The Friedman rule and inflation targeting

Qian Guo[a*], Huw Rhys[b], Xiaojing Song[c] and Mark Tippett[d,e]

[a] *Department of Management, Birkbeck College, University of London, WC1E 7HU, UK*
[b] *School of Management and Business, University of Aberystwyth, Ceredigion, SY23 3AL, UK*
[c] *Norwich Business School, University of East Anglia, NR4 7TJ, UK*
[d] *Victoria Business School, Victoria University of Wellington, New Zealand*
[e] *Business School, University of Sydney, Australia*

We use concepts from the financial economics discipline – and in particular the methods of continuous time finance – to develop a monetarist model under which the rate of inflation evolves in terms of a first order mean reversion process based on a "white noise" error structure. The Fokker-Planck (that is, the Chapman-Kolmogorov) equation is then invoked to retrieve the steady state (that is, unconditional) probability distribution for the rate of inflation. Monthly data for the U.K. Consumer Price Index (CPI) covering the period from 1988 until 2012 are then used to estimate the parameters of the probability distribution for the U.K. inflation rate. The parameter estimates are compatible with the hypothesis that the U.K. inflation rate evolves in terms of a slightly skewed and highly leptokurtic probability distribution that encompasses non-convergent higher moments. We then determine the Hamilton-Jacobi-Bellman fundamental equation of optimality corresponding to a monetary policy loss function defined in terms of the squared difference between the targeted rate of inflation and the actual inflation rate. Optimising and then solving the Hamilton-Jacobi-Bellman equation shows that the optimal control for the rate of increase in the money supply will be a linear function of the difference between the current rate of inflation and the targeted inflation rate. The conditions under which the optimal control will lead to the Friedman Rule are then determined. These conditions are used in conjunction with the Fokker-Planck equation and the mean reversion process describing the evolution of the inflation rate to determine the probability distribution for the inflation rate under the Friedman Rule. This shows that whilst the empirically determined probability distribution for the U.K. inflation rate meets some of the conditions required for the application of the Friedman Rule, it does not meet them all.

Key Words: Friedman rule; Hamilton-Jacobi-Bellman equation; inflation; White noise process

JEL Classification: C14; C61; E31; E51; E52

---

* Corresponding author. Email: m.tippett@lboro.ac.uk

1. Introduction

Persistently high inflation is a relatively recent phenomenon for many western countries. In the United Kingdom, for example, the price level remained broadly stable between 1660 and 1930. In contrast, the UK price level in 2011 was about fifty times what it had been in 1930 (Miles and Scott 2002, 287-288; O'Donoghue, Goulding and Allen 2004). Likewise, prices were broadly stable in the United States between 1665 and 1930. However, the US price level in 2011 was about thirteen times what it had been in 1930 (Sahr 2011; US Bureau of Labor Statistics 2012). In his *A Program for Monetary Stability* (1960, 90) Milton Friedman proposes that inflationary pressures be addressed by imposing a requirement under which the monetary authorities increase the "… stock of money … at a fixed rate year-in and year-out without any variation in the rate of increase to meet cyclical needs." This is known as "Friedman's k% Rule" and would effectively deprive the monetary authorities of all discretionary powers; in particular, the discretionary powers needed to adjust monetary policy so as to accommodate current economic circumstances and events. The "no feedback" aspect of the Friedman rule has been heavily criticised; in particular, that the volatility in the demand for money will mean that a constant rate of increase in the money supply will accentuate the variability in the rate of inflation and real output as well. It is because of this that many economists – and in particular, the neo-Keynesians – believe that a discretionary monetary policy that responds to the state of the economy will perform much better than the naively simple Friedman rule (Giannoni and Woodford 2005; Svensson 2010). In response Friedman (1960, 98) acknowledges that whilst there "are persuasive theoretical grounds for desiring to vary the rate of growth [in the stock of money] to offset other factors … in practice, we do not know when to do so and by how much …. [T]herefore, deviations from th[is] simple rule have been destabilizing rather than the reverse."

Our purpose here is to formulate a monetarist model of inflation targeting and then to use it to assess whether the Friedman Rule might represent an optimal control within the model. We preface the development of our model by noting that monetarists are of the view that long run real output hinges on the productive capacity of the economy and that government initiatives can at best, have only a trivial and more generally, an adverse impact on economic activity (Friedman 1972, 28; Friedman 1988; Miles and Scott 2002, 416-418). Support for this proposition is to be found in finance theory

where the Miller and Modigliani (1961) dividend policy irrelevance theorem articulates that a firm's value is determined by discounting the future cash flows arising from its investment opportunity set. Whilst firms may use dividends to redistribute their operating cash flows across time, a firm's dividend policy itself cannot alter the present value of the future cash flows that arise from its underlying investment opportunity set. Similarly, governments may use their tax, borrowing and spending powers to re-distribute an economy's real output across time. However, the inter-temporal re-distribution policies invoked by the government cannot boost the present value of the future real output produced by the economy (Fama 1981, Stiglitz, 1983; 1988). What determines real output depends on real factors such as the economy's natural endowments; the enterprise, ingenuity and industry of its people; the extent of thrift; the structure of industry and its competitive institutions and so on (Friedman 1971, 847). Monetarists argue that these factors taken in conjunction with the quantity theory of money will mean that the most significant factor influencing the rate of inflation over the longer term is the rate of growth in the money supply. In particular, in the long run, the instantaneous rate of inflation will be equal to the average rate of growth in the money supply less the average rate of growth in real output (Friedman 1972, 27-28; Miles and Scott 2002, 310-312; McCallum and Nelson 2010, 21; Mishkin 2013, 51-52).

The methodological framework employed in our analysis is based on what Miller (1999, 96) calls the typical "business school" approach of "maximising some objective function … taking the prices of securities in the market as given". This contrasts with "the characteristic economics department approach" which uses Walrasian general equilibrium analysis to deduce "how the market prices, which the micro maximizers take as given, actually evolve" (Miller, 1999, 96). The differing nature of these two approaches to finance theory means that there will be subtle differences in the way they model the inflation process – although we would here emphasise that Friedman's preferences were much more akin to those of the business school approach than that which characterises the economics department approach (Friedman 1955). Given this, in the next section we invoke the classical business school approach of modelling instantaneous increments in the inflation rate in terms of a first order mean reversion process in continuous time with a white noise error structure. We then use the mean reversion process in conjunction with the Fokker-Planck (that is, the Chapman-Kolmogorov) equation to retrieve the steady state (that is, the unconditional)

probability distribution for the rate of inflation (Cox and Miller 1965, 213-215). In section 3 we use monthly data for the U.K. Consumer Price Index (CPI) covering the period from January, 1988 until December, 2012 to estimate the parameters of the steady state probability distribution for the U.K. inflation rate.[1] The parameter estimates based on this data are compatible with the hypothesis that the U.K. inflation rate evolves in terms of a negatively skewed and highly leptokurtic probability distribution that encompasses higher moments which are non-convergent. In section 4 we formulate an inflation targeting model under which monetary policy is determined by minimising a loss function defined in terms of the squared difference between the targeted rate of inflation and the actual inflation rate. The monetary policy loss function can then be used in conjunction with the mean reversion process describing the evolution of the inflation rate to determine the Hamilton-Jacobi-Bellman fundamental equation of optimality for the inflation targeting problem. Optimising and then solving the Hamilton-Jacobi-Bellman equation shows that the optimal control for the rate of increase in the money supply will be a linear function of the difference between the current rate of inflation and the targeted inflation rate. The conditions under which the optimal control will lead to the Friedman Rule are then determined. These latter conditions are then used in conjunction with the Fokker-Planck equation and the mean reversion process describing the evolution of the inflation rate to determine the steady state probability distribution for the inflation rate under the Friedman Rule. This shows that whilst the empirically determined probability distribution for the U.K. inflation rate meets some of the conditions necessary for the application of the Friedman Rule, it does not meet them all. This in turn will mean it is likely that a discretionary monetary policy will be more successful in controlling inflation than the strict application of the Friedman Rule. Section 5 closes the paper with our summary conclusions.

2. Monetarist model of inflation

We begin our analysis with a formal statement of the assumptions on which our model of the inflation process is based.

[1] The Bank of England (2014) is required to implement monetary policy with the objective of meeting an "inflation target of 2% [as] expressed in terms of an annual rate of inflation based on the Consumer Prices Index (CPI)."

Assumption 1: There is a single physical consumption good (that is, real output) which is in aggregate supply of *y(t)* units at time *t*. The instantaneous rate of growth (per unit time) in the supply of the consumption good evolves in terms of the following process:

$$\frac{1}{y(t)}.\frac{dy(t)}{dt} = \bar{\mu} + \eta.\frac{dz_1(t)}{dt} \tag{1}$$

where $\bar{\mu}$ is the expected instantaneous proportionate increase (per unit time) in the supply of the consumption good and η is an intensity parameter defined on a white noise process, $\frac{dz_1(t)}{dt}$, with unit variance parameter.[2]

Assumption 2: The unit price, *I(t)*, of the consumption good is stated in terms of a monetary unit which is in aggregate supply of *m(t)* units at time *t*. The instantaneous proportionate increase (per unit time) in the money supply evolves in terms of the following process:

$$\frac{1}{m(t)}.\frac{dm(t)}{dt} = \bar{\alpha} + \delta.\frac{dz_2(t)}{dt} \tag{2}$$

where and $\bar{\alpha}$ is the expected instantaneous proportionate increase in the money supply (per unit time) and δ is an intensity parameter defined on a white noise process, $\frac{dz_2(t)}{dt}$, with unit variance parameter.

Assumption 3: The instantaneous rate of inflation (per unit time) in the price of the consumption good is given by (Friedman 1956 1970 1976):

$$i(t) = \frac{1}{I(t)}.\frac{dI(t)}{dt} = \frac{d\log[I(t)]}{dt} \tag{3}$$

---

[2] A Wiener process, *z(t)*, is a stochastic variable that is normally distributed with a mean of zero and variance of t (Hoel, Port and Stone 1987, 122-124). The derivative of a Wiener process is called a white noise process. Hoel, Port and Stone (1987, 142) note that the white noise process "is not a stochastic process in the usual sense. Rather *dz(t) = z'(t)dt* is a 'functional' that … can be used to define certain stochastic differential equations."

Moreover, instantaneous changes in the inflation rate evolve in terms of the following process:

$$\frac{di(t)}{dt} = \beta\{\frac{1}{m(t)}.\frac{dm(t)}{dt} - \frac{1}{y(t)}.\frac{dy(t)}{dt} - i(t)\} + \sigma\{\frac{1}{m(t)}.\frac{dm(t)}{dt} + \theta - \frac{1}{y(t)}.\frac{dy(t)}{dt} - i(t)\}.\frac{dz_3(t)}{dt} \tag{4}$$

where β is a speed of adjustment coefficient, σ is a constant of proportionality, θ is a parameter that captures the skewness in the distribution of the inflation rate and $\frac{dz_3(t)}{dt}$ is a white noise process with unit variance parameter.

Under this process the instantaneous rate of inflation will gravitate towards an equilibrium given by the difference between the instantaneous rate of growth in the money supply and the instantaneous rate of growth in real output (Friedman 1972, 49; Miles and Scott 2002, 310-313; Lee and Chang 2007, McCallum and Nelson 2010, 21; Mishkin 2013, 51-52; Chang et. al 2013). Moreover, the rate of convergence will hinge on the speed of adjustment coefficient, β, as well as stochastic perturbations that become more volatile the greater the disequilibrium:

$$\{\frac{1}{m(t)}.\frac{dm(t)}{dt} + \theta - \frac{1}{y(t)}.\frac{dy(t)}{dt} - i(t)\}$$

in the "skewness adjusted" rate of inflation (Friedman 1977, 278; Miles and Scott 2002, 296). More detailed information about the nature of the stochastic perturbations that arise as a result of disequilibrium in the inflation rate can be obtained by substituting the instantaneous rate of growth in real output as summarised by equation (1) and the instantaneous proportionate increase in the money supply as summarised by equation (2) into the expression for instantaneous changes in the rate of inflation as summarised by equation (4), or:

$$\frac{di(t)}{dt} = \beta\{\bar{\alpha} + \delta.\frac{dz_2(t)}{dt} - \bar{\mu} - \eta.\frac{dz_1(t)}{dt} - i(t)\} +$$

$$\sigma\{\bar{\alpha} + \delta.\frac{dz_2(t)}{dt} + \theta - \bar{\mu} - \eta.\frac{dz_1(t)}{dt} - i(t)\}.\frac{dz_3(t)}{dt}$$

Now, one can simplify this latter expression by assuming that all white noise processes are uncorrelated or (Arnold 1974, 91):[3]

$$\frac{dz_1(t)}{dt}.\frac{dz_2(t)}{dt}=\frac{dz_1(t)}{dt}.\frac{dz_3(t)}{dt}=\frac{dz_2(t)}{dt}.\frac{dz_3(t)}{dt}=0$$

It then follows that the differential equation describing the evolution of the inflation rate can be stated as:

$$\frac{di(t)}{dt}=\beta\{\overline{\alpha}-\overline{\mu}-i(t)\}+\beta\delta.\frac{dz_2(t)}{dt}-\beta\eta.\frac{dz_1(t)}{dt}+\sigma\{\overline{\alpha}+\theta-\overline{\mu}-i(t)\}.\frac{dz_3(t)}{dt}$$

Here one can define the standardised variable:

$$z(t)\equiv\frac{\beta\delta.dz_2(t)-\beta\eta.dz_1(t)+\sigma\{\overline{\alpha}+\theta-\overline{\mu}-i(t)\}dz_3(t)}{\sqrt{Var[\beta\delta.dz_2(t)-\beta\eta.dz_1(t)+\sigma\{\overline{\alpha}+\theta-\overline{\mu}-i(t)\}dz_3(t)]}}$$

where *Var(.)* is the variance operator. Evaluating the variance in the denominator of the above expression then shows that the standardised variable can be restated as:

$$z(t)=\frac{\beta\delta.dz_2(t)-\beta\eta.dz_1(t)+\sigma\{\overline{\alpha}+\theta-\overline{\mu}-i(t)\}dz_3(t)}{\sqrt{[\beta^2(\delta^2+\eta^2)+\sigma^2\{\overline{\alpha}+\theta-\overline{\mu}-i(t)\}^2]dt}}$$

where $dz(t)\equiv z(t).\sqrt{dt}$ is a white noise process with unit variance parameter. It then follows that the rate of inflation will evolve in terms of the following differential equation: [4]

---

[3] This is a particularly simple specification, yet as we shall see in subsequent sections, one that is adequate for explaining the functional form of the probability distribution of the rate of inflation. The parsimonious nature of Assumptions 1, 2 and 3 on which our model rests will mean that there will be a merging of errors of measurement with the unexpected rate of inflation, the unexpected proportionate increase in real output and the unexpected proportionate increase in the money supply and these will contribute to the plausibility that the affected white noise processes are all uncorrelated. Moreover, the relaxing of the assumption that the white noise processes are all uncorrelated will not change the functional form of the probability distribution of the rate of inflation, although it will considerably complicate the algebra involved in deriving the given probability distribution.

[4] Note that setting $\sigma^2 = 0$ in this differential equation leads to the well known Uhlenbeck and Ornstein (1930) process which is one of the most widely used stochastic processes for the modelling of mean reversion processes in the financial economics literature (Gibson and Schwartz 1990; Barndorff-

$$\frac{di(t)}{dt} = \beta\{\overline{\alpha} - \overline{\mu} - i(t)\} + \sqrt{\omega^2 + \sigma^2\{\overline{\alpha} + \theta - \overline{\mu} - i(t)\}^2}.\frac{dz(t)}{dt} \qquad (5)$$

where $\omega^2 = \beta^2(\delta^2 + \eta^2)$ is the variance parameter associated with increments in the inflation rate when the rate of inflation is equal to its skewness adjusted long run mean; that is, when $i(t) = \{\overline{\alpha} - \overline{\mu} + \theta\}$. Moreover, this will mean that the instantaneous inflation rate will gravitate towards a long run mean equal to the difference between the expected rate of growth in the money supply and the expected rate of growth in real output, or:

$$E[di(t)] = \beta\{\overline{\alpha} - \overline{\mu} - i(t)\}dt \qquad (6)$$

where *E(.)* is the expectation operator. The speed with which the current rate of inflation will converge towards its long run mean, $(\overline{\alpha} - \overline{\mu})$, will hinge on the speed of adjustment coefficient, β.[5] Larger values of β will mean that the rate of inflation

---

Nielsen and Shephard 2001; Hong and Satchell 2012). Moreover setting $\theta = 0$ leads to the differential equation of the scaled "t" distribution which Praetz (1972) and Blattberg and Gonedes (1974) use to provide an early example of the modelling of stock returns.

[5] The conditional expected rate of inflation is given by:

$$M(t) \equiv E[i(t)] = \int_{-\infty}^{\infty} ig(i,t)di$$

where *g(i,t)* is the conditional probability distribution for the inflation rate at time t. One can differentiate through this expression in which case we have:

$$M'(t) = \int_{-\infty}^{\infty} i.\frac{\partial g(i,t)}{\partial t}.di$$

Here, the Fokker-Planck equation shows (Karlin and Taylor 1981, 219-221; Cox and Miller 1965, 213-215):

$$\frac{\partial g(i,t)}{\partial t} = \frac{1}{2}.\frac{\partial^2}{\partial i^2}\{\frac{Var[di(t)]}{dt}.g(i,t)\} - \frac{\partial}{\partial i}\{\frac{E[di(t)]}{dt}.g(i,t)\}$$

or upon substituting equations (6) and (7):

$$\frac{\partial g(i,t)}{\partial t} = \frac{1}{2}.\frac{\partial^2}{\partial i^2}[\{\omega^2 + \sigma^2(\overline{\alpha} + \theta - \overline{\mu} - i)^2\}g(i,t)] - \frac{\partial}{\partial i}[\beta(\overline{\alpha} - \overline{\mu} - i)g(i,t)]$$

It then follows that:

converges more quickly towards its long run mean than will be the case with lower values of β. Furthermore, the stochastic component of the instantaneous increment in the inflation rate will have a variance of:

$$Var[di(t)] = \omega^2 + \sigma^2\{\overline{\alpha} - \overline{\mu} + \theta - i(t)\}^2 dt \qquad (7)$$

Note that if the current inflation rate differs from its skewness adjusted long run mean then the instantaneous variance associated with increments in the inflation rate increases by $\sigma^2\{\overline{\alpha} - \overline{\mu} + \theta - i(t)\}^2$. Here, σ^2 is a parameter that captures the magnitude of the additional uncertainty that arises because of disequilibrium in the skewness adjusted inflation rate (Friedman 1977, 278; Miles and Scott 2002, 296).

Now, let $x \equiv i - (\overline{\alpha} - \overline{\mu})$ be defined as the "abnormal" or "unexpected" rate of inflation. Then one can use the Fokker-Planck (that is, the Chapman-Kolmogorov) equation in conjunction with the instantaneous mean (6) and instantaneous variance (7) of increments in the inflation rate to retrieve the steady state (that is, unconditional) probability distribution of the unexpected rate of inflation; namely (Merton 1975, 389-390; Karlin and Taylor 1981, 219-221):[6]

$$M'(t) = \int_{-\infty}^{\infty} i.\{\frac{1}{2}.\frac{\partial^2}{\partial i^2}[(\omega^2 + \sigma^2(\overline{\alpha} + \theta - \overline{\mu} - i)^2)g(i,t)] - \frac{\partial}{\partial i}[\beta(\overline{\alpha} - \overline{\mu} - i)g(i,t)]\}di$$

One can then use appropriate high order contact conditions to apply integration by parts to both terms on the right hand side of the above equation in which case it follows that the conditional expectation for the inflation rate satisfies the following ordinary differential equation:

$$M'(t) + \beta \mathrm{M}(t) = \beta(\overline{\alpha} - \overline{\mu})$$

Solving the above differential equation under the initial condition *M(0) = i(0)* shows that the conditional expected rate of inflation is given by:

$$E[i(t)] = [i(0) - (\overline{\alpha} - \overline{\mu})]e^{-\beta t} + (\overline{\alpha} - \overline{\mu})$$

Note how this result shows that the expected inflation rate converges more quickly towards its long run mean, $(\overline{\alpha} - \overline{\mu})$, as the speed of adjustment coefficient, β, grows in magnitude.

[6] We have previously shown that the Fokker-Planck equation for the stochastic differential equation (5) is given by (Karlin and Taylor 1981, 219-221; Cox and Miller 1965, 213-215):

$$\frac{\partial g(i,t)}{\partial t} = \frac{1}{2}.\frac{\partial^2}{\partial i^2}[\{\omega^2 + \sigma^2(\overline{\alpha} + \theta - \overline{\mu} - i)^2\}g(i,t)] - \frac{\partial}{\partial i}[\beta(\overline{\alpha} - \overline{\mu} - i)g(i,t)]$$

$$g(x) = c[1 + \frac{(\theta - x)^2}{\nu_1}]^{-(1+\nu_2)}.\exp[\frac{2\nu_2\theta}{\sqrt{\nu_1}}.\tan^{-1}(\frac{\theta - x}{\sqrt{\nu_1}})] \tag{8}$$

where $-\infty < x < \infty$ and (Jeffreys 1961, 75; Yan 2005, 6):

$$c = \frac{\Gamma(\nu_2 + 1)}{\sqrt{\pi}.\Gamma(\nu_2 + \frac{1}{2})} \left| \frac{\Gamma(\nu_2 + 1 + \frac{j\nu_2\theta}{\sqrt{\nu_1}})}{\Gamma(\nu_2 + 1)} \right|^2$$

is the normalising constant. Moreover, $j = \sqrt{-1}$ is the pure imaginary number, $\Gamma(.)$ is the gamma function, $|.|$ is the modulus of a complex number, $\nu_1 = \frac{\omega^2}{\sigma^2} > 0$ and $\nu_2 = \frac{\beta}{\sigma^2} > 0$. Here, however, Yan (2005, 6) notes that a serious "obstacle" with the empirical implementation of the above "probability distribution has been the evaluation of the normalising constant, c." Likewise, Kendall and Stuart (1977, 163) note that the probability distribution "is very difficult to handle in practice … owing to the impossibility of expressing the distribution function in terms of ordinary functions." Hence, given the difficulties associated with the direct evaluation of the normalising constant one can make the substitution $z = \frac{-2}{\pi}.\tan^{-1}(\frac{\theta - x}{\sqrt{\nu_1}})$ into equation (8). It then follows *-1 < z < 1* and that the probability distribution (8) may be re-stated as:

---

where, as previously, *g(i,t)* is the conditional probability distribution for the inflation rate at time t. Now suppose *i* has a steady-state probability distribution independent of *i(0)* by which we mean:

$$\underset{t\to\infty}{Limit}\, g(i,t;i(0)) = g(i)$$

Then, $\underset{t\to\infty}{Limit} \frac{\partial g(i,t)}{\partial t} = 0$, and the unconditional probability distribution for the inflation rate, *g(i),* will satisfy the ordinary differential equation (Merton 1975, 389-390; Karlin and Taylor 1981, 220):

$$\frac{1}{2}.\frac{d^2}{di^2}[\{\omega^2 + \sigma^2(\bar{\alpha} + \theta - \bar{\mu} - i)^2\}g(i,t)] = \frac{d}{di}[\beta(\bar{\alpha} - \bar{\mu} - i)g(i,t)]$$

Solving this differential equation under the requirement $\int_{-\infty}^{\infty} g(i)di = 1$ leads to the steady state probability distribution stated as equation (8) of the text.

$$g(z) = c.\frac{\pi}{2}\sqrt{\nu_1}\cos^{(2\nu_2)}(\frac{\pi}{2}.z).\exp(\frac{-\nu_2\theta\pi}{\sqrt{\nu_1}}.z) \tag{9}$$

Integrating across the above expression then shows that the normalising constant can be expressed as:[7]

$$c = \frac{2}{\pi\sqrt{\nu_1}}[\int_{-1}^{1}\cos^{(2\nu_2)}(\frac{\pi}{2}.z).\exp(\frac{-\nu_2\theta\pi}{\sqrt{\nu_1}}.z)dz]^{-1} = \frac{\Gamma(\nu_2+1)}{\sqrt{\pi}.\Gamma(\nu_2+\frac{1}{2})}|\frac{\Gamma(\nu_2+1+\frac{j\nu_2\theta}{\sqrt{\nu_1}})}{\Gamma(\nu_2+1)}|^2 \tag{10}$$

However, since the above integral cannot be evaluated in terms of elementary functions we apply numerical estimation using 15 point Gauss-Legendre quadrature to determine the normalising constant, *c*. This procedure is exact for integrals comprised of polynomials of order 29 or less (Carnahan, Luther and Wilkes 1969, 101-105).

3. Parameter estimation

Having resolved the difficulties associated with the evaluation of the normalising constant, *c*, we now address the issue of parameter estimation for the probability

---

[7] We show below that $\nu_2 = 1$ is a necessary condition for the Friedman Rule to be an optimal monetary targeting rule. This in turn will mean:

$$\int_{-1}^{1}\cos^{(2\nu_2)}(\frac{\pi}{2}.z).\exp(\frac{-\nu_2\theta\pi}{\sqrt{\nu_1}}.z)dz = \int_{-1}^{1}[\cos(\frac{\pi}{2}.z)]^2.\exp(\frac{-\theta\pi}{\sqrt{\nu_1}}.z)dz$$

Here, however, one can substitute the trigonometric identity $[\cos(\frac{\pi}{2}.z)]^2 = \frac{1}{2}\{1+\cos(\pi z)\}$ to give:

$$\int_{-1}^{1}[\cos(\frac{\pi}{2}.z)]^2.\exp(\frac{-\theta\pi}{\sqrt{\nu_1}}.z)dz = \frac{1}{2}\int_{-1}^{1}\{1+\cos(\pi z)\}\exp(\frac{-\theta\pi}{\sqrt{\nu_1}}.z)dz = \frac{\nu_1}{\theta\pi(\theta^2+\nu_1)}.\sinh(\frac{\theta\pi}{\sqrt{\nu_1}})$$

where sinh(.) is the hyperbolic sine function. It then follows:

$$c = \frac{2}{\pi\sqrt{\nu_1}}.\frac{\theta\pi(\theta^2+\nu_1)}{\nu_1}.\frac{1}{\sinh(\frac{\theta\pi}{\sqrt{\nu_1}})} = \frac{2\theta(\theta^2+\nu_1)}{\nu_1.\sqrt{\nu_1}}.\operatorname{csc}h(\frac{\theta\pi}{\sqrt{\nu_1}})$$

where csch(.) is the hyperbolic cosecant function.

distribution (8). We begin by noting that the numerical procedure used to evaluate the normalising constant (which suppresses the parameters, θ, ν_1 and ν_2) will mean that it is not possible to use the method of maximum likelihood (ML) for parameter estimation. Moreover, Ashton and Tippett (2006, 1591) show that the variance of the probability distribution (8) is given by:

$$E(x^2) = \frac{\nu_1 + \theta^2}{2\nu_2 - 1} \tag{11}$$

whilst its third moment is:

$$E(x^3) = \frac{-4\theta(\nu_1 + \theta^2)}{(2\nu_2 - 1)(2\nu_2 - 2)} \tag{12}$$

Note how this latter result implies that the third (and higher) moments of the probability distribution (8) will be non-convergent when $\nu_2 \leq 1$. When this latter circumstance prevails, the Generalised Method of Moments (GMM) will return inefficient (that is, inconsistent) estimates of the relevant parameters. Given the difficulties associated with the ML and GMM techniques, parameter estimation was implemented using the "χ^2 minimum method" based on the Cramér-von Mises goodness of fit statistic as summarised by Cramér (1946, 426-427).[8]

To implement parameter estimation using the χ^2 minimum method we first determined the monthly rates of inflation implied by the U.K. Consumer Price Index (CPI) covering the period from 31 January, 1988 until 31 December, 2012. Table 1 provides a summary of the distributional properties of the monthly inflation rate over this period. Note how the average annualised monthly rate of inflation amounts to 2.82% with a standard deviation of 5.14%.[9] The median rate of inflation amounted to 3.27%, the

[8] Mood, Graybill and Boes (1974, 286-287) also refer to this procedure but call it the "minimum distance method". Avni (1976) treats the problem of finding confidence limits for parameters when the "best values" are calculated using the χ^2 minimum method.

[9] We are here following Friedman (1956 1970 1976) in defining the inflation rate in terms of the instantaneous logarithmic derivative of a specified price index. Thus, the annual rate of inflation over the instantaneous period from time t until time $(t + dt)$ is defined in Assumption 3 and takes the form:

INSERT TABLE ONE ABOUT HERE

minimum inflation rate was -11.67% and the maximum rate of inflation amounted to 39.82%. Finally, the standardised skewness measure is 1.02 whilst the standardised kurtosis measure is 8.97. The latter statistic evidences a significant degree of leptokursis and thereby shows that it is unlikely the distributional properties of the inflation rate can be characterised as Gaussian.

Further information about the distributional properties of the rate of inflation can be obtained by applying the Cramér-von Mises goodness of fit statistic to the probability density (8). To calculate the Cramér-von Mises goodness of fit statistic one must first order the $N = 299$ monthly rates of inflation comprising our sample from the most negative rate of inflation up to the most positive rate of inflation. We then have $i_1 = -11.67\%$ (per annum) to be the most negative rate of inflation in our sample, $i_2 = -10.29\%$ to be the second most negative rate of inflation, $i_3 = -9.27\%$ to be the third most negative rate of inflation and so on, right up to the most positive rate of

$$i(t) = \underset{\Delta t \to 0}{Limit} \frac{\log[\frac{I(t+\Delta t)}{I(t)}]}{\Delta t} = \frac{\log[I(t+dt)] - \log[I(t)]}{dt} = \frac{d\log[I(t)]}{dt}$$

where *I(t)* is the unit price of the consumption good defined in Assumption 2 as approximated by the U.K. Consumer Price Index. Since the U.K. Consumer Price Index is published on a monthly basis we let $\Delta t = {}^{1}/_{12}$ in which case we approximate the instantaneous annual rate of inflation through the formula:

$$i(t) \approx \frac{\log[\frac{I(t+\Delta t)}{I(t)}]}{\Delta t} = 12\log[\frac{I(t+\Delta t)}{I(t)}]$$

As an example, on 30 November, 2006 the U.K. CPI stood at 103.4. By 31 December, 2006 the CPI had increased to 104.0. This means that over the month of December, 2006 the approximation to the instantaneous annual rate of inflation turns out to be:

$$12\log(\frac{104.0}{103.4}) = 0.0694$$

or 6.94%.

inflation which is $i_{299} = 39.82\%$ (per annum). The Cramér-von Mises goodness of fit statistic, T_3, is then determined from the following formula (Kendall and Stuart 1979, 476):

$$T_3 = \frac{1}{12N} + \sum_{p=1}^{N} [\int_{-1}^{z_p} g(z)dz - \frac{2p-1}{2N}]^2 \qquad (13)$$

Here $z_p = \frac{-2}{\pi}.\tan^{-1}(\frac{\theta - x_p}{\sqrt{\nu_1}})$ and $x_p = i_p - (\bar{\alpha} - \bar{\mu})$ is the abnormal component of the pth ordered instantaneous rate of inflation comprising our sample. Moreover, $\int_{-1}^{z_p} g(z)dz$ is the accumulated area under the probability distribution (9) below the pth ordered inflation rate – which was again evaluated using the 15 point Gauss-Legendre quadrature formula (Carnahan, Luther and Wilkes 1969, 101-105). Table 2 summarises the estimates of the parameters θ, ν_1 and ν_2 obtained from this procedure for long run expected rates of inflation, $(\bar{\alpha} - \bar{\mu})$, which vary from 1% to 3% (per annum). Thus, if one assumes a long run expected rate of inflation of $(\bar{\alpha} - \bar{\mu}) = 1\%$ then the Cramér-von Mises goodness of fit statistic is minimised when $T_3 = 0.1251$ and the estimated parameters are $\theta = 0.0372$, $\nu_1 = 0.0011$ and $\nu_2 = 0.2424$. Note that as the long run expected rate of inflation gradually increases from 1% to 2.4% (per annum), the minimised Cramér-von Mises goodness of fit statistic gradually declines to $T_3 = 0.0657$ with estimated parameters of $\theta = 0.0337$, $\nu_1 = 0.0031$ and $\nu_2 = 1.1056$.

In contrast, as the long run expected rate of inflation grows beyond 2.4% the minimised Cramér-von Mises goodness of fit statistic gradually increases in magnitude, reaching a value of $T_3 = 0.1388$ when the long run expected rate of inflation is 3% (per annum).

INSERT TABLE TWO ABOUT HERE

Here we need to note that Anderson and Darling (1952, 203) have computed quantiles of the probability distribution for the Cramér-von Mises statistic, T_3. These show that if the data on which T_3 is based are drawn from the hypothesised probability distribution, then the probability of T_3 exceeding 0.4614 is 5% whilst the probability of T_3 exceeding 0.7435 amounts to 1%.[10] Hence, our minimised test statistic of $T_3 = 0.0657$ when the long run expected rate of inflation is $(\bar{\alpha} - \bar{\mu}) = 2.4\%$ (per annum) would appear to confirm that the probability distribution with parameter values of $\theta = 0.0337$, $\nu_1 = 0.0031$ and $\nu_2 = 1.1056$ provides a better description of the way the inflation rate evolves than for any of the other long run expected rates of inflation on which Table 2 is based.[11] Unfortunately, Anderson and Darling (1952) determine the probability distribution of T_3 only for the case of a completely specified hypothesised probability distribution; that is, under the assumption that none of the parameters of the hypothesised probability distribution have had to be estimated. When, as in the present case, the calculation of T_3 is based on estimated parameters, Anderson (2010, 6) notes that the quantiles for the probability distribution of the Cramér-von Mises statistic are both distribution specific and "much smaller than those given above for the case when parameters are known." Fortunately, Cramér (1946, 506) shows that when the parameters of an hypothesised probability distribution must be estimated, one can apply the χ^2 goodness of fit test using optimally estimated parameters but with the degrees of freedom reduced by one unit for each parameter estimated.

The ordered inflation data comprising our sample were thus divided into eleven groups of approximately equal size and the χ^2 goodness of fit test applied. Since three parameters were estimated before the test was applied – θ, ν_1 and ν_2 – it follows that

---

[10] The probability distribution of the Cramér-von Mises statistic, T_3, takes the form of an infinite series expansion stated in terms of the Bessel function of order $^1/_4$ (Anderson and Darling 1952, 202). One can use this probability distribution to show that the Cramér-von Mises statistic possesses a mean of $E(T_3) = {}^1/_6$ whilst its variance is $Var(T_3) = {}^1/_{45}$. Likewise, the third and fourth central moments of T_3 are given by $\mu_3 = {}^8/_{945}$ and $\mu_4 = {}^{31}/_{4725}$, respectively. It is a fairly simple exercise to use these moments in conjunction with a Gram-Charlier (or Edgeworth) expansion to obtain non-tabulated values of the probability distribution for T_3.

[11] Here we would emphasise that whilst Table 1 shows that the average (annualised) rate of inflation over the period from January, 1988 until December, 2012 was 2.82% there are epochs over this period where the average rate of inflation is much lower than this. For example, over the fifteen year period from January, 1991 until December, 2005 the average rate of inflation amounts to 2.1%. Hence, the long-term expected rate of inflation of 2.4% obtained under the χ^2minimum technique is not significantly different in a statistical sense (t score = 1.42) from the average rate of inflation over our entire sample period.

the test statistic will be approximately distributed as a χ^2 variate with 11 – 3 = 8 degrees of freedom. The final column of Table 2 summarises the χ^2 goodness of fit test statistics when the long run expected rate of inflation varies from $(\bar{\alpha} - \bar{\mu}) = 1\%$ to $(\bar{\alpha} - \bar{\mu}) = 3\%$ (per annum). Notice how the test statistic initially falls from a statistically significant 16.28 when the long run expected rate of inflation is 1%[12] to a statistically insignificant 7.27 when the long run expected rate of inflation is 2.4%[13] (per annum). Beyond this point the χ^2 test statistic gradually grows in magnitude reaching a statistically significant 18.31 when the long run expected rate of inflation is 3% (per annum). Thus, the χ^2 goodness of fit test statistics summarised in Table 2 confirm that the inflation data comprising our sample are most compatible with the probability distribution (8) when the parameter values are $\theta = 0.0337$, $\nu_1 = 0.0031$ and $\nu_2 = 1.1056$, and the long run expected rate of inflation is 2.4% (per annum).

Figure 1 provides a graphical representation of the estimated probability distribution for the U.K. inflation rate based on CPI data covering the period from January, 1988 until December, 2012. The first panel of this figure summarises the difference between the actual distribution function for the U.K. inflation rate and the empirically estimated distribution function with parameters $\theta = 0.0337$, $\nu_1 = 0.0031$ and $\nu_2 = 1.1056$, and a long run expected rate of inflation of 2.4% (per annum). The second panel is a graph of the probability density for the U.K. inflation rate with the above parameter values. This latter graph depicts the U.K. inflation rate in terms of a slightly skewed and highly leptokurtic probability density function that encompasses non-convergent higher

---

INSERT FIGURE ONE ABOUT HERE

---

[12] Thereby indicating we reject the null hypothesis that the empirical probability distribution for our sample of inflation data is compatible with the probability distribution defined by equation (8).

[13] Thereby indicating we fail to reject the null hypothesis that the empirical probability distribution for our sample of inflation data is compatible with the probability distribution defined by equation (8).

moments. Now here it is important to emphasise that the χ^2 minimum method employed in our empirical analysis returns an estimate of $\nu_2 = \frac{\beta}{\sigma^2}$ which is not significantly different from unity.[14] Substituting $\nu_2 = 1$ into equation (8) then shows that the probability density for the U.K. rate of inflation can be re-stated in the following canonical form:[15]

$$g(x) = \frac{2\theta(\theta^2 + \nu_1)}{\nu_1 . \sqrt{\nu_1}} . \csc h(\frac{\pi\theta}{\sqrt{\nu_1}})[1 + \frac{(\theta - x)^2}{\nu_1}]^{-2} . \exp[\frac{2\theta}{\sqrt{\nu_1}} . \tan^{-1}(\frac{\theta - x}{\sqrt{\nu_1}})] \qquad (14)$$

where $\csc h(.)$ is the hyperbolic cosecant function, $x = i - (\bar{\alpha} - \bar{\mu})$ is the abnormal or unexpected rate of inflation, $\bar{\alpha}$ is the expected instantaneous proportionate increase in

---

[14] When the long run expected rate of inflation is 2.4% it will be recalled that $\theta = 0.0337$, $\nu_1 = 0.0031$ and $\nu_2 = 1.1056$. Moreover, the χ^2 goodness of fit test statistic is 7.2695 with eight degrees of freedom. Imposing the constraint $\nu_1 = 1$ returns a χ^2 goodness of fit test statistic of 7.7631 with nine degrees of freedom. It then follows that $^{7.2695\times 9}/_{7.7631\times 8} = 1.0534$ will be distributed as an F variate with 8 and 9 degrees of freedom (Cochran 1934). This value of the F statistic falls well below those that apply at conventional levels of statistical significance. This in turn means that the estimate of $\nu_2 = 1.1056$ obtained under the χ^2 minimum method is not significantly different from $\nu_2 = 1$ in a statistical sense.

[15] Since $\nu_2 = 1$ it follows from equation (8) that the normalising constant for the probability density (14) will be:

$$c = \frac{\Gamma(\nu_2 + 1)}{\sqrt{\pi}.\Gamma(\nu_2 + \frac{1}{2})} \left| \frac{\Gamma(\nu_2 + 1 + \frac{j\nu_2\theta}{\sqrt{\nu_1}})}{\Gamma(\nu_2 + 1)} \right|^2 = \frac{\Gamma(2)}{\sqrt{\pi}\Gamma(\frac{3}{2})} \left| \frac{\Gamma(2 + \frac{j\theta}{\sqrt{\nu_1}})}{\Gamma(2)} \right|^2 = \frac{2}{\pi} \left| \Gamma(2 + \frac{j\theta}{\sqrt{\nu_1}}) \right|^2$$

It is well known, however, that for integral n the following identity holds (Abramowitz and Stegun 1964, 257):

$$\left|\Gamma(n + 1 + jx)\right| = \left|(n + jx)!\right| = \sqrt{\frac{\pi x}{\sinh(\pi x)}} . \prod_{s=1}^{n} \sqrt{s^2 + x^2}$$

Thus, if one lets $n = 1$ and $x = \frac{\theta}{\sqrt{\nu_1}}$ it follows:

$$c = \frac{2}{\pi} \left|\Gamma(2 + \frac{j\theta}{\sqrt{\nu_1}})\right|^2 = [(1 + \frac{j\theta}{\sqrt{\nu_1}})!]^2 = \frac{2}{\pi} . \frac{\frac{\theta\pi}{\sqrt{\nu_1}}}{\sinh(\frac{\theta\pi}{\sqrt{\nu_1}})} (1 + \frac{\theta^2}{\nu_1}) = \frac{2\theta(\theta^2 + \nu_1)}{\nu_1 . \sqrt{\nu_1}} . \csc h(\frac{\pi\theta}{\sqrt{\nu_1}})$$

the money supply and $\bar{\mu}$ is the expected instantaneous proportionate increase in real output. Now, one can let $\nu_2 = 1$ in equation (11) and thereby show that the variance of the abnormal rate of inflation will be:

$$Var(x) = \nu_1 + \theta^2 \tag{15}$$

where as previously, *Var(.)* is the variance operator. More important, however, is that substituting $\nu_2 = 1$ into equation (12) shows that the moments of order three and higher, are all non-convergent. This in turn confirms our earlier stated suspicion that the GMM technique may lead to inefficient and inconsistent parameter estimation procedures if applied to the inflation data on which are empirical analysis is based. [16]

Our analysis to date provides estimates of the skewness parameter, θ, the ratio of the two variance parameters, $\nu_1 = \frac{\omega^2}{\sigma^2}$, and the ratio of the speed of adjustment coefficient to the variance parameter associated with disequilibrium in the skewness adjusted rate of inflation, $\nu_2 = \frac{\beta}{\sigma^2}$. In other words, we do not as yet have estimates of the speed of adjustment coefficient, β, nor of the variance parameters, ω^2 and σ^2 themselves; only the various ratios of them. However, one can address this issue by recalling from equation (5) that:

$$\nu_1 = \frac{\omega^2}{\sigma^2} = \frac{\beta^2}{\sigma^2}(\delta^2 + \eta^2) = \beta\nu_2(\delta^2 + \eta^2)$$

where δ^2 is the instantaneous variance of the proportionate increase in the money supply (as in Assumption #2) and η^2 is the instantaneous variance of the proportionate increase in real output (as in Assumption #1). Empirical estimates of δ^2 =0.0143 and

[16] Inflation targeting was first applied in the U.K. in 1992 and was initially based on the Retail Price Index (excluding mortgage interest payments). Given this, we replicated our empirical analysis on the monthly inflation rates implied by the Retail Price Index for the period from 31 January, 1988 until 31 December, 2012. This returns parameter estimates of θ =0.0258, ν_1 = 0.0032 and ν_2 = 0.9340, with the latter parameter estimate being insignificantly different from unity at all conventional levels. The long run expected rate of inflation amounts to $(\bar{\alpha} - \bar{\mu}) = 3.1\%$ (per annum), the minimized Cramér-von Mises statistic is T_3 = 0.0622 whilst the Chi-square goodness of fit test statistic amounts to χ^2 = 6.08 with eight degrees of freedom. Hence, the results for the Retail Price Index are very similar to those summarised in the text for the CPI - with the exception that the long run expected rate of inflation amounts to 2.4% for the CPI as compared to 3.1% for the RPI.

η^2 =0.0005 were obtained using data for the M2: LPMVWYH monetary aggregate covering the period from 1988 until 2011 as summarised on the Bank of England website and data for U.K. real GDP covering the period from 1988 until 2011 as summarised on the "ukpublicspending" website, respectively.[17] These estimates and those for ν_1 = 0.0031 and ν_2 = 1.1056, obtained under the χ^2 minimum method taken in conjunction with the above formula show that the estimate of the speed of adjustment coefficient implied by our empirical analysis turns out to be:

$$\beta = \frac{\nu_1}{\nu_2(\delta^2 + \eta^2)} = \frac{0.0031}{1.1056(0.0143 + 0.0005)} = 0.1895 \qquad (16)$$

The relatively small value of the estimate obtained for the speed of adjustment parameter, β, suggests that whilst there might be (Friedman 1962, 54) "… a close connection between monetary actions and the price level … the connection is not so close, so invariable, or so direct that the objective of achieving a stable price level is an appropriate guide to the day-to-day activities of the [monetary] authorities." It is to this issue that we now turn and in particular, to the problem of determining the instantaneous rate of increase in the money supply that will lead to a low and relatively stable rate of inflation.

4. An optimal inflation targeting rule

We begin by emphasising that our analysis of this problem is based on previously chronicled assumptions (as in section 2) except that we now assume that the monetary

[17] We use M2 as the monetary base in our empirical analysis because it was Friedman's preferred measure of the money supply. Here Nelson (2007, 163) notes that "Friedman and Schwartz's *A Monetary History of the United States, 1867-1960* had used M2 as their monetary series and they had defended this choice at length in their 1970 *Monetary Statistics of the United States*." Nelson (2007, 173) then goes on to note that one of "the most important changes in Friedman's outlook over the 1980's, 1990's and 2000's [was] a firming of his preference for M2 as the definition of money ...." It is also interesting to note that between January, 2007 and December, 2008 the M2 monetary aggregate in the U.K. increased at an average annual rate of 9.78%. This contrasts with an average annual increase in the M2 aggregate between January, 2009 and December, 2012 of just 2.76%. The former figure is well above and the latter figure below the 3 to 5% range that had always been advocated by Friedman (1962, 54) for advanced industrialised countries like the United Kingdom and the United States. Nelson (2011) summarises the various policy responses made to the global financial crisis and their compatibility or otherwise with the monetary policies articulated by Friedman.

authorities must determine the rate of increase in the money supply so as to optimise an exogenously specified "monetary policy loss function". Given this, let:

$$\alpha(t) = \frac{1}{m(t)} . \frac{dm(t)}{dt} \tag{17}$$

be the instantaneous rate of increase in the money supply as determined by the monetary authorities. Substitution into equation (4) then shows the rate of inflation will evolve in accordance with the following differential equation:

$$\frac{di(t)}{dt} = \beta\{\alpha(t) - \bar{\mu} - \eta . \frac{dz_1(t)}{dt} - i(t)\} + \sigma\{\alpha(t) + \theta - \bar{\mu} - \eta . \frac{dz_1(t)}{dt} - i(t)\} . \frac{dz_3(t)}{dt} \tag{18}$$

Moreover, one can simplify this latter expression by again assuming (as in section 2) that the two white noise processes are uncorrelated, or (Arnold 1974, 91):

$$\frac{dz_1(t)}{dt} . \frac{dz_3(t)}{dt} = 0$$

It then follows that the above differential equation describing the evolution of the inflation rate may be re-stated as:

$$\frac{di(t)}{dt} = \beta\{\alpha(t) - \bar{\mu} - i(t)\} - \beta\eta . \frac{dz_1(t)}{dt} + \sigma\{\alpha(t) + \theta - \bar{\mu} - i(t)\} . \frac{dz_3(t)}{dt}$$

Now, here one can define the standardised variable:

$$q(t) = \frac{-\beta\eta . dz_1(t) + \sigma\{\alpha(t) + \theta - \bar{\mu} - i(t)\} . dz_3(t)}{\sqrt{\beta^2\eta^2 + \sigma^2\{\alpha(t) + \theta - \bar{\mu} - i(t)\}^2 dt}}$$

where $dq(t) \equiv q(t).\sqrt{dt}$ is a white noise process with unit variance parameter. It then follows that the rate of inflation will evolve in terms of the following differential equation:

$$\frac{di(t)}{dt} = \beta\{\alpha(t) - \bar{\mu} - i(t)\} + \sqrt{\omega^2 + \sigma^2\{\alpha(t) - \bar{\mu} + \theta - i(t)\}^2} . \frac{dq(t)}{dt} \tag{19}$$

where, as previously, $\bar{\mu}$ is the long run rate of growth in real output for the economy, β is a speed of adjustment coefficient, θ captures the skewness in the distribution of the inflation rate and $\omega^2 = \beta^2\eta^2$ is the variance parameter when the inflation rate is equal to its skewness adjusted long run mean – that is, when $i(t) = \{\bar{\alpha} - \bar{\mu} + \theta\}$ and η^2 is the variance parameter associated with instantaneous proportionate increases in real output. Finally, σ^2 is a parameter that captures the magnitude of the additional uncertainty that arises because of disequilibrium in the skewness adjusted inflation rate.

Now suppose the monetary authorities determine $\alpha(t)$ so as to minimise a monetary loss function defined in terms of the sum of the discounted squared difference between the targeted rate of inflation, λ, and the actual inflation rate, $i(s)$, or (Svensson 2010):

$$J(i,t) = \underset{\alpha}{Max}\, E[-\int_t^{\infty} e^{-\rho s}(\lambda - i(s))^2 ds] \tag{20}$$

where ρ is the (real) rate of social discount.[18] One can then use the differential equation (19) describing the evolution of the inflation rate in conjunction with the above monetary policy loss function to determine the Hamilton-Jacobi-Bellman fundamental equation of optimality; namely (Arnold 1974, 211-213):

$$\underset{\alpha}{Max}\, \phi(\alpha) = 0 =$$

$$-e^{-\rho t}(\lambda - i)^2 + \beta(\alpha - \bar{\mu} - i)\frac{\partial J}{\partial i} + \frac{1}{2}\{\omega^2 + \sigma^2(\alpha - \bar{\mu} + \theta - i)^2\}\frac{\partial^2 J}{\partial i^2} - \frac{\partial J}{\partial t} \tag{21}$$

Differentiating through the Hamilton-Jacobi-Bellman equation then shows that the first-order (or Euler) condition for a regular interior maximum is given by:

$$\frac{d\phi(\alpha)}{d\alpha} = \beta.\frac{\partial J}{\partial i} + \sigma^2(\alpha - \bar{\mu} + \theta - i)\frac{\partial^2 J}{\partial i^2} = 0 \tag{22}$$

or, equivalently:

[18] We have previously noted how the Bank of England (2014) is required to implement an inflation target of λ = 2% based on the CPI. The Bank of England (2014) also emphasizes, however, that this remit does not envisage "the lowest possible inflation rate. Inflation below the target of 2% is judged to be just as bad as inflation above the target. The inflation target is therefore symmetrical." This latter requirement is reflected in the objective function formalized through equation (20).

$$(\alpha - \overline{\mu} + \theta - i) = -\frac{\beta}{\sigma^2} \cdot \frac{\frac{\partial J}{\partial i}}{\frac{\partial^2 J}{\partial i^2}} \tag{23}$$

Moreover, a sufficient condition for a regular interior maximum is given by:

$$\frac{d^2\phi(\alpha)}{d\alpha^2} = \sigma^2 . \frac{\partial^2 J}{\partial i^2} < 0 \tag{24}$$

Now, one can substitute the Euler condition for a regular interior maximum into $\phi(\alpha)$ and thereby obtain the optimised Hamilton-Jacobi-Bellman equation; namely:

$$-e^{-\rho t}(\lambda - i)^2 - \beta\theta . \frac{\partial J}{\partial i} + \frac{1}{2}\omega^2 . \frac{\partial^2 J}{\partial i^2} - \frac{1}{2} . \frac{\beta^2}{\sigma^2} . \frac{(\frac{\partial J}{\partial i})^2}{\frac{\partial^2 J}{\partial i^2}} - \frac{\partial J}{\partial t} = 0 \tag{25}$$

Moreover, one can further simplify the above expression for the optimised Hamilton-Jacobi-Bellman equation by making the substitution:

$$J(i,t) = e^{-\rho t} G(i) \tag{26}$$

where *G(i)* is a twice continuously differentiable function of the instantaneous inflation rate, *i*. It then follows that the optimised Hamilton-Jacobi-Bellman equation will assume the canonical form:

$$-(\lambda - i)^2 - \beta\theta . G'(i) + \frac{1}{2}\omega^2 . G''(i) - \frac{1}{2} . \frac{\beta^2}{\sigma^2} . \frac{[G'(i)]^2}{G''(i)} - \rho G(i) = 0 \tag{27}$$

One can solve this equation by taking the trial solution:[19]

[19] The boundary conditions $\underset{i \to \infty}{Limit}\, G(i) \to -\infty$ and $\underset{i \to -\infty}{Limit}\, G(i) \to -\infty$ are sufficient to guarantee that the trial solution given here is the unique solution of the optimised Hamilton-Jacobi-Bellman equation. Note also that the trial solution satisfies the required condition for a regular interior maximum in the rate of growth of the money supply given earlier; namely:

$$\frac{\partial^2 J}{\partial i^2} = e^{-\rho t} G''(i) = -2e^{-\rho t} \mathrm{C} < 0$$

$$G(i) = -A + Bi - Ci^2 \tag{28}$$

where $A > 0$, $B > 0$ and $C > 0$ are constants. Substitution will then show that the trial solution satisfies the optimised Hamilton-Jacobi-Bellman equation for the following parameter values:

$$C = \frac{1}{(\frac{\beta^2}{\sigma^2} + \rho)}$$

and: (29)

$$B = \frac{2\lambda}{(\frac{\beta^2}{\sigma^2} + \rho)} + \frac{2\beta\theta}{(\frac{\beta^2}{\sigma^2} + \rho)^2}$$

One can then substitute the trial solution with these parameter values into the Euler condition determined earlier and thereby show that the optimal rate of growth in the money supply, α, will be given by:

$$(\alpha - \bar{\mu} + \theta - i) = -\frac{\beta}{\sigma^2} \cdot \frac{\frac{\partial J}{\partial i}}{\frac{\partial^2 J}{\partial i^2}} = -\frac{\beta}{\sigma^2} \cdot \frac{G'(i)}{G''(i)} = \frac{\beta}{\sigma^2} \cdot \frac{B - 2Ci}{2C} \tag{30}$$

Substituting the expressions for B and C given earlier will then show that the optimal rate of growth in the money supply (that is, the optimal control) can be expressed as:

$$\alpha = (\bar{\mu} + \lambda) - \frac{\theta\rho}{(\frac{\beta^2}{\sigma^2} + \rho)} + (1 - \frac{\beta}{\sigma^2})(i - \lambda) \tag{31}$$

Note how the optimal rate of increase in the money supply hinges on a weighting factor $(1 - \frac{\beta}{\sigma^2})$ that is applied to the difference between the current rate of inflation, i, and the targeted rate of inflation, λ. Hence, when the weighting factor is zero or equivalently,

$\nu_2 = \frac{\beta}{\sigma^2} = 1$, the optimal rate of growth in the money supply will be given by the Friedman (1960, 90) Rule, namely:

$$\alpha = (\bar{\mu} + \lambda) - \frac{\theta\rho}{(\beta + \rho)} \tag{32}$$

Substituting this latter result into the expression for the instantaneous increment in the rate of inflation as given by equation (19) shows that under the Friedman Rule the rate of inflation will evolve in terms of the following stochastic differential equation:

$$\frac{di(t)}{dt} = \beta\{\lambda - \frac{\theta\rho}{\beta + \rho} - i(t)\} + \sqrt{\omega^2 + \beta\{\lambda + \frac{\theta\beta}{\beta + \rho} - i(t)\}^2}.\frac{dq(t)}{dt} \tag{33}$$

Note how the above result implies that the expected optimised rate of inflation will be equal to the targeted rate of inflation, λ, less a component that hinges on the skewness parameter, θ, or:

$$E(i) = \lambda - \frac{\theta\rho}{\beta + \rho} \tag{34}$$

Thus, if the probability distribution for the inflation rate is symmetric – that is, $\theta = 0$ – then under the Friedman Rule the expected instantaneous rate of inflation will be equal to the target rate of inflation. If there is positive skewness in the probability distribution for the inflation rate – that is, $\theta < 0$ – then the expected instantaneous rate of inflation will exceed the target rate. Finally, if there is negative skewness in the probability distribution for the inflation rate – that is, $\theta > 0$ – then the expected instantaneous rate of inflation will be lower than the target rate of inflation. Moreover, equation (33) shows that the component of the instantaneous variance attributable to the skewness adjusted disequilibria in the inflation rate grows in magnitude with the speed of adjustment coefficient.[20] Hence, whilst larger values of the speed of adjustment

[20] From equation (33), we see that under the Friedman Rule the variance of the instantaneous change in the rate of inflation is given by:

$$Var[di(t)] = [\omega^2 + \beta\{\lambda + \frac{\theta\beta}{\beta + \rho} - i(t)\}^2]dt$$

coefficient will mean that the inflation rate will converge more quickly towards its long run mean, it will do so with greater noise thereby confirming the primary objection made to monetary policy based on the Friedman Rule; namely, that it accentuates the variability in the rate of inflation when compared to more finely honed discretionary monetary policies (Giannoni and Woodford 2005; Svensson 2010).

One can also determine the steady state probability distribution for the optimised rate of inflation under the Friedman Rule by solving the Fokker-Planck (that is, the Chapman-Kolmogorov) equation in conjunction with the above differential equation (Merton 1975, 389-390; Karlin and Taylor 1981, 219-221). To facilitate this we first define:

$$x = i + \frac{\theta\rho}{\beta + \rho} - \lambda \tag{35}$$

to be the abnormal or unexpected instantaneous rate of inflation. Solving the Fokker-Planck equation in terms of x will then show that the steady state probability distribution, g(x), for the unexpected instantaneous rate of inflation will be: [21]

$$g(x) = \frac{2\theta(\theta^2 + \nu_1)}{\nu_1 \cdot \sqrt{\nu_1}} . \csc h(\frac{\pi\theta}{\sqrt{\nu_1}})[1 + \frac{(\theta - x)^2}{\nu_1}]^{-2} . \exp[\frac{2\theta}{\sqrt{\nu_1}} . \tan^{-1}(\frac{\theta - x}{\sqrt{\nu_1}})] \tag{36}$$

where $-\infty < x < \infty$, and as previously $\csc h(.)$ is the hyperbolic cosecant function whilst $\nu_1 = \frac{\omega^2}{\sigma^2} = \frac{\omega^2}{\beta}$.

Here it is important to note that monetary targeting rules of the kind envisaged by the Friedman Rule have never been implemented. This in turn means that the data required to assess the descriptive validity of the above probability distribution are not available.

Note how the component of the variance attributable to the skewness adjusted disequilibria in the inflation rate – $\beta\{\lambda + \frac{\theta\beta}{\beta + \rho} - i(t)\}^2$ – grows in magnitude with the speed of adjustment coefficient, β.

[21] The reader will be able to confirm that:

$$\theta - x = \theta - (i + \frac{\theta\rho}{\beta + \rho} - \lambda) = \frac{\theta\beta}{\beta + \rho} + \lambda - i$$

Here we should note, however, that the empirical analysis summarised in section 3 shows that the probability distribution for the U.K. inflation rate, as given by equation (14), is identical in functional form to the probability distribution for the inflation rate under the Friedman rule, as given by equation (36). This has the important implication that a necessary condition for the application of the Friedman Rule – namely, that the parameter $\nu_2 = \frac{\beta}{\sigma^2} = 1$ – is compatible with the empirically estimated probability distribution for the U.K. inflation rate. However, the Fokker-Planck equation from which the probability distribution (14) for the U.K. inflation rate is determined is based on different expressions for the mean and variance of instantaneous changes in the inflation rate to those applied to the Fokker-Planck equation in determining the probability distribution (36) for the inflation rate under the Friedman Rule. This in turn will mean that the two probability distributions will be based on generally different estimates of the parameters θ and $\nu_1 = \frac{\omega^2}{\sigma^2}$. Thus, whilst that the probability distribution for the U.K. inflation rate may assume the same functional form as the probability distribution for the inflation rate under the Friedman rule, this will not generally imply that the Friedman Rule represents an optimal control for the rate of increase in the money supply for the U.K. economy.

5. Summary conclusions

Our purpose here is to formulate a monetarist model of inflation targeting and then to use it to assess whether the Friedman Rule might represent an optimal control within the model. Our methodological framework follows what Miller (1999, 96) describes as the typical "business school" approach of "maximising some objective function … taking the prices of securities in the market as given". We implement this methodology by specifying a set of assumptions under which the rate of inflation will evolve in terms of a first order mean reversion process in continuous time based on a white noise error structure. We then use the Fokker-Planck equation to determine the steady state (that is, unconditional) probability distribution for the rate of inflation.

The χ^2 minimum method is used in conjunction with inflation rates implied by the U.K. Consumer Price Index covering the period from 1988 until 2012 to estimate the parameters of the steady state probability distribution. The resulting parameter

estimates are compatible with the hypothesis that the U.K. inflation rate evolves in terms of a slightly skewed and highly leptokurtic probability distribution that encompasses higher moments which are non-convergent. This in turn means that the widely employed Generalised Method of Moments will be an inefficient procedure for parameter estimation when applied to the probability distribution on which our empirical analysis is based. Moreover, our empirical analysis shows that the lags in U.K. monetary policy are considerable.

We then formulate an inflation targeting model under which monetary policy is determined by minimising a loss function defined in terms of the squared difference between the targeted rate of inflation and the actual inflation rate. One can then use the mean reversion process describing the evolution of the inflation rate in conjunction with the monetary policy loss function to determine the Hamilton-Jacobi-Bellman fundamental equation of optimality for the inflation targeting problem. Optimising and then solving the Hamilton-Jacobi-Bellman equation shows that the optimal control for the rate of increase in the money supply will be a linear function of the difference between the current rate of inflation and the targeted inflation rate. The conditions under which the optimal control will lead to the Friedman Rule are then determined. These conditions are used in conjunction with the Fokker-Planck equation and the mean reversion process describing the evolution of the inflation rate to determine the steady state probability distribution for the inflation rate under the Friedman Rule. This shows that whilst the empirically determined probability distribution for the U.K. inflation rate meets some of the conditions required for the application of the Friedman Rule, it does not meet all of them.

Our final point relates to the fact that the differential equation (5) for instantaneous changes in the inflation rate can be embedded in a Walrasian general equilibrium economy by stating the relative prices in terms of the monetary unit defined in Assumption 2. One can then follow Rhys and Tippett (2001) in letting the inflation rate evolve in terms of a binomial filtration with state probabilities that upon taking limits, lead to the differential equation (5) given here for the inflation rate. This means that our analysis has the potential to transform Walrasian general equilibrium based asset pricing models in which time zero consumption acts as the numéraire commodity and

all returns are therefore stated in real terms, into asset pricing models that are based on nominal returns.

References

Abramowitz, M. and I. Stegun. 1964. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. Washington, DC: NBS Applied Mathematics Series 55, National Bureau of Standards.

Anderson, T. 2010. Anderson-Darling tests of goodness-of-fit. Working Paper, Department of Statistics and Economics, Stanford University.

Anderson, T. and D. Darling. 1952. Asymptotic theory of certain "goodness of fit" criteria based on stochastic processes. *Annals of Mathematical Statistics* 23: 193-212.

Arnold, L. 1974. *Stochastic differential equations*. New York: Wiley.

Ashton, D. and M. Tippett. 2006. Mean reversion and the distribution of United Kingdom stock index returns. *Journal of Business Finance and Accounting* 33, Nos. 9&10: 1586-1609.

Avni,Y. 1976. Energy spectra of X-Ray clusters of galaxies. *Astrophysical Journal* 210: 642-646.

Bank of England. 2014. *Monetary Policy Framework*. http://www.bankofengland.co.uk/monetarypolicy/Pages/framework/framework.aspx.

Barndorff-Nielsen, O. and N. Shephard. 2001. Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics. *Journal of the Royal Statistical Society, Series B (Statistical Methodology)* 63, No. 2: 167-241.

Blattberg, R. and N. Gonedes. 1974. A Comparison of the stable and student distributions as statistical models for stock prices. *Journal of Business* 47, No. 2: 244-280.

Carnahan, B., H. Luther and J. Wilkes. 1969. *Applied numerical analysis*. New York: Wiley.

Chang, T., Ranjbar, O. and D. Tang. 2013. Revisiting the mean reversion of inflation rates for 22 OECD countries. *Economic Modelling* 30: 245-252.

Cochran, W. 1934. The distribution of quadratic forms in a normal system, with applications to the analysis of covariance. *Mathematical Proceedings of the Cambridge Philosophical Society* 30, No. 2: 178-191.

Cox, D. and H. Miller. 1965. *Theory of stochastic processes*. London: Chapman and Hall.

Cramér, H. 1946. *Mathematical methods of statistics*. Princeton: Princeton University Press.

Fama, E. 1981. Stock returns, real activity, inflation, and money. *American Economic Review* 71, No. 4: 545-565.

Federal Reserve Board of New York. (2012), http://www.newyorkfed.org/aboutthefed/fedpoint/fed49.html.

Friedman, M. 1955. Leon Walras and his economic system: A review article. *American Economic Review* 45, No. 5: 900-909.

Friedman, M. 1956. The Quantity Theory of Money - A Restatement. In Friedman, M. Ed. *Studies in the quantity theory of money*. Chicago: University of Chicago Press.

Friedman, M. 1960. *A program for monetary stability*. New York: Fordham University Press.

Friedman, M. 1962. *Capitalism and freedom*. Chicago: University of Chicago Press.

Friedman, M. 1970. A theoretical framework for monetary analysis. *Journal of Political Economy* 78, No. 2: 193-238.

Friedman, M. 1971. Government revenue from inflation. *Journal of Political Economy* 79, No. 4: 846-856.

Friedman, M. 1972. *Money and economic development*. New York: Praeger Publishers

Friedman, M. (1977), 'Inflation and Unemployment', http://www.nobelprize.org/nobel_prizes/economics/laureates/1976/friedman-lecture.pdf.

Friedman, M. 1988. Money and the stock market. *Journal of Political Economy* 96, No. 2: 221-245.

Friedman, M. and A. Schwartz.1963. *A monetary history of the United States, 1867-1960*. Princeton, N.J: Princeton University Press.

Friedman, M. and A. Schwartz.1970. *Monetary Statistics of the United States*. New York: National Bureau of Economic Research.

Freund, J. 1971. *Mathematical statistics*. Englewood Cliffs, N.J.: Prentice-Hall.

Giannoni, M. and M. Woodford, M. 2005. Optimal Inflation Targeting Rules. In Bernanke, B. and M. Woodford. Eds. *The inflation-targeting debate*. Chicago: University of Chicago Press.

Gibson, R., and E. Schwartz. 1990. Stochastic convenience yield and the pricing of oil contingent claims. *Journal of Finance* 45, No. 3: 959-976.

Hoel, P., S. Port and C. Stone. 1987. *Introduction to stochastic processes*. Prospect Heights, Illinois: Waveland Press.

Hong, K. and S, Satchell. 2012. Defining single asset price momentum in terms of a stochastic process. *Theoretical Economics Letters*, 2, No. 3: 274-277.

Jeffreys, H. 1961. *Theory of probability*. Oxford: Oxford University Press.

Karlin, S. and H. Taylor. (1981), *A Second Course in Stochastic Processes.* London: Academic Press.

Kendall, M. and A. Stuart. 1977. *Advanced theory of statistics: volume 1*. London: Charles Griffin & Company.

Kendall, M. and A. Stuart. 1979. *Advanced theory of statistics*: *volume II*. London: Charles Griffin & Company.

Lee, C. and C.-P. Chang. 2007. Mean reversion of inflation rates in 19 OECD countries: evidence from Panel LM unit root tests with structural breaks. *Economics Bulletin* 3, No. 13: 1-15.

Lee, H.-Y. and J-L. Wu. 2001. Mean reversion of inflation rates: Evidence from 13 OECD countries. *Journal of Macroeconomics* 23, No. 3: 477-487.

McCallum, B. and E. Nelson. 2010. *Money and inflation: some critical issues: Discussion Paper 2010-57*. Washington, D.C.: Divisions of Research & Statistics and Monetary Affairs, Federal Reserve Board.

Merton, R. 1975. An asymptotic theory of growth under uncertainty. *Review of Economic Studies* 42, No. 3: 375-393.

Miles, D. and A. Scott. 2002. *Macroeconomics: understanding the wealth of nations*. New York: Wiley.

Miller, M. 1999. The history of finance. *Journal of Portfolio Management* 25, No. 4: 95-101.

Miller, M. and F. Modigliani. 1961. Dividend policy, growth and the valuation of shares. *Journal of Business* 34, No. 4: 411-33.

Mishkin, F. 2013. *Economics of money, banking and financial markets*. Edinburgh Gate, Harlow: Pearson Education Limited.

Mood, A., F. Graybill and D. Boes. 1974. *Introduction to the theory of statistics*. New York: McGraw-Hill.

Nelson, E. 2007. Milton Friedman and U.S. monetary history: 1961-2006. *Federal Reserve Bank of St. Louis Review* 89, No. 3: 153-182.

Nelson, E. 2011. Friedman's monetary economics in practice. Finance and Economics Discussion paper 2011-26, Federal Reserve Board, Washington, D.C.

O'Donoghue, J., L. Goulding and G. Allen. 2004. Consumer price inflation since 1750. *Economic Trends* 604: 38-46.

Praetz, P. 1972. The distribution of share price changes. *Journal of Business*, 45, No. 1: 49-55.

Rhys, H. and M. Tippett. 2001. A binomial basis for the Cox, Ingersoll and Ross model of the term structure of interest rates. *Journal of Business Finance and Accounting* 28, Nos. 3&4: 379-405.

Sahr, R. 2011. Political Science Department, Oregon State University, http:// www.orst. edu/Dept/pol_ sci/fac/ sahr/sah rhome.html.

Stiglitz, J. 1983. On the relevance or irrelevance of public financial policy: indexation, price rigidities and optimal monetary policy. in R. Dornbusch and M. Simonsen. Eds. *Inflation, debt and indexation*. Cambridge, Massachusetts: MIT Press.

Stiglitz, J. 1988. On the relevance or irrelevance of public financial policy. In Proceedings of the 1986 International Economics Association Meeting. *The Economics of public debt*. London: McMillan.

Svensson, L. 2010. *Inflation targeting: working paper 16654*. Cambridge, Massachusetts: National Bureau of Economic Research.

Uhlenbeck, G. and L. Ornstein. 1930. On the theory of the brownian motion. *Physical Review* 36, No. 5: 823-841.

US Bureau of Labor Statistics. 2012. *Table containing history of CPI-U U.S. all items indexes and annual percent changes from 1913 to present*. Washington, DC: Division of Consumer Prices and Price Indexes.

Yan, J. 2005. Asymmetry, fat-tail, and autoregressive conditional density in Financial return data with systems of frequency curves. University of Iowa:Department of Statistics and Actuarial Science.

Table 1. Distributional properties of the U.K. monthly rate of inflation based on the U.K. Consumer Price Index covering the period from 31 January, 1988 until 31 December, 2012.

Average	2.82%
Standard Deviation	5.14%
Standardised Skewness	1.02
Standardised Kurtosis	8.97
Median	3.27%
Minimum	-11.67%
Maximum	39.82%

Notes: The above table is based on $N = 299$ monthly continuously compounded rates of inflation based on the U.K. Consumer Price Index covering the period from 31 January, 1988 until 31 December, 2012. Each inflation rate was obtained by taking the natural logarithm of the ratio of the consumer price index at the end of the month to the consumer price index at the beginning of the month. The monthly continuously compounded rates of inflation were then multiplied by 1,200 in order to state them on an annual basis.

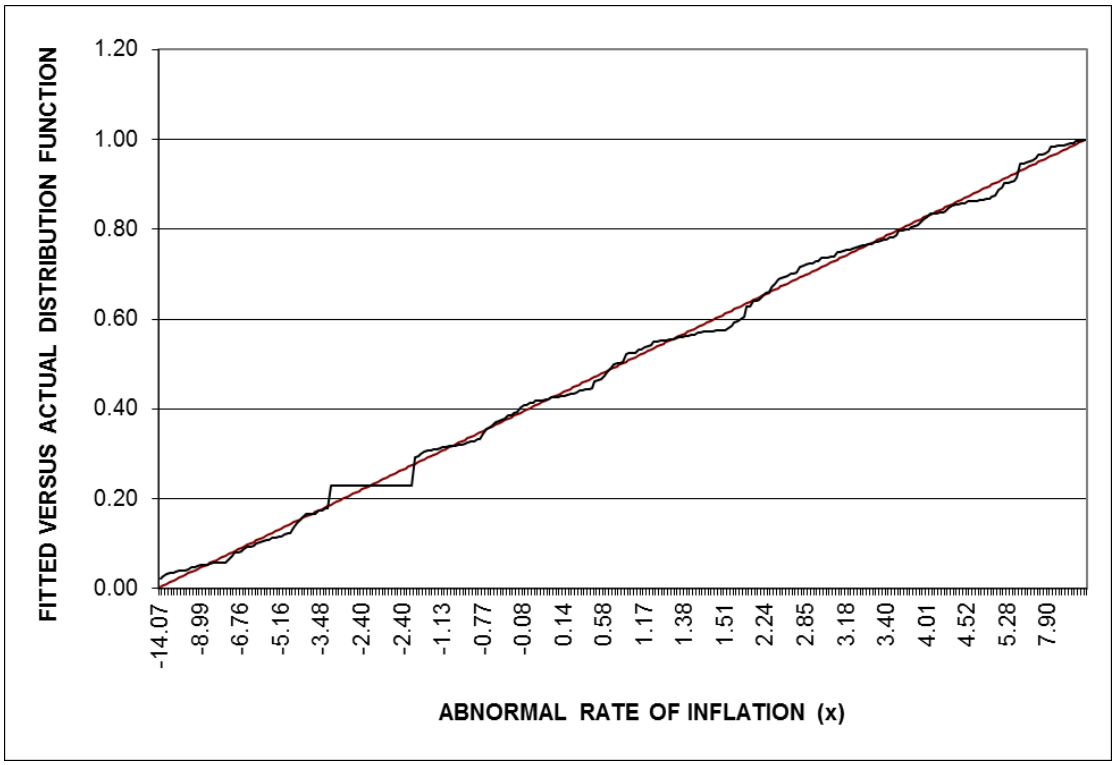
Table 2. Estimated parameters using χ^2 minimum method when long-term expected rate of inflation assumes values of $(\overline{\alpha} - \overline{\mu}) = 1\%$ to $(\overline{\alpha} - \overline{\mu}) = 3\%$ (per annum) based on the U.K. Consumer Price Index covering the period from 31 January, 1988 until 31 December, 2012.

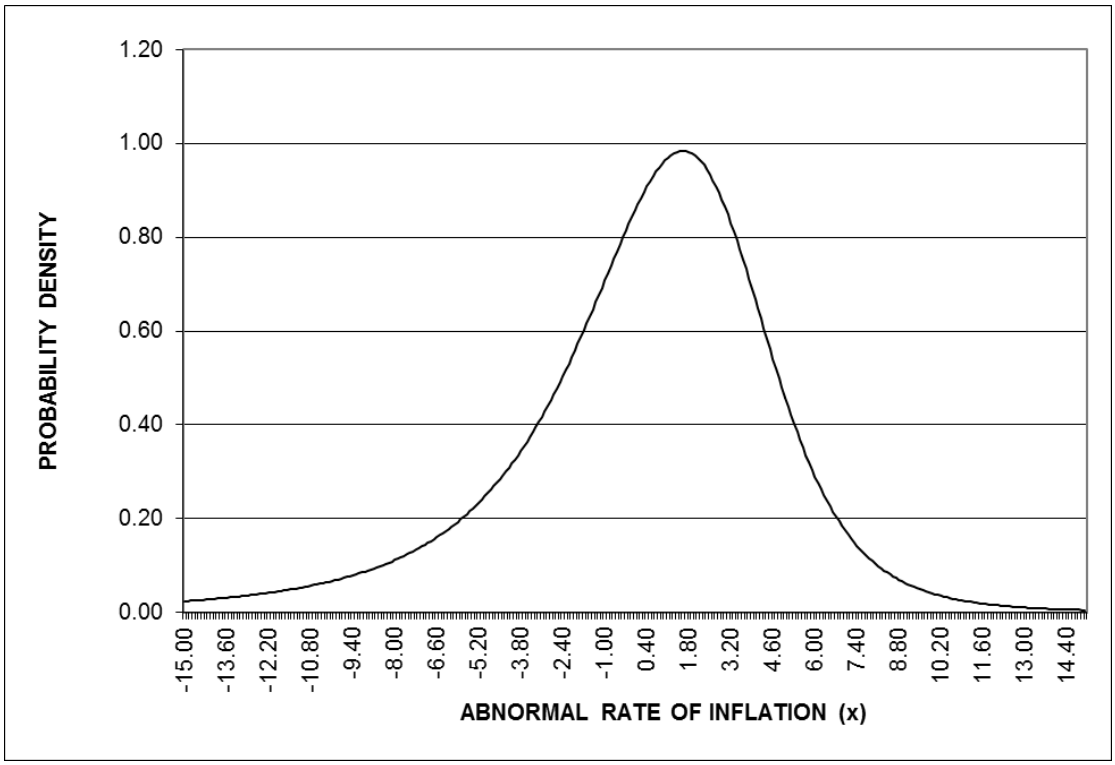
$(\overline{\alpha} - \overline{\mu})$	θ	ν_1	ν_2	Cramér-von Mises Statistic T_3	χ^2 Goodness of fit Statistic
1.0	0.0372	0.0011	0.2424	0.1251	16.28*
1.2	0.0361	0.0011	0.2800	0.1170	14.98
1.4	0.0350	0.0012	0.3290	0.1081	13.56
1.6	0.0342	0.0014	0.3946	0.0983	12.05
1.8	0.0337	0.0016	0.4858	0.0879	10.49
2.0	0.0335	0.0019	0.6177	0.0777	8.99
2.2	0.0337	0.0024	0.8152	0.0692	7.75
2.4	0.0337	0.0031	1.1056	0.0657	7.27
2.6	0.0307	0.0040	1.4304	0.0729	8.44
2.8	0.0199	0.0043	1.4448	0.0974	12.33
3.0	0.0062	0.0033	1.0042	0.1388	18.31*

* signifies that the relevant statistic falls in the upper 5% of the relevant probability distribution.

** signifies that the relevant statistic falls in the upper 1% of the relevant probability distribution.

Figure 1. (a) Difference between actual distribution function and empirically estimated distribution function for U.K. inflation rate. (b) Estimated probability density of the U.K. rate of inflation.





The above graphs are based on the probability density:

$$g(x) = c[1 + \frac{(\theta - x)^2}{\nu_1}]^{-(1+\nu_2)} . \exp[\frac{2\nu_2\theta}{\sqrt{\nu_1}} . \tan^{-1}(\frac{\theta - x}{\sqrt{\nu_1}})]$$

where:

$$c = \frac{\Gamma(\nu_2 + 1)}{\sqrt{\pi}.\Gamma(\nu_2 + \frac{1}{2})} \left| \frac{\Gamma(\nu_2 + 1 + \frac{j\nu_2\theta}{\sqrt{\nu_1}})}{\Gamma(\nu_2 + 1)} \right|^2$$

is the normalising constant, x is the abnormal rate of inflation, $j = \sqrt{-1}$ is the pure imaginary number, $|.|$ is the modulus of a complex number, $\Gamma(.)$ is the gamma function, $\theta = 0.0337$, $\nu_1 = 0.0031$ and $\nu_2 = 1.1056$.