THE EQUATIONAL THEORIES OF REPRESENTABLE RESIDUATED SEMIGROUPS

SZABOLCS MIKULÁS

Abstract. We show that the equational theory of representable lower semilattice-ordered residuated semigroups is finitely based. We survey related results.

Keywords: finite axiomatizability, relation algebras, residuation, free algebra

Residuated algebras and their equational theories have been investigated on their own right and also in connection with substructural logics. The reason for the latter is that the algebraizations of substructural logics like relevance logic [AB75, ABD92] and the Lambek calculus (LC) [La58] yield residuated algebras. Indeed, for these logics, the Lindenbaum–Tarski algebras are residuated algebras and sound relational semantics can be provided using families of binary relations, i.e., representable residuated algebras. These connections are explained in detail in [Mik??] and the references therein. In particular, we show in [Mik??] completeness of an expansion of LC with meet w.r.t. binary relational semantics. This completeness result states that that derivability in LC augmented with derivation rules for meet coincides with semantic validity, i.e., completeness is stated in its weak form and does not capture general semantic consequence. The proof uses cut-elimination. In algebraic terms this result means that the equational theories of abstract (related to the syntactic calculus) and representable (related to binary semantics) algebras coincide. In other words, the free abstract algebra is representable.

In this paper we provide an alternative, purely algebraic, proof of this result. We will define the variety of lower semilattice-ordered residuated semigroups using finitely many equations. The subclass of representable algebras is given by the isomorphs of families of binary relations. Using a step-by-step construction we show that the free algebra of the variety of lower semilattice-ordered residuated semigroups is representable. On the other hand, there might be algebras in this variety that are not representable; we leave this as an open problem. Hopefully the technique we use for the representation of the *free* algebra could be used in other cases as well when

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the variety generated by representable algebras is finitely based (but the quasivariety of representable algebras may not have a finite axiomatization).

1. Algebras of relations

In this paper we will focus on the following operations: join $+$, meet $\cdot$, relation composition ;, right $\backslash$ and left $/$ residuals of composition. We recall the interpretations of the operations in an algebra $\mathfrak{C}$ of binary relations with base $U_{\mathfrak{C}}$. Join $+$ is union, meet $\cdot$ is intersection, and

$$\begin{aligned} x \,;\, y &= \{(u,v) \in U_{\mathfrak{C}} \times U_{\mathfrak{C}} : (u,w) \in x \text{ and } (w,v) \in y \text{ for some } w\} \\ x \backslash y &= \{(u,v) \in U_{\mathfrak{C}} \times U_{\mathfrak{C}} : \text{for every } w, (w,u) \in x \text{ implies } (w,v) \in y\} \\ x \,/\, y &= \{(u,v) \in U_{\mathfrak{C}} \times U_{\mathfrak{C}} : \text{for every } w, (v,w) \in y \text{ implies } (u,w) \in x\} \end{aligned}$$

and we may also need the identity constant interpreted as

$$1' = \{(u,v) \in U_{\mathfrak{C}} \times U_{\mathfrak{C}} : u = v\}$$

although usually we will not assume that $1'$ is an element of $\mathfrak{C}$.

Let $\mathsf{R}(\Lambda)$ denote the class of algebras of binary relations for similarity type Λ, the *representable* algebras, and let $\mathsf{V}(\Lambda)$ be the variety generated by $\mathsf{R}(\Lambda)$.

2. Lower semilattice-ordered residuated semigroups

In this section we look at $\Lambda = (\cdot, ;, \backslash, /)$. As usual $x \le y$ is defined by $x \cdot y = x$. We will say that x is a *residuated term* if it has the form $y \backslash z$ or $y \,/\, z$, and a residuated term is *reflexive* if $y = z$, since terms of the form $y \backslash y$ and $y \,/\, y$ include the identity relation in representable algebras.

We define $\mathrm{Ax}(\cdot, ;, \backslash, /)$ as the collection of the following axioms.

Semilattice axioms (for meet).
Semigroup axiom (for composition).
Monotonicity:

$$(x \cdot x') \,;\, (y \cdot y') \le x \,;\, y \tag{1}$$

Residuation:

$$x \backslash (y \cdot y') \le x \backslash y \qquad (x \cdot x') \,/\, y \le x \,/\, y \tag{2}$$

$$x \,;\, (x \backslash y) \le y \qquad (x \,/\, y) \,;\, y \le x \tag{3}$$

$$y \le x \backslash (x \,;\, y) \qquad x \le (x \,;\, y) \,/\, y \tag{4}$$

"Reflexivity":

$$y \le x \,;\, y \qquad y \le y \,;\, x \tag{5}$$

if x is a reflexive residuated term.
"Idempotency":

$$(x \cdot y) \backslash (x \cdot y) = x \cdot y = (x \cdot y) \,/\, (x \cdot y) \tag{6}$$

if x, y are reflexive residuated terms.

A model $\mathfrak{A} = (A, \cdot, ;, \backslash, /)$ of these axioms is a *lower semilattice-ordered residuated semigroup.*

The reader may be more familiar with the following quasiequations

$$y \leq x \setminus z \text{ iff } x \,;\, y \leq z \text{ iff } x \leq z \,/\, y \tag{7}$$

expressing the residual property. But [Pr90] observed that equations (2)–(4) in fact imply (7), hence we have a variety when meet is present.

It is easily checked that the above axioms are valid in representable algebras. We just note, in connection with the last two axioms, that the interpretation of reflexive residuated elements x must include the identity (they are reflexive) and they are transitive ($x \,;\, x \leq x$).

Theorem 2.1. *The equational theory of* $\mathsf{R}(\cdot, ;, \backslash, /)$ *is finitely axiomatized by* $\mathrm{Ax}(\cdot, ;, \backslash, /)$.

Proof. We will use a modification of the step-by-step construction of [AM94, Theorem 3.2]. Let $\mathfrak{F}_X$ be the free lower semilattice-ordered residuated semigroup freely generated by a set X of variables. We show that $\mathfrak{F}_X \in \mathsf{R}(\cdot, ;, \backslash, /)$. It follows that any equation $\sigma \leq \tau$ is valid in $\mathsf{R}(\cdot, ;, \backslash, /)$ if and only if it is derivable from $\mathrm{Ax}(\cdot, ;, \backslash, /)$ using equational logic.

Let T_X be the set of $(\cdot, ;, \backslash, /)$-terms using the variables from X. When no confusion is likely, we may blur the distinction between terms and the elements of $\mathfrak{F}_X$, the equivalence classes of terms under derivability from $\mathrm{Ax}(\cdot, ;, \backslash, /)$. By a filter $\mathcal{F}$ of $\mathfrak{F}_X$ we mean a subset of terms closed upward and under meet. That is, if $\tau, \sigma \in \mathcal{F}$, then $\rho \in \mathcal{F}$ whenever $\mathrm{Ax}(\cdot, ;, \backslash, /) \vdash \tau \leq \rho$ and also $\tau \cdot \sigma \in \mathcal{F}$. For a subset S, let $\mathcal{F}(S)$ denote the filter generated by S. In particular, for a term τ, $\mathcal{F}(\tau)$ denotes the principal filter generated by $\{\tau\}$, i.e., the upward closure of the singleton set $\{\tau\}$. We will need $\mathcal{E}$, the filter generated by reflexive residuated terms (terms of the form $x \setminus x$ and y/y). Observe that the set of reflexive residuated terms is closed under meet by axiom (6). Hence $\mathcal{E}$ is given by the upward closure of these elements. Also note that $\mathcal{E}$ is closed under composition by axiom (5).

We will define labelled, directed graphs $G_\alpha = (U_\alpha, \ell_\alpha)$ where U_α is the set of nodes and $\ell_\alpha \colon U_\alpha \times U_\alpha \to \wp(T_X)$ is a labelling function. We will use the notation $E_\alpha \subseteq U_\alpha \times U_\alpha$ for the set of edges with non-empty labels. We will make sure that E_α is reflexive, transitive, antisymmetric. Furthermore, for every $(u, v) \in E_\alpha$ with $u \neq v$, we will choose $\ell_\alpha(u, v)$ be a principal filter. We will also maintain the following coherence condition.

Coherence: for all $u, v, w \in U_\alpha$, we have $\ell_\alpha(u, w) \,;\, \ell_\alpha(w, v) \subseteq \ell_\alpha(u, v)$ where $\ell_\alpha(u, w) \,;\, \ell_\alpha(w, v) = \{\sigma \,;\, \tau : \sigma \in \ell_\alpha(u, w), \tau \in \ell_\alpha(w, v)\}$.

In the 0th step of the step-by-step construction we define $G_0 = (U_0, \ell_0)$. We define U_0 by choosing distinct u_τ, v_τ for distinct terms τ, and define

$$\ell_0(u_\tau, u_\tau) = \ell_0(v_\tau, v_\tau) = \mathcal{E}$$
$$\ell_0(u_\tau, v_\tau) = \mathcal{F}(\tau)$$

and we label all other edges by $\emptyset$. Observe that E_0 is reflexive, transitive, antisymmetric. Note that the non-empty labels on irreflexive edges are principal filters and that they are coherent, e.g., for every $\epsilon \in \ell_0(u_\tau, u_\tau)$ and $\sigma \in \ell_0(u_\tau, v_\tau)$, we have $\epsilon ; \sigma \in \ell_0(u_\tau, v_\tau)$ by axiom (5).

In the $(\alpha+1)$th step we have three subcases. To deal with the residual $\backslash$ we choose a fresh point z, for every point $x \in U_\alpha$ and term τ, and define

$$\begin{aligned}
\ell_{\alpha+1}(z,z) &= \mathcal{E}\\
\ell_{\alpha+1}(z,x) &= \mathcal{F}(\tau)\\
\ell_{\alpha+1}(z,p) &= \mathcal{F}(\tau ; \ell_\alpha(x,p)) \qquad p \neq x, z
\end{aligned}$$

when $(x,p) \in E_\alpha$. For all other edges (u,v), we let $\ell_{\alpha+1}(u,v) = \ell_\alpha(u,v)$ if $\ell_\alpha(u,v) \in E_\alpha$ and $\ell_{\alpha+1}(u,v) = \emptyset$ if $\ell_\alpha(u,v) \notin E_\alpha$. See Figure 1. Note that

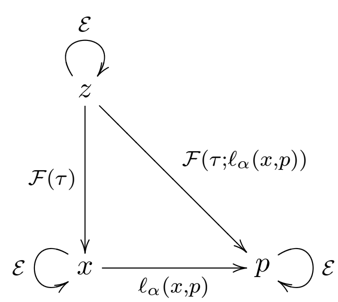

FIGURE 1. Step for the residual

the labels on irreflexive edges are indeed principal filters, since $\ell_{\alpha+1}(z,p) = \mathcal{F}(\tau ; \ell_\alpha(x,p))$ and $\ell_\alpha(x,p)$ is a principal filter by the induction hypothesis. Coherence is easy to check as are the properties on non-empty edges $E_{\alpha+1}$. The case for $/$ is completely analogous; we leave the details to the reader.

To deal with composition $;$ we choose a fresh point z, for every $\tau ; \sigma \in \ell_\alpha(x,y)$ and $x \neq y$, and define

$$\begin{aligned}
\ell_{\alpha+1}(z,z) &= \mathcal{E}\\
\ell_{\alpha+1}(x,z) &= \mathcal{F}(\tau)\\
\ell_{\alpha+1}(z,y) &= \mathcal{F}(\sigma)\\
\ell_{\alpha+1}(r,z) &= \mathcal{F}(\ell_\alpha(r,x) ; \tau) \qquad r \neq x, z\\
\ell_{\alpha+1}(z,s) &= \mathcal{F}(\sigma ; \ell_\alpha(y,s)) \qquad s \neq y, z
\end{aligned}$$

whenever $(r,x),(y,s) \in E_\alpha$. For all other edges (u,v), we let $\ell_{\alpha+1}(u,v) = \ell_\alpha(u,v)$ if $\ell_\alpha(u,v) \in E_\alpha$ and $\ell_{\alpha+1}(u,v) = \emptyset$ if $\ell_\alpha(u,v) \notin E_\alpha$. See Figure 2. Observe that the labels on irreflexive edges are indeed principal filters. Then checking coherence and the properties on $E_{\alpha+1}$ are routine.

Limit step of the construction: take the union of the constructed labelled structures.

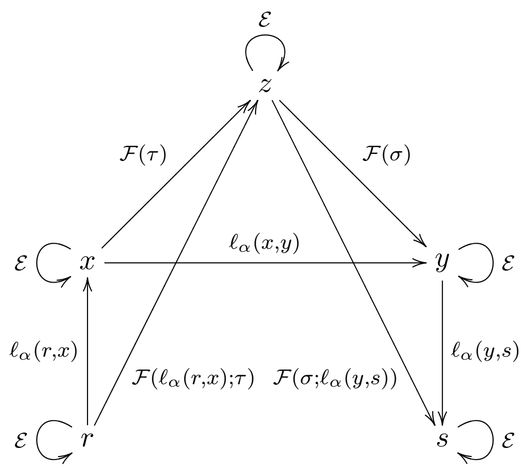

FIGURE 2. Step for composition

After the construction terminates we end up with a labelled structure $G_\infty = (U_\infty, \ell_\infty)$. Note that G_∞ is coherent, the set E_∞ of non-empty edges is a reflexive, transitive, antisymmetric relation and the non-empty labels on irreflexive edges are principal filters.

Recall that we made the step for composition only if $x \neq y$. So, in principle, it might happen that $\tau\,;\sigma \in \ell_\infty(u,u)$, but there is no $v \in U_\infty$ such that $\tau \in \ell_\infty(u,v)$ and $\sigma \in \ell_\infty(v,u)$. We will see that this in fact cannot arise, since we will have $\tau, \sigma \in \ell_\infty(u,u)$ in this case, see Lemma 2.2 below.

Next we define a valuation ι of variables. We let

$$\iota(x) = \{(u,v) \in U_\infty \times U_\infty : x \in \ell_\infty(u,v)\}$$

for every variable $x \in X$. Note that $\iota(x) \subseteq E_\infty$ is an irreflexive relation. Indeed, $x \notin \mathcal{E}$, since $\tau \setminus \tau \leq x$ is not valid, hence is not derivable from $\mathrm{Ax}(\cdot, ;, \setminus, /)$, for any term τ and variable x. Let $\mathfrak{A} = (A, \cdot, ;, \setminus, /)$ be the subalgebra of the full algebra $(\wp(U_\infty \times U_\infty), \cdot, ;, \setminus, /)$ generated by $\{\iota(x) : x \in X\}$.

Lemma 2.2. *For every term τ and $(u,v) \in U_\infty \times U_\infty$,*

$$(u,v) \in \tau^{\mathfrak{A}} \textit{ iff } \tau \in \ell_\infty(u,v)$$

where $\tau^{\mathfrak{A}}$ is the interpretation of τ in $\mathfrak{A}$ under the valuation ι.

Proof. We prove the lemma by induction on terms. The case when τ is a variable is straightforward by the definition of the valuation ι. The case $\tau = \sigma \cdot \rho$ easily follows from the induction hypothesis (IH), since the labels are filters.

Next consider the case $\tau = \sigma\,;\rho$ and assume that $(u,v) \in (\sigma\,;\rho)^{\mathfrak{A}}$. Then $(u,w) \in \sigma^{\mathfrak{A}}$ and $(w,v) \in \rho^{\mathfrak{A}}$ for some $w \in U_\infty$. By IH we have $\sigma \in \ell_\infty(u,w)$ and $\rho \in \ell_\infty(w,v)$. By the coherence of G_∞ we get that $\sigma\,;\rho \in \ell_\infty(u,v)$ as desired.

Now assume that $\sigma\,;\rho \in \ell_\infty(u,v)$. First consider the case when $u \neq v$. During the construction, we put $z \in U_\infty$ such that $\sigma \in \ell_\infty(u,z)$ and $\rho \in \ell_\infty(z,v)$. By IH we get $(u,z) \in \sigma^{\mathfrak{A}}$ and $(z,v) \in \rho^{\mathfrak{A}}$, whence $(u,v) \in (\sigma\,;\rho)^{\mathfrak{A}}$ as desired. Next assume that $u = v$, i.e., $\sigma\,;\rho \in \ell_\infty(u,u) = \mathcal{E}$. Since $\mathfrak{A}$ is a representable algebra, we have $(u,u) \in \epsilon^{\mathfrak{A}}$ for every $\epsilon \in \mathcal{E}$ and, in particular, $(u,u) \in (\sigma\,;\rho)^{\mathfrak{A}}$.

The final case is when τ is a residuated term, say, $\sigma \setminus \rho$. First assume that $(u,v) \in (\sigma \setminus \rho)^{\mathfrak{A}}$. Then for every $w \in U_\infty$, $(w,u) \in \sigma^{\mathfrak{A}}$ implies $(w,v) \in \rho^{\mathfrak{A}}$. During the construction we created $z \in U_\infty$ such that $\sigma \in \ell_\infty(z,u) = \mathcal{F}(\sigma)$, whence $(z,u) \in \sigma^{\mathfrak{A}}$ by IH. Then, by the definition of $\setminus$ in representable algebras, $(z,v) \in \rho^{\mathfrak{A}}$, whence $\rho \in \ell_\infty(z,v)$ by IH. We distinguish two cases according to whether u and v are different. If $u \neq v$, then $\ell_\infty(z,v) = \mathcal{F}(\sigma\,;\ell_\infty(u,v))$ by the construction. Let γ be a term such that $\ell_\infty(u,v) = \mathcal{F}(\gamma)$. Since $\rho \in \ell_\infty(z,v)$, we have $\rho \geq \sigma\,;\gamma$. Thus $\gamma \leq \sigma \setminus \rho$ by the axioms for the residuals, whence $\sigma \setminus \rho \in \ell_\infty(u,v)$. If $u = v$, then $\rho \in \ell_\infty(z,u) = \mathcal{F}(\sigma)$. Thus $\sigma \leq \rho$. By axiom (2) we have $\sigma \setminus \rho \geq \sigma \setminus \sigma \in \mathcal{E}$, whence $\sigma \setminus \rho \in \mathcal{E} = \ell_\infty(u,u)$ as desired.

Finally assume that $\sigma \setminus \rho \in \ell_\infty(u,v)$. Let $w \in U_\infty$ such that $(w,u) \in \sigma^{\mathfrak{A}}$. We have to show $(w,v) \in \rho^{\mathfrak{A}}$. By IH we have $\sigma \in \ell_\infty(w,u)$. By coherence of G_∞ we get $\sigma\,;\sigma \setminus \rho \in \ell_\infty(w,v)$. Hence $\rho \in \ell_\infty(w,v)$ by $\sigma\,;\sigma \setminus \rho \leq \rho$. By IH we get $(w,v) \in \rho^{\mathfrak{A}}$, finishing the proof of Lemma 2.2. □

Define

$$\mathrm{rep}(\tau) = \{(u,v) \in U_\infty \times U_\infty : \tau \in \ell_\infty(u,v)\}$$

for every term τ. Then rep is an isomorphism between $\mathfrak{F}_X$ and $\mathfrak{A}$ by Lemma 2.2. That is, $\mathfrak{F}_X$ is representable, finishing the proof of Theorem 2.1. □

Remark 2.3. The reader may wonder whether there is a finite axiomatization for the quasivariety $\mathsf{R}(\cdot, ;, \setminus, /)$ of representable algebras. The problem with representing an arbitrary algebra $\mathfrak{B}$ satisfying the axioms is as follows. Assume that $a \setminus a \leq b\,;c$ in $\mathfrak{B}$, for some elements b, c that are not in $\mathcal{E}$, and we are in a step-by-step construction dealing with composition for $a \setminus a \in \ell_\alpha(u,u)$. Then we need v such that $b \in \ell_{\alpha+1}(u,v)$ and $c \in \ell_{\alpha+1}(v,u)$. These labels are not difficult to find, but we need an appropriate label for (v,v) as well. The label $\ell_{\alpha+1}(v,v)$ should include $c;b$ and all reflexive residuated terms, and hence their meets as well. There are valid quasiequations that guarantee the existence of suitable labels, see below, but it is an open problem whether there is a finite base for all these quasiequations.

Consider the following quasiequations q_n for $n \in \omega \smallsetminus \{0\}$:

$$a \setminus a \leq b\,;c \Rightarrow d \leq d\,;(b\,;[(c\,;b) \cdot (a \setminus a)]^n\,;c)$$

where $x^1 = x$ and $x^{n+1} = x\,;x^n$. We claim that, for every $n \geq 1$, we have $\mathsf{R}(\cdot, ;, \setminus) \models q_n$. Let $\mathfrak{C} \in \mathsf{R}(\cdot, ;, \setminus)$ be an algebra represented on a set U. Assume that $(u,v) \in d$. Since $a \setminus a$ contains the identity on U, we have $(v,v) \in a \setminus a$. By $a \setminus a \leq b\,;c$, we get $(v,w) \in b$ and $(w,v) \in c$ for some

$w \in U$. Also $(w, w) \in a \setminus a$. Then $(w, w) \in [(c\,;b) \cdot (a \setminus a)]^n$, for every $n \geq 1$. Thus $(v, v) \in b\,;[(c\,;b) \cdot (a \setminus a)]^n\,;c$, whence $(u, v) \in d\,;(b\,;[(c\,;b) \cdot (a \setminus a)]^n\,;c)$ as desired.

Problem 2.4. *Are the representation classes* $\mathsf{R}(\cdot, ;, \setminus)$ *and* $\mathsf{R}(\cdot, ;, \setminus, /)$ *finitely axiomatizable?*

Interestingly, if we assume commutativity $(x\ ;\ y = y\ ;\ x)$ as an additional axiom, we have finite axiomatization of the commutative subclass of $\mathsf{R}(\cdot, ;, \setminus, /)$, see [Mik??].

3. Conclusion

The class of representable ordered residuated semigroups, i.e., algebraic structures of similarity type $(;, \setminus, /, \leq)$, is finitely axiomatizable, [AM94]. On the other hand, we have negative results when join is included into the signature. The (quasi)equational theories of representable upper semilattice-ordered and distributive lattice-ordered residuated semigroups, $\mathsf{R}(+, ;, \setminus, /)$ and $\mathsf{R}(+, \cdot, ;, \setminus, /)$, are not finitely based, [AMN12, Mik11].

We conclude with an open problem.

Problem 3.1. *Is (the equational theory of)* $\mathsf{R}(\cdot, ;, \setminus, /, 1')$ *finitely axiomatizable?*

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Department of Computer Science and Information Systems, Birkbeck, University of London, Malet Street, London WC1E 7HX, UK