Involution Statistics in Finite Coxeter Groups

By

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Birkbeck Pure Mathematics Preprint Series | Preprint Number 4

www.bbk.ac.uk/ems/research/pure/preprints

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June 2014

Abstract

Let W be a finite Coxeter group and X a subset of W. The length polynomial $L_{W,X}(t)$ is defined by $L_{W,X}(t) = \sum_{x \in X} t^{\ell(x)}$, where ℓ is the length function on W. In this article we derive expressions for the length polynomial where X is any conjugacy class of involutions, or the set of all involutions, in any finite Coxeter group W. In particular, these results correct errors in [6] for the involution length polynomials of Coxeter groups of type B_n and D_n. Moreover, we give a counterexample to a unimodality conjecture stated in [6].
Mathematics Subject Classification: 20F55, 05E15
Keywords: Coxeter group, permutation statistics, inversions

1 Introduction

The purpose of this article is to derive statistics about the distribution of lengths of involutions in Coxeter groups. The *length polynomial*, $L_{W,X}(t)$, where W is a finite Coxeter group and $X \subseteq W$, is the principal object of study here. It is defined by

$$L_{W,X}(t) = \sum_{x \in X} t^{\ell(x)},$$

where ℓ is the length function on W. If $X = W$, this is the well-known Poincaré polynomial (see 1.11 of [7]). For the special case where $X = \{x \in W : x^2 = 1\}$ is the set of involutions in W together with the identity element, we write $L_W(t)$ for the polynomial

$$L_W(t) = L_{W,X}(t) = \sum_{x \in X} t^{\ell(x)}.$$

We refer to $L_W(t)$ as the *involution length polynomial* of W.

In this article we obtain expressions for the length polynomials of all conjugacy classes of involutions in finite Coxeter groups (and hence for the sets of all involutions in these groups). For type A this is known [5], but we could only find statements, not proofs, in the literature, so we have included a proof here. In [6] expressions for $L_{W(B_n)}(t)$ and $L_{W(D_n)}(t)$ are given, but unfortunately the proofs contain errors which lead to the results being incorrect. Finally we remark that the length polynomial for the special case where X is the set of reflections in a Coxeter group W has been studied in another guise [4].

*The authors wish to acknowledge support from a London Mathematical Society 'Research in Pairs' grant and from the Department of Economics, Mathematics and Statistics at Birkbeck.

Theorem 4.1 of [4] gives the generating function for counting the depth of roots of a Coxeter system of finite rank. The number of reflections of length ℓ in a Coxeter group equals the number of positive roots of depth $\frac{\ell+1}{2}$, and so the polynomial $L_{W,X}(t)$, where X is the set of reflections, may be obtained.

There are two main motivations for this work. The first is that there are known results counting involutions in the symmetric group that have a given number of inversions. Of course the symmetric group is a Coxeter group of type A, and it is well known that the number of inversions of an element is equal to its length, in the Coxeter group context. Thus it is natural to wonder about generalisations of results on inversion numbers in symmetric groups to results relating to lengths in Coxeter groups.

The second motivation is that Coxeter groups have a very special relationship with involutions. They are generated by involutions (known as fundamental reflections). The set of reflections of a Coxeter group (conjugates of fundamental reflections) is in one-to-one correspondence with its set of positive roots. For a Coxeter group of rank n, every involution can be expressed as a product of at most n orthogonal reflections. Every element of a finite Coxeter group can be expressed as a product of at most two involutions [2]. Moreover, the conjugacy classes of involutions have a particularly nice structure. Due to a result of Richardson [9], it is straightforward to determine them from the Coxeter graph. The involutions of minimal and maximal length in a conjugacy class are well understood [8], and the lengths of involutions behave well with respect to conjugation, in that if x is an involution and r is a fundamental reflection, then either $\ell(rxr) = \ell(x) + 2$, or $\ell(rxr) = \ell(x) - 2$, or $rxr = x$. The involutions of minimal length in a conjugacy class are central elements of parabolic subgroups, and so are fairly easily counted. So again it is natural to ask what can be determined about the length distribution of involutions, both in a conjugacy class and in a Coxeter group as a whole.

We may now state our results. Let n, m and ℓ be integers. We define $\alpha_{n,m,\ell}$ to be the number of involutions in $W(A_{n-1}) \cong \mathrm{Sym}(n)$ of length ℓ, having m transpositions. We adopt the convention that the identity element is an involution, so that we have $\alpha_{n,0,0} = 1$. Note that $\alpha_{n,0,\ell} = 0$ for all $\ell \neq 0$ and $\alpha_{n,m,\ell} = 0$ for any $\ell < 0$.

We define the following polynomial

$$L_{n,m}(t) = \sum_{\ell=0}^{\infty} \alpha_{n,m,\ell} t^{\ell}.$$

This is essentially a shorthand for $L_{W(A_{n-1}),X}(t)$ where X is the set of involutions (including the identity) whose expression as a product of disjoint cycles contains precisely m transpositions.

Theorem 1.1 *Let $n \geq 3$ and $m \geq 1$. If $\ell < m$ then $\alpha_{n,m,\ell} = 0$. If $\ell \geq m$ then*

$$\alpha_{n,m,\ell} = \alpha_{n-1,m,\ell} + \sum_{k=1}^{n-1} \alpha_{n-2,m-1,\ell+1-2k}.$$

Moreover $L_{n,0}(t) = 1$, $L_{1,1}(t) = 0$, $L_{2,1}(t) = t$, $L_{n,m}(t) = 0$ for $n < 2m$, and for all other *positive integers* n, m,

$$L_{n,m}(t) = L_{n-1,m}(t) + \frac{t(t^{2n-2}-1)}{t^2-1} L_{n-2,m-1}(t).$$

For example, to find the number of double transpositions of length 6 in $W(A_4) \cong \mathrm{Sym}(5)$, we calculate

$$\begin{aligned}\alpha_{5,2,6} &= \alpha_{4,2,6} + [\alpha_{3,1,5} + \alpha_{3,1,3} + \alpha_{3,1,1} + \alpha_{3,1,-1}] \\ &= 1 + [0+1+2+0] = 4.\end{aligned}$$

Note that [5], for example, defines a related polynomial $I_n(x,q)$ in two variables, x and q, where the coefficient of $q^j x^k$ is the number of involutions in Sym_n with k fixed points and j inversions (that is, length j). He states "on vérifie aisément que ces polynômes satisfont la récurrence $I_0(x,q) = 1$, $I_1(x,q) = x$, $I_{n+1}(x,q) = xI_n(x,q) + \frac{1-q^{2n}}{1-q^2} qI_{n-1}(x,q)$, $n \geq 1$". Of course one may derive Theorem 1.1 from this statement, but we felt it may be helpful to include a direct proof here, especially as our result for $W(A_{n-1}) \cong \mathrm{Sym}(n)$ is arrived at as a subcase of a general argument that also covers types B_n and D_n.

Our results for types B_n and D_n are as follows. (A detailed description of these groups as groups of permutations will be given in Section 2.) Let $\beta_{n,m,e,\ell}$ be the number of involutions of length ℓ in $W(B_n)$ whose expression as a product of disjoint signed cycles contains m transpositions and e negative 1-cycles. Also let $\delta_{n,m,e,\ell}$ be the number of involutions of length ℓ in $W(D_n)$ whose expression as a product of disjoint signed cycles contains m transpositions and e negative 1-cycles. Note that $\beta_{n,0,0,0} = \delta_{n,0,0,0} = 1$, as we include the identity element in our count. Let

$$L_{n,m,e}(t) = \begin{cases} \sum_{\ell=0}^{\infty} \beta_{n,m,e,l} t^{\ell} & \text{when } W = W(B_n); \\ \sum_{\ell=0}^{\infty} \delta_{n,m,e,\ell} t^{\ell} & \text{when } W = W(D_n). \end{cases}$$

Again, $L_{n,m,e}(t)$ is another way of writing $L_{W,X}(t)$ where X is the appropriate involution class of W, for W of type B_n or D_n.

It is not common to work with the groups of type B_1, D_1, D_2 and D_3, because they are isomorphic to certain groups of type A (or in the case of D_2, to $A_1 \times A_1$), but there is a natural definition in line with the usual definitions (see Section 2) of $W(B_n)$ and $W(D_n)$ (for example $W(D_3)$ is the group of positive elements of $W(B_3)$). Therefore, for the purposes of recursion, we will work with $L_{n,m,e}(t)$ for W of types B_1, D_1, D_2, and D_3. We have the following two theorems.

Theorem 1.2 *Suppose* $W = W(B_n)$, *and let* $n \geq 3$, $m \geq 0$ *and* $e \geq 0$. *If* $\ell < m+e$ *then* $\beta_{n,m,e,\ell} = 0$. *If* $\ell \geq m+e$ *then*

$$\beta_{n,m,e,\ell} = \beta_{n-1,m,e,\ell} + \beta_{n-1,m,e-1,\ell+1-2n} + \sum_{k=1}^{2n-2} \beta_{n-2,m-1,e,\ell+1-2k}.$$

Moreover $L_{n,0,0}(t) = 1$ *for all positive integers* n, $L_{1,0,1}(t) = t$, $L_{2,1,0}(t) = t + t^3$, $L_{2,0,1}(t) = t + t^3$, $L_{2,0,2}(t) = t^4$, $L_{n,m,e}(t) = 0$ *whenever* $n < 2m+e$, *and for all* $n \geq 3$ *and* $m, e \geq 0$,

$$L_{n,m,e}(t) = L_{n-1,m,e}(t) + t^{2n-1} L_{n-1,m,e-1}(t) + \frac{t(t^{4n-4}-1)}{t^2-1} L_{n-2,m-1,e}(t).$$

Before we can state the next theorem we define a further polynomial $D_{n,m,e}(t)$, where n, m and e are non-negative integers. Set $D_{n,0,0}(t) = 1$ and if $2m + e > n$, define $D_{n,m,e}(t) = 0$. Set $D_{1,0,1}(t) = t$, $D_{2,1,0}(t) = 2t$, $D_{2,0,1}(t) = 1 + t^2$, $D_{2,0,2}(t) = t^2$, and for all $n \geq 3$,

$$D_{n,m,e}(t) = D_{n-1,m,e}(t) + t^{2n-2}D_{n-1,m,e-1}(t) + \frac{t(1+t^{2n-4})(t^{2n-2}-1)}{t^2-1}D_{n-2,m-1,e}(t).$$

Theorem 1.3 *Suppose $W = W(D_n)$ and let ℓ be a non-negative integer. Then $\delta_{n,m,e,\ell}$ is the coefficient of t^ℓ in $D_{n,m,e}(t)$ when e is even, and zero otherwise. Furthermore $L_{n,m,e}(t) = D_{n,m,e}(t)$ when e is even and $L_{n,m,e,}(t) = 0$ when e is odd.*

By summing the involutions of various types of a given length, we can obtain the polynomials $f_W(t)$ for W of types A, B and D. This result is Corollary 4.1. We postpone the statement and proof until Section 4 because the D_n case requires extra notation.

This article is structured as follows: in Section 2 we give the basic facts about presentations and root systems of the classical Weyl groups, and describe the relationship between roots and length. In Section 3 we prove the general results for classical Weyl groups, which are then applied in Section 4 to each of types A, B and D in turn. In Section 5 we give the length polynomials for involution conjugacy classes in the exceptional finite Coxeter groups. Finally in Section 6 we discuss briefly some conjectures about unimodality of the sequences of coefficients of the polynomials $L_{W,X}(t)$, where X is either a conjugacy class of involutions, or the set of involutions of even length, or the set of involutions of odd length, in a finite Coxeter group W.

2 Root Systems and Length

Suppose W is a Coxeter group, and R its set of fundamental reflections. Then $W = \langle R \rangle$. For any $w \in W$, the *length* of w, denoted $\ell(w)$ is the length of any shortest expression for w as a product of fundamental reflections. That is

$$\ell(w) = \min\{k \in \mathbb{Z} : w = r_1 r_2 \cdots r_k, \text{ some } r_1, \ldots, r_k \in R\}.$$

By convention the identity element has length zero. If the choice of Coxeter group W is not clear from context, we will write either ℓ_W or sometimes ℓ_Y if W is of type Y. The length function has been extensively studied, not least because of its link to the root system, which we will now describe.

To every Coxeter group W we may assign a root system Φ, a set of positive roots Φ^+ and a set of negative roots $\Phi^- := -\Phi^+$, such that Φ is the disjoint union of the positive and negative roots. Frequently we shall write $\rho > 0$ to mean $\rho \in \Phi^+$ and $\rho < 0$ to mean $\rho \in \Phi^-$. The Coxeter group W acts faithfully on its root system. For $w \in W$, define

$$N(w) = \{\alpha \in \Phi^+ : w \cdot \alpha \in \Phi^-\}.$$

One of many connections between Φ and the length function ℓ is the fact that $\ell(w) = |N(w)|$ (see, for example, Section 5.6 of [7]).

Every finite Coxeter group is a direct product of irreducible finite Coxeter groups. The finite irreducible Coxeter groups were classified by Coxeter [3] (see also [7]).

Theorem 2.1 *An irreducible finite Coxeter group is either of type $A_n (n \geq 1)$, $B_n (n \geq 2)$, $D_n (n \geq 4)$, E_6, E_7, E_8, F_4, H_3, H_4 or I_n $(n \geq 5)$.*

The next result (the proof of which is straightforward) shows that for questions about conjugacy classes of involutions, or the set of all involutions, or indeed the set of involutions of odd or even length, in a finite Coxeter group, it is sufficient to analyse the irreducible cases only.

Lemma 2.2 *Let W be a finite Coxeter group, and X a union of involution conjugacy classes of W. Suppose that W is isomorphic to the direct product $W_1 \times W_2 \times \cdots \times W_k$, where each W_i is an irreducible Coxeter group. Then X is the direct product of sets X_1, X_2, ..., X_k. Each X_i is either $\{1\}$ or is a union of involution conjugacy classes of W_i, and $L_{W,X}(t) = \prod_{i=1}^{k} L_{W_i,X_i}(t)$.*

We next discuss concrete descriptions of the Coxeter groups of types A_n, B_n and D_n which will feature in a number of our proofs. First, $W(A_n)$ may be viewed as being $\mathrm{Sym}(n+1)$. The elements of $W(B_n)$ can be thought of as signed permutations of $\mathrm{Sym}(n)$. For example, the element $w = (\overset{+}{1}\overset{-}{2})$ of $W(B_n)$ is given by $w(1) = 2$, $w(-1) = -2$, $w(2) = -1$ and $w(-2) = 1$. We say a cycle in an element of $W(B_n)$ is of negative sign type if it has an odd number of minus signs, and positive sign type otherwise. An element w expressed as a product $g_1 g_2 \cdots g_k$ of disjoint signed cycles is *positive* if the product of all the sign types of the cycles is positive, and negative otherwise. The group $W(D_n)$ consists of all positive elements of $W(B_n)$, while the group $W(A_{n-1})$ consists of all elements of $W(B_n)$ whose cycles contain only plus signs. Even if w is positive, it may contain negative cycles, which we wish on occasion to consider separately, so when considering elements of $W(D_n)$ we often work in the environment of $W(B_n)$ to avoid ending up with non-group elements.

Let V be an n-dimensional real vector space with orthonormal basis $\{e_1, \ldots, e_n\}$. Then we may take the root system B_n to have positive roots of the form $e_j \pm e_i$ for $1 \leq i < j \leq n$ and e_i for $1 \leq i \leq n$. The positive roots of D_n are of the form $e_j \pm e_i$ for $1 \leq i < j \leq n$. The positive roots of A_{n-1} are of the form $e_j - e_i$ for $1 \leq i < j \leq n$. Therefore the root system D_n consists of the long roots of the root system B_n, and the root system A_{n-1} is contained in the set of long roots of B_n.

From the fact that for an element w of a Coxeter group W we have $\ell(w) = |N(w)|$, we deduce the following.

Theorem 2.3 *Let n be a positive integer greater than 1, and let W be of type A_{n-1}, B_n or D_n. Set*

$$\begin{aligned} \Lambda &= \{e_j - e_i, e_j + e_i : 1 \leq i < j \leq n\} \text{ and} \\ \Sigma &= \{e_i : 1 \leq i \leq n\}. \end{aligned}$$

For $w \in W$, set

$$\begin{aligned} \Lambda(w) &= \{\alpha \in \Lambda : w \cdot \alpha < 0\} \quad \text{and} \\ \Sigma(w) &= \{\alpha \in \Sigma : w \cdot \alpha < 0\}. \end{aligned}$$

Then $\ell_{B_n}(w) = |\Lambda(w)| + |\Sigma(w)|$. *If* w *is an element of* $W(D_n)$, *then* $\ell_{D_n}(w) = |\Lambda(w)|$, *and if* w *is an element of* $W(A_{n-1})$, *then* $\ell_{A_{n-1}}(w) = |\Lambda(w)|$.

Proof Note that $\Lambda(w)$ is the set of long positive roots taken negative by w, and $\Sigma(w)$ is the set of short positive roots taken negative by w. The results for $\ell_{B_n}(w)$ and $\ell_{D_n}(w)$ are now immediate. For the case where $w \in W(A_{n-1})$, note that $w \cdot (e_j + e_i) = e_{w(j)} + e_{w(i)} > 0$ for all i, j, so $|\Lambda(w)| = |\{e_j - e_i : 1 \leq i < j < n, w \cdot (e_j - e_i) < 0\}| = \ell_{A_{n-1}}(w)$. So the result follows. □

In the proof of the next lemma we will require the elementary observation that $-h \cdot \Lambda(h) = \Lambda(h^{-1})$, and hence $|\Lambda(h^{-1})| = |\Lambda(h)|$.

Lemma 2.4 *Let* W *be of type* A_{n-1}, B_n *or* D_n. *Let* $g, h \in W$. *Then*

$$|\Lambda(gh)| = |\Lambda(g)| + |\Lambda(h)| - 2|\Lambda(g) \cap \Lambda(h^{-1})|.$$

Proof Suppose $\alpha \in \Lambda(gh)$. Then either $h \cdot \alpha \in \Lambda(g)$ or $\alpha \in \Lambda(h)$ and $h \cdot \alpha \notin -\Lambda(g)$. That is, $\Lambda(gh)$ is the disjoint union of $h^{-1} \cdot (\Lambda(g) \setminus \Lambda(h^{-1}))$ and $\Lambda(h) \setminus (h^{-1} \cdot (-\Lambda(g)))$. But

$$|h^{-1} \cdot (\Lambda(g) \setminus \Lambda(h^{-1}))| = |\Lambda(g)| - |\Lambda(g) \cap \Lambda(h^{-1})|$$

and

$$\begin{aligned} |\Lambda(h) \setminus (h^{-1} \cdot (-\Lambda(g)))| &= |(-h \cdot \Lambda(h)) \setminus \Lambda(g)| \\ &= |\Lambda(h^{-1}) \setminus \Lambda(g)| \\ &= |\Lambda(h^{-1})| - |\Lambda(h^{-1}) \cap \Lambda(g)| \\ &= |\Lambda(h)| - |\Lambda(g) \cap \Lambda(h^{-1})|. \end{aligned}$$

Therefore $|\Lambda(gh)| = |\Lambda(g)| + |\Lambda(h)| - 2|\Lambda(g) \cap \Lambda(h^{-1})|$, as required. □

Since this paper is concerned with involutions, let us say a few words about conjugacy classes of involutions. It is well known that involutions in $W(A_{n-1}) \cong \mathrm{Sym}(n)$ are parameterised by cycle type. That is, involutions are conjugate if and only if they have the same number of transpositions m. Therefore Theorem 1.1 is giving the length polynomials for individual conjugacy classes of involutions. For type B_n, the situation is similar. It is clear that an element w of $W(B_n)$ is an involution precisely when each of its signed cycles is either a positive transposition, a negative 1-cycle or a positive 1-cycle. It can be shown that conjugacy classes are parameterised by signed cycle type, so that involutions are conjugate if and only if they have the same number, m, of transpositions (all of which must be positive), and the same number, e, of negative 1-cycles. So again, Theorem 1.2 gives the length polynomials for individual conjugacy classes of involutions. The case for type D_n is slightly more involved. Involutions again consist of 1-cycles and positive transpositions, and since we are in type D_n there must be an even number of negative 1-cycles. There is exactly one conjugacy class of involutions in $W(D_n)$ with m transpositions and e negative 1-cycles (with e even), *except* when $n = 2m$. In that case, there are two conjugacy classes. However, there is a length preserving automorphism of the Coxeter graph which interchanges these classes. Therefore if X is either class in $W(D_n)$, we see that $L_{2m,m,0}(t) = 2L_{2m,X}(t)$, and so the information for any conjugacy class of involutions can be retrieved from Theorem 1.3. This means that Theorems 1.1, 1.2 and 1.3 give full information about the length polynomials of conjugacy classes of involutions in the classical Weyl groups. This is the motivation for finding, in Section 4, corresponding information for conjugacy classes of involutions in the remaining finite irreducible Coxeter groups.

Here is a rough outline of the method we will follow in the next two sections. Let W_n be of type A_{n-1}, B_n or D_n, where we are viewing types A_{n-1} and D_n as subgroups of $W(B_n)$ as described above. For an involution x of W, we will define τ to be the cycle containing n in the expression for x as a product of disjoint cycles. Since x is an involution, τ will be either a 1-cycle or a transposition. Then set $y = \tau x$. By calculating the length of y in terms of the lengths of τ and x, we may hope to obtain a recursive formula for our length polynomial. However if τ is a transposition, we cannot retrieve $\ell(x)$ from $\ell(y)$ without knowing y itself, and thus we must perform another step. That step is to 'compress' y by conjugating by $(\overset{+}{n}\ \overset{+}{n-1}\ \cdots \overset{+}{r})$. This results in an involution z whose length depends only on n, r and $\ell(x)$.

3 Results for Classical Weyl groups

Throughout this section, W will be a Coxeter group of type A_{n-1}, B_n or D_n.

Theorem 3.1 *Let x be an involution in W, τ be the cycle containing n in the expression for x as a product of disjoint cycles, and set $y = x\tau$. Then either $\tau = (\overset{+}{n})$, $\tau = (\overset{-}{n})$ or there is some r with $1 \leq r < n$ for which τ equals $(\overset{+}{r}\overset{+}{n})$ or $(\overset{-}{r}\overset{-}{n})$. Moreover, writing*

$$\Delta_r(y) = |\{k \in \{1,\ldots,n\} : |y(k)| < r < k\}| + |\{k \in \{1,\ldots,n\} : y(k) < 0, r < k, r < |y(k)|\}|,$$

we have

$$|\Sigma(x)| = \begin{cases} |\Sigma(y)| & \text{if } \tau = (\overset{+}{n}) \\ |\Sigma(y)| + 1 & \text{if } \tau = (\overset{-}{n}) \\ |\Sigma(y)| & \text{if } \tau = (\overset{+}{r}\overset{+}{n}) \\ |\Sigma(y)| + 2 & \text{if } \tau = (\overset{-}{r}\overset{-}{n}) \end{cases}$$

$$|\Lambda(x)| = \begin{cases} |\Lambda(y)| & \text{if } \tau = (\overset{+}{n}) \\ 2(n-1) + |\Lambda(y)| & \text{if } \tau = (\overset{-}{n}) \\ 2(n-r) - 1 + |\Lambda(y)| - 2\Delta_r(y) & \text{if } \tau = (\overset{+}{r}\overset{+}{n}) \\ 2(n+r) - 5 + |\Lambda(y)| - 2\Delta_r(y) & \text{if } \tau = (\overset{-}{r}\overset{-}{n}) \end{cases}$$

Proof Since for any $w \in W$, $\Sigma(w)$ is just the number of minus signs in the expression for w as a product of disjoint signed cycles, the result for $\Sigma(x)$ is clear. Therefore we will now concentrate on finding $\Lambda(x)$ in terms of $\Lambda(y)$. By Lemma 2.4 we observe that

$$|\Lambda(x)| = |\Lambda(\tau)| + |\Lambda(y)| - 2|\Lambda(y) \cap \Lambda(\tau)|. \tag{1}$$

Now $y(n) = n$ and, if τ is not $(\overset{+}{n})$ or $(\overset{-}{n})$, then $y(r) = r$. Therefore for $1 \leq i < n$ we see that $y \cdot (e_n \pm e_i)$ is either $e_n + e_j$ or $e_n - e_j$ for some $j < n$. Thus $e_n \pm e_i \notin \Lambda(y)$. We now work through the possibilities for τ.

If $\tau = (\overset{+}{n})$, then $x = y$ and there is nothing to prove.

Suppose that $\tau = (\bar{n})$. Then

$$\Lambda(\tau) = \{e_n \pm e_i : 1 \leq i < n\}.$$

Thus $|\Lambda(\tau)| = 2(n-1)$ and $\Lambda(\tau) \cap \Lambda(y) = \emptyset$. Substituting these values into Equation 1 gives

$$|\Lambda(x)| = 2(n-1) + |\Lambda(y)|.$$

We move on to the case when $\tau = (\overset{+}{r}\overset{+}{n})$. Then

$$\Lambda(\tau) = \{e_j - e_r : r < j < n\} \cup \{e_n - e_i : i \leq r < n\}.$$

Thus $|\Lambda(\tau)| = 2(n-r) - 1$. We have already noted that $e_n - e_i \notin \Lambda(y)$. Now $y \cdot (e_j - e_r)$ is $e_{|y(j)|} - e_r$ if $y(j) > 0$ and $-e_{|y(j)|} - e_r$ if $y(j) < 0$. If $|y(j)| < r < j$, then $y \cdot (e_j - e_r) = \pm e_{|y(j)|} - e_r < 0$ and so $e_j - e_r \in \Lambda(y)$. If $r < |y(j)|$, then $e_j - e_r \in \Lambda(y)$ if and only if $r < j$ and $y(j) < 0$. Hence

$$|\Lambda(y) \cap \Lambda(\tau)| = |\{j : |y(j)| < r < j\}| + |\{j : r < j, r < |y(j)|, y(j) < 0\}| = \Delta_r(y).$$

Substitution into Equation 1 now produces the expression

$$|\Lambda(x)| = 2(n-r) - 1 + |\Lambda(y)| - 2\Delta_r(y).$$

Finally we set $\tau = (\bar{r}\bar{n})$. Then

$$\begin{aligned}\Lambda(\tau) = \{e_n \pm e_i : 1 \leq i < r\} &\cup \{e_n - e_i : r < i < n\} \\ &\cup \{e_r \pm e_i : 1 \leq i < r\} \cup \{e_j + e_r : r < j \leq n\}.\end{aligned}$$

Hence $|\Lambda(\tau)| = 2(n+r) - 5$, and for $\Lambda(\tau) \cap \Lambda(y)$ we need only consider roots of the form $e_r \pm e_i$, for $i < r$, and roots $e_j + e_r$ for $r < j$. For $e_r \pm e_i$, we have $y \cdot \{e_r + e_i, e_r - e_i\} = \{e_r + e_{|y(i)|}, e_r - e_{|y(i)|}\}$. Now $e_r + e_{|y(i)|}$ is certainly positive, and $e_r - e_{|y(i)|}$ is negative precisely when $r < |y(i)|$. Hence

$$|\Lambda(y) \cap \{e_r \pm e_i : 1 \leq i < r\}| = |\{i : i < r < |y(i)|\}|.$$

Moreover, we see that $e_j + e_r \in \Lambda(y)$ precisely when $r < j$ (to ensure $e_j + e_r$ is positive), $y(j) < 0$ and $r < |y(j)|$. Therefore

$$|\Lambda(y) \cap \Lambda(\tau)| = |\{i : i < r < |y(i)|\}| + |\{j : r < j, y(j) < 0, r < |y(j)|\}| = \Delta_r(y).$$

A final substitution into Equation 1 gives

$$|\Lambda(x) = 2(n+r) - 5 + |\Lambda(y)| - 2\Delta_r(y),$$

and this completes the proof of Theorem 3.1. □

The next result uses $\Delta_r(y)$ again. The definition is the same as that given in Theorem 3.1, but is included in the statement of Theorem 3.3 for ease of reference. We need one more definition.

Definition 3.2 *For $r < n$, we define c_r to be the cycle $(\overset{+}{n}\ \overset{+}{n-1}\ \cdots\ \overset{+}{r})$.*

In Theorem 3.3 note that by y^{c_r} we mean $c_r y c_r^{-1}$.

Theorem 3.3 *Let y be an involution in W with the property that $y(r) = r$ for some $r < n$. Then $|\Sigma(y^{c_r})| = |\Sigma(y)|$. Moreover, writing*

$$\Delta_r(y) = |\{k \in \{1, \ldots, n\} : |y(k)| < r < k\}| + |\{k \in \{1, \ldots, n\} : y(k) < 0, r < k, r < |y(k)|\}|,$$

we have

$$|\Lambda(y^{c_r})| = |\Lambda(y)| - 2\Delta_r(y).$$

Proof Let y be an involution in W with $y(r) = r$. We will write c instead of c_r for ease of notation. Since c contains no minus signs, clearly $|\Sigma(y)| = |\Sigma(y^c)|$. To derive the result for $\Lambda(y)$, we will consider two subsets V_r and U_r of $\Lambda(y)$, where

$$V_r(y) = \Lambda(y) \cap \{e_n \pm e_r, \ldots, e_{r+1} \pm e_r, e_r \pm e_{r-1}, \ldots, e_r \pm e_1\} \text{ and}$$
$$U_r(y) = \Lambda(y) \setminus V_r.$$

Note that $\Lambda(y)$ is the disjoint union of $U_r(y)$ and $V_r(y)$. We claim that $\Lambda(y^c) = c \cdot U_r(y)$. Firstly, consider $c \cdot U_r(y)$. A root $e_j \pm e_i$ is in $U_r(y)$ if and only if $r \notin \{i, j\}$, $j > i$ and $y \cdot (e_j \pm e_i) < 0$. Now c is order preserving on $\{1, \ldots, r, r+1, \ldots, n\}$, and c^{-1} is order preserving on $\{1, \ldots, n-1\}$. Hence $e_j \pm e_i \in U_r(y)$ if and only if $n \notin \{c(i), c(j)\}$, $c(j) > c(i)$ and $y \cdot (e_j \pm e_i) < 0$. Now $y(r) = r$, so if $r \notin \{i, j\}$, then $r \notin \{|y(i)|, |y(j)|\}$. Hence $e_j \pm e_i \in U_r(y)$ if and only if $n \notin \{c(i), c(j)\}$, $c(j) > c(i)$ and $y^c \cdot (e_{c(j)} \pm e_{c(i)}) = cy \cdot (e_j \pm e_i) < 0$. That is, $e_j \pm e_i \in U_r(y)$ if and only if $n \notin \{c(i), c(j)\}$ and $e_{c(j)} \pm e_{c(i)} \in \Lambda(y^c)$. Observe though that $y^c(n) = n$, and so $\Lambda(y^c)$ contains no elements of the form $e_n \pm e_i$. Therefore the restriction $n \notin \{c(i), c(j)\}$ is redundant for elements of $\Lambda(y^c)$. Therefore $e_j \pm e_i \in U_r(y)$ if and only if $e_{c(j)} \pm e_{c(i)} \in \Lambda(y^c)$. That is, $\Lambda(y^c) = c \cdot U_r(y)$, as claimed.

We have shown so far that $|\Lambda(y)| = |\Lambda(y^c)| + |V_r(y)|$. So it remains to find $|V_r(y)|$. Unfortunately there are eight possibilities. For $r < j$, we must look at the positive roots $e_j - e_r$ and $e_j + e_r$. For $i < r$, we must look at $e_r - e_i$ and $e_r + e_r$. The following tables give the outcome in each case. Firstly, we consider j where $r < j$.

$y(j)$	$\lvert y(j)\rvert$	$y \cdot (e_j - e_r)$	$y \cdot (e_j + e_r)$	# Roots in $V_r(y)$
< 0	$< r$	$-e_{\lvert y(j)\rvert} - e_r$	$-e_{\lvert y(j)\rvert} + e_r$	1
< 0	$> r$	$-e_{\lvert y(j)\rvert} - e_r$	$-e_{\lvert y(j)\rvert} + e_r$	2
> 0	$< r$	$e_{\lvert y(j)\rvert} - e_r$	$e_{\lvert y(j)\rvert} + e_r$	1
> 0	$> r$	$e_{\lvert y(j)\rvert} - e_r$	$e_{\lvert y(j)\rvert} + e_r$	0

Therefore the number of roots in $V_r(y)$ of the form $e_j \pm e_r$ for some $j > r$ is

$$|\{j : |y(j)| < r < j\}| + 2|\{j : r < j, y(j) < 0, r < |y(j)|\}|. \tag{2}$$

Now we consider $i < r$ in the following table.

$y(i)$	$\lvert y(i)\rvert$	$y \cdot (e_r - e_i)$	$y \cdot (e_r + e_i)$	# Roots in $V_r(y)$
< 0	$< r$	$e_r + e_{\lvert y(i)\rvert}$	$e_r - e_{\lvert y(i)\rvert}$	0
< 0	$> r$	$e_r + e_{\lvert y(i)\rvert}$	$e_r - e_{\lvert y(i)\rvert}$	1
> 0	$< r$	$e_r - e_{\lvert y(i)\rvert}$	$e_r + e_{\lvert y(i)\rvert}$	0
> 0	$> r$	$e_r - e_{\lvert y(i)\rvert}$	$e_r + e_{\lvert y(i)\rvert}$	1

Therefore the number of roots in $V_r(y)$ of the form $e_r \pm e_i$ for some $i < r$ is $|\{i : i < r < |y(i)|\}|$. Writing $k = |y(i)|$, we note that $\{i : i < r < |y(i)|\} = \{k : |y(k)| < r < k\}$. Therefore the number of roots in $V_r(y)$ of the form $e_r \pm e_i$ for some $i < r$ is

$$|\{k : |y(k)| < r < k\}|. \tag{3}$$

Combining (2) and (3) we get that

$$|V_r(y)| = 2|\{k \in \{1, \ldots, n\} : |y(k)| < r < k\}| + 2|\{k \in \{1, \ldots, n\} : y(k) < 0, r < k, r < |y(k)|\}|,$$

which is just $2\Delta_r(y)$. Recalling that $|\Lambda(y)| = |U_r(y)| + |V_r(y)|$ gives $|\Lambda(y)| = |\Lambda(y^c)| + 2\Delta_r(y)$, and the proof is complete. □

Corollary 3.4 *Let x be an involution in W such that $\tau = (\overset{+}{r}\overset{+}{n})$ is the cycle containing n in the expression of x as a product of disjoint signed cycles. Let $z = (x\tau)^{c_r} = c_r(x\tau)c_r^{-1}$. Then $|\Sigma(x)| = |\Sigma(z)|$ and $|\Lambda(x)| = |\Lambda(z)| + 2(n - r) - 1$.*

Proof Let $y = x\tau$. Then by Theorem 3.1, $|\Sigma(y)| = |\Sigma(x)|$ and $\Lambda(x) = 2(n - r) - 1 + |\Lambda(y)| - 2\Delta_r(y)$. By Theorem 3.3, $|\Sigma(z)| = |\Sigma(y)|$ and $|\Lambda(z)| = |\Lambda(y)| - 2\Delta_r(y)$. The result follows immediately. □

An almost identical argument to the proof of Corollary 3.4 gives Corollary 3.5.

Corollary 3.5 *Let x be an involution in W such that $\tau = (\overset{-}{r}\overset{-}{n})$ is the cycle containing n in the expression of x as a product of disjoint signed cycles. Let $z = (x\tau)^{c_r} = c_r(x\tau)c_r^{-1}$. Then $|\Sigma(x)| = |\Sigma(z)| + 2$ and $|\Lambda(x)| = |\Lambda(z)| + 2(n + r) - 5$.*

4 Proof of the Main Theorems

Proof of Theorem 1.1 Let $X_{A_{n-1},m,\ell}$ be the set of involutions in $W(A_{n-1}) \cong \mathrm{Sym}(n)$ with m transpositions and length ℓ. Let $X_{A_{n-1},m,\ell,r}$ be the set of involutions x in $W(A_{n-1})$ with m transpositions and length ℓ such that $x(n) = r$. Then $X_{A_{n-1},m,\ell} = \cup_{r=1}^{n} X_{A_{n-1},m,\ell,r}$. Clearly $X_{A_{n-1},m,\ell,n} = X_{A_{n-2},m,\ell}$. If $r < n$, then the cycle containing n is $(\overset{+}{r}\overset{+}{n})$. Therefore, by Theorem 2.3 and Corollary 3.4, the map $x \mapsto (x(\overset{+}{r}\overset{+}{n}))^{c_r}$ is a bijection between $X_{A_{n-1},m,\ell,r}$ and $X_{A_{n-3},m-1,\ell-2(n-r)+1}$. Hence

$$|X_{A_{n-1},m,\ell}| = |X_{A_{n-2},m,\ell}| + \sum_{r=1}^{n-1} |X_{A_{n-3},m-1,\ell-2(n-r)+1}|.$$

From this, setting $k = n - r$, we immediately get

$$\alpha_{n,m,\ell} = \alpha_{n-1,m,\ell} + \sum_{k=1}^{n-1} \alpha_{n-2,m-1,\ell+1-2k},$$

which is the first part of Theorem 1.1, and

$$\begin{aligned} L_{n,m}(t) &= L_{n-1,m}(t) + \sum_{k=1}^{n-1} t^{2k-1} L_{n-2,m-1}(t) \\ &= L_{n-1,m}(t) + (t + t^3 + \cdots + t^{2n-3}) L_{n-2,m-1}(t) \\ &= L_{n-1,m}(t) + \frac{t(t^{2(n-1)} - 1)}{t^2 - 1} L_{n-2,m-1}(t). \end{aligned}$$

This gives the second statement in Theorem 1.1. $\square$

We note that the statement relating to type A_{n-1} in Corollary 4.1 is a simple consequence of Theorem 1.1 and the fact that $L_{W(A_{n-1})}(t) = \sum_{m=1}^{\lfloor n/2 \rfloor} L_{n,m}(t)$.

Proof of Theorem 1.2 Let $X_{B_n,m,e,\ell}$ be the set of involutions of length ℓ in B_n whose expression as a product of disjoint signed cycles has m transpositions and e negative 1-cycles. Let $X_{B_n,m,e,\ell,\rho}$ be the set of involutions x in $X_{B_n,m,e,\ell}$ such that $x(n) = \rho$. Then $X_{B_n,m,\ell} = \bigcup_{r=1}^{n} \left(X_{B_n,m,e,\ell,r} \cup X_{B_n,m,e,\ell,(-r)} \right)$. Clearly $X_{B_n,m,e,\ell,n} = X_{B_{n-1},m,e,\ell}$. If $\rho = -n$, then by Theorem 2.3 and Theorem 3.1, the map $x \mapsto x(\bar{n})$ is a bijection between $X_{B_n,m,e,\ell,(-n)}$ and $X_{B_{n-1},m,e-1,\ell+1-2n}$. If $\rho = r < n$, then by Theorem 2.3 and Corollary 3.4, $x \mapsto (x(\overset{+}{r}\overset{+}{n}))^{c_r}$ is a bijection between $X_{B_n,m,e,\ell,r}$ and $X_{B_{n-2},m-1,e,\ell+1-2(n-r)}$. Finally if $\rho = -r \neq -n$, then by Theorem 2.3 and Corollary 3.5, $x \mapsto (x(\overset{-}{r}\overset{-}{n}))^{c_r}$ is a bijection between $X_{B_n,m,e,\ell,r}$ and $X_{B_{n-2},m-1,e,\ell+3-2(n+r)}$.

Hence

$$\begin{aligned} |X_{B_n,m,e,\ell}| &= |X_{B_{n-1},m,e,\ell}| + |X_{B_{n-1},m,e-1,\ell+1-2n}| \\ &\quad + \sum_{r=1}^{n-1} \left(|X_{B_{n-2},m-1,e,\ell+1-2(n-r)}| + |X_{B_{n-2},m-1,e,\ell+3-2(n+r)}| \right) \\ &= |X_{B_{n-1},m,e,\ell}| + |X_{B_{n-1},m,e-1,\ell+1-2n}| + \sum_{k=1}^{2n-2} |X_{B_{n-2},m-1,e,\ell+1-2k}|. \end{aligned}$$

From the definitions of $\beta_{n,m,e,\ell}$ and $L_{n,m,e}(t)$ we now get

$$\beta_{n,m,e,\ell} = \beta_{n-1,m,e,\ell} + \beta_{n-1,m,e-1,\ell+1-2n} + \sum_{k=1}^{2n-2} \beta_{n-2,m-1,e,\ell+1-2k},$$

which is the first part of Theorem 1.2, and

$$\begin{aligned} L_{n,m,e}(t) &= L_{n-1,m,e}(t) + t^{2n-1} L_{n-1,m,e-1}(t) + \sum_{k=1}^{2n-2} t^{2k-1} L_{n-2,m-1,e}(t) \\ &= L_{n-1,m,e}(t) + t^{2n-1} L_{n-1,m,e-1}(t) + (t + t^3 + \cdots + t^{4n-5}) L_{n-2,m-1,e}(t) \\ &= L_{n-1,m,e}(t) + t^{2n-1} L_{n-1,m,e-1}(t) + \frac{t(t^{4n-4}-1)}{t^2-1} L_{n-2,m-1,e}(t). \end{aligned}$$

This gives the second statement in Theorem 1.2. $\square$

Before embarking on the case of D_n, we define yet another polynomial. Let $Y_{B_n,m,e}$ be the set of involutions in $W(B_n)$ whose expression as a product of disjoint signed cycles has m transpositions and e negative 1-cycles. Set

$$E_{n,m,\ell}(t) = \sum_{x \in Y_{B_n,m,e}} t^{|\Lambda(x)|}.$$

Proof of Theorem 1.3 We need to work within the environment of B_n for the moment, because of the risk that when we remove the cycle containing n from an involution that happens to be in the subgroup $W(D_n)$, we end up with an involution outside of $W(D_n)$. We get round this by working in $W(B_n)$, but instead of considering $\ell_{B_n}(x)$ or $\ell_{D_n}(x)$ for an involution x, we consider $|\Lambda(x)|$. If x happens to be an element of $W(D_n)$, then by Theorem 2.3, $\ell_{D_n}(x) = |\Lambda(x)|$. Therefore we will be able, with care, to retrieve the length polynomial for involutions in $W(D_n)$ at the end of the process.

Let $Y_{B_n,m,e,\ell}$ be the set of involutions x of $W(B_n)$ satisfying $|\Lambda(y)| = \ell$, whose expression as a product of disjoint signed cycles has m transpositions and e negative 1-cycles. Let $Y_{B_n,m,e,\ell,\rho}$ be the set of involutions x in $Y_{B_n,m,e,\ell}$ such that $x(n) = \rho$. Then

$$Y_{B_n,m,e,\ell} = \bigcup_{r=1}^{n} \left(Y_{B_n,m,e,\ell,r} \cup Y_{B_n,m,e,\ell,(-r)}\right).$$

Clearly $|Y_{B_n,m,e,\ell,n}| = |Y_{B_{n-1},m,e,\ell}|$. If $\rho = -n$, then by Theorem 2.3 and Theorem 3.1, the map $x \mapsto x(\bar{n})$ is a bijection between $Y_{B_n,m,e,\ell,(-n)}$ and $Y_{B_{n-1},m,e-1,\ell+2-2n}$. If $\rho = r < n$, then by Theorem 2.3 and Corollary 3.4, $x \mapsto (x(\overset{+}{r}\overset{+}{n}))^{c_r}$ is a bijection between $Y_{B_n,m,e,\ell,r}$ and $Y_{B_{n-2},m-1,e,\ell+1-2(n-r)}$. Finally if $\rho = -r \neq -n$, then by Theorem 2.3 and Corollary 3.5, $x \mapsto (x(\bar{r}\bar{n}))^{c_r}$ is a bijection between $Y_{B_n,m,e,\ell,r}$ and $Y_{B_{n-2},m-1,e,\ell+5-2(n+r)}$.

Hence

$$\begin{aligned}
|Y_{B_n,m,e,\ell}| &= |Y_{B_{n-1},m,e,\ell}| + |Y_{B_{n-1},m,e-1,\ell+2-2n}| \\
&\quad + \sum_{r=1}^{n-1} \left(|Y_{B_{n-2},m-1,e,\ell+1-2(n-r)}| + |Y_{B_{n-2},m-1,e,\ell+5-2(n+r)}|\right) \\
&= |Y_{B_{n-1},m,e,\ell}| + |Y_{B_{n-1},m,e-1,\ell+2-2n}| \\
&\quad + \sum_{k=1}^{n-1} \left(|Y_{B_{n-2},m-1,e,\ell+1-2k}| + |Y_{B_{n-2},m-1,e,\ell+5-2n-2k}|\right).
\end{aligned}$$

We now consider the polynomial $E_{n,m,e}(t)$ with the aim of showing that $E_{n,m,e}(t)$ is precisely the $D_{n,m,e}(t)$ defined just before Theorem 1.3. We observe that

$$E_{n,m,e}(t) = \sum_{\ell=0}^{\infty} |Y_{B_n,m,e,\ell}| t^{\ell}.$$

Certainly $E_{n,0,0}(t) = 1$ and if $2m + e > n$, then $E_{n,m,e}(t) = 0$. Now $W(B_1) = \{(\overset{+}{1}), (\bar{1})\}$, so $E_{1,0,1}(t) = t$. The set of involutions in $W(B_2)$ is $\{(\overset{+}{1}\overset{+}{2}), (\bar{1}\bar{2}), (\bar{1}), (\bar{2}), (\bar{1})(\bar{2})\}$. Hence $E_{2,1,0}(t) = 2t$, $E_{2,0,1}(t) = 1 + t^2$, $E_{2,0,2}(t) = t^2$. Moreover, from our recurrence relation for $|Y_{B_n,m,e,\ell}|$ above, we get that for $n \geq 3$,

$$\begin{aligned}E_{n,m,e}(t) &= E_{n-1,m,e}(t) + t^{2n-2}E_{n-1,m,e-1}(t) + \sum_{k=1}^{n-1}\left(t^{2k-1} + t^{2k+2n-5}\right)E_{n-2,m-1,e}(t)\\ &= E_{n-1,m,e}(t) + t^{2n-2}E_{n-1,m,e-1}(t) + (1+t^{2n-4})\sum_{k=1}^{n-1}t^{2k-1}E_{n-2,m-1,e}(t)\\ &= E_{n-1,m,e}(t) + t^{2n-2}E_{n-1,m,e-1}(t) + \frac{t(1+t^{2n-4})(t^{2n-2}-1)}{t^2-1}E_{n-2,m-1,e}(t).\end{aligned}$$

That is,

$$E_{n,m,e}(t) = E_{n-1,m,e}(t) + t^{2n-2}E_{n-1,m,e-1}(t) + \frac{t(1+t^{2n-4})(t^{2n-2}-1)}{t^2-1}E_{n-2,m-1,e}(t). \quad (4)$$

Now $E_{n,m,e}(t)$ has exactly the same initial conditions and recurrence relation as $D_{n,m,e}(t)$. So the two polynomials are the same. In particular, the coefficient of t^ℓ in $D_{n,m,e}(t)$ equals the coefficient of t^ℓ in $E_{n,m,e}(t)$, which by definition is $|Y_{B_n,m,e,\ell}|$. But we know that when e is even, the elements of $Y_{B_n,m,e,\ell}$ are precisely the elements of $W(D_n)$ with length ℓ that have m transpositions and e negative 1-cycles. Therefore $|Y_{B_n,m,e,\ell}| = \delta_{n,m,e,\ell}$ and hence the coefficient of t^ℓ in $D_{n,m,e}$ is $\delta_{n,m,e,\ell}$ when e is even. If e is odd, then there are no involutions in $W(D_n)$ with e negative 1-cycles, so $\delta_{n,m,e,\ell} = 0$. Similarly, when e is odd we have $L_{n,m,e}(t) = 0$. When e is even, the fact that $|Y_{B_n,m,e,\ell}| = \delta_{n,m,e,\ell}$ implies that $D_{n,m,e}(t) = E_{n,m,e}(t) = L_{n,m,e}(t)$. This completes the proof of Theorem 1.3. □

Given that we now have expressions for the length polynomials for involutions of every signed cycle type in $W(A_n)$, $W(B_n)$ and $W(D_n)$, we can now produce recurrence relations for the length polynomials $L_W(t)$ for the sets of all involutions in these groups. The only potential stumbling block is $W(D_n)$. Here our recurrence relation for $L_{n,m,e}(t)$ involves involutions with $e-1$ negative 1-cycles, which of course are not elements of $W(D_n)$. We work round this as follows. Define

$$L_{(B\setminus D)_n}(t) = \sum_{m,j\geq 0} E_{n,m,2j+1}(t).$$

Observe that the coefficient of t^ℓ in $L_{(B\setminus D_n)}(t)$ is the number of involutions x in $W(B_n)$ for which $|\Lambda(x)| = \ell$ whose expression as a product of disjoint signed cycles contains an odd number of negative 1-cycles. These are precisely the involutions of $W(B_n)$ which are not contained in $W(D_n)$. We may now state and prove Corollary 4.1. Note that part (a) of Corollary 4.1 is known; it is the first part of Proposition 2.8 in [6].

Corollary 4.1 *(a)* $L_{W(A_1)}(t) = 1+t$, $L_{W(A_2)}(t) = 1+2t+t^3$, *and for* $n \geq 3$,

$$L_{W(A_n)}(t) = L_{W(A_{n-1})}(t) + \frac{t(t^{2n}-1)}{t^2-1}L_{W(A_{n-2})}(t).$$

(b) $L_{W(B_1)}(t) = 1+t$, $L_{W(B_2)}(t) = 1+2t+2t^3+t^4$ *and for* $n \geq 3$,

$$L_{W(B_n)}(t) = (1+t^{2n-1})L_{W(B_{n-1})}(t) + \frac{t(t^{4n-4)}-1)}{t^2-1}L_{W(B_{n-2})}(t).$$

(c) $L_{W(D_1)}(t) = 1$, $L_{(B\backslash D)_1}(t) = 1$, $L_{W(D_2)}(t) = 1 + 2t$, $L_{(B\backslash D)_2}(t) = 1 + t^2$ *and for* $n \geq 3$,

$$L_{D_n}(t) = L_{W(D_{n-1})}(t) + t^{2n-2}L_{(B\backslash D)_{n-1}}(t) + \frac{t(1+t^{2n-4})(t^{2n-2}-1)}{t^2-1}L_{W(D_{n-2})}(t)$$

and

$$L_{(B\backslash D)_n}(t) = L_{(B\backslash D)_{n-1}}(t) + t^{2n-2}L_{W(D_{n-1})}(t) + \frac{t(1+t^{2n-4})(t^{2n-2}-1)}{t^2-1}L_{(B\backslash D)_{n-2}}(t).$$

Proof For parts (a) and (b), we simply observe that $L_{W(A_n)}(t) = \sum_m L_{n+1,m}(t)$, and similarly $L_{W(B_n)}(t) = \sum_{m,e} L_{n,m,e}(t)$, and apply Theorems 1.1 and 1.2. For part (c), the initial values $L_{W(D_1)}(t)$, $L_{(B\backslash D)_1}(t)$, $L_{W(D_2)}(t)$ and $L_{(B\backslash D)_2}(t)$ are easy to calculate. For the recurrence relations we use Equation 4:

$$E_{n,m,e}(t) = E_{n-1,m,e}(t) + t^{2n-2}E_{n-1,m,e-1}(t) + \frac{t(1+t^{2n-4})(t^{2n-2}-1)}{t^2-1}E_{n-2,m-1,e}(t).$$

Hence

$$\sum_{m,j} E_{n,m,2j}(t) = \sum_{m,j} E_{n-1,m,2j}(t) + t^{2n-2}\sum_{m,j} E_{n-1,m,2j-1}(t) + \frac{t(1+t^{2n-4})(t^{2n-2}-1)}{t^2-1}\sum_{m,j} E_{n-2,m-1,2j}(t)$$

and

$$\sum_{m,j} E_{n,m,2j+1}(t) = \sum_{m,j} E_{n-1,m,2j+1}(t) + t^{2n-2}\sum_{m,j} E_{n-1,m,2j}(t) + \frac{t(1+t^{2n-4})(t^{2n-2}-1)}{t^2-1}\sum_{m,j} E_{n-2,m-1,2j+1}(t).$$

Now the involutions in $W(D_n)$ are precisely the involutions in $W(B_n)$ whose signed cycle expression has an even number $e = 2j$ of negative 1-cycles. Therefore $L_{W(D_n)}(t) = \sum_{m,j} E_{n,m,2j}(t)$ and $L_{(B\backslash D)_n}(t) = \sum_{m,j} E_{n,m,2j+1}(t)$. Hence we can immediately conclude that

$$L_{W(D_n)}(t) = L_{W(D_{n-1})}(t) + t^{2n-2}L_{(B\backslash D)_{n-1}}(t) + \frac{t(1+t^{2n-4})(t^{2n-2}-1)}{t^2-1}L_{W(D_{n-2})}(t)$$

and

$$L_{(B\backslash D)_n}(t) = L_{(B\backslash D)_{n-1}}(t) + t^{2n-2}L_{W(D_{n-1})}(t) + \frac{t(1+t^{2n-4})(t^{2n-2}-1)}{t^2-1}L_{(B\backslash D)_{n-2}}(t).$$

□

5 The remaining finite Coxeter groups

We first deal with the groups of type I_n. Here $W(I_n)$ is the dihedral group of order $2n$. The following lemma is easy to prove.

Lemma 5.1 *Let $W = W(I_n)$. If n is odd, then there is one conjugacy class of involutions, and its length polynomial is $t^n + \frac{2t(1-t^n)}{1-t^2}$. If n is even, there are three conjugacy classes of involutions. One consists of the unique central involution and has length polynomial t^n. The other two conjugacy classes both have length polynomial $\frac{t(1-t^n)}{1-t^2}$. Therefore*

$$L_W(t) = 1 + t^n + \frac{2t(1-t^n)}{1-t^2}.$$

The remaining exceptional finite Coxeter groups are types E_6, E_7, E_8, F_4, H_3 and H_4. We have used the computer algebra package MAGMA[1] to calculate the length polynomials here. For a conjugacy class X in W, where W is one of these groups, we write $a_{W,X,\ell}$ for the coefficent of t^ℓ in the length polynomial. That is,

$$L_{W,X}(t) = \sum_{x \in X} t^{\ell(x)} = \sum a_{W,X,\ell} t^\ell.$$

The sequence $[a_{W,X,\ell}]$ (starting at the smallest nonzero term) we refer to as the 'length profile' of X in W. Involutions in a given conjugacy class have lengths of the same parity (either all odd length or all even length), and so when writing down the length profiles we would get alternating zeros. We suppress these, and write the 'odd length profile' (the sequence $[a_{W,X,2k+1}]$) or 'even length profile' (the sequence $[a_{W,X,2k}]$) as appropriate. As a small example, in the dihedral group of order 8, each conjugacy class of reflections has length profile $[1, 0, 1]$, where the smallest length is 1. So its odd length profile is $[1, 1]$. The involution length profile (including the identity) of the whole dihedral group of order 8 is $[1, 2, 0, 2, 1]$, so its odd length profile is $[2, 2]$ and its even length profile is $[1, 0, 1]$.

The length profiles of conjugacy classes in the exceptional groups of types E_6, E_7, E_8, F_4, H_3 and H_4 are given in Tables 1 – 6. For these groups W, it is well known that every nontrivial involution in W is conjugate to the central involution of some standard parabolic subgroup. For each conjugacy class X of involutions, the class is indicated by giving (up to isomorphism) the relevant standard parabolic subgroup, the size of the class, and the minimum length $\ell_{\min}$ of elements in the class. This information is nearly always enough to specify X uniquely. Where it is not (and this occurs in type F_4), the length profiles of the given conjugacy classes are happily identical, by virtue of the length-preserving automorphism of the Coxeter graph.

We begin with the table for $W(E_6)$.

class	size	$\ell_{\min}$	odd/even length profile
A_1	36	1	[6,5,5,5,4,3,3,2,1,1,1]
A_1^2	270	2	[10,15,21,28,31,30,31,28,22,18,16,10,6,3,1]
A_1^3	540	3	[5, 10, 17, 28, 40, 48, 56, 60, 58, 53, 49, 41, 32, 22, 13, 6, 2]
D_4	45	12	[1,2,3,4,5,5,5,5,5,4,3,2,1]

Table 1: Involutions in $W(E_6)$

Note that types E_7, E_8, F_4, H_3 and H_4 all have non-trivial centres. This means that multiplication by the central involution will map a given conjugacy class X to one of equal size with the length profile reversed. We exploit this in Tables 2 – 6.

class	size	$\ell_{\min}$	odd/even length profile
A_1	63	1	[7,6,6,6,6,5,5,4,4,3,3,2,2,1,1,1,1]
A_1^2	945	2	[15, 24, 34, 44, 55, 60, 67, 68, 71, 68, 68, 62, 59, 50, 44, 38, 35, 26, 20, 14, 10, 6, 4, 2, 1]
A_1^3	315	3	[1,2,4,6,9,11,14,16,19,20,22,22,23,22,22,20,19,16,14,11,9,6,4,2,1]
A_1^3	3780	3	[10,22,39,61,91,119,152,180,209,228,248,257,265,259,251,235, 222,198,175,147,122,94,72,51,35,20,11,5,2]
A_1^4	3780	4	reverse of class A_1^3, size 3780
D_4	315	12	reverse of class A_1^3, size 315
$D_4 \times A_1$	945	13	reverse of class A_1^2
D_6	63	30	reverse of class A_1
E_7	1	63	[1]

Table 2: Involutions in $W(E_7)$

class	size	$\ell_{\min}$	odd/even length profile
A_1	120	1	[8, 7, 7, 7, 7, 7, 7, 6, 6, 6, 6, 5, 5, 4, 4, 4, 4, 3, 3, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1]
A_1^2	3780	2	[21,35,50,65,80,95,111,120,130,140,151,155,161, 160,161,162, 164,159,157,148,141,134,129,117,108,99,92,85,80,68,59,50, 42,34,28,22,18,14,11,8,6,4,3,2,1]
A_1^3	37800	3	[21, 49, 89, 141, 205, 279, 369, 460, 556, 656, 766, 868, 973, 1065, 1154, 1237, 1320, 1383, 1443, 1482, 1510, 1521, 1528, 1510, 1483, 1442, 1399, 1346, 1295, 1225, 1153, 1072, 989, 896, 805, 711, 625, 540, 464, 391, 326, 265, 215, 170, 131, 95, 67, 45, 30, 18, 10, 5, 2]
A_1^4	113400	4	[7, 20, 43, 80, 135, 207, 303, 420, 559, 719, 907, 1112, 1337, 1571, 1819, 2078, 2352, 2621, 2892, 3152, 3404, 3634, 3849, 4027, 4175, 4283, 4365, 4412, 4434, 4412, 4365, 4283, 4175, 4027, 3849, 3634, 3404, 3152, 2892, 2621, 2352, 2078, 1819, 1571, 1337, 1112, 907, 719, 559, 420, 303, 207, 135, 80, 43, 20, 7]
D_4	3150	12	[1, 2, 4, 7, 11, 15, 21, 27, 34, 41, 49, 56, 65, 73, 82, 90, 99,105, 112, 117, 122, 124, 127, 127, 128, 127, 127, 124, 122, 117, 112, 105, 99, 90, 82, 73, 65, 56, 49, 41, 34, 27, 21, 15, 11, 7, 4, 2, 1]
$D_4 \times A_1$	37800	13	reverse of class A_1^3
D_6	3780	30	reverse of class A_1^2
E_7	120	63	reverse of class A_1
E_8	1	120	[1]

Table 3: Involutions in $W(E_8)$

The table for $W(E_8)$ is Table 3. Note that the longest element maps the class corresponding to D_4 to itself, and the class corresponding to A_1^4 to itself. Therefore the length profile for these classes are symmetric. The same occurs in the class corresponding to A_1^2 in $W(F_4)$, as shown in Table 4.

class	size	$\ell_{\min}$	odd/even length profile
A_1	12	1	[2,2,2,2,1,1,1,1] (two such classes)
A_1^2	72	2	[3,4,6,10,9,8,9,10,6,4,3]
B_2	18	4	[1,2,3,2,2,2,3,2,1]
B_3	12	9	reverse of class A_1 (two such classes)
F_4	1	24	[1]

Table 4: Involutions in $W(F_4)$

Finally, we deal with $W(H_3)$ and $W(H_4)$. In $W(H_4)$ the class corresponding to A_1^2 is mapped to itself by the longest element, so the length profile for this class is symmetric.

class	size	$\ell_{\min}$	odd/even length profile
A_1	15	1	[3, 3, 3, 2, 2, 1, 1]
A_1^2	15	2	reverse of class A_1
H_3	1	15	[1]

Table 5: Involutions in $W(H_3)$

class	size	$\ell_{\min}$	odd/even length profile
A_1	60	1	[4,4,4,4,4,4,4,3,3,3,3,3,3,2,2,2,2,1,1,1,1,1,1]
A_1^2	450	2	[3,5,7,9,11,14,18,18,18,19,21,23,25,23,22, 23,25,23,21,19,18,18,18,14,11,9,7,5,3]
H_3	60	15	reverse of class A_1
H_4	1	15	[1]

Table 6: Involutions in $W(H_4)$

There are various conjectures and some results about the odd and even involution length profiles for classical Weyl groups. We will discuss these in Section 6. For that reason, we include here the tables of odd and even length profiles for the exceptional groups. The odd involution length profiles of the exceptional groups are given in Table 7. The even involution length profiles of the exceptional groups are given in Table 8.

Type	Profile
E_6	[6,10,15,22,32,43,51,58,61,59,54,49,41,32,22,13,6,2]
E_7	[7,17,30,49,73,105,136,172,204,237,261,286,301,315,317,312, 300,291,273,251,226,199,171,144,120,96,75,55,39,26,15,1]
E_8	[8,28,56,96,148,212,288,380,476,580,692,816,940,1072,1200,1328,1456, 1588,1712,1836,1948,2052,2148,2240,2316,2380, 2432,2472,2500, 2520, 2520,2500,2472,2432,2380,2316,2240,2148,2052,1948,1836,1712,1588, 1456,1328,1200,1072,940,816,692,580,476,380,288,212,148, 96,56,28,8]
F_4	[4,4,4,4,4,4,4,4,4,4,4,4]
H_3	[3,3,3,2,2,1,1,1]
H_4	[4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4]
I_{2k}	[2,2, ..., 2,2]
I_{2k+1}	[2,2,...,2,1]

Table 7: Odd involution length profiles in exceptional groups

Type	Profile
E_6	[1,10,15,21,28,31,31,33,31,26,23,21,15,11,8,5,3,2,1]
E_7	[1,15,26,39,55,75,96,120,144,171,199,226,251,273,291,300,312, 317,315,301,286,261,237,204,172,136,105,73,49,30,17,7]
E_8	[1,21,35,50,65,80,96,113,124,137,151,166,176,188,194,203,213, 223,228,236,238,242,247,252,251,253,255,258,262, 266,264, 266, 262,258,255,253,251,252,247,242,238,236,228,223,213,203,194, 188,176,166,151,137,124,113,96,80,65,50,35,21,1]
F_4	[1,3,5,8,13,11,10,11,13,8,5,3,1]
H_3	[1,1,1,2,2,3,3,3]
H_4	[1, 3, 5, 7, 9, 11, 14, 18, 18, 18, 19, 21, 23, 25, 23, 22, 23, 25, 23, 21, 19, 18, 18, 18, 14, 11, 9, 7, 5, 3, 1]
I_{2k}	[1,0,...,0,1]
I_{2k+1}	[1]

Table 8: Even involution length profiles (including the identity element) in exceptional groups

6 Unimodality

Various conjectures have been made concerning the unimodality and/or log-concavity of the coefficients of involution length polynomials in the classical groups. Note that a sequence $(x_i)_{i=1}^N$ of non-negative integers is *unimodal* if for some j, $x_1 \leq \cdots \leq x_{j-1} \leq x_j \geq x_{j+1} \geq \cdots \geq x_N$, and *log-concave* if for all i between 2 and $N-1$ we have $x_i^2 \geq x_{i-1}x_{i+1}$ (see [10]). A log-concave sequence is always unimodal, but not vice versa.

Conjecture 6.1 ([6] Conjecture 4.1 (ii) – (iv)) *(a) The even involution length profile and the odd involution length profile of $W(A_n)$ are log-concave.*

(b) The even involution length profile and the odd involution length profile in $W(B_n)$ are unimodal.

(c) The even involution length profile and the odd involution length profile in $W(D_n)$ are unimodal.

To test these conjectures, and to see if they can be generalised, we have checked the classical groups of types A_n, B_n and D_n up to $n = 10$, and all the exceptional groups.

These are the only examples we know of for even or odd involution length profiles in finite irreducible Coxeter groups that are not unimodal:

1. Even length involutions in $W(B_6)$. Using Corollary 4.1 the even length profile is calculated to be

$$[1, 10, 20, 27, 35, 41, 49, 51, 55, 54, 55, 51, 49, 41, 35, 27, 20, 10, 1].$$

2. Even length involutions in types E_8, F_4, H_4 and I_n, for n even.

These are the only examples we know of for conjugacy classes of involutions in finite irreducible Coxeter groups whose even or odd length profiles are not unimodal:

1. The class corresponding to A_1^2 in $W(E_6)$.
2. The classes corresponding to A_1^2 and D_6 in $W(E_8)$.
3. The classes corresponding to A_1^2 and B_2 in $W(F_4)$.
4. The class corresponding to A_1^2 in $W(H_4)$.
5. The conjugacy classes of $(\overset{-}{1})(\overset{-}{2})(\overset{++}{3\,4})$ and of $(\overset{-}{1})(\overset{-}{2})(\overset{-}{3})(\overset{-}{4})(\overset{++}{5\,6})$ in $W(D_8)$.

We can therefore immediately say that not all conjugacy classes of involutions in finite irreducible Coxeter have unimodal length profiles, and that the even involution length profile of a Coxeter group is not always unimodal. In particular, Conjecture 6.1(b) is false. However, on current data, we can make the following conjectures.

Conjecture 6.2 *(i) If X is a conjugacy class of involutions in $W(A_n)$ or $W(B_n)$, then the even/odd length profile of X is unimodal.*

(ii) If X is the set of involutions of odd length in a finite Coxeter group, then the odd length profile of X is unimodal.

We end with a small step (extending results of [6]) towards addressing these questions.

Theorem 6.3 *Let $n \geq 2$ be even. Let W be of type A_{n-1}, B_n or D_n, and let X be the set of involutions in W with no 1-cycles. Then*

$$L_{W(A_{n-1}),X}(t) = t^{n/2} \prod_{k=1}^{n/2} \frac{t^{4k-2} - 1}{t^2 - 1};$$

$$L_{W(B_n),X}(t) = t^{n/2} \prod_{k=1}^{n/2} \frac{t^{8k-4} - 1}{t^2 - 1};$$

$$L_{W(D_n),X}(t) = t^{n/2} \prod_{k=1}^{n/2} \frac{(1 + t^{4k-4})(t^{4k-2} - 1)}{t^2 - 1}.$$

Hence the sequences of odd-power and even-power coefficients of $L_{W,X}(t)$ are symmetric, unimodal and, in the case of $W(A_{n-1})$ and $W(B_n)$, log-concave.

The result for $L_{W(A_n),X}(t)$ is the second part of Proposition 2.8 and Theorem 2.10 of [6]. The result for $L_{W(D_n),X}(t)$ is Theorems 3.3 and 3.4 of [6]. The following proof is simply an extension of the arguments given there to include the case of $W(B_n)$, though our derivation of the formula for $W(D_n)$ is somewhat shorter.

Proof Note that X is just the conjugacy class (or union of two classes for $W(D_n)$) where $e = 0$ and $n = 2m$. Exactly one of the sequences will consist entirely of zeros and so will be trivially log-concave. The expressions for $L_{W,X}(t)$ follow from Theorems 1.1, 1.2 and 1.3, because of the fact that $L_{W,X}(t) = 0$ whenever $e < 0$ or $2m < n$ leaves just one term in the recursion. So we have that

$$L_{W(A_{n-1}),X}(t) = \frac{t(t^{2n-2} - 1)}{t^2 - 1} L_{W(A_{n-3}),X}(t);$$

$$L_{W(B_n),X}(t) = \frac{t(t^{4n-4} - 1)}{t^2 - 1} L_{W(B_{n-2}),X}(t);$$

$$L_{W(D_n),X}(t) = \frac{t(1 + t^{2n-4})(t^{2n-2} - 1)}{t^2 - 1} L_{W(D_{n-2}),X}(t).$$

The product expression for each polynomial now follows by induction, noting that for the base case $n = 2$ we have $L_{W(A_1),X}(t) = t = \frac{t(t^{4\times1-2}-1)}{t^2-1}$, $L_{W(B_2),X}(t) = t + t^3 = \frac{t(t^{8\times1-4}-1)}{t^2-1}$ and $L_{W(D_2),X}(t) = 2t = \frac{t(1+t^{4\times1-4})(t^{4\times1-2}-1)}{t^2-1}$.

It remains to show that the sequences of coefficients of these polynomials are symmetric, unimodal and, for types A_{n-1} and B_n, log-concave. Write $s = t^2$. Then

$$t^{-n/2} L_{W,X}(t) = \prod_{k=1}^{n/2} Q_k(s)$$

where $Q_k(s)$ is either (for $W(A_{n-1})$)

$$\frac{s^{2k-1} - 1}{s - 1} = 1 + s + \cdots + s^{2k-2}$$

or (for $W(B_n)$)

$$\frac{s^{4k-2} - 1}{s - 1} = 1 + s + \cdots + s^{4k-3}$$

or (for $W(D_n)$)

$$\frac{(1+s^{2k-2})(s^{2k-1}-1)}{s-1} = 1+s+\cdots+s^{2k-3}+2s^{2k-2}+s^{2k-1}+\cdots+s^{4k-3}.$$

For all three of these, the sequence of coefficients of $Q_k(s)$ is symmetric and unimodal with non-negative coefficients. Proposition 1 of [10] states that the product of any such polynomials is also symmetric and unimodal with non-negative coefficients. For the first two of these cases, corresponding to types A_{n-1} and B_n, the sequence of coefficients of $Q_k(s)$ is always a non-negative log-concave sequence with no internal zero coefficients. Proposition 2 of [10] states that the product of any such polynomials is also log-concave with no internal zero coefficients. Theorem 6.3 now follows. □

Note that $L_{W(D_n),X}(t)$ is not log-concave in general. For example, the length profile for X, the set of involutions with no 1-cycles in $W(D_4)$, is $[2,2,4,2,2]$, and $2^2 < 2 \times 4$. So the sequence is not log-concave.

References

[1] W. Bosma, J. Cannon, and C. Playoust. *The Magma algebra system. I. The user language*, J. Symbolic Comput., 24 (1997), 235–265.

[2] R.W. Carter. *Conjugacy Classes in the Weyl Group*, Compositio. Math. 25, Facs. 1 (1972), 1–59.

[3] H.S.M. Coxeter. *The complete enumeration of finite groups of the form* $R_i^2 = (R_iR_j)^{k_{ij}} = 1$, J. London Math. Soc. 10 (1935), 21–25.

[4] R. De Man. *The Generating Function for the Number of Roots of a Coxeter Group* J. Symbolic Computation (1999) 27, 535–541.

[5] J. Désarménien. *Les q-analogues des polynômes d'Hermite*, Sém. Lothar. Combin. 6 (1982) Art. B06b, 12pp.

[6] W.M.B. Dukes. *Permutation statistics on involutions*, European J. Combin. 28 (2007), 186–198.

[7] J.E. Humphreys. *Reflection Groups and Coxeter Groups*, Cambridge Studies in Advanced Mathematics, 29 (1990).

[8] S.B. Perkins and P.J. Rowley. *Minimal and maximal length involutions in finite Coxeter groups*, Comm. Algebra 30 (2002), no. 3, 1273–1292.

[9] R.W. Richardson. *Conjugacy classes of involutions in Coxeter groups*, Bull. Austral. Math. Soc. 26 (1982), 1–15.

[10] R. P. Stanley. *Log-concave and unimodal sequences in Algebra, Combinatorics and Geometry*, Ann. New York Acad. Sci. 576 (1989), 500–534.