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S.B. Hart and P.J. Rowley*

Abstract

The maximal lengths of elements in each of the conjugacy classes of Coxeter groups of types B_n, D_n and E_6 are determined. Additionally, representative elements are given that attain these maximal lengths.
(MSC2000: 20F55; keywords: Coxeter group, length, conjugacy class)

1 Introduction

Conjugacy classes of finite Coxeter groups have long been of interest, the correspondence between partitions and the conjugacy classes for the symmetric groups having been observed by Cauchy [4] in the early days of group theory. For Coxeter groups of type B_n and D_n, descriptions of their conjugacy classes, by Specht [11] and Young [12], have also been known for a long time. In 1972, Carter [2] gave a uniform and systematic treatment of the conjugacy classes of Weyl groups. More recently, Geck and Pfeiffer [6] reworked Carter's descriptions from more of an algorithmic standpoint. Motivation for investigating the conjugacy classes of finite Coxeter groups, and principally those of the irreducible finite Coxeter groups, has come from many directions, for example in the representation theory of these groups and the classification of maximal tori in groups of Lie type (see [3]). The behaviour of lengths in a conjugacy class is frequently important. Of particular interest are those elements of minimal and maximal lengths in their class. Minimal length elements have received considerable attention – see [6]. Here we look at maximal length elements. Now every finite irreducible Coxeter group W possesses a (unique) element w_0 of maximal length in W. For C a conjugacy class of W, set $C' = Cw_0 = \{ww_0 : w \in C\}$. If, as happens in many cases, $w_0 \in Z(W)$, then C' is also a conjugacy class of W. Moreover, $w \in C$ has minimal length in C if and only if ww_0 has maximal length in C'. Thus information about maximal length elements in a conjugacy class may be obtained from that known about minimal length elements. Among the finite irreducible Coxeter groups, only those of type I_m, m odd, A_n, D_n, n odd and E_6 have $w_0 \notin Z(W)$. The first of these, being just dihedral groups, are quickly dealt with. Descriptions of maximal length elements in conjugacy classes of type A_n were given by Kim [8]. It is the purpose of this note to deal with the remaining two cases. Representatives of maximal length for type D_n can be extracted from Section 4 of [5], however the proofs given here are more direct.

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Suppose $\hat{W}$ is of type D_n. Then we may regard $\hat{W}$ as a canonical index 2 subgroup of W where W is a Coxeter group of type B_n. The conjugacy classes of W are parameterized by signed cycle type. So for X a subset of a conjugacy class of W, this data may be encoded by

$$\lambda(X) = (\lambda_1, \ldots, \lambda_{\nu_X}; \lambda_{\nu_X+1}, \ldots, \lambda_{z_X})$$

where ν_X is the number of negative cycles, z_X is the total number of cycles, and $\lambda_1 \leq \ldots \leq \lambda_{\nu_X}$, respectively $\lambda_{\nu_X+1} \leq \cdots \leq \lambda_{z_X}$, are the lengths of the negative, respectively positive, cycles of X. So any element of X has $\lambda(X)$ as its signed cycle type.

Theorem 1.1 *Suppose $\hat{W}$ is a Coxeter group of type D_n, and let $\hat{C}$ be a conjugacy class of $\hat{W}$. Set $C = \hat{C}w_0$, where w_0 is the longest element of W, and assume that $\lambda(C) = (\lambda_1, \ldots, \lambda_\nu; \lambda_{\nu+1}, \ldots, \lambda_z)$. Then the maximal length in $\hat{C}$ is*

$$n^2 + z - 2\sum_{i=1}^{\nu}(\nu - i)\lambda_i.$$

Theorem 1.1 is a consequence of a more general result, Theorem 2.5, concerning D-lengths and B-lengths of elements in W (D-length and B-length will be defined in Section 2). In Theorem 1.1, in the case when n is odd, $w_0 \neq \hat{w}_0$ (the longest element of $\hat{W}$) and consequently C is not even a subset of $\hat{W}$, much less a conjugacy class of $\hat{W}$. However, working in the wider context of W, we are able to obtain elements of maximal D-length in $\hat{C}$ from suitable elements of minimal B-length in C. Therefore, in the course of establishing Theorem 1.1, we also produce representative elements of maximal length in their conjugacy class. Representatives of minimal length in types B_n and D_n appear in Theorems 3.4.7 and 3.4.12 of [6]. However we need additional information about elements of minimal length in W conjugacy classes, which gives as a byproduct (in Corollaries 2.3 and 2.4) an alternative proof that the representatives given in [6] are indeed of minimal length.

The maximal lengths in conjugacy classes of the Coxeter group of type E_6, together with additional information, is tabulated in Section 3. These were calculated with the aid of Magma [1]. In the case of conjugacy classes of involutions, a rule to determine the minimal and maximal length in the class is given in [10], and a complete description of the set of elements of maximal and minimal length appears in [9].

2 B-length and D-length

Throughout this section, W is assumed to be a Coxeter group of type B_n containing $\hat{W}$, the canonical index 2 subgroup of type D_n. We will view elements of W as signed cycles. Given $w \in W$, let $\overline{w}$ be the corresponding element of S_n. So, for example, if $w = (\overset{-}{1}\overset{+}{3}\overset{+}{8})$, then $\overline{w} = (138)$. Thus, for any $w \in W$,

$$w = \overline{w}\left(\prod\nolimits_{e_i \in \Sigma(w)}(\overset{-}{i})\right).$$

Hence, in our above example, $(\overset{-}{1}\overset{+}{3}\overset{+}{8}) = (138)(\overset{-}{1})$.

Let Φ be the root system of W. We employ the usual description of Φ (as given, for example in [7]). So the positive long roots are $\Phi^+_{\text{long}} = \{e_i \pm e_j : 1 \leq i < j \leq n\}$, the negative long roots are $\Phi^-_{\text{long}} = -\Phi_{\text{long}}$ and $\Phi_{\text{long}} = \Phi^+_{\text{long}} \cup \Phi^-_{\text{long}}$. The short roots are $\Phi^+_{\text{short}} = \{e_i : 1 \leq i \leq n\}$, $\Phi^-_{\text{short}} = -\Phi^+_{\text{short}}$ and $\Phi_{\text{short}} = \Phi^+_{\text{short}} \cup \Phi^-_{\text{short}}$. While the positive roots are $\Phi^+ = \Phi^+_{\text{long}} \cup \Phi^+_{\text{short}}$, the negative roots are $\Phi^- = \Phi^-_{\text{long}} \cup \Phi^-_{\text{short}}$ and $\Phi = \Phi^+ \cup \Phi^-$. We note that the positive roots for $\hat{W}$ are Φ^+_{long}. Our convention will be that the action of a group element is on the left of the root, so that for example $(\overset{-}{1}\overset{+}{3}\overset{+}{8}) \cdot e_1 = (138)(\overset{-}{1}) \cdot e_1 = -e_3$.

For $w \in W$, we define the following two sets.

$$\Lambda(w) = \{\alpha \in \Phi^+_{\text{long}} : w \cdot \alpha \in \Phi^-\};$$

$$\Sigma(w) = \{\alpha \in \Phi^+_{\text{short}} : w \cdot \alpha \in \Phi^-\}.$$

Set $l_B(w) = |\Lambda(w)| + |\Sigma(w)|$ and $l_D(w) = |\Lambda(w)|$. By [7] $l_B(w)$ is the length of w and, should $w \in \hat{W}$, then $l_D(w)$ is the length of w viewed as an element of $\hat{W}$. We call $l_B(w)$ the B-length of w and $l_D(w)$ the D-length of w.

Letting C be a conjugacy class of W, we write ν_C for the number of negative cycles in the signed cycle type of C, and z_C for the total number of cycles in the signed cycle type of C (including 1-cycles). As noted earlier, conjugacy classes are parameterized by signed cycle type. Given a conjugacy class C, the signed cycle type of (elements of) C can be written as a partition $\lambda(C)$ where

$$\lambda(C) = (\lambda_1, \ldots, \lambda_{\nu_C}; \lambda_{\nu_C+1}, \ldots, \lambda_{z_C}).$$

Here, $\lambda_1 \leq \ldots \leq \lambda_{\nu_C}$ are the lengths of the negative cycles of C and $\lambda_{\nu_C+1} \leq \cdots \leq \lambda_{z_C}$ are the lengths of the positive cycles of C.

In $\hat{W}$, conjugacy classes are also parameterized by signed cycle type, with the exception that there are two classes for each signed cycle type consisting only of even length, positive cycles. The length profiles in each pair of split classes are identical.

Lemma 2.1 *Let C be a conjugacy class of W, and $w \in C$. Set $\nu = \nu_C$ and $z = z_C$. Suppose that $\lambda(C) = (\lambda_1, \ldots, \lambda_\nu; \lambda_{\nu+1}, \ldots, \lambda_z)$. Then $|\Lambda(w)| \geq n - z_C + 2\sum_{i=1}^{\nu}(\nu - i)\lambda_i$. Moreover $|\Sigma(w)| \geq \nu$.*

Proof Write w as a product of disjoint cycles, $w = \sigma_1\sigma_2\cdots\sigma_z$, where $\sigma_1, \ldots, \sigma_\nu$ are negative cycles and the remaining cycles are positive. Also, order the negative cycles such that $i < j$ if and only if the minimal element in (the support of) σ_i is smaller than the minimal element in (the support of) σ_j. Our approach is to consider certain $\langle w \rangle$-orbits of roots.

Firstly, let σ be a positive k-cycle of w and consider the orbits consisting of roots of the form $e_a - e_b$, for $a, b \in \sigma$ and $a \neq b$. Each such orbit has length k. There are $2\binom{k}{2}$ roots of this form, and hence $k - 1$ such orbits. Let c be the maximal element in σ. Then each orbit contains both $e_a - e_c$ and $e_c - e_b$ for some $a, b \in \sigma$. Now $e_a - a_c \in \Phi^+$ and $e_c - e_b \in \Phi^-$. Therefore each orbit includes a transition from positive to negative (that is, a positive root α for which $w \cdot \alpha$ is negative). Therefore each orbit contributes at least one root to $\Lambda(w)$. Therefore each positive k-cycle contributes at least $k-1$ roots to $\Lambda(w)$.

Next suppose σ is a negative k-cycle of w. This time we consider orbits consisting of roots of the form $\pm e_a \pm e_b$, for $a, b \in \sigma$ and $a \neq b$. Each such orbit has length $2k$. There are $4\binom{k}{2}$ roots of this form, and hence $k-1$ such orbits. Moreover if α lies in one of these orbits, then $-\alpha$ lies in the same orbit. Thus again each orbit includes a transition from positive to negative and hence contributes at least one root to $\Lambda(w)$. Therefore each negative k-cycle contributes at least $k-1$ roots to $\Lambda(w)$.

Now suppose σ_i and σ_j are negative cycles, with $i < j$, and consider the union of all orbits consisting of roots of the form $\pm e_a \pm e_b$, where $a \in \sigma_i$ and $b \in \sigma_j$. Suppose $|\sigma_i| = k$ and $|\sigma_j| = l$. Let c be minimal in σ_i. Then every orbit contains some $\pm e_c \pm e_b$ for some $b \in \sigma_j$. For every root of the form $e_c \pm e_b$, we have $w^k(e_c \pm e_b) = -e_c \pm e_{b'}$ and $w^{2k}(e_c \pm e_{b'}) = e_c \pm e_{b''}$ for some $b', b'' \in \sigma_j$. Now $e_c \pm e_b$ and $e_c \pm e_{b''}$ are positive roots, but $-e_c \pm e_{b'}$ is negative. Therefore in this orbit or part of orbit there is at least one transition from positive to negative. There are $2l$ roots of the form $e_c \pm e_b$, and hence each pair σ_i, σ_j of negative cycles with $i < j$ contributes at least $2|\sigma_j|$ roots to $\Lambda(w)$. For example, letting i range from 1 to $\nu - 1$, we get a total of $(\nu - 1) \times 2|\sigma_\nu|$ roots from pairs σ_i and σ_ν.

Combining these three observations and writing k_i for $|\sigma_i|$, we see that

$$\Lambda(w) \geq \sum_{i=1}^{z} k_i + 2\sum_{i=1}^{\nu}(\nu - i)k_i.$$

Since $\{k_1, \ldots, k_\nu\} = \{\lambda_1, \ldots, \lambda_\nu\}$, and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_\nu$, it is clear that

$$2\sum_{i=1}^{\nu}(\nu - i)k_i \geq 2\sum_{i=1}^{\nu}(\nu - i)\lambda_i.$$

Therefore

$$|\Lambda(w)| \geq n - z_C + 2\sum_{i=1}^{\nu}(\nu - i)\lambda_i.$$

It only remains to show that $|\Sigma(w)| \geq \nu$. This trivially follows from the fact that there are ν negative cycles and each negative cycle must contain at least one minus sign. Therefore there are at least ν roots e_a for which $w(e_a) \in \Phi^-$. Thus $|\Sigma(w)| \geq \nu$ and the proof of the lemma is complete. □

Next, given a conjugacy class C of W we define a particular element $u = u_C$ of C (which will turn out to have minimal B-length). Let the signed cycle type of C be

$$\lambda(C) = (\lambda_1, \lambda_2, \ldots, \lambda_{\nu_C}; \lambda_{\nu_C+1}, \ldots, \lambda_{z_C})$$

as before, and write $\mu_i = n - \sum_{j=1}^{i} \lambda_j$ for $1 \leq i \leq z_C$. Then define $u = u_C$

to be the following element of C.

$$\begin{aligned} u_C &= (\overset{-}{n}\ \overset{+}{n-1}\cdots \overset{+}{\mu_1+1})(\overset{-}{\mu_1}\ \overset{+}{\mu_1-1}\cdots \overset{+}{\mu_2+1})\cdots(\overset{-}{\mu_{\nu_C-1}}\ \overset{+}{\mu_{\nu_C-1}-1}\cdots \overset{+}{\mu_{\nu_C}+1})\cdot \\ &\qquad \cdot(\overset{+}{\mu_{\nu_C}}\ \overset{+}{\mu_{\nu_C}-1}\cdots \overset{+}{\mu_{\nu_C+1}+1})\cdots(\overset{+}{\mu_{z_C-1}}\ \overset{+}{\mu_{z_C-1}-1}\cdots \overset{+}{1}). \end{aligned}$$

As an example, let $w = (\overset{-+--}{1729})(\overset{-+-}{346})(\overset{+-}{58})$ and let C be the conjugacy class of w in type B_9. Then $\lambda_C = (2,4;3)$, $\nu_C = 2$, $\mu_1 = 7$, $\mu_2 = 3$ and $\mu_3 = 9$. This gives $u_C = (\overset{-+}{98})(\overset{-+++}{7654})(\overset{+++}{321})$.

Lemma 2.2 *Suppose $w = u_C$ for some conjugacy class C of W. Then $|\Sigma(w)| = \nu_C$ and $|\Lambda(w)| = n - z_C + 2\sum_{i=1}^{\nu_C}(\nu_C - i)\lambda_i$*

Proof Set $z = z_C$ and $\nu = \nu_C$. The size of $\Sigma(w)$ is simply the number of minus signs appearing in the expression for w. Here, $\Sigma(w) = \{e_n, e_{\mu_1}, \ldots, e_{\mu_{\nu-1}}\}$ and $|\Sigma(w)| = \nu$.

To find $\Lambda(w)$, consider pairs (i,j) with $1 \leq i < j \leq n$. Suppose first that i and j are in the same cycle of $\overline{w}$. Then $e_i \notin \Sigma(w)$ because only the maximal element of each negative cycle has a minus sign above it. If $j = \mu_k$ for some k, or if $j = n$, then exactly one of $e_i + e_j \in \Lambda(w)$ or $e_i - e_j \in \Lambda(w)$ occurs (depending if $k \leq \nu$). Otherwise, $e_i - e_j \notin \Lambda(w)$ and $e_i + e_j \notin \Lambda(w)$. Hence a cycle $(\overset{\pm}{\mu_k}\ \overset{+}{\mu_k - 1}\cdots \overset{+}{\mu_{k+1}+1})$ contributes exactly $\lambda_{k+1} - 1$ roots to $\Lambda(w)$.

Now suppose that i and j are in different cycles. Hence $\overline{w}(i) < \overline{w}(j)$. It is a simple matter to check that if $e_i \in \Sigma(w)$, then $\{e_i + e_j, e_i - e_j\} \subseteq \Lambda(w)$, whereas if $e_i \notin \Sigma(w)$, then $e_i - e_j$ and $e_i + e_j$ are not in $\Sigma(w)$. Therefore each i with $e_i \in \Sigma(w)$ contributes exactly $2(n-i)$ additional roots to $\Lambda(w)$, and no roots are contributed when $e_i \notin \Sigma(w)$.

Therefore

$$\begin{aligned} |\Lambda(w)| &= \sum_{k=1}^{z}(\lambda_{k+1} - 1) + \sum_{k:e_k\in\Sigma(w)} 2(n-k) \\ &= (n-z) + 2\left((n-n) + (n-\mu_1) + (n-\mu_2) + \cdots + (n-\mu_{\nu-1})\right) \\ &= (n-z) + 2\sum_{i=1}^{\nu-1}\sum_{j=1}^{i}(\lambda_j) \\ &= n - z + 2\sum_{i=1}^{\nu-1}(\nu - i)\lambda_i \\ &= n - z_C + 2\sum_{i=1}^{\nu_C}(\nu_C - i)\lambda_i. \end{aligned}$$

Therefore $|\Lambda(w)| = n - z_C + 2\sum_{i=1}^{\nu_C}(\nu_C - i)\lambda_i$ and $|\Sigma(w)| = \nu_C$. □

Corollary 2.3 *Let C be a conjugacy class of W. Then the minimal B-length in C is $n + \nu_C - z_C + 2\sum_{i=1}^{\nu_C}(\nu_C - i)\lambda_i$. If $w \in C$ has minimal B-length, then $|\Lambda(w)| = n - z_C + 2\sum_{i=1}^{\nu_C}(\nu_C - i)\lambda_i$ and $|\Sigma(w)| = \nu_C$. Moreover u_C is a representative of minimal B-length in C.*

In the next corollary the element u_C^t is the element obtained from u_C by taking its shortest positive cycle (which in this context will be the cycle $(n\ n-1\ \ldots m)$ for some odd m), and putting minus signs over n and $n-1$. In other words it is the conjugate of u_C by $t = (\bar{n})$.

Corollary 2.4 *Let C be a conjugacy class of W. If C is also a conjugacy class, or a union of conjugacy classes, of $\hat{W}$, then the minimal D-length of elements in the class(es) is $n - z_C + 2\sum_{i=1}^{\nu_C}(\nu_C - i)\lambda_i$. Moreover u_C and u_C^t are representatives of minimal D-length in the class(es), with one in each $\hat{W}$-class if the class C splits.*

Theorem 2.5 *Let C be a conjugacy class of W and $w \in C$. Let $\hat{C}$ be the conjugacy class of ww_0 where w_0 is the longest element of W. Then the maximal B-length of elements of $\hat{C}$ is $n^2 - |\Lambda(u_{\hat{C}})| - |\Sigma(u_{\hat{C}})|$, and if $\hat{C}$ is a conjugacy class or union of conjugacy classes of $\hat{W}$, the maximal D-length of elements of $\hat{C}$ is $n^2 - n - |\Lambda(u_{\hat{C}})|$.*

Proof Let C be a conjugacy class of W. Since w_0 is central, the conjugacy class $\hat{C}$ of ww_0 is just Cw_0. Moreover, for any root α we have $w_0(\alpha) = -\alpha$. Therefore for all $x \in W$, $|\Lambda(xw_0)| = (n^2 - n) - |\Lambda(x)|$ and $|\Sigma(xw_0)| = n - |\Sigma(x)|$. (Note that there are $n^2 - n$ long positive roots and n short positive roots.) Let $u = u_{\hat{C}}$. Then by Lemma 2.1 and Lemma 2.2, we have that for all $v \in \hat{C}$, $|\Lambda(v)| \geq |\Lambda(u)|$ and $|\Sigma(v)| \geq |\Sigma(u)|$. Now every $x \in C$ is of the form xw_0 for some $v \in \hat{C}$. Hence for every $x \in C$, we have:

$$\begin{aligned} |\Lambda(x)| &\leq n^2 - n - |\Lambda(u)| \quad \text{and} \\ |\Sigma(v)| &\leq n - |\Sigma(u)| \end{aligned}$$

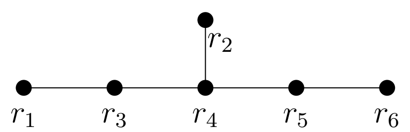
Also $|\Lambda(uw_0)| = n^2 - n - |\Lambda(u)|$ and $|\Sigma(uw_0)| = n - |\Sigma(u)|$. Therefore the maximal B-length in C is $n^2 - n - |\Lambda(u)| + n - |\Sigma(u)| = n^2 - |\Lambda(u)| - |\Sigma(u)|$ and this is attained by the element uw_0. Moreover, if C is a conjugacy class (or union of conjugacy classes) of $\hat{W}$, then the maximal D-length is $n^2 - n - |\Lambda(u)|$ and this is attained by uw_0 (or $(uw_0)^t$ if the class splits). □

Theorem 1.1 follows immediately from Theorem 2.5 and Lemma 2.2.

3 Maximal lengths in type E_6

Table 1 details information about maximal length elements in conjugacy classes of the Coxeter group of type E_6. Each class is labelled using the system in Carter [2]. In the final column, to save space, rather than write a representative w of $X_{\max}$, we have written $w_0 w$, where, as usual, w_0 is the longest element. The labelling of the E_6 diagram is shown below and we shorten each generator r_i to i.

E_6



Class X	$\|X\|$	$\|X_{\max}\|$	$l(w)$	$w_0 w$ $(w \in X_{\max})$
$\emptyset$	1	1	0	w_0
A_1	36	1	21	131431543165431
A_1^2	270	1	30	343543
A_2	240	20	20	3431542314565431
A_1^3	540	2	35	4
$A_2 \times A_1$	1440	8	29	3543654
A_3	1620	6	29	4234543
A_1^4	45	1	36	$\mathrm{Id}(W)$
$A_2 \times A_1^2$	2160	4	34	45
A_2^2	480	146	22	13143542654234
$A_3 \times A_1$	3240	6	34	25
A_4	5184	40	28	23435431
D_4	1440	2	34	24
$D_4(a_1)$	540	120	18	234315465423143546
$A_2^2 \times A_1$	1440	22	31	34356
$A_3 \times A_1^2$	540	4	35	3
$A_4 \times A_1$	5184	12	33	346
A_5	4320	6	33	123
D_5	6480	8	33	542
$D_5(a_1)$	4320	4	33	246
A_2^3	80	80	24	314231565423
$A_5 \times A_1$	1440	6	34	15
E_6	4320	56	36	423143
$E_6(a_1)$	5760	24	32	1234
$E_6(a_2)$	720	180	20	1425423165423145

Table 1: Maximal Length Elements in Classes of E_6

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