are, at least weakly, better than one.

While a general analysis of the effect of varying altruism is not straightforward, but numerical analysis is feasible for particular examples. We considered the case where \( b(x) \) is logarithmic and \( F(\theta) \) and \( G(\alpha) \) are uniform. In our numerical simulations as the distribution of \( \alpha \) became less dispersed, the optimal choice of tax-exempt \( \pi^* \) was found to fall (so that optimal policy in effect became ‘closer’ to price regulation). In contrast as the distribution of \( \theta \) became less dispersed, the optimal regime was found to involve higher level of taxation beyond \( \pi^* \) (closer to quantity regulation).

6 Thoughts on Two Issues

6.1 Tradability of the quantity instrument

Much of the ‘prices versus quantities’ literature has focused implicitly or explicitly on major industrial pollutants, say, as sulphur dioxide or carbon emissions. In those settings, the difficulties of choosing optimal quantitative restrictions for a heterogeneous population can be mitigated by allowing regulated firms to trade quantitative caps. Not surprisingly, a lot of scholarly attention in recent years has focused on the implications of tradable allocations.

Our decision to restrict our model to non-tradable quantity instruments can be justified at three levels. Firstly, in many contexts, the relevant externality-generating activity may not be amenable to trade. Our focus on altruism as a restraining motivation points to individuals rather than firms as the unit of analysis. In the motivating examples that we have provided it would be difficult – for a variety of reasons ranging from practicality and relatively high transactions costs – to contemplate tradable quantitative limits.

Secondly, we observe that policy generally involves legal standards or limits that are not tradable. As Brozovic et al (2004: 1) note: “Our profession’s fondness for the use of economic incentives notwithstanding, regulation via direct quantitative control is commonplace in the real world. Diverse consumer and producer activities ... are controlled using maximum allowable limits.” Even in the context of environmental regulation tradability remains the exception rather than the rule.
Whilst there are important and high-profile exceptions in which quantitative limits are allowed to be traded (sulphur emissions in the US, carbon emissions in the EU, etc.) the great majority of quantity-based regulation do not incorporate tradability. This is the case even with new regulations as followers of EU legislative output (for example) will attest.\(^\text{13}\)

Thirdly, where individual choices are conditioned by altruism, that altruism may itself interfere with willingness to trade and likely market outcomes. Considering limits rather than tradable permits meant that we were not obliged to address the question of how an impurely altruistic agent would feel about selling his permit, knowing that the buyer might herself use it as the basis for engaging in the externality-generating activity. This sort of ‘complicity with’ anti-social behavior would have required additional ad hoc modeling assumptions and would have served to make predicting market outcomes and developing normative conclusions more difficult.\(^\text{14}\)

### 6.2 How do these results articulate with Weitzman?

We did not aim to stick to the framework set out in Weitzman’s (1974) seminal paper on comparing price- and quantity-regulation under uncertainty. The analysis here can, however, be related to that paper and the associated literature that followed.

Our model departs from the standard Weitzman approach in two significant ways. Firstly, in our model quantity regulation sets a quantitative upper bound on activity levels but does not require, as in Weitzman’s model, that agents set quantities exactly at the prescribed level. Indeed, an important and novel element

\(^\text{13}\) A few randomly-selected recent examples of rule-making based on quantitative limits of these sorts include: (a) The London Airports Noise Restrictions Notice (February 2007) placed an upper bound on the noisiness of aircraft allowed to land at London’s Heathrow and Gatwick Airports; (b) Authorizations to operate power stations in the UK are issued by the Environment Agency and set limits on the amounts of a variety of air and water pollutants that the station can emit; (c) The EC Waste Incineration Directive (2000/76/EC) limits emissions of heavy metals from incinerators; (d) the 2001 EC Battery Directive (2001/338EC) limits the cadmium content of batteries; (e) In 2007 the European Commission proposed a move to mandatory standards for the fuel-efficiency of new cars under which all cars sold in the EU must produce no more than 120 grams of carbon dioxide per kilometer traveled.

\(^\text{14}\) It could be speculated that introducing tradability into the current set-up would allow the quantity instrument to achieve first best. However, this is not straightforward: without sufficient restrictions trade could transfer the externality from relatively altruistic agents to those less so, even when the former has a higher \(\theta\) than the latter.
of our model is that a cap will not necessarily bind on everyone – for instance, agents with sufficiently high $\alpha$’s will choose to stay strictly within it, and in setting optimal regulatory standards we must recognize this possibility.

This modeling difference complicates comparison with Weitzman, and other models in that tradition. Nonetheless, we could ask how welfare under quantity regulation of the conventional kind (that is, when regulation prescribed a level rather than an upper bound) would compare with price regulation in our setting. Adapting the formal structure of our model to the Weitzman analysis, it is possible to show that the preference between price and quantity regulation depends upon the comparison of the variances of the two parameters, $\theta$ and $\alpha$.\textsuperscript{15} This is consistent in spirit with the intuitive discussion following Proposition 5 – price regulation is better at handling technological heterogeneity (variation in $\theta$), quantity regulation better at handling motivational heterogeneity (variation in $\alpha$).

Secondly, our model assumes that the external cost is linear in the activity level (that is, the external damage is $cx$) while benefits are concave in the activity level. The standard approach assumes damages are convex – indeed this more general structure is essential to these models, as preference between price and quantity regulation rests on a comparison of the second derivative of the benefit and cost curves.

While we have reworked the model more generally our preferred specification remains the one presented. Absent altruism (setting all $\alpha$’s equal to 0) linearity of the damage function creates an unambiguous preference for price regulation. So we can be clear that the potential superiority of the quantity instrument identified in Section 5 is because of the introduction of altruism. Incorporating convex external costs would complicate the analysis by weakening the efficiency of price regulation.

\textsuperscript{15}Weitzman’s analysis involves quadratic approximations of the cost and benefit functions, which is more appropriate if the uncertain parameter $\theta$ has an additive structure rather than the multiplicative structure used here. With that modification we can show that the welfare-superiority of price regulation over quantity regulation equals

$$\Delta = \frac{1}{2b''\theta} [c^2 \sigma_\theta^2 - \sigma_\alpha^2],$$

where $\sigma_\alpha^2$ and $\sigma_\theta^2$ are the variances of the parameters $\alpha$ and $\theta$, $c$ is the constant marginal cost associated with the externality and $b''\theta$ the slope of the marginal benefit curve is evaluated at the mean value of $\theta$. The extreme cases discussed in the proof of Proposition 5 emerge as special cases: Noting that $b''\theta < 0$, price regulation is superior when $\sigma_\alpha^2 \to 0$, while quantity regulation is preferable when $\sigma_\theta^2 \to 0$.\textsuperscript{20}
But it is worth noting why that should be the case. With constant marginal external costs, the efficiency of price regulation depends only on the variance of the altruism parameter and not on its mean. The efficient tax \((1-\pi)c\) corrects for the insufficiency of altruism, and works equally well regardless of the value that \(\pi\) takes. If external costs were convex, the efficient corrective tax would itself vary with the extent of altruism, even if that level were uniform in the population. While this is an interesting observation it would confound the clarity of our main results – those with regard to variability in the population.

7 Conclusions

Our analysis builds on the evidence that people take into account the impact that their actions have upon others around them then. They may feel displeasure (‘cold prickle’) from imposing unpleasantness upon others even when they may be within their legal rights to do so and this may cause them to moderate their engagement in externality-imposing activities. At the same time those activities may be subject to some regulatory or legal control. The question we have addressed in this paper is the following: How does introducing impure altruism into the regulated population change regulatory prescriptions?

We have been able to draw a number of conclusions both about the appropriate calibration of any particular instrument and about the comparative merits of alternative instrument classes.

In general, if individuals’ externality-generating activities are partially restrained by altruistic motivations, optimal regulatory standards need to be less stringent than otherwise. If tax is the policy instrument of choice, the existence of altruistic agents in the population leads to a lower tax being optimal. If quantitative caps are in use then the optimal cap is higher.

The choice between regulatory instruments depends on the nature of the heterogeneity in the population. Loosely speaking, we can think of the model presented here as featuring two types of heterogeneity – technological (variation in \(\theta\)’s) and motivational (variation in \(\alpha\)’s). In our set-up price instruments tend to be good at handling the former but less so at handling the latter. When there is no motivational heterogeneity the price instrument is unambiguously preferred, but once
motivational heterogeneity is introduced the ranking of instruments becomes more complex.

The complexity is best illustrated through our simple example that considers the optimal regulatory choice when there are just two types of individuals: selfish and altruistic. Under optimal quantity-based regulation welfare is everywhere improving in the altruistic proportion in the population. Under optimal price-based regulation the relationship is non-monotonic. If so, price-based regulation may be preferable when the regulated population is either predominantly altruistic or predominantly selfish, quantity-based regulation for intermediate cases.

Our model assumes that the distribution of altruistic tendencies in the population is exogenously specified and, crucially, is independent of the choice of regulatory instrument. A recent line of research relating to – but quite distinct from – ours suggests that the use of price incentives may itself diminish pro-social behavior, perhaps by crowding-out intrinsic motivation. The works of Frey and Oberholzer-Gee (1997), Gneezy and Rustichini (2000a, 2000b) and Brekke et al. (2003) provide a representative sample. In terms of our model, this creates the possibility that the choice of instrument might affect the distribution of $\alpha$’s in the regulated population. While we acknowledge the value of these lines of inquiry, we have abstracted from any such effects in our model, taking the attitudes of individuals towards their behavior as fixed but the behavior itself subject to influence by policy. This allows us to conclude that it is not the effects identified in that line of research that generate the apparent under-performance of the tax instrument in the model presented here.
References


