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X -POSETS OF CERTAIN COXETER GROUPS

Sarah B. Hart* and Peter J. Rowley

Abstract. Let X be a subgroup of a Coxeter group W . In [5], the authors developed the notion of X -posets, which are defined on certain equivalence classes of the (right) cosets of X in W . These posets can be thought of as a generalization of the well-known Bruhat order of W . This article provides a catalogue of all the X -posets for various small Coxeter groups.

1. INTRODUCTION

Suppose that W is a Coxeter group and let X be a subgroup of W . Then, in [5], the authors introduced and began the investigation of X -posets, which are defined on certain equivalence classes of the (right) cosets of X in W . The special case when X is the trivial subgroup yields the well known, and important, Bruhat order [6]. While taking X to be a standard parabolic subgroup of W delivers us the (generalized) Bruhat order defined on the cosets in W of that standard parabolic subgroup [3]. The study of X -posets is in its infancy and will benefit greatly from a well organized collection of examples. The aim here is to provide a catalogue of all the X -posets for various small Coxeter groups. Specifically we look at the Coxeter groups of type A_2 , A_3 , A_4 , A_5 , B_2 , B_3 and D_4 .

We now describe X -posets in more detail as well as establishing our notation and briefly recapping some basic facts about Coxeter groups. Assume that W is a finite Coxeter group with X a subgroup of W . Then, by definition, W has a presentation

$$W = \langle R \mid (rs)^{m_{rs}} = 1, r, s \in R \rangle$$

where $m_{rs} \in \mathbb{N}$, $m_{rr} = 1$ and for $r, s \in R$, $r \neq s$, $m_{rs} = m_{sr} \geq 2$. Let V be a real vector space with basis $\Pi = \{\alpha_r \mid r \in R\}$, upon which we define the symmetric bilinear form $\langle \cdot, \cdot \rangle$ by

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$$\langle \alpha_r, \alpha_s \rangle = -\cos\left(\frac{\pi}{m_{rs}}\right) \quad \text{where } r, s \in R.$$

For $r, s \in R$ we also define

$$r \cdot \alpha_s = \alpha_s - 2\langle \alpha_r, \alpha_s \rangle \alpha_r.$$

This extends to give an action of W which is faithful and respects $\langle \cdot, \cdot \rangle$ (see [6]). The root system Φ (of W) is the following subset of V

$$\Phi = \{w \cdot \alpha_r \mid r \in R, w \in W\},$$

with $\Phi^+ = \{\sum_{r \in R} \lambda_r \alpha_r \in \Phi \mid \lambda_r \geq 0 \text{ for all } r \in R\}$ and $\Phi^- = -\Phi^+$ being, respectively, the positive and negative roots of Φ . As is well-known $\Phi = \Phi^+ \cup \Phi^-$. The elements in R are called the fundamental reflections of W and $\text{Ref}(W)$, the set of reflection of W , consists of all W -conjugates of the fundamental reflections.

For Y a subset of W define

$$N(Y) = \{\alpha \in \Phi^+ \mid w \cdot \alpha \in \Phi^- \text{ for some } w \in Y\}$$

and $l(Y) = |N(Y)|$. We call $l(Y)$ the Coxeter length of Y . This is a generalization of the usual length function in Coxeter groups, first defined in [7].

For right cosets Xg and Xh of X we write $Xg \sim Xh$ whenever $Xgt = Xh$ for some $t \in \text{Ref}(W)$ and Xg and Xh have the same Coxeter length. Let \approx be the equivalence relation generated by \sim on the set of right cosets of X in W and let \mathfrak{X} be the set of \approx equivalence classes. (We remark that our choice of right, as opposed to left, cosets is due to the fact that W acts on the right of Φ – see [5] for more on this.) Now let $\mathbf{x}, \mathbf{x}' \in \mathfrak{X}$. We write $\mathbf{x} \rightsquigarrow \mathbf{x}'$ if there is a right coset Xg in \mathbf{x} and a right coset Xh in \mathbf{x}' such that $Xgt = Xh$ for some $t \in \text{Ref}(W)$ and $l(Xg) < l(Xh)$. The partial order \preceq on \mathfrak{X} is defined by $\mathbf{x} \preceq \mathbf{x}'$ if and only if there exist $\mathbf{x}_1, \dots, \mathbf{x}_m \in \mathfrak{X}$ such that $\mathbf{x} \rightsquigarrow \mathbf{x}_1 \rightsquigarrow \dots \rightsquigarrow \mathbf{x}_m \rightsquigarrow \mathbf{x}'$ and we call \mathfrak{X} the X -poset (of W).

A standard parabolic subgroup of W is a subgroup generated by S where $S \subseteq R$ and is usually denoted by W_S . The (generalized) Bruhat order defined on the cosets of W_S will be denoted by $\mathcal{B}(W_S)$. Any conjugate of a standard parabolic subgroup is called a parabolic subgroup of W . For $X \leq W$ let $\langle X \rangle_{\text{sp}}$ denote the standard parabolic closure of X which is the intersection of all standard parabolic subgroups of W containing X . The following three results from [5] have a bearing on our calculations here.

Theorem 1.1. ([5], Corollary 1.4). *Suppose that $X \leq W$ where W is finite. If $N(X) = N(\langle X \rangle_{\text{sp}})$, then the X -poset \mathfrak{X} is poset isomorphic to $\mathcal{B}(\langle X \rangle_{\text{sp}})$.*

Theorem 1.2. ([5], Proposition 4.1). *Suppose that $X \leq W$ where W is finite. If X is not contained in any proper parabolic subgroup of W , then $|\mathfrak{X}| = 1$. In particular, \mathfrak{X} is poset isomorphic to $\mathcal{B}(W_R)$.*

Theorem 1.3. ([5], Theorem 3.8). *Let $X \leq Y \leq W$ where X and Y are finite and Y is a reflection subgroup of W . Let \mathfrak{X} and \mathfrak{Y} denote, respectively, the X -poset and the Y -poset. If $N(X) = N(Y)$ then \mathfrak{X} is poset isomorphic to \mathfrak{Y} . If W is finite and \mathfrak{X} is poset isomorphic to \mathfrak{Y} then $N(X) = N(Y)$.*

Accompanying each Coxeter group of rank n is its Coxeter graph with the nodes labelled $\{1, \dots, n\}$ (in one-to-one correspondence with elements R). Let $R = \{r_1, \dots, r_n\}$. To compress our tabular information in Section 2 when giving elements of W we suppress the symbol “ r ”; so, for example, for W of type A_4 instead of $r_1r_2r_4r_3$ we shall write [1243]. We use $\mathbb{Z}_m, 2^m, \text{Dih}(m), \text{Alt}(m), \text{Sym}(m)$ to denote, respectively, the cyclic group of order m , the elementary abelian group of order 2^m , the dihedral group of order m , the alternating group of degree m and the symmetric group of degree m .

Our next section describes the structure of various X -posets – this information was obtained with the assistance of Magma [2]. Our third section draws some lessons from these examples.

2. X-POSETS FOR SMALL COXETER GROUPS

In compiling the data below on X -posets where $X \leq W$, we take the view that the (generalized) Bruhat order on cosets of a standard parabolic subgroup is “known”. Thus, because of Theorem 1.2, we only need concern ourselves with subgroups contained in parabolic subgroups of W . Also Theorem 1.1 tells us that we can ignore any X for which $N(X) = N(\langle X \rangle_{\text{sp}})$.

For $m \in \mathbb{N}$, \mathcal{C}_m will denote the totally ordered set with m elements – \mathcal{C}_m is sometimes called the m -chain poset. In the posets presented in the figures below we have only joined \mathbf{x}_1 and \mathbf{x}_2 (where $\mathbf{x}_1, \mathbf{x}_2 \in \mathfrak{X}$) if $\mathbf{x}_1 \prec \mathbf{x}_2$ and $l(\mathbf{x}_2) = l(\mathbf{x}_1) + 1$. For $\mathbf{x} \in \mathfrak{X}$ the length of \mathbf{x} , $l(\mathbf{x})$ is defined to be $l(Xg)$ where Xg is any coset in \mathbf{x} – clearly $l(\mathbf{x})$ is well defined. Also, in these figures we have indicated on the right-hand side the lengths of the poset elements. Here we cover the Coxeter groups of type $A_2, A_3, A_4, A_5, B_2, B_3$ and D_4 . An example of an X -poset for type F_4 is given in [4].

(2.1) W of type A_2 , $\begin{matrix} \bullet & \text{---} & \bullet \\ 1 & & 2 \end{matrix}$. For $X \leq W$, $\mathfrak{X} \cong \mathcal{B}(\langle X \rangle_{\text{sp}})$.

(2.2) W of type A_3 , $\begin{matrix} \bullet & \text{---} & \bullet & \text{---} & \bullet \\ 1 & & 2 & & 3 \end{matrix}$. For $X \leq W$, either $\mathfrak{X} \cong \mathcal{B}(\langle X \rangle_{\text{sp}})$ or one of the following holds:-

X	\mathfrak{X}
$\langle [32123] \rangle \cong \mathbb{Z}_2$	\mathcal{C}_2
$\langle [2132] \rangle \cong \mathbb{Z}_2$	\mathcal{C}_3

(2.3) W of type A_4 , $\bullet_1 - \bullet_2 - \bullet_3 - \bullet_4$. For $X \leq W$, either $\mathfrak{X} \cong \mathcal{B}(\langle X \rangle_{\text{sp}})$ or one of the following holds:-

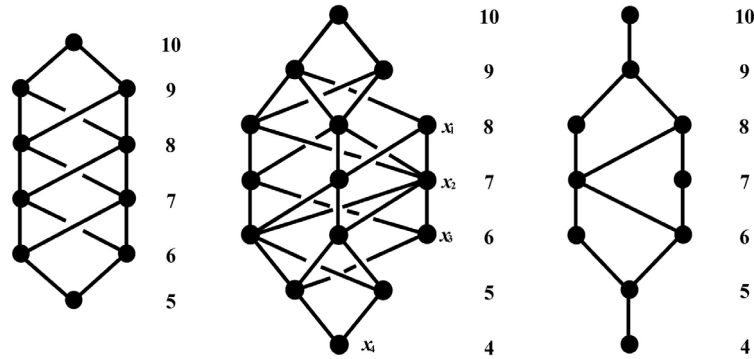


Fig. 1. $A_4(i)$ (left), $A_4(ii)$ (centre) and $A_4(iii)$ (right).

Next we look at the case when W is of type A_5 – so we have the Coxeter diagram: $\bullet_1 - \bullet_2 - \bullet_3 - \bullet_4 - \bullet_5$. Unlike for W of type A_4 (in (2.3)), here we shall take advantage of the graph automorphism of order 2 (which interchanges vertices 1 and 5, vertices 2 and 4, and fixes vertex 3) and only consider subgroups of W conjugate to subgroups of W_{1234} ($= W_{\{1,2,3,4\}}$), W_{1235} ($= W_{\{1,2,3,5\}}$) and W_{1245} ($= W_{\{1,2,4,5\}}$).

In (2.4) we first list the subgroups X which are contained in W_{1234} and whose poset is not of the form $\mathcal{B}(\langle X \rangle_{\text{sp}})$ (note that these are given in the same order as in (2.3)). Then we consider similar subgroups of W_{1235} (which is of type $A_3 \times A_1$). Let X_{123} , respectively X_5 , denote the projection of X in W_{123} , respectively W_5 . If the X_{123} -poset in W_{123} is $\mathcal{B}(\langle X_{123} \rangle_{\text{sp}})$, and the X_5 -poset in W_5 is $\mathcal{B}(\langle X_5 \rangle_{\text{sp}})$, then $\mathfrak{X} \cong \mathcal{B}(\langle X \rangle_{\text{sp}})$. Hence, consulting (2.2), we see that in this subcase we only need examine $X = \langle [321235] \rangle$ and $X = \langle [21325] \rangle$. Similar considerations apply to the subgroups of W_{1245} with (2.1) showing that we need not consider any subgroups of W_{1245} .

A number of X -posets we encounter when W is of type A_5 have quite a large number of elements – too large to draw a comprehensible lattice. To describe these larger posets we use the following scheme. For $i \in \mathbb{N}$ and X -poset \mathfrak{X} we set $\mathfrak{X}_i = \{x \in \mathfrak{X} \mid l(x) = i\}$. If $|\mathfrak{X}| = t$, then we will label the elements of \mathfrak{X} by $1, 2, \dots, t$. In $A_5(vii)$, $A_5(viii)$, $A_5(ix)$ and $A_5(x)$, for each non-empty \mathfrak{X}_i and $i < |\Phi^+|$ we give the element \mathbf{x} of \mathfrak{X}_i followed by the set of all elements \mathbf{y} in \mathfrak{X}_{i+1} , with the property that $\mathbf{x} \prec \mathbf{y}$. So, for example, in $A_5(vii)$ we see that $\mathfrak{X}_7 = \{8, 9, 10, 11, 12, 13\}$ and that the element 8 is less than 14 and 16, the element 9 is less than 14, 15, 17, and so on. One further point, we may reduce our computational labours by using Theorem 1.3. Thus we observe in (2.4) that for $X \leq Y \leq W$ with $N(X) = N(Y)$ and Y a reflection subgroup we have $\mathfrak{X} \cong \mathfrak{Y}$ for the pairs (X, Y) :-

$(\langle [12324321] \rangle, \langle [4321234], [43421234] \rangle), (\langle [12134321] \rangle, \langle [1234321], [12134321] \rangle),$
 $(\langle [1214] \rangle, \langle [4], [212] \rangle), (\langle [1343] \rangle, \langle [1], [343] \rangle), (\langle [134321] \rangle, \langle [121], [343] \rangle), (\langle [234321] \rangle,$
 $\langle [1], [23432] \rangle), (\langle [124321] \rangle, \langle [12321], [4] \rangle), (\langle [2321] \rangle, \langle [1], [232] \rangle), (\langle [1321] \rangle,$
 $\langle [121], [3] \rangle), (\langle [3432] \rangle, \langle [2], [343] \rangle), (\langle [2432] \rangle, \langle [232], [4] \rangle).$

X	\mathfrak{X}
$\langle [12324321] \rangle \cong \mathbb{Z}_2$	\mathcal{C}_3
$\langle [21321432] \rangle \cong \mathbb{Z}_2$	\mathcal{C}_3
$\langle [12134321] \rangle \cong \mathbb{Z}_2$	\mathcal{C}_3
$\langle [4321234] \rangle \cong \mathbb{Z}_2$	\mathcal{C}_4
$\langle [132143] \rangle \cong \mathbb{Z}_2$	\mathcal{C}_5
$\langle [213432] \rangle \cong \mathbb{Z}_2$	\mathcal{C}_5
$\langle [1214] \rangle \cong \mathbb{Z}_2$	$ \mathfrak{X} = 10$; see $A_4(iii)$
$\langle [1343] \rangle \cong \mathbb{Z}_2$	$ \mathfrak{X} = 10$; see $A_4(iii)$
$\langle [32123] \rangle \cong \mathbb{Z}_2$	$ \mathfrak{X} = 10$; see $A_4(i)$
$\langle [23432] \rangle \cong \mathbb{Z}_2$	$ \mathfrak{X} = 10$; see $A_4(i)$
$\langle [2132] \rangle \cong \mathbb{Z}_2$	$ \mathfrak{X} = 15$; see $A_4(ii)$
$\langle [3243] \rangle \cong \mathbb{Z}_2$	$ \mathfrak{X} = 15$; see $A_4(ii)$
$\langle [234321] \rangle \cong \mathbb{Z}_3$	\mathcal{C}_2
$\langle [124321] \rangle \cong \mathbb{Z}_3$	\mathcal{C}_2
$\langle [134321] \rangle \cong \mathbb{Z}_3$	\mathcal{C}_2
$\langle [1321] \rangle \cong \mathbb{Z}_3$	\mathcal{C}_5
$\langle [3432] \rangle \cong \mathbb{Z}_3$	\mathcal{C}_5
$\langle [2321] \rangle \cong \mathbb{Z}_3$	\mathcal{C}_5
$\langle [2432] \rangle \cong \mathbb{Z}_3$	\mathcal{C}_5
$\langle [4321234], [43421234] \rangle \cong 2^2$	\mathcal{C}_3
$\langle [32123], [342123] \rangle \cong 2^2$	\mathcal{C}_3
$\langle [1234321], [12134321] \rangle \cong 2^2$	\mathcal{C}_3
$\langle [23432], [213432] \rangle \cong 2^2$	\mathcal{C}_3
$\langle [4], [212] \rangle \cong 2^2$	$ \mathfrak{X} = 10$; see $A_4(iii)$
$\langle [1], [343] \rangle \cong 2^2$	$ \mathfrak{X} = 10$; see $A_4(iii)$
$\langle [121], [343] \rangle \cong \text{Sym}(3)$	\mathcal{C}_2
$\langle [1], [23432] \rangle \cong \text{Sym}(3)$	\mathcal{C}_2
$\langle [12321], [4] \rangle \cong \text{Sym}(3)$	\mathcal{C}_2
$\langle [1], [232] \rangle \cong \text{Sym}(3)$	\mathcal{C}_5
$\langle [121], [3] \rangle \cong \text{Sym}(3)$	\mathcal{C}_5
$\langle [2], [343] \rangle \cong \text{Sym}(3)$	\mathcal{C}_5
$\langle [232], [4] \rangle \cong \text{Sym}(3)$	\mathcal{C}_5

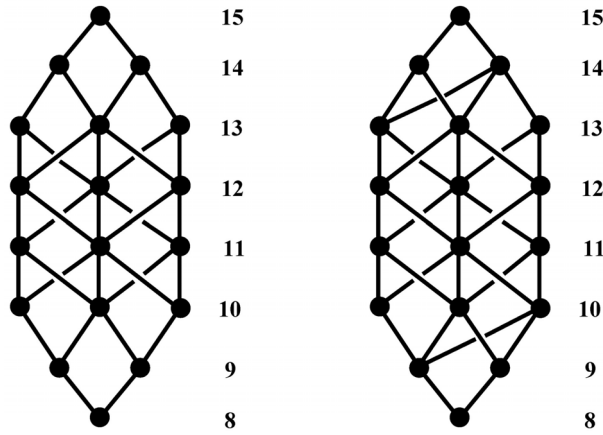


Fig. 2. $A_5(i)$ (left) and $A_5(ii)$.

(2.4) W of type A_5 , $\bullet_1 - \bullet_2 - \bullet_3 - \bullet_4 - \bullet_5$. For $X \leq W$, either $\mathfrak{X} \cong \mathcal{B}(\langle X \rangle_{\text{sp}})$ or one of the following holds:-

	X	\mathfrak{X}
X a subgroup of W_{1234}	$\langle [12324321] \rangle \cong \mathbb{Z}_2$	$ \mathfrak{X} = 18$; see $A_5(i)$
	$\langle [21321432] \rangle \cong \mathbb{Z}_2$	$ \mathfrak{X} = 18$; see $A_5(ii)$
	$\langle [12134321] \rangle \cong \mathbb{Z}_2$	$ \mathfrak{X} = 18$; see $A_5(iii)$
	$\langle [4321234] \rangle \cong \mathbb{Z}_2$	$ \mathfrak{X} = 24$; see $A_5(iv)$
	$\langle [132143] \rangle \cong \mathbb{Z}_2$	$ \mathfrak{X} = 30$; see $A_5(v)$
	$\langle [213432] \rangle \cong \mathbb{Z}_2$	$ \mathfrak{X} = 30$; see $A_5(vi)$
	$\langle [1214] \rangle \cong \mathbb{Z}_2$	$ \mathfrak{X} = 60$; see $A_5(vii)$
	$\langle [1343] \rangle \cong \mathbb{Z}_2$	$ \mathfrak{X} = 60$; see $A_5(viii)$
	$\langle [32123] \rangle \cong \mathbb{Z}_2$	$ \mathfrak{X} = 60$; see $A_5(ix)$
	$\langle [23432] \rangle \cong \mathbb{Z}_2$	$ \mathfrak{X} = 60$; see $A_5(x)$
	$\langle [2132] \rangle \cong \mathbb{Z}_2$	$ \mathfrak{X} = 90$; see $A_5(xi)$
	$\langle [3243] \rangle \cong \mathbb{Z}_2$	$ \mathfrak{X} = 90$; see $A_5(xii)$
	$\langle [234321] \rangle \cong \mathbb{Z}_3$	$ \mathfrak{X} = 12$; see $A_5(xiii)$
	$\langle [124321] \rangle \cong \mathbb{Z}_3$	$ \mathfrak{X} = 12$; see $A_5(xiii)$
	$\langle [134321] \rangle \cong \mathbb{Z}_3$	$ \mathfrak{X} = 12$; see $A_5(xiii)$
	$\langle [1321] \rangle \cong \mathbb{Z}_3$	$ \mathfrak{X} = 30$; see $A_5(xiv)$
	$\langle [3432] \rangle \cong \mathbb{Z}_3$	$ \mathfrak{X} = 30$; see $A_5(xv)$
	$\langle [2321] \rangle \cong \mathbb{Z}_3$	$ \mathfrak{X} = 30$; see $A_5(xiv)$
	$\langle [2432] \rangle \cong \mathbb{Z}_3$	$ \mathfrak{X} = 30$; see $A_5(xv)$
	$\langle [4321234], [43421234] \rangle \cong 2^2$	$ \mathfrak{X} = 18$; see $A_5(i)$
	$\langle [32123], [342123] \rangle \cong 2^2$	$ \mathfrak{X} = 18$; see $A_5(i)$

$\langle [1234321], [12134321] \rangle \cong 2^2$	$ \mathfrak{X} = 18$; see $A_5(iii)$
$\langle [23432], [213432] \rangle \cong 2^2$	$ \mathfrak{X} = 18$; see $A_5(iii)$
$\langle [4], [212] \rangle \cong 2^2$	$ \mathfrak{X} = 60$; see $A_5(vii)$
$\langle [1], [343] \rangle \cong 2^2$	$ \mathfrak{X} = 60$; see $A_5(viii)$
$\langle [121], [343] \rangle \cong \text{Sym}(3)$	$ \mathfrak{X} = 12$; see $A_5(xiii)$
$\langle [1], [23432] \rangle \cong \text{Sym}(3)$	$ \mathfrak{X} = 12$; see $A_5(xiii)$
$\langle [12321], [4] \rangle \cong \text{Sym}(3)$	$ \mathfrak{X} = 12$; see $A_5(xiii)$
$\langle [1], [232] \rangle \cong \text{Sym}(3)$	$ \mathfrak{X} = 30$; see $A_5(xiv)$
$\langle [121], [3] \rangle \cong \text{Sym}(3)$	$ \mathfrak{X} = 30$; see $A_5(xiv)$
$\langle [2], [343] \rangle \cong \text{Sym}(3)$	$ \mathfrak{X} = 30$; see $A_5(xv)$
$\langle [232], [4] \rangle \cong \text{Sym}(3)$	$ \mathfrak{X} = 30$; see $A_5(xv)$
X a subgroup of W_{1235}	
$\langle [321235] \rangle \cong \mathbb{Z}_2$	$ \mathfrak{X} = 30$; see $A_5(xvi)$
$\langle [21325] \rangle \cong \mathbb{Z}_2$	$ \mathfrak{X} = 45$; see $A_5(xvii)$

X conjugate to a subgroup of W_{1234} and X not contained in any proper standard parabolic subgroup of W .

	X	\mathfrak{X}
$X \cong \mathbb{Z}_2$	$\langle [12132432154321] \rangle$	\mathcal{C}_2
	$\langle [121321454321] \rangle$	\mathcal{C}_4
	$\langle [213243215432] \rangle$	
	$\langle [123243254321] \rangle$	
	$\langle [2132145432] \rangle$	\mathcal{C}_6
	$\langle [1232454321] \rangle$	
	$\langle [1324321543] \rangle$	
$X \cong \mathbb{Z}_3$	$\langle [21345432] \rangle$	\mathcal{C}_8
	$\langle [13214543] \rangle$	
	$\langle [12432154] \rangle$	
$X \cong \mathbb{Z}_4$	$\langle [1213454321] \rangle$	$ \mathfrak{X} = 8$; see $A_5(xviii)$
	$\langle [1234354321] \rangle$	
	$\langle [123454321] \rangle$	$ \mathfrak{X} = 8$; see $A_5(xix)$
$X \cong \mathbb{Z}_3$	$\langle [12134543] \rangle$	\mathcal{C}_4
	$\langle [12321454] \rangle$	
	$\langle [12345432] \rangle$	
	$\langle [12343215] \rangle$	
$X \cong \mathbb{Z}_4$	18 possible X 's	\mathcal{C}_2
$X \cong 2^2$	X conjugate to $\langle [13], [2132] \rangle$ (6 possibilities)	\mathcal{C}_2
	$\langle [23432], [12132432154321] \rangle$	
	$\langle [1234321], [213243215432] \rangle$	\mathcal{C}_4
	$\langle [232], [121321454321] \rangle$	
	$\langle [12321], [2132145432] \rangle$	
$\langle [343], [123243254321] \rangle$		
$\langle [1234321], [1324321543] \rangle$	\mathcal{C}_6	
$\langle [3], [1232454321] \rangle$		
	$\langle [12321], [13214543] \rangle$	

	$\left. \begin{array}{l} \langle [121], [21345432] \rangle \\ \langle [2], [1213454321] \rangle \\ \langle [4], [1234354321] \rangle \\ \langle [1234321], [12432154] \rangle \end{array} \right\}$	$ \mathfrak{X} = 8$; see $A_5(xviii)$
$X \cong \mathbb{Z}_6$	$\left. \begin{array}{l} \langle [2432154] \rangle \\ \langle [1214354] \rangle \\ \langle [1243254] \rangle \\ \langle [1432154] \rangle \end{array} \right\}$	C_3
	$\left. \begin{array}{l} \langle [3214543] \rangle \\ \langle [1324543] \rangle \end{array} \right\}$	C_4
$X \cong \text{Dih}(8)$	18 possible X 's	C_2
$X \cong \text{Alt}(4)$	3 possible X 's	C_2
$X \cong \text{Sym}(4)$	6 possible X 's	C_2

X conjugate to a subgroup of W_{1245} which is not a subgroup of W_{1234} and X not contained in any proper standard parabolic subgroup of W .

	X	\mathfrak{X}
$X \cong \mathbb{Z}_3$	$\left. \begin{array}{l} \langle [13243254] \rangle \\ \langle [21324325] \rangle \\ \langle [21324354] \rangle \\ \langle [23214325] \rangle \\ \langle [32432154] \rangle \end{array} \right\}$	C_4
	$\left. \begin{array}{l} \langle [321435] \rangle \\ \langle [132435] \rangle \end{array} \right\}$	C_7

X conjugate to a subgroup of W_{1235} which is not a subgroup of W_{1234} nor W_{1245} and X not contained in any proper standard parabolic subgroup of W .

	X	\mathfrak{X}
$X \cong \mathbb{Z}_2$	$\left. \begin{array}{l} \langle [2132143215432] \rangle \\ \langle [1232143254321] \rangle \end{array} \right\}$	C_3
	$\left. \begin{array}{l} \langle [13214321543] \rangle \\ \langle [23214325432] \rangle \end{array} \right\}$	C_5
	$\left. \begin{array}{l} \langle [121432154] \rangle \\ \langle [321432543] \rangle \\ \langle [213435432] \rangle \end{array} \right\}$	C_7
	$\langle [2143254] \rangle$	C_9
	$\langle [12134354321] \rangle$	$ \mathfrak{X} = 6$; see $A_5(xx)$

$X \cong Z_4$	8 possible X 's	C_5
$X \cong 2^2$	$\langle [32432543], [321432543] \rangle$ $\langle [21321432], [321432543] \rangle$ $\langle [132143], [23214325432] \rangle$ $\langle [2132145432], [23214325432] \rangle$ $\langle [324543], [13214321543] \rangle$ $\langle [1324321543], [13214321543] \rangle$ $\langle [213243215432], [2132143215432] \rangle$ $\langle [13214543], [321432543] \rangle$ $\langle [3243], [1232143254321] \rangle$	C_3
	$\langle [121432154], [14354] \rangle$ $\langle [13214321543], [321432543] \rangle$ $\langle [121432154], [2143254] \rangle$ $\langle [213435432], [21325] \rangle$ $\langle [2143254], [14354] \rangle$ $\langle [2143254], [21325] \rangle$ $\langle [213435432], [2143254] \rangle$ $\langle [23214325432], [321432543] \rangle$	C_5
ctd.	X	\mathfrak{X}
$X \cong 2^2$	$\langle [213432], [243254] \rangle$ $\langle [12134321], [12432154] \rangle$ $\langle [21345432], [23435432] \rangle$ $\langle [1213454321], [1234354321] \rangle$ $\langle [12134321], [121432154] \rangle$ $\langle [1213454321], [12134354321] \rangle$ $\langle [24], [12134354321] \rangle$ $\langle [121454], [2143254] \rangle$ $\langle [1214], [213435432] \rangle$ $\langle [23435432], [213435432] \rangle$ $\langle [1234354321], [12134354321] \rangle$ $\langle [2454], [121432154] \rangle$	$ \mathfrak{X} = 6$; see $A_5(xx)$
	$\langle [213432], [2143254] \rangle$ $\langle [21345432], [213435432] \rangle$ $\langle [12432154], [121432154] \rangle$ $\langle [243254], [2143254] \rangle$	C_7
$X \cong 2^3$	$\langle [121], [4], [2345432] \rangle$ $\langle [2], [4], [123454321] \rangle$ $\langle [2], [454], [1234321] \rangle$ $\langle [121], [454], [23432] \rangle$	$ \mathfrak{X} = 6$; see $A_5(xx)$

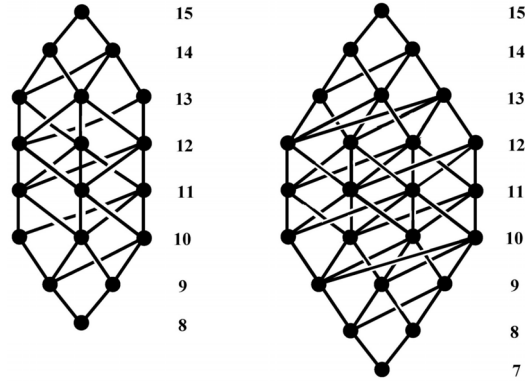


Fig. 3. $A_5(iii)$ (left) and $A_5(iv)$ (right).

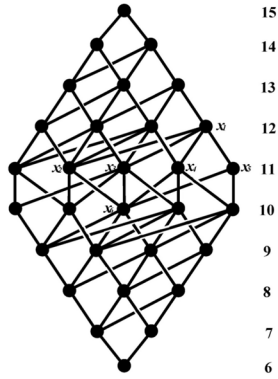


Fig. 4. $A_5(v)$.

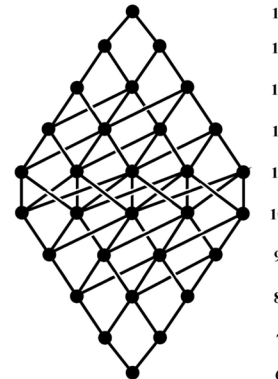


Fig. 5. $A_5(vi)$.

Table 1. $A_5(vii)$

\mathfrak{X}_i	$j\{k \in \mathfrak{X}_{i+1} \mid j \prec k\} (j \in \mathfrak{X}_i)$
\mathfrak{X}_4	$1\{2, 3\}$
\mathfrak{X}_5	$2\{4, 5, 6\}; 3\{6, 7\}$
\mathfrak{X}_6	$4\{9, 10, 12\}; 5\{8, 9, 11\}; 6\{10, 11, 12, 13\}; 7\{12, 13\}$
\mathfrak{X}_7	$8\{14, 16\}; 9\{14, 15, 17\}; 10\{17, 18\}; 11\{16, 17, 19, 21\}; 12\{18, 19, 20\}; 13\{20, 21\}$
\mathfrak{X}_8	$14\{23, 25\}; 15\{22, 24, 28\}; 16\{23, 25, 30\}; 17\{23, 24, 26\}; 18\{26, 27\}; 19\{25, 26, 28, 29\}; 20\{27, 28, 29\}; 21\{29, 30\}$
\mathfrak{X}_9	$22\{31, 32, 35\}; 23\{32, 33\}; 24\{32, 34, 36\}; 25\{33, 35, 39\}; 26\{33, 34, 36, 37\}; 27\{36, 37\}; 28\{35, 36, 38\}; 29\{37, 38, 39\}; 30\{39\}$
\mathfrak{X}_{10}	$31\{40, 44\}; 32\{40, 41, 42\}; 33\{41, 42, 46\}; 34\{41, 43\}; 35\{42, 44, 47\}; 36\{42, 43, 45\}; 37\{45, 46\}; 38\{44, 45, 47\}; 39\{46, 47\}$
\mathfrak{X}_{11}	$40\{48, 50\}; 41\{48, 49\}; 42\{49, 50, 53\}; 43\{49, 51\}; 44\{50, 52\}; 45\{50, 51, 53\}; 46\{53\}; 47\{52, 53\}$
\mathfrak{X}_{12}	$48\{54, 55\}; 49\{54, 55, 57\}; 50\{55, 56\}; 51\{55, 57\}; 52\{56\}; 53\{56, 57\}$
\mathfrak{X}_{13}	$54\{58\}; 55\{58, 59\}; 56\{59\}; 57\{59\}$
\mathfrak{X}_{14}	$58\{60\}; 59\{60\}$

Table 2. $A_5(viii)$

\mathfrak{X}_i	$j\{k \in \mathfrak{X}_{i+1} \mid j \prec k\} (j \in \mathfrak{X}_i)$
\mathfrak{X}_4	1{2, 3}
\mathfrak{X}_5	2{4, 5, 6}; 3{6, 7}
\mathfrak{X}_6	4{9, 10}; 5{8, 9, 11}; 6{10, 11, 12}; 7{12, 13}
\mathfrak{X}_7	8{14, 16, 19}; 9{14, 15, 17}; 10{17, 18}; 11{16, 17, 19, 20}; 12{18, 19, 20, 21}; 13{20, 21}
\mathfrak{X}_8	14{22, 23, 26}; 15{22, 24}; 16{23, 25}; 17{23, 24, 26, 27}; 18{26, 27, 30}; 19{25, 26, 28}; 20{27, 28, 29}; 21{29, 30}
\mathfrak{X}_9	22{31, 32, 34}; 23{32, 33}; 24{32, 34, 37}; 25{33, 35}; 26{33, 34, 36}; 27{36, 37, 39}; 28{35, 36, 38}; 29{37, 38, 39}; 30{39}
\mathfrak{X}_{10}	31{40, 43}; 32{40, 41}; 33{41, 42}; 34{41, 43, 45}; 35{42, 44}; 36{42, 43, 45, 47}; 37{45, 46}; 38{44, 45, 47}; 39{46, 47}
\mathfrak{X}_{11}	40{48, 49}; 41{48, 49, 50}; 42{49, 50, 52}; 43{49, 51}; 44{50, 52}; 45{50, 51, 53}; 46{53}; 47{52, 53}
\mathfrak{X}_{12}	48{54}; 49{54, 55}; 50{55, 56}; 51{55, 57}; 52{56}; 53{56, 57}
\mathfrak{X}_{13}	54{58}; 55{58, 59}; 56{59}; 57{59}
\mathfrak{X}_{14}	58{60}; 59{60}

Table 3. $A_5(ix)$

\mathfrak{X}_i	$j\{k \in \mathfrak{X}_{i+1} \mid j \prec k\} (j \in \mathfrak{X}_i)$
\mathfrak{X}_5	1{2, 3, 4}
\mathfrak{X}_6	2{5, 6, 7, 9}; 3{5, 6, 8}; 4{7, 8, 9}
\mathfrak{X}_7	5{10, 12, 14}; 6{10, 13, 15}; 7{11, 12, 13, 16}; 8{12, 13, 14, 15}; 9{11, 14, 15, 16}
\mathfrak{X}_8	10{20, 21, 22}; 11{17, 18, 19, 23}; 12{19, 20, 24}; 13{18, 20, 22, 25}; 14{19, 21, 22, 24}; 15{18, 21, 25}; 16{17, 22, 23, 24, 25}
\mathfrak{X}_9	17{26, 27, 30}; 18{26, 29, 32}; 19{27, 28, 29, 33}; 20{28, 29, 34, 35}; 21{29, 31, 34}; 22{28, 31, 35}; 23{28, 30, 32, 33}; 24{27, 33, 34, 35}; 25{26, 31, 32, 34}
\mathfrak{X}_{10}	26{37, 39}; 27{36, 37, 40}; 28{36, 38, 41, 42}; 29{37, 38, 43}; 30{36, 39, 40}; 31{38, 41, 44}; 32{38, 39, 43}; 33{40, 41, 42, 43}; 34{37, 41, 43, 44}; 35{36, 42, 44}
\mathfrak{X}_{11}	36{45, 46, 48}; 37{45, 47}; 38{45, 49, 50}; 39{45, 47}; 40{46, 47, 48}; 41{48, 49, 51}; 42{46, 50, 51}; 43{47, 49, 50}; 44{45, 48, 50, 51}
\mathfrak{X}_{12}	45{52, 53}; 46{52, 54, 55}; 47{52, 53}; 48{53, 54, 55}; 49{53, 55, 56}; 50{52, 55, 56}; 51{54, 56}
\mathfrak{X}_{13}	52{57, 58}; 53{57, 58}; 54{57, 59}; 55{58, 59}; 56{57, 59}
\mathfrak{X}_{14}	57{60}; 58{60}; 59{60}

Table 4. $A_5(x)$

\mathfrak{X}_i	$j\{k \in \mathfrak{X}_{i+1} \mid j \prec k\} (j \in \mathfrak{X}_i)$
\mathfrak{X}_5	1{2, 3, 4}
\mathfrak{X}_6	2{5, 7, 9}; 3{5, 6, 8}; 4{6, 7, 8, 9}
\mathfrak{X}_7	5{11, 12, 13}; 6{11, 14, 15}; 7{10, 11, 12, 16}; 8{12, 13, 14, 15}; 9{10, 13, 16}
\mathfrak{X}_8	10{17, 19, 22}; 11{20, 21, 23}; 12{19, 20, 23, 24}; 13{19, 21, 24}; 14{18, 20, 25}; 15{18, 21, 25}; 16{17, 21, 22, 24}
\mathfrak{X}_9	17{26, 28, 30}; 18{27, 29, 31}; 19{26, 30, 32, 34}; 20{27, 31, 32, 33}; 21{30, 31, 32, 35}; 22{28, 32, 34}; 23{27, 30, 33, 35}; 24{26, 31, 34, 35}; 25{19, 32, 33}
\mathfrak{X}_{10}	26{36, 38}; 27{39, 40}; 28{38, 41, 43}; 29{37, 39, 42}; 30{36, 41, 43}; 31{37, 40, 42}; 32{37, 41, 44}; 33{39, 41, 42, 44}; 34{37, 38, 43, 44}; 35{36, 40, 42, 43, 44}

\mathfrak{X}_{11}	36{47, 48}; 37{45, 46, 49}; 38{47, 48}; 39{46, 50}; 40{46, 50}; 41{45, 47, 51}; 42{45, 49, 40}; 43{45, 48, 51}; 44{46, 47, 49, 51}
\mathfrak{X}_{12}	45{52, 53, 54}; 46{52, 55}; 47{53, 56}; 48{53, 56}; 49{53, 54, 55}; 50{52, 55}; 51{52, 54, 56}
\mathfrak{X}_{13}	52{57, 58}; 53{57, 59}; 54{58, 59}; 55{57, 58}; 56{57, 59}
\mathfrak{X}_{14}	57{60}; 58{60}; 59{60}

Table 5. $A_5(xi)$

\mathfrak{X}_i	$j\{k \in \mathfrak{X}_{i+1} \mid j \prec k\} (j \in \mathfrak{X}_i)$
\mathfrak{X}_4	1{2, 3, 4}
\mathfrak{X}_5	2{5, 6, 7, 8}; 3{6, 8, 9, 10}; 4{7, 9, 10}
\mathfrak{X}_6	5{11, 12, 13}; 6{11, 13, 14, 15, 16}; 7{12, 14, 15, 17, 19}; 8{13, 16, 17, 19}; 9{14, 17, 18}; 10{15, 18, 19}
\mathfrak{X}_7	11{20, 21, 22, 23}; 12{21, 22, 24, 26}; 13{20, 23, 24, 26, 27, 29}; 14{21, 24, 25, 28}; 15{22, 25, 26, 29, 31}; 16{23, 27, 28, 31}; 17{24, 28, 29, 30}; 18{25, 29, 30}; 19{26, 30, 31}
\mathfrak{X}_8	20{32, 33, 35, 36, 38}; 21{33, 34, 37}; 22{34, 35, 38, 40}; 23{32, 36, 37, 40, 42}; 24{33, 37, 38, 39, 41}; 25{34, 38, 39, 42, 44}; 26{35, 39, 40, 43, 45}; 27{36, 41, 45}; 28{37, 41, 42, 44}; 29{38, 42, 43}; 30{39, 43, 44}; 31{40, 44, 45}
\mathfrak{X}_9	32{46, 47, 50, 52}; 33{47, 48, 49, 50}; 34{48, 49, 52, 54}; 35{49, 50, 53, 55}; 36{46, 51, 55, 58}; 37{47, 51, 52, 54, 57}; 38{48, 52, 53, 58}; 39{49, 53, 54, 57, 59}; 40{50, 54, 55, 56}; 41{51, 58, 59}; 42{52, 56, 57, 58}; 43{53, 56, 57}; 44{54, 56, 59}; 45{55, 59}
\mathfrak{X}_{10}	46{60, 64, 70}; 47{60, 61, 63, 67}; 48{61, 62, 70}; 49{62, 63, 67, 71}; 50{63, 64, 66}; 51{60, 69, 70, 71}; 52{61, 66, 67, 70}; 53{62, 66, 67, 68, 69}; 54{63, 65, 66, 71}; 55{64, 68, 71}; 56{65, 66, 68}; 57{65, 67, 69}; 58{68, 69, 70}; 59{68, 71}
\mathfrak{X}_{11}	60{78, 79, 80}; 61{74, 76, 79}; 62{74, 76, 77, 78}; 63{73, 74, 80}; 64{77, 80}; 65{72, 73, 75}; 66{72, 73, 74, 77}; 67{73, 76, 78}; 68{75, 77}; 69{72, 75, 78}; 70{72, 77, 78, 79}; 71{75, 77, 80}
\mathfrak{X}_{12}	72{81, 82}; 73{81, 83, 84}; 74{81, 83}; 75{82, 84}; 76{83, 86}; 77{82, 84, 85}; 78{84, 86}; 79{81, 85, 86}; 80{84, 85}
\mathfrak{X}_{13}	81{87, 88}; 82{88}; 83{87, 89}; 84{88, 89}; 85{88, 89}; 86{87, 89}
\mathfrak{X}_{14}	87{90}; 88{90}; 89{90}

Table 6. $A_5(xii)$

\mathfrak{X}_i	$j\{k \in \mathfrak{X}_{i+1} \mid j \prec k\} (j \in \mathfrak{X}_i)$
\mathfrak{X}_4	1{2, 3, 4}
\mathfrak{X}_5	2{5, 6, 7, 8, 10}; 3{6, 8, 9}; 4{7, 9, 10}
\mathfrak{X}_6	5{11, 12, 13, 15}; 6{11, 13, 14, 16, 18}; 7{12, 14, 15, 17, 19}; 8{13, 16, 17}; 9{14, 17, 18}; 10{15, 18, 19}
\mathfrak{X}_7	11{20, 21, 23, 25}; 12{21, 22, 24, 26}; 13{20, 23, 24, 27, 29}; 14{21, 24, 25, 28, 30}; 15{22, 25, 26, 29, 31}; 16{23, 27, 28}; 17{24, 28, 29}; 18{25, 29, 30}; 19{26, 30, 31}
\mathfrak{X}_8	20{32, 33, 36, 38}; 21{33, 34, 37, 39}; 22{34, 35, 38, 40}; 23{32, 36, 37, 43}; 24{33, 37, 38, 41, 44}; 25{34, 38, 39, 42, 45}; 26{35, 39, 40, 43}; 27{36, 41, 42}; 28{37, 41, 42, 43}; 29{38, 42, 43, 44}; 30{39, 43, 44, 45}; 31{40, 44, 45}
\mathfrak{X}_9	32{46, 47, 53}; 33{47, 48, 51, 54}; 34{48, 49, 52, 55}; 35{49, 50, 53}; 36{46, 51, 52, 58}; 37{47, 51, 52, 53, 56}; 38{48, 52, 53, 54, 57, 59}; 39{49, 53, 54, 55, 58}; 40{50, 54, 55, 56}; 41{51, 59}; 42{52, 58, 59}; 43{53, 56, 58}; 44{54, 56, 57}; 45{55, 57}

\mathfrak{X}_{10}	46{60, 61, 70}; 47{60, 61, 62, 65}; 48{61, 62, 63, 69, 71}; 49{62, 63, 64, 70}; 50{63, 64, 65}; 51{60, 68, 71}; 52{61, 66, 70, 71}; 53{62, 65, 67, 68, 70}; 54{63, 65, 66, 69}; 55{64, 67, 69}; 56{65, 66, 67}; 57{67, 69}; 58{66, 68, 70}; 59{68, 71}
\mathfrak{X}_{11}	60{76, 80}; 61{74, 79, 80}; 62{73, 75, 76, 79}; 63{73, 74, 78}; 64{75, 78}; 65{72, 73, 74, 75}; 66{72, 74, 77}; 67{75, 77}; 68{72, 76}; 69{75, 77, 78}; 70{74, 76, 77, 79}; 71{72, 76, 80}
\mathfrak{X}_{12}	72{81, 82}; 73{81, 83, 84}; 74{81, 83, 86}; 75{82, 84, 86}; 76{81, 82, 85}; 77{82, 86}; 78{84, 86}; 79{83, 85, 86}; 80{81, 85}
\mathfrak{X}_{13}	81{87, 88}; 82{88}; 83{87, 89}; 84{88, 89}; 85{87, 88}; 86{88, 89}
\mathfrak{X}_{14}	87{90}; 88{90}; 89{90}

Table 7. $A_5(xvii)$

\mathfrak{X}_i	$j\{k \in \mathfrak{X}_{i+1} \mid j \prec k\} (j \in \mathfrak{X}_i)$
\mathfrak{X}_5	1{2, 3}
\mathfrak{X}_6	2{4, 5, 7}; 3{4, 5, 6};
\mathfrak{X}_7	4{8, 9, 10, 11}; 5{10, 11, 12}; 6{8, 12}; 7{9, 10}
\mathfrak{X}_8	8{13, 16, 18}; 9{15, 16, 17}; 10{13, 14, 15, 17, 19}; 11{17, 18, 19}; 12{13, 14, 18}
\mathfrak{X}_9	13{20, 21, 23, 25}; 14{20, 26}; 15{20, 22, 23, 24}; 16{21, 23}; 17{21, 22, 24, 26}; 18{21, 25, 26}; 19{24, 25}
\mathfrak{X}_{10}	20{27, 28, 31}; 21{28, 29, 30, 32}; 22{28, 29, 33}; 23{27, 29, 30}; 24{30, 31, 33}; 25{30, 31}; 26{28, 31, 32}
\mathfrak{X}_{11}	27{34, 37}; 28{34, 35, 37}; 29{34, 35, 36}; 30{36, 37, 38}; 31{37, 38}; 32{35, 38}; 33{36, 37}
\mathfrak{X}_{12}	34{39, 42}; 35{41, 42}; 36{39, 41}; 37{39, 40, 41}; 38{40, 41}
\mathfrak{X}_{13}	39{43, 44}; 40{43}; 41{43, 44}; 42{44}
\mathfrak{X}_{14}	43{45}; 44{45}

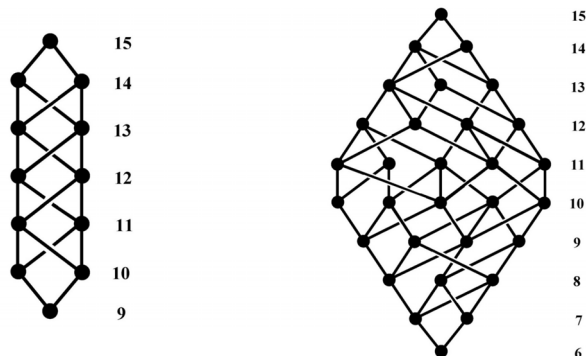


Fig. 6. $A_5(xiii)$.

Fig. 7. $A_5(xiv)$.

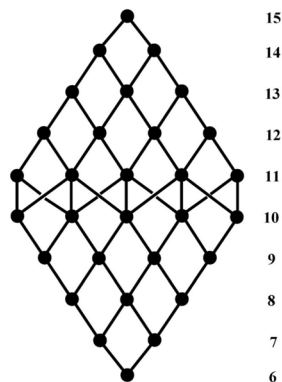


Fig. 8. $A_5(xv)$.

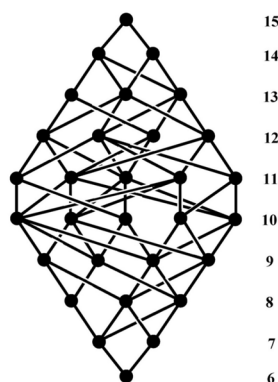


Fig. 9. $A_5(xvi)$.

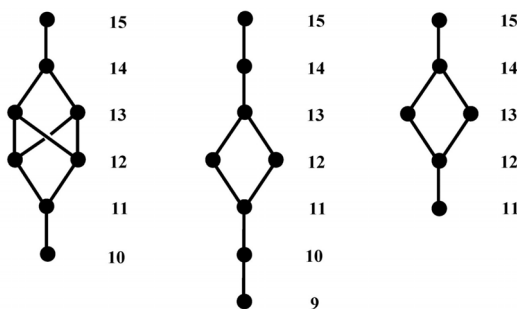


Fig. 10. $A_5(xviii)$ (left), $A_5(xix)$ (centre) and $A_5(xx)$ (right).

(2.5) W of type B_2 , $\bullet \text{---} \bullet$. For $X \leq W$ either $\mathfrak{X} \cong \mathcal{B}(\langle X \rangle_{\text{sp}})$ or one of the following holds:-

X	\mathfrak{X}
$\langle [212] \rangle \cong \mathbb{Z}_2$	\mathcal{C}_2
$\langle [121] \rangle \cong \mathbb{Z}_2$	\mathcal{C}_2

(2.6) W of type B_3 , $\bullet \text{---} \bullet \text{---} \bullet$. For $X \leq W$ either $\mathfrak{X} \cong \mathcal{B}(\langle X \rangle_{\text{sp}})$ or one of the following holds:-

X	\mathfrak{X}
$\langle [21232123] \rangle \cong \mathbb{Z}_2$	\mathcal{C}_2
$\langle [12123212] \rangle \cong \mathbb{Z}_2$	\mathcal{C}_2
$\langle [12132123] \rangle \cong \mathbb{Z}_2$	\mathcal{C}_2

$\langle [2123212] \rangle \cong \mathbb{Z}_2$	C_3
$\langle [132123] \rangle \cong \mathbb{Z}_2$	C_4
$\langle [121321] \rangle \cong \mathbb{Z}_2$	C_4
$\langle [232123] \rangle \cong \mathbb{Z}_2$	C_4
$\langle [32123] \rangle \cong \mathbb{Z}_2$	$ \mathcal{X} = 6$; see $B_3(v)$
$\langle [2132] \rangle \cong \mathbb{Z}_2$	$ \mathcal{X} = 8$; see $B_3(vi)$
$\langle [12321] \rangle \cong \mathbb{Z}_2$	$ \mathcal{X} = 8$; see $B_3(iii)$
$\langle [232] \rangle \cong \mathbb{Z}_2$	$ \mathcal{X} = 8$; see $B_3(iv)$
$\langle [212] \rangle \cong \mathbb{Z}_2$	$ \mathcal{X} = 12$; see $B_3(i)$
$\langle [121] \rangle \cong \mathbb{Z}_2$	$ \mathcal{X} = 12$; see $B_3(ii)$
$\langle [1213] \rangle \cong \mathbb{Z}_3$	C_4
$\langle [2123] \rangle \cong \mathbb{Z}_4$	C_2
$\langle [1232] \rangle \cong \mathbb{Z}_4$	C_2
$\langle [2123], [21232123] \rangle \cong 2^2$	C_2
$\langle [1232], [132123] \rangle \cong 2^2$	C_2
$\langle [212], [21232123] \rangle \cong 2^2$	C_2
$\langle [1], [2123212] \rangle \cong 2^2$	C_2
$\langle [212], [12321] \rangle \cong 2^2$	C_2
$\langle [32123], [121] \rangle \cong 2^2$	C_2
$\langle [212], [232] \rangle \cong 2^2$	C_4
$\langle [32123], [2] \rangle \cong 2^2$	C_4
$\langle [1], [132123] \rangle \cong 2^2$	C_4
$\langle [121], [3] \rangle \cong \text{Sym}(3)$	C_4
$X = W_{12}^g \neq W_{12}$ (twice) $\cong \text{Dih}(8)$	C_2
$X = W_{12}^h \neq W_{12}$ (twice) $\cong \text{Dih}(8)$	C_4

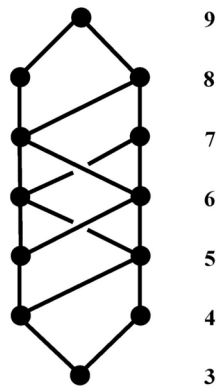


Fig. 11. $B_3(i)$.

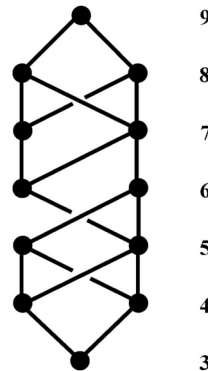


Fig. 12. $B_3(ii)$.

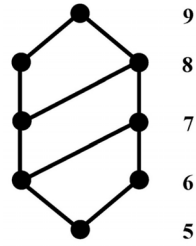


Fig. 13. $B_3(iii)$.

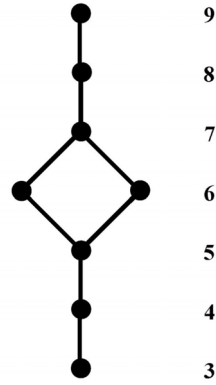


Fig. 14. $B_3(iv)$.

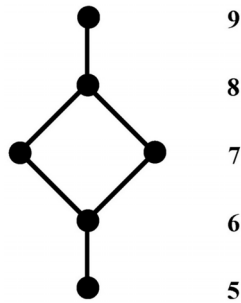


Fig. 15. $B_3(v)$.

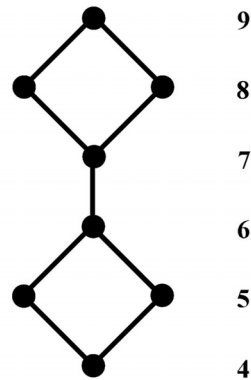


Fig. 16. $B_3(vi)$.

(2.7) W of type D_4 , $\begin{matrix} & 1 & & \\ & \bullet & & \\ & / \quad \backslash & & \\ 2 & \bullet & 3 & \bullet & 4 \\ & \backslash \quad / & & \\ & & & \end{matrix}$. On account of the graph automorphisms we shall only consider conjugates of subgroups of $W_{234} = \langle [2], [3], [4] \rangle$ and $W_{124} = \langle [1], [2], [4] \rangle$. For $X \leq W$, either $\mathfrak{X} \cong \mathcal{B}(\langle X \rangle_{sp})$ or one of the following holds:-

X	\mathfrak{X}
$\left. \begin{aligned} &\langle [12312343123] \rangle, \langle [12312431234] \rangle, \\ &\langle [13123431234] \rangle, \langle [23123431234] \rangle \end{aligned} \right\}$	\mathcal{C}_2
$\langle [1312343123] \rangle$	\mathcal{C}_3
$\left. \begin{aligned} &\langle [123124312] \rangle, \langle [132431234] \rangle, \\ &\langle [231431234] \rangle, \langle [312343123] \rangle \end{aligned} \right\}$	\mathcal{C}_4

$X \cong \mathbb{Z}_2$	$\langle [12324312] \rangle$	\mathcal{C}_5
	$\langle [1312431] \rangle, \langle [2312432] \rangle, \langle [3431234] \rangle, \langle [1234312] \rangle \}$	\mathcal{C}_6
	$\langle [132431] \rangle$	\mathcal{C}_7
	$\langle [31243] \rangle$	\mathcal{C}_8
	$\langle [43234] \rangle$ $\langle [3243] \rangle$	$ \mathfrak{X} = 16$; see $D_4(i)$ $ \mathfrak{X} = 24$; see $D_4(ii)$
$X \cong \mathbb{Z}_3$	$\langle [124312] \rangle$	\mathcal{C}_4
	$\langle [123432] \rangle$	
	$\langle [134312] \rangle$	
$X \cong \mathbb{Z}_4$	$\langle [12431] \rangle$	\mathcal{C}_3
	$\langle [32431] \rangle$	
	$\langle [14312] \rangle$	
	$\langle [1324312] \rangle$	
$X \cong 2^2$	X conjugate to $Y_1 = \langle [12], [124] \rangle,$ $X \neq Y_1, X_1$ or X_2 (9 conjugates)	\mathcal{C}_2
	X conjugate to $Y_2 = \langle [14], [24] \rangle,$ $X \neq Y_2, X_3$ (10 conjugates)	\mathcal{C}_2
	$\langle [132431], [3243] \rangle$	\mathcal{C}_3
	$\langle [13431], [132431] \rangle$	
	$\langle [1234312], [12324312] \rangle$	
	$\langle [312343123], [1312343123] \rangle$	\mathcal{C}_5
	$\langle [132431], [24] \rangle$	
	$X_1 = \langle [3123], [31243] \rangle$	
	$X_2 = \langle [431234], [3431234] \rangle$	
	$X_3 = \langle [3243], [3143] \rangle$	\mathcal{C}_6
$X \cong \text{Sym}(3)$	$\langle [4], [123124] \rangle$	\mathcal{C}_4
	$\langle [23432], [123432] \rangle$	
	$\langle [13431], [123431] \rangle$	
$X \cong \text{Dih}(8)$	$\langle [1234312], [24] \rangle$	\mathcal{C}_3
	$\langle [312343123], [3243] \rangle$	
	$\langle [13431], [24] \rangle$	
	$\langle [4], [132431] \rangle$	

3. SOME OBSERVATIONS

3.1. Möbius Functions

It was first shown by Deodhar [3] that the Möbius function of the Bruhat order of any Coxeter group takes values $-1, 1$. Generalized quotients, introduced by Björner and Wachs [1] and having the Bruhat order of a Coxeter group as a special case, have

Möbius functions which only take values $-1, 0, 1$. This is not true for X -posets as may be seen by looking at $A_5(v)$ where W is of type A_5 and $X = \langle [132143] \rangle \cong \mathbb{Z}_2$. The elements $\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5$ are all the elements of length 11 which are greater than \mathbf{x}_6 (of length 10) and less than \mathbf{x}_1 (of length 12). So (where μ denotes the Möbius function of \mathfrak{X})

$$\mu(\mathbf{x}_6, \mathbf{x}_1) + \mu(\mathbf{x}_2, \mathbf{x}_1) + \mu(\mathbf{x}_3, \mathbf{x}_1) + \mu(\mathbf{x}_4, \mathbf{x}_1) + \mu(\mathbf{x}_5, \mathbf{x}_1) + \mu(\mathbf{x}_1, \mathbf{x}_1) = 0$$

whence $\mu(\mathbf{x}_6, \mathbf{x}_1) - 1 - 1 - 1 - 1 + 1 = 0$, and hence $\mu(\mathbf{x}_6, \mathbf{x}_1) = 3$. In particular, \mathfrak{X} cannot be a generalized quotient.

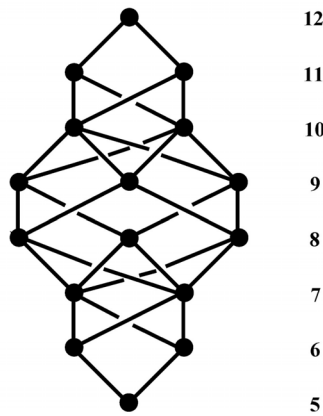


Fig. 17. $D_4(i)$.

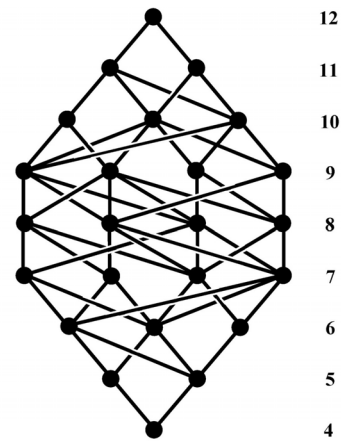


Fig. 18. $D_4(ii)$.

3.2. Odd and even elements in intervals

In any interval of the Bruhat order of a Coxeter group there are the same number of odd (length) elements as there are even (length) elements [6]. This property is not shared by X -posets. For example in $A_4(ii)$ the interval between (and including) \mathbf{x}_1 and \mathbf{x}_3 has two even elements and only one odd element (as a more substantial example take the interval between \mathbf{x}_1 and \mathbf{x}_4).

3.3. X -posets in standard parabolic subgroups

Suppose that X is a subgroup of Y where Y is a standard parabolic subgroup of W . Let \mathfrak{X}_Y be the X -poset in Y . Then there are a number of connections between \mathfrak{X}_Y and \mathfrak{X} (see [5]) Does \mathfrak{X}_Y exert even greater control upon the structure of \mathfrak{X} ? The answer appears to be a resounding no. Looking at (2.3) and (2.4) and taking W of type A_5 with $Y = W_{1234}$ we have that the X -posets for $\langle [12324321] \rangle$ and $\langle [21321432] \rangle$ in Y are isomorphic (both are \mathcal{C}_2) but the X -posets in W are not isomorphic. Note that both $\langle [12324321] \rangle$ and $\langle [21321432] \rangle$ also have the same standard parabolic closure in Y . There are other examples like this for W of type A_5 as well as for W of type B_3 .

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