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New examples of mixed Beauville groups

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Abstract. We generalise a construction of mixed Beauville groups first given by Bauer, Catanese and Grunewald. We go on to give several examples of infinite families of characteristically simple groups that satisfy the hypotheses of our theorem and thus provide a wealth of new examples of mixed Beauville groups.

1 Introduction

We recall the following definition first made by Bauer, Catanese and Grunewald in [5, Definition 4.1].

Definition 1.1. Let G be a finite group and for $x, y \in G$ define

$$\Sigma(x, y) = \bigcup_{i=1}^{|G|} \bigcup_{g \in G} \{(x^i)^g, (y^i)^g, ((xy)^i)^g\}.$$

A *mixed Beauville quadruple* for G is a quadruple $(G^0; a, c; g)$ consisting of a subgroup G^0 of index 2 in G , of elements $a, c \in G^0$ and of an element $g \in G$ such that

- (M1) G^0 is generated by a and c ,
- (M2) $g \notin G^0$,
- (M3) for every $\gamma \in G^0$ we have that $(g\gamma)^2 \notin \Sigma(a, c)$,
- (M4) $\Sigma(a, c) \cap \Sigma(a^g, c^g) = \{e\}$.

If G has a mixed Beauville quadruple we say that G is a *mixed Beauville group* and call $(G^0; a, c; g)$ a *mixed Beauville structure* on G .

We will not discuss the corresponding ‘unmixed’ Beauville groups in this paper. Beauville groups were originally introduced in connection with a class of complex surfaces of general type, known as Beauville surfaces. These surfaces possess

many useful geometric properties: their automorphism groups [26] and fundamental groups [10] are relatively easy to compute and these surfaces are rigid in the sense of admitting no non-trivial deformations [6] and thus correspond to isolated points in the moduli space of surfaces of general type. Early motivation came from providing cheap counterexamples to the so-called ‘Friedman–Morgan speculation’ [18] but they also provide a wide class of surfaces that are unusually easy to deal with to test conjectures and provide counterexamples. A number of excellent surveys on these and closely related matters have appeared in recent years – see any of [4, 8, 9, 16, 29, 39] and the references therein.

We will require the following definition.

Definition 1.2. Let G be a finite group. For $x, y \in G$ we write

$$v(x, y) = o(x)o(y)o(xy).$$

To construct examples of mixed Beauville groups, Bauer, Catanese and Grunewald proved the following in [5, Lemma 4.5].

Theorem 1.3. Let H be a perfect finite group and $a_1, c_1, a_2, c_2 \in H$. Assume that

- (1) $o(a_1)$ and $o(c_1)$ are even,
- (2) $\langle a_1, c_1 \rangle = H$,
- (3) $\langle a_2, c_2 \rangle = H$,
- (4) $v(a_1, c_1)$ is coprime to $v(a_2, c_2)$.

Set $G := (H \times H) : \langle g \rangle$ where g is an element of order 4 that acts by interchanging the two factors; $G^0 = H \times H \times \langle g^2 \rangle$; $a := (a_1, a_2, g^2)$ and $c := (c_1, c_2, g^2)$. Then $(G^0; a, c; g)$ is a mixed Beauville structure on G .

The only examples of groups satisfying the hypotheses of Theorem 1.3 given by Bauer, Catanese and Grunewald were the alternating groups A_n ‘for large n ’ (proved using the heavy duty machinery first employed to verify Higman’s conjecture concerning alternating groups as images of triangle groups) and the groups $\mathrm{SL}_2(p)$ with $p \neq 2, 3, 5, 17$ prime (though their argument also does not apply to the case $p = 7$).

More generally, mixed Beauville groups have proved extremely difficult to construct. Bauer, Catanese and Grunewald show [7] that there are two groups of order 2^8 which admit a mixed Beauville structure but no group of smaller order, however, the method they use is that of checking every group computationally. Indeed, any p -group admitting a mixed Beauville structure must be a 2-group. Barker, Boston, Peyerimhoff and Vdovina construct five new examples of mixed Beauville 2-groups in [2] and an infinite family in [3] and the aforementioned

examples account for all presently known mixed Beauville groups. Non-examples of mixed Beauville groups, however, are in abundance. The following result is due to Fuertes and González-Diez [20, Lemma 5].

Lemma 1.4. *Let $(C \times C)/G$ be a Beauville surface of mixed type and G^0 the subgroup of G consisting of the elements which preserve each of the factors. Then the order of any element $f \in G \setminus G^0$ is divisible by 4.*

Fuertes and González-Diez used the above to prove that no symmetric group is a mixed Beauville group. It is easy to see, however that this result actually rules out most almost simple groups [15] (though, as the groups $P\Sigma L_2(p^2)$ with p prime show, not quite all of them). Bauer, Catanese and Grunewald also show in [5, Theorem 4.3] that G^0 must be non-abelian.

In light of the above discussion we make the following definition.

Definition 1.5. Let H be a group satisfying the hypotheses of Theorem 1.3. Then we say that H is a *mixable Beauville group*. More specifically, H is a mixable Beauville group if H is a perfect group and there exist $x_1, y_1, x_2, y_2 \in H$ such that $o(x_1)$ and $o(y_1)$ are even, $\langle x_1, y_1 \rangle = \langle x_2, y_2 \rangle = H$ and $\nu(x_1, y_1)$ is coprime to $\nu(x_2, y_2)$. We call (x_1, y_1, x_2, y_2) a *mixable Beauville structure* for H of type $(o(x_1), o(y_1), o(x_1 y_1); o(x_2), o(y_2), o(x_2 y_2))$.

Thus any mixable Beauville group automatically gives us a mixed Beauville group by Theorem 1.3. In Section 2 we will prove a generalisation of Theorem 1.3 that shows a given mixable Beauville group actually provides us with a wealth of mixed Beauville groups.

We remark that finding 2-generated non-mixable Beauville groups is not difficult. Any group with a subgroup of index 2 cannot be mixable (since any generating set must contain elements of even order and so we cannot satisfy condition (4) of Theorem 1.3); p -groups are not mixable (to have an index 2 subgroup we must have $p = 2$ and then again the conditions in Definition 1.5 cannot be satisfied) and even among the finite simple groups, $L_2(2^n)$ for $n \geq 2$ are not mixable (the only elements of even order have order 2 and thus condition (2) of Theorem 1.3 cannot be satisfied). Despite this, we prove the following.

Theorem 1.6. *If G belongs to any of the following families of simple groups*

- *the alternating groups A_n , $n \geq 6$,*
- *the linear groups $L_2(q)$ with $q \geq 7$ odd,*
- *the unitary groups $U_3(q)$ with $q \geq 3$,*
- *the Suzuki groups ${}^2B_2(2^{2n+1})$ with $n \geq 1$,*

- the small Ree groups ${}^2G_2(3^{2n+1})$ with $n \geq 1$,
- the exceptional groups $G_2(q)$ with $q \geq 3$,
- the large Ree groups ${}^2F_4(2^{2n+1})$ with $n \geq 1$,
- the Steinberg triality groups ${}^3D_4(q)$ with $q \geq 2$, or
- the sporadic groups (including the Tits group ${}^2F_4(2)'$),

then G is a mixable Beauville group. Furthermore, if G is any of the above or $L_2(2^n)$ with $n \geq 3$ then $G \times G$ is also a mixable Beauville group.

In light of the above theorem we make the following conjecture.

Conjecture 1.7. If G is a non-abelian finite simple group not isomorphic to $L_2(2^n)$ for $n \geq 2$ then G is a mixable Beauville group.

Groups of the form G^n where G is a finite simple group are called characteristically simple groups. The study of characteristically simple Beauville groups has recently been initiated by Jones in [27,28]. It is easy to show that the characteristically simple group $L_2(7) \times L_2(7) \times L_2(7)$ is not a mixable Beauville group. We thus ask the following natural question.

Question 1.8. If G is a simple group then for which n is G^n a mixable Beauville group?

Throughout we use the standard ‘Atlas’ notation for finite groups and related concepts as described in [12]. In Section 2 we discuss a generalisation of Theorem 1.3 which enables mixable Beauville groups to define several mixed Beauville groups before turning our attention in the remaining sections to the proof of Theorem 1.6.

2 Generalisation and auxiliary results

We begin with the following definition which will be key to the generating structures we demonstrate.

Definition 2.1. Let G be a finite group and $x, y, z \in G$. A *hyperbolic generating triple* for G is a triple $(x, y, z) \in G \times G \times G$ such that

- (1) $\frac{1}{o(x)} + \frac{1}{o(y)} + \frac{1}{o(z)} < 1$,
- (2) $\langle x, y, z \rangle = G$, and
- (3) $xyz = 1$.

The *type* of a hyperbolic generating triple (x, y, z) is the triple $(o(x), o(y), o(z))$. If at least two of $o(x), o(y)$ and $o(z)$ are even then we call (x, y, z) an *even triple*. If $o(x), o(y)$ and $o(z)$ are all odd then we call (x, y, z) an *odd triple*. It is clear that since $z = (xy)^{-1}$, being generated by x, y and z is equivalent to being generated by x and y . If the context is clear we may refer to a hyperbolic generating triple simply as a ‘triple’ for brevity and write (x, y, xy) or even just (x, y) .

Remark 2.2. Occasionally we may need to specify which conjugacy class an element of a certain order belongs to. For example, if a group G has a hyperbolic generating triple of type $(6, 6, 7)$, where the elements of order 6 belong to the conjugacy class $6C$ and the element of order 7 belongs to the conjugacy class $7A$, then we may simply write $(6C, 6C, 7A)$ as the type of our triple.

For a positive integer k let Q_{4k} be the dicyclic group of order $4k$ with presentation

$$Q_{4k} = \langle p, q \mid p^{2k} = q^4 = 1, p^q = p^{-1}, p^k = q^2 \rangle.$$

Let $G = (H \times H) : Q_{4k}$ with the action of Q_{4k} defined as follows. For any element $(g_1, g_2) \in H \times H$ let $p(g_1, g_2) = (g_1, g_2)$ and $q(g_1, g_2) = (g_2, g_1)$. Then $G^0 = H \times H \times \langle p \rangle$ is a subgroup of index 2 inside G .

Theorem 2.3. *Let H be a perfect finite group and $a_1, c_1, a_2, c_2 \in H$. Assume that*

- (1) *the orders of a_1, c_1 are even,*
- (2) $\langle a_1, c_1 \rangle = H,$
- (3) $\langle a_2, c_2 \rangle = H,$
- (4) $v(a_1, c_1)$ *is coprime to* $v(a_2, c_2).$

Set $k > 1$ to be any integer that divides $\gcd(o(a_1), o(c_1))$, $G := (H \times H) : Q_{4k}$, $G^0 := H \times H \times \langle p \rangle$, $a := (a_1, a_2, p)$ and $c := (c_1, c_2, p^{-1})$. Then $(G^0; a, c; q)$ is a mixed Beauville structure on G .

Proof. We verify that the conditions of Definition 1.1 are satisfied. Since k divides $v(a_1, c_1)$ it is coprime to $v(a_2, c_2)$ so we can clearly generate the elements $(1, a_2, 1)$ and $(1, c_2, 1)$ giving us the second factor. This also shows that we can generate the elements $a' = (a_1, 1, p)$ and $c' = (c_1, 1, p^{-1})$. Since H is perfect we can then generate the first factor. Finally, since we can generate $H \times H$ we can clearly generate $\langle p \rangle$, hence we satisfy condition (M1).

Now let $g \in G \setminus G^0$ and $\gamma \in G^0$. Then $g\gamma$ is of the form $(h_1, h_2, q^i p^j)$ for some $h_1, h_2 \in H, i = 1, 3$ and $1 \leq j \leq 2k$. Then

$$(g\gamma)^2 = (h_1 h_2, h_2 h_1, (q^i p^j)^2) = (h_1 h_2, h_2 h_1, p^k).$$

For a contradiction, suppose that $(g\gamma)^2 \in \Sigma(a, c)$. Then since $h_1 h_2$ has the same

order as h_2h_1 condition (4) implies that $(g\gamma)^2 = (1, 1, p^k) \in \Sigma(a, c)$ if and only if k does not divide $o(a)$ or k does not divide $o(c)$. Note that if $(g\gamma)^2$ were a power of ac by construction it would be 1_G . Since by hypothesis k divides $\gcd(o(a_1), o(c_1))$ we satisfy conditions (M2) and (M3).

To show that condition (M4) is satisfied, suppose $g' \in \Sigma(a, c) \cap \Sigma(a^g, c^g)$ for $g \in G \setminus G^0$. Since conjugation by such an element g interchanges the first two factors of any element we again have from condition (4) that g' is of the form $(1, 1, p^i)$ for some power of p , but from our previous remarks it is clear that $p^i = 1_H$ and so $g' = 1_G$. \square

Remark 2.4. In the proof of Theorem 2.3 above we chose $a := (a_1, a_2, p)$ and $c := (c_1, c_2, p^{-1})$ but in principle we could have chosen the third factor of a or c to be 1 and the third factor in their product ac to be p or p^{-1} as appropriate. If we then require k to divide $\gcd(o(a), o(ac))$ or $\gcd(o(ac), o(c))$ as necessary this gives rise to further examples of mixed Beauville groups.

We conclude this section with a number of results regarding hyperbolic generating triples for $G \times G$. The number of prime divisors of the order of a group will pose an obvious restriction on finding a mixable Beauville structure and so naturally we would like to know for how large an n we can generate G^n with a triple of a given type. Hall treats this question in much more generality in [25], here we simply mention that for a hyperbolic generating triple (x, y, z) of a given type this depends on how many orbits there are of triples of the same type under the action of $\text{Aut}(G)$.

Definition 2.5. Let G be a finite group and let $(a_1, b_1, c_1), (a_2, b_2, c_2)$ be hyperbolic generating triples for G . We call these two triples *equivalent* if there exists an element $g \in \text{Aut}(G)$ such that $\{a_1^g, b_1^g, c_1^g\} = \{a_2, b_2, c_2\}$.

Since conjugate elements must have the same order the following is immediate.

Lemma 2.6. *Let G be a finite group and let (a_1, b_1, c_1) be a hyperbolic generating triple of G of type (l_1, m_1, n_1) and (a_2, b_2, c_2) a hyperbolic generating triple of G of type (l_2, m_2, n_2) . If $\{l_1, m_1, n_1\} \neq \{l_2, m_2, n_2\}$ then (a_1, b_1, c_1) and (a_2, b_2, c_2) are inequivalent triples.*

Lemma 2.7. *Let G be a non-abelian finite group. Let (a_1, b_1, c_1) and (a_2, b_2, c_2) be equivalent hyperbolic generating triples for G for some $g \in \text{Aut}(G)$. Then*

- (a) if $a_1^g = a_2$ then $b_1^g = b_2$ and $c_1^g = c_2$,
- (b) if $a_1^g = b_2$ then $b_1^g = c_2$ and $c_1^g = a_2$, and
- (c) if $a_1^g = c_2$ then $b_1^g = a_2$ and $c_1^g = b_2$.

Proof. Note that in this instance we take $a_i b_i c_i = 1_G$ for $i = 1, 2$. Let $a_1^g = a_2$ and suppose $b_1^g = c_2 = (a_2 b_2)^{-1}$ and $c_1^g = b_2$. Then

$$b_2 = c_1^g = (b_1^{-1} a_1^{-1})^g = c_2^{-1} a_2^{-1} = a_2 b_2 a_2^{-1}$$

which implies that G is generated by elements that commute, a contradiction since G is non-abelian. The proof is analogous for the remaining two statements. \square

Lemma 2.8. *Let G be a finite group, (x, y, z) a hyperbolic generating triple for G and $\gcd(o(x), o(y)) = 1$. Then $((x, y), (y, x^y), (z, z))$ is a hyperbolic generating triple for $G \times G$.*

Proof. If (x, y, z) is a hyperbolic generating triple of G then so is (y, x^y, z) since $x^y = y^{-1}xy$. Then, since the orders of x and y are coprime we can produce the elements $(x, 1_G), (y, 1_G), (1_G, y)$ and $(1_G, x^y)$ which generate $G \times G$. \square

Remark 2.9. The proof of the preceding lemma naturally generalises to any pair, or indeed triple, of elements in a hyperbolic generating triple whose orders are coprime.

3 The alternating groups

We make heavy use of the following theorem due to Jordan.

Theorem 3.1 (Jordan). *Let G be a primitive permutation group of finite degree n , containing a cycle of prime length fixing at least three points. Then $G \geq A_n$.*

Lemma 3.2. *The alternating group A_6 and $A_6 \times A_6$ are mixable.*

Proof. For our even triples we take the following elements in the natural representation of A_6 ,

$$\begin{aligned} x_1 &= (1, 2)(3, 4, 5, 6), & y_1 &= (1, 5, 6, 4)(2, 3), \\ x'_1 &= (1, 2)(3, 4, 5, 6), & y'_1 &= (1, 5, 6). \end{aligned}$$

It can easily be checked in GAP [22] that $(x_1, y_1, x_1 y_1)$ is an even triple for A_6 of type $(4, 4, 4)$ and that $((x_1, x'_1), (y_1, y'_1), (x_1 y_1, x'_1 y'_1))$ is an even triple for $A_6 \times A_6$ of type $(4, 12, 12)$.

For our odd triples, let $x_2 = (1, 2, 3, 4, 5)$, $y_2 = x_2^{(1,3,6)}$ and $y'_2 = x_2^{(1,2,3,4,6)}$. Then it can be checked that $(x_2, y_2, x_2 y_2)$ is an odd triple of type $(5, 5, 5)$ for A_6 and $((x_2, x_2), (y_2, y'_2), (x_2 y_2, x_2 y'_2))$ is an odd triple for $A_6 \times A_6$ also of type $(5, 5, 5)$. Therefore we have a mixable Beauville structure of type $(4, 4, 4; 5, 5, 5)$ for A_6 and type $(4, 12, 12; 5, 5, 5)$ for $A_6 \times A_6$. \square

Lemma 3.3. *The alternating group A_7 and $A_7 \times A_7$ are mixable.*

Proof. For our even triples we take the following elements in the natural representation of A_7 ,

$$\begin{aligned} x_1 &= (1, 2)(3, 4)(5, 6, 7), & y_1 &= (1, 2, 3)(4, 5)(6, 7), \\ x'_1 &= (1, 6)(2, 4, 5)(3, 7), & y'_1 &= (1, 6, 2)(3, 7, 4). \end{aligned}$$

It can easily be checked in GAP that $(x_1, y_1, x_1 y_1)$ is an even triple for A_7 of type $(6, 6, 5)$ and that $((x_1, x'_1), (y_1, y'_1), (x_1 y_1, x'_1 y'_1))$ is an even triple for $A_7 \times A_7$ of type $(6, 6, 5)$.

For our odd triples let $x_2 = (1, 2, 3, 4, 5, 6, 7)$, $y_2 = x_2^{(1,2,3)}$ and $y'_2 = x_2^{(1,3,2)}$. Again, it can be checked that $(x_2, y_2, x_2 y_2)$ is an odd triple of type $(7, 7, 7)$ for A_7 and that $((x_2, x_2), (y_2, y'_2), (x_2 y_2, x_2 y'_2))$ is an odd triple also of type $(7, 7, 7)$ for $A_7 \times A_7$. Therefore both A_7 and $A_7 \times A_7$ admit a mixable Beauville structure of type $(6, 6, 5; 7, 7, 7)$. \square

Lemma 3.4. *The alternating group A_{2m} and $A_{2m} \times A_{2m}$ are mixable for $m \geq 4$.*

Proof. For $m \geq 4$ let $G = A_{2m}$ under its natural representation and consider the elements

$$\begin{aligned} a_1 &= (1, 2)(3, \dots, 2m), \\ b_1 &= a_1^{(1,3,4)} = (1, 5, 6, \dots, 2m, 4)(2, 3), \\ a_1 b_1 &= (1, 3)(2, 5, 7, \dots, 2m-3, 2m-1, 4, 6, 8, \dots, 2m). \end{aligned}$$

The subgroup $H_1 = \langle a_1, b_1 \rangle$ is clearly transitive and the elements

$$\begin{aligned} a_1^2 &= (3, 5, \dots, 2m-1)(4, 6, \dots, 2m), \\ b_1^2 &= (1, 6, 8, \dots, 2m)(5, 7, \dots, 2m-1, 4) \end{aligned}$$

fix the point 2 and act transitively on the remaining points. Finally,

$$a_1^2 b_1^{-2} = (1, 2m, 2m-1, 3, 4)$$

is a cycle of length 5, which is prime, fixing at least three points for all m and so by Jordan's theorem $H_1 = G$. This gives us our first hyperbolic generating triple of type $(2m-2, 2m-2, 2m-2)$ for G . For our second, we show that there is a similar triple which is inequivalent to the first under the action of $\text{Aut}(G) = S_{2m}$. Consider the elements

$$\begin{aligned} a'_1 &= (1, 2)(3, \dots, 2m), \\ b'_1 &= a_1^{(1,4,3)} = (1, 3, 5, 6, \dots, 2m)(2, 4), \\ a'_1 b'_1 &= (1, 4, 6, \dots, 2m, 5, 7, \dots, 2m-1)(2, 3) \end{aligned}$$

and note that $a'_1 = a_1$. For the same argument as before we have that $\langle a'_1, b'_1 \rangle = G$. Now suppose that $(a_1, b_1, a_1 b_1)$ is equivalent to $(a'_1, b'_1, a'_1 b'_1)$ for some automorphism $g \in \text{Aut}(G)$. If $a_1^g = a'_1$ then by Lemma 2.7 we have that $b_1^g = b'_1$ and $(a_1 b_1)^g = a'_1 b'_1$. Then $(1, 2)^g = (1, 2)$ and $(2, 3)^g = (2, 4)$ but these are incompatible with $(1, 3)^g = (2, 3)$ since for the former to hold 3 must map to 4 which is incompatible with the latter. Now suppose $a_1^g = b'_1$ implying $b_1^g = a'_1 b'_1$ and $a_1 b_1 = a_1$. Then similarly we have $(1, 2)^g = (2, 4)$ and $(2, 3)^g = (2, 3)$ forcing g to map 1 to 4 which is incompatible with requiring that $(1, 3)^g = (1, 2)$. Finally, if $a_1^g = a'_1 b'_1$, forcing $b_1^g = a'_1$ and $a_1 b_1 = b'_1$, we get that $(1, 2)^g = (2, 3)$ and $(2, 3)^g = (1, 2)$ and we find this is incompatible with $(1, 3)^g = (2, 4)$. Hence these two hyperbolic generating triples are inequivalent under the action of the automorphism group of G and so $((a_1, a'_1), (b_1, b'_1), (a_1 b_1, a'_1 b'_1))$ is a hyperbolic generating triple for $G \times G$ of type $(2m - 2, 2m - 2, 2m - 2)$.

For our first odd triple consider the elements

$$\begin{aligned} a_2 &= (1, 2, \dots, 2m - 1), \\ b_2 &= a_2^{(1, 2m, 3)} = (1, 4, 5, \dots, 2m - 1, 2m, 2), \\ a_2 b_2 &= (2, 3, 5, 7, \dots, 2m - 1, 4, 6, 8, \dots, 2m - 2, 2m) \end{aligned}$$

and let $H_2 = \langle a_2, b_2 \rangle$. We clearly have transitivity and 2-transitivity, hence H_2 is primitive. Since $a_2 b_2^{-1} = (1, 2m, 2m - 1, 2, 3)$ is a prime cycle fixing at least three points for all m , again we can apply Jordan's theorem and we have that $H_2 = G$. Then (a_1, b_1, a_2, b_2) is a mixable Beauville structure on G of type $(2m - 2, 2m - 2, 2m - 2; 2m - 1, 2m - 1, 2m - 1)$. For our second odd triple consider the elements

$$\begin{aligned} a'_2 &= (1, 2, \dots, 2m - 1), \\ b'_2 &= a'^2_{2}^{(1, 3, 2m)} = (2, 2m, 4, \dots, 2m - 1, 3), \\ a'_2 b'_2 &= (1, 2m, 4, 6, \dots, 2m - 2, 3, 5, \dots, 2m - 1) \end{aligned}$$

and note that $a'_2 = a_2$. It follows that $\langle a_2, b_2 \rangle = G$ from a similar argument as before and so it remains to show that $\langle (a_2, a'_2), (b_2, b'_2) \rangle = G \times G$. Let $g \in \text{Aut}(G)$ and suppose that $a_2^g = a'_2$. Then by Lemma 2.7 and inspection of the fixed points of these triples we have g fixes $2m$ and maps $3 \rightarrow 1$ and $1 \rightarrow 2$ which is incompatible with $a_2^g = a'_2$. Similarly, if $a_2^g = b'_2$ then $2m \rightarrow 1, 3 \rightarrow 2$ and $1 \rightarrow 2m$; but, from $b_2^g = a'_2 b'_2$, g must then map $3 \rightarrow 5$, a contradiction. Finally, if $a_2^g = a'_2 b'_2$ we get the mappings $2m \rightarrow 2, 3 \rightarrow 2m$ and $1 \rightarrow 1$; but, since $a_2^g = a'_2 b'_2$, g must then also map $3 \rightarrow 4$, a final contradiction. Then $(a_2, b_2, a_2 b_2)$ and $(a'_2, b'_2, a'_2 b'_2)$ are inequivalent hyperbolic generating triples both of type $(2m - 1, 2m - 1, 2m - 1)$ on G and so we get a mixable Beauville structure on $G \times G$ of type $(2m - 2, 2m - 2, 2m - 2; 2m - 1, 2m - 1, 2m - 1)$. \square

Lemma 3.5. *The alternating group A_{2m+1} and $A_{2m+1} \times A_{2m+1}$ are mixable for $m \geq 4$.*

Proof. For $m \geq 4$ let $G = A_{2m+1}$ under its natural representation and consider the elements

$$\begin{aligned} a_1 &= (1, 2)(3, 4)(5, \dots, 2m + 1), \\ b_1 &= (1, 2, \dots, 2m - 3)(2m - 2, 2m - 1)(2m, 2m + 1), \\ a_1 b_1 &= (1, 3, 5, \dots, 2m - 1, 2m + 1, 6, 8, \dots, 2m - 4). \end{aligned}$$

The subgroup $G_1 = \langle a_1, b_1 \rangle$ is clearly transitive and the elements

$$\begin{aligned} b_1 a_1^2 b_1^{-1} &= (4, 6, 8, \dots, 2m, 5, 7, \dots, 2m - 5, 2m - 1, 2m + 1), \\ a_1 b_1^2 a_1^{-1} &= (2, 4, 2m + 1, 6, 8, \dots, 2m - 4, 1, 3, \dots, 2m - 5) \end{aligned}$$

both fix the point $2m - 3$ and act transitively on the remaining points; hence G_1 acts primitively. Finally, the element $a_1 b_1^{-1} = (2, 2m - 3, 2m - 1, 2m + 1, 4)$ is a prime cycle fixing at least three points for all $m \geq 4$ and so by Jordan's theorem we have that $G_1 = G$. This gives our first hyperbolic generating triple of type $(2(2m - 3), 2(2m - 3), 2m - 3)$ for G . For our second even triple, we manipulate the first in the following way. Let

$$\begin{aligned} a'_1 &= (1, 2m - 4, \dots, 6, 2m + 1, 2m - 1, \dots, 3), \\ b'_1 &= (1, 2)(3, 4)(5, \dots, 2m + 1), \\ a'_1 b'_1 &= (1, 2m - 3, \dots, 2)(2m - 2, 2m - 1)(2m, 2m + 1). \end{aligned}$$

Since we have $b'_1 = a_1$ and $a'_1 b'_1 = b_1^{-1}$ it is clear that $\langle a'_1, b'_1 \rangle = G$. Note also that $a'_1 = b_1^{-1} a_1^{-1}$. To see that $(a_1, b_1, a_1 b_1)$ and $(a'_1, b'_1, a'_1 b'_1)$ are inequivalent, suppose for a contradiction there exists $g \in \text{Aut}(G)$ for which these triples are equivalent. Since conjugation preserves cycle type it must be that $(a_1 b_1)^g = a'_1$ which, by Lemma 2.7, implies that $a_1^g = b'_1$ and $b_1^g = a'_1 b'_1$. This gives the equality

$$a_1 b_1^{-1} = b'_1 (a'_1 b'_1) = a_1^g b_1^g = (a_1 b_1)^g = a'_1 = b_1^{-1} a_1,$$

a contradiction since otherwise G would be abelian. Then, $((a_1, a'_1), (b_1, b'_1))$ is an even triple for $G \times G$ of type $(2(2m - 3), 2(2m - 3), 2(2m - 3))$.

For our first odd triple consider the elements

$$\begin{aligned} a_2 &= (1, 2, \dots, 2m + 1), \\ b_2 &= a_2^{(1, 2, 3)} = (1, 4, 5, \dots, 2m, 2m + 1, 2, 3), \\ a_2 b_2 &= (1, 3, 5, \dots, 2m - 1, 2m + 1, 4, 6, \dots, 2m - 2, 2m, 2). \end{aligned}$$

The subgroup $G_2 = \langle a_2, b_2 \rangle$ is clearly transitive while the elements

$$b_2^{-1}a_2^2 = (1, 5, 6, \dots, 2m + 1)(3, 4), \quad a_2b_2^{-1} = (1, 2m + 1, 3)$$

fix the point 2 and act transitively on the remaining points. Hence G_2 is primitive with a prime cycle fixing at least three points, then by Jordan's theorem $G_2 = A_{2m+1}$. This gives us an odd triple of type $(2m + 1, 2m + 1, 2m + 1)$ for G and so since we have $\gcd(2(2m - 3), 2m + 1) = 1$ it follows that (a_1, b_1, a_2, b_2) is a mixable Beauville structure for A_{2m+1} of type $(2(2m - 3), 2(2m - 3), 2m - 3; 2m + 1, 2m + 1, 2m + 1)$. For our second odd triple consider the cycles

$$\begin{aligned} x_2 &= (1, 2, \dots, 2m - 1), \\ y_2 &= x_2^{(1, 2m, 2, 2m+1, 3)} = (1, 4, 5, \dots, 2m - 1, 2m, 2m + 1), \\ x_2y_2 &= (1, 2, 3, 5, \dots, 2m - 1, 4, 6, \dots, 2m, 2m + 1) \end{aligned}$$

and let $H_2 = \langle x_2, y_2 \rangle$. We clearly have transitivity while the elements

$$\begin{aligned} [x_2, y_2] &= (1, 2m, 4, 5, 2), \\ x_2[x_2, y_2] &= (2, 3, 5, 6, \dots, 2m - 1, 2m, 4) \end{aligned}$$

show that H_2 acts transitively on the stabiliser of the point $2m + 1$ and contains a prime cycle fixing at least 3 points for all m . Then by Jordan's theorem $H_2 = G$ and this gives us a second odd triple of type $(2m - 1, 2m - 1, 2m - 1)$. Since it is clear that $2(2m - 3)$ is coprime to both $2m - 1$ and $2m + 1$ we then have a mixable Beauville structure on $G \times G$ of type $(2(2m - 3), 2(2m - 3), 2(2m - 3); 4m^2 - 1, 4m^2 - 1, 4m^2 - 1)$. \square

4 The groups of Lie type

We make use of theorems due to Zsigmondy, generalising a theorem of Bang, and Gow which we include here for reference. Throughout this section $q = p^e$ will denote a prime power for a natural number $e \geq 1$.

Theorem 4.1 (Zsigmondy [45] or Bang [1], as appropriate). *For any positive integer $a > 1$ and $n > 1$ there is a prime number that divides $a^n - 1$ and does not divide $a^k - 1$ for any positive integer $k < n$, with the following exceptions:*

- (1) $a = 2$ and $n = 6$,
- (2) $a + 1$ is a power of 2, and $n = 2$.

We denote a prime with such a property $\Phi_n(a)$.

Remark 4.2. The case where $a = 2, n > 1$ and not equal to 6 was proven by Bang in [1] while the general case was proven by Zsigmondy in [45]. We shall refer to this as Zsigmondy's Theorem. A more recent account of a proof is given by Lüneburg in [32]. An even more recent account in English is given by Roitman in [36].

Definition 4.3. Let G be a group of Lie type defined over a field of characteristic $p > 0$, prime. A *semisimple* element is one whose order is coprime to p . A semisimple element is *regular* if p does not divide the order of its centraliser in G .

Theorem 4.4 (Gow [24]). *Let G be a finite simple group of Lie type of characteristic p , and let g be a non-identity semisimple element in G . Let L_1 and L_2 be any conjugacy classes of G consisting of regular semisimple elements. Then g is expressible as a product xy , where $x \in L_1$ and $y \in L_2$.*

Remark 4.5. A slight generalisation of this result to quasisimple groups appears in [17, Theorem 2.6].

4.1 Projective special linear groups $L_2(q) \cong A_1(q)$

The projective special linear groups $L_2(q)$ are defined over fields of order q and have order $q(q+1)(q-1)/d$ where $d = \gcd(2, q+1)$. Their maximal subgroups are listed in [21].

Lemma 4.6. *Let $G = L_2(7)$. Then G^n admits a mixable Beauville structure if and only if $n = 1, 2$.*

Proof. The maximal subgroups of G are known [12, p. 2] and these are subgroups isomorphic to S_4 or point stabilisers in the natural representation of G on eight points. Hyperbolic generating triples cannot have type $(3, 3, 3)$, since we have that $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} \not\leq 1$, and similarly for types $(2, 2, 2)$, $(2, 2, 4)$ or $(2, 4, 4)$. The number of hyperbolic generating triples of type $(7, 7, 7)$ can be computed using GAP, but since it is equal to the order of $\text{Aut}(G)$ we see from [25] that there is no triple of type $(7, 7, 7)$ for G^n when $n > 1$. Triples of type $(4, 4, 4)$ exist and any such triple generates G since elements of order 4 are not contained in point stabilisers and inside a subgroup isomorphic to S_4 the product of three elements of order 4 cannot be equal to the identity. We can compute the number of such triples from the structure constants and since this is twice the order of $\text{Aut}(G)$ we have that there exists a hyperbolic generating triple of type $(4, 4, 4)$ on G and on $G \times G$. We then see that this is the maximum number of direct copies of G for which there exists a mixable Beauville structure.

For our odd triple we then take a triple of type $(7, 7, 3)$ which can be shown to exist by computing their structure constants and are seen to generate G since if they were to belong to a maximal subgroup then the product of two elements of order 7 would again have order 7. This gives a mixable Beauville structure of type $(4, 4, 4; 7, 7, 3)$ on G . Finally, we then have mixable Beauville structures on $G \times G$ of type $(4, 4, 4; 7, 7, 21)$ by Lemma 2.6 or alternatively of type $(4, 4, 4; 7, 21, 21)$ by Lemma 2.8. \square

Lemma 4.7. *Let $G = L_2(8)$. Then $G \times G$ admits a mixable Beauville structure.*

Proof. It can easily be checked in GAP that for G there exist hyperbolic generating triples of types $(2, 7, 7)$, $(3, 3, 9)$ and $(3, 9, 9)$. The two odd triples are inequivalent by Lemma 2.6 and by Lemma 2.8 we have that there exists a mixable Beauville structure of type $(14, 14, 7; 3, 9, 9)$ on $G \times G$. \square

Lemma 4.8. *Let $G = L_2(9)$. Then both G and $G \times G$ admit a mixable Beauville structure.*

Proof. This follows directly from the exceptional isomorphism $L_2(9) \cong A_6$ and Lemma 3.2. \square

We make use of the following lemmas:

Lemma 4.9. *Let $G = L_2(q)$ for $q = 7, 8$ or $q \geq 11$. Let $d = \gcd(2, q + 1)$ and $\phi(n)$ be Euler's totient function. Then, under the action of $\text{Aut}(G) = P\Gamma L_2(q)$ the number of conjugacy classes of elements of order $\frac{q+1}{d}$ in G is $\phi(\frac{q+1}{d})/2e$.*

Proof. Elements of order $\frac{q+1}{d}$ are conjugate to their inverse so there are $\phi(\frac{q+1}{d})/2$ conjugacy classes of elements of order $\frac{q+1}{d}$ in $L_2(q)$. The only outer automorphisms of G come from the diagonal automorphisms and the field automorphisms, but since diagonal automorphisms do not fuse conjugacy classes of semisimple elements we examine the field automorphisms. These come from the action of the Frobenius automorphism on the elements of the field \mathbb{F}_q sending each entry of the matrix to its p -th power. The only fixed points of this action are the elements of the prime subfield \mathbb{F}_p and so, since the entries on the diagonal of the elements of order $\frac{q+1}{d}$ are not both contained in the prime subfield we have that the orbit under this action has length e , the order of the Frobenius automorphism. We then get e conjugacy classes of elements of order $\frac{q+1}{d}$ inside of G fusing under this action. Hence under the action of the full automorphism group there are $\phi(\frac{q+1}{d})/2e$ conjugacy classes of elements of order $\frac{q+1}{d}$. \square

Lemma 4.10. *For a prime power, $q = p^e \geq 13$, $q \neq 27$, let $q^+ = \frac{q+1}{d}$ where $d = \gcd(2, q + 1)$. Then $\phi(q^+)/(2e) > 1$ where $\phi(n)$ is Euler's totient function.*

Proof. Let $S = \{p^i, q^+ - p^i \mid 0 \leq i \leq e - 1\}$ be a set of $2e$ positive integers less than and coprime to q^+ and whose elements are distinct when $q \geq 13$. To this set we add k which will depend on q . When $p = 2$ we let $k = 7$ since for all $e > 3$, $7 \notin S$ and $\gcd(7, q^+) = 1$. When $p = 3$ we let $k = 11$, then for $e > 3$, $11 \notin S$ and $\gcd(11, q^+) = 1$. Now consider the cases $q \equiv \pm 1 \pmod{4}$ for $p \neq 2, 3$. When $q \equiv 1 \pmod{4}$ we let $k = q^+ - 2$. Since $p \neq 2$ we have $k \notin S$ and since q^+ is odd when $q \equiv 1 \pmod{4}$ we have $\gcd(k, q^+) = 1$. Finally, when $q \equiv 3 \pmod{4}$ then e must be odd. When $e > 2$ then $k = \frac{p-1}{2} \notin S$ and is coprime to q^+ . When $e = 1$, $q^+ = 2^i m$ where $i > 0$ and m is odd. Then

$$\phi(q^+) = \phi(2^i)\phi(m) = 2^{i-1}\phi(m) > 2$$

since $p > 11$. This completes the proof. \square

Lemma 4.11. *Let G be the projective special linear group $L_2(q)$ where $q \geq 7$. Then,*

- (1) *there is a mixable Beauville structure for $G \times G$,*
- (2) *when $p \neq 2$ there is also a mixable Beauville structure for G .*

Proof. In light of Lemmas 4.6–4.8 we can assume that $q \geq 11$. Define $q^+ = \frac{q+1}{d}$, where $d = \gcd(2, q + 1)$, and similarly for q^- . Jones proves in [28] that hyperbolic generating triples of type (p, q^-, q^-) exist for G when $q \geq 11$ and since $\gcd(p, q^-) = 1$ we immediately have, by Lemma 2.8, a hyperbolic generating triple for $G \times G$. We proceed to show that there exists a hyperbolic generating triple (x, y, z) for G of type (q^+, q^+, q^+) and note that both p and q^- are coprime to q^+ . The only maximal subgroups containing elements of order q^+ are the dihedral groups of order $2q^+$ which we denote by D_{q^+} . By Gow's theorem, for a conjugacy class, C , of elements of order q^+ there exist $x, y, z \in C$ such that $xyz = 1$. Since inside D_{q^+} any conjugacy class of elements of order q^+ contains only two elements, x, y and z can not all be contained in the same maximal subgroup of G . Hence (x, y, z) is a hyperbolic generating triple for G of type (q^+, q^+, q^+) . When the number of conjugacy classes of elements of order q^+ in G under the action of $\text{Aut}(G)$ is strictly greater than 1 we can apply Gow's theorem a second time to give a hyperbolic generating triple of type (q^+, q^+, q^+) for $G \times G$. This follows from Lemmas 4.9 and 4.10 with the exceptions of $q = 11$ or 27. For $G = L_2(11)$ we have that a triple of type (p, q^-, q^-) exists by [28] or alternatively the words ab and $[a, b]$ in the standard generators for G ([44]) give an odd triple of type $(11, 5, 5)$. In both cases we have, by Lemma 2.8, an odd triple of type $(55, 55, 5)$ for $G \times G$. For our even triple, the structure constants for the number of triples of type $(6, 6, 6)$ can be computed and is seen to be twice the order of $\text{Aut}(G)$ and so we have an even triple for G and $G \times G$. For $G = L_2(27)$

we take the words in the standard generators ([44]) $(ab)^2(abb)^2, a^{b^2}$ which give an even triple of type $(2, 14, 7)$ and the words b^2, b^a which give an odd triple of type $(3, 3, 13)$. Again, by Lemma 2.8, these give a mixable Beauville structure on $G \times G$. Finally, we remark that when $q \equiv \pm 1 \pmod 4$ we have that q^- and q^+ have opposite parity and this determines the parity of our triples. When $q \equiv 1 \pmod 4$, (p, q^-, q^-) becomes our even triple, (q^+, q^+, q^+) our odd triple, and vice versa when $q \equiv 3 \pmod 4$. \square

4.2 Projective special unitary groups $U_3(q) \cong {}^2A_2(q)$

The projective special unitary groups $U_3(q)$ are defined over fields of order q^2 and have order $q^3(q^3 + 1)(q^2 - 1)/d$ where $d = (3, q + 1)$. Their maximal subgroups can be found in [34] and we refer to the character table and notation in [38].

Lemma 4.12. *Let $G = U_3(q)$ for $q = 4$ or $q \geq 7$. Let $d = \gcd(3, q + 1)$ and $t' = (q^2 - q + 1)/d$. Then there exists a hyperbolic generating triple of type (t', t', t') for G .*

Proof. Let G, d and t' be as in the hypothesis. Let C be a conjugacy class of elements of order t' in G and for $g \in C$, let $T = \langle g \rangle$. When $q \geq 7$ the unique maximal subgroup of G containing g is $N_G(T)$ of order $3t'$. Since $\gcd(6, t') = 1$ and since T is by definition normal in $N_G(T)$ the Sylow p -subgroup in $N_G(T)$ for any prime $p|t'$ is contained in T and is therefore unique. Hence for all $x \in N_G(T) \setminus T$ the order of x is 3. Then for $g \in C$ since g is conjugate to g^{-q} and g^{q^2} ([38]) we have $C \cap N_G(T) = \{g, g^{-q}, g^{q^2}\}$. This partitions C into $|C|/3$ disjoint triples. Using the structure constants obtained from the character table of G we show that it is possible to find a hyperbolic generating triple of G entirely contained within C . Our method is to count the total number of triples $(x, y, z) \in C \times C \times C$ such that $xyz = 1$, which we denote $n(C, C, C)$, and show that there exists at least one such triple where x, y, z come from distinct maximal subgroups. Let S be a triple of the form $\{g, g^{-q}, g^{q^2}\} \subset C$. Then for $s_1, s_2, s_3 \in S, s_1s_2s_3 = 1$ if and only if all three elements are distinct. Therefore the contribution to $n(C, C, C)$ from triples contained within a single maximal subgroup of G is $2|C|$. Using the formula for structure constants as found in [19] we have that

$$n(C, C, C) = \frac{|C|^3}{|G|} \left(1 - \frac{1}{q(q-1)} - \frac{1}{q^3} - \sum_{u=1}^{t'-1} \frac{(\zeta_{t'}^u + \zeta_{t'}^{-uq} + \zeta_{t'}^{uq^2})^3}{3(q+1)^2(q-1)} \right)$$

where $\zeta_{t'}$ is a primitive t' -th root of unity. From the triangle inequality we have

$$|(\zeta_{t'}^u + \zeta_{t'}^{-uq} + \zeta_{t'}^{uq^2})|^3 \leq 27$$

so we can bound $n(C, C, C)$ from below and for $q \geq 8$ the following inequality holds:

$$n(C, C, C) \geq \frac{|C|^3}{|G|} \left(1 - \frac{1}{q(q-1)} - \frac{1}{q^3} - \frac{9(t'-1)}{(q+1)^2(q-1)} \right) > 2|C|.$$

For $q = 4$ or 7 direct computation of the structure constants shows that we can indeed find a hyperbolic generating triple of the desired type. \square

Lemma 4.13. *Let $G = U_3(q)$ for $q = 4$ or $q \geq 7$. Let $c = \gcd(3, q^2 - 1)$, $d = \gcd(3, q + 1)$ and $t' = (q^2 - q + 1)/d$. Then:*

(1) *for $p = 2$ there exists a mixable Beauville structure on G of type*

$$\left(2, 4, \frac{q^2 - 1}{c}; t', t', t' \right),$$

(2) *for $p \neq 2$ there exists a mixable Beauville structure on G of type*

$$\left(p, q + 1, \frac{q^2 - 1}{d}; t', t', t' \right).$$

Proof. The existence of hyperbolic generating triples of type (t', t', t') in both even and odd characteristic is given in Lemma 4.12 and so we turn to the even triples. When $p = 2$ we have the existence of our even triples from [17, Lemma 4.20 and Theorem 4.22] and so we now assume that $G = U_3(q)$ where p is odd, $q \geq 7$, letting $r = q + 1$ and $s = q - 1$.

From the list of maximal subgroups of G , elements of order rs/d exist and can belong to subgroups corresponding to stabilisers of isotropic points, stabilisers of non-isotropic points and possibly one of the maximal subgroups of a fixed order which can occur is $U_3(q)$ for certain q . Stabilisers of isotropic points have order q^3rs/d whereas stabilisers of non-isotropic points have order qr^2s/d . There exist $1 + d$ conjugacy classes of elements of order p in G which are as follows. The unique conjugacy class, C_2 , of elements whose centralisers have order q^3r/d ; and d conjugacy classes, $C_3^{(l)}$ for $0 \leq l \leq d - 1$, of elements whose centralisers have order q^2 . Since an element of order p which stabilises a non-isotropic point belongs to a subgroup of G isomorphic to $SL_2(q)$, the order of its centraliser in G must be a multiple of $2p$, hence must belong to C_2 . In particular, elements of C_3^0 do not belong to stabilisers of non-isotropic points. There exists a conjugacy class, C_6 , of elements of order r whose centralisers have order r^2/d . An element of order r contained in the stabiliser of an isotropic point must be contained in a cyclic subgroup of order rs/d , hence rs/d must divide the order of its centraliser and so elements of C_6 are not contained in the stabilisers of isotropic points. If C_7 is a conjugacy class of elements of order rs/d then any triple of

elements $(x, y, z) \in C_3^0 \times C_6 \times C_7$, such that $xyz = 1$, will not be entirely contained within the stabiliser of an isotropic or non-isotropic point. If $n(C_3^0, C_6, C_7)$ is the number of such triples then using the structure constant formula from [19] and the character table for G we have

$$n(C_3^0, C_6, C_7) = \frac{|C_3^0||C_6||C_7|}{|G|} \left(1 + \sum_{u=1}^{\frac{r}{d}-1} \frac{\epsilon^{3u}(\epsilon^{3u} + \epsilon^{6u} + \epsilon^{(r-3)u})}{t} \right)$$

where ϵ is a primitive r -th root of unity. Using the triangle inequality we can bound the absolute value of the summation by $3q/(q^2 - q + 1)$ which, for $q \geq 7$, is strictly less than 1. In order to show that such an (x, y, z) is not contained in any of the possible maximal subgroups of order 36, 72, 168, 216, 360, 720 or 2520, notice that the subgroup generated by (x, y, z) has order divisible by $n = p(q^2 - 1)/d$, hence this can only occur when $p = 3, 5$ or 7 . The only cases where $n \leq 2520$ and divides one of the possible subgroup orders are the cases $q = 7$ or 9 , but none of these subgroups contain elements of order 48 or 80 so we see that this is indeed an even triple for G . Finally, we must show that $\gcd(\frac{4rs}{c}, t') = 1$ when p is even and $\gcd(\frac{prs}{d}, t') = 1$ when p is odd. For all p it is clear that $\gcd(p, t') = 1$ and so it suffices to show that $\gcd(rs, t') = 1$. We have that $t'd - s = q^2$ so t' is coprime to s and since $r^2 - t'd = 3q$ and t' is coprime to 3 we have that t' is coprime to r . □

In order to extend this to a mixable Beauville structure on the product $G \times G$ where $G = U_3(q)$ we will need the following lemma:

Lemma 4.14. *Let $G = U_3(q)$ for $q \geq 3$ and $d = (3, q + 1)$. Then the number of conjugacy classes of elements of order $t' = (q^2 - q + 1)/d$ in G under the action of $\text{Aut}(G)$ is $\phi(t')/6e$ where $\phi(n)$ is Euler's totient function.*

Proof. Let $x \in G$ have order t' . Then x is conjugate to x^{-q} and x^{q^2} and so the number of conjugacy classes of order t' in G is $\phi(t')/3$. Then since the field automorphism has order $2e$ and the diagonal entries of an element of order t' are not all contained in the prime subfield its orbit has length $2e$. This gives the desired result. □

Lemma 4.15. *Let G be the projective special unitary group $U_3(q)$ for $q = 7$ or $q \geq 9$. Let $c = \gcd(3, q^2 - 1)$, $d = \gcd(3, q + 1)$ and $t' = (q^2 - q + 1)/d$. Then:*

(1) *for $p = 2$ there exists a mixable Beauville structure on $G \times G$ of type*

$$\left(2\frac{q^2 - 1}{c}, 2\frac{q^2 - 1}{c}, 4; t', t', t' \right),$$

(2) for $p \neq 2$ there exists a mixable Beauville structure on $G \times G$ of type

$$\left(p(q+1), p(q+1), \frac{q^2-1}{d}; t', t', t' \right).$$

Proof. Let the conditions of the hypothesis be satisfied with $p = 2$. Then, as in the proof of Lemma 4.13, there exists a hyperbolic generating triple of type $(2, 4, (q^2 - 1)/c)$ on G and by Lemma 2.8 this yields an even triple of type $(2(q^2 - 1)/c, 2(q^2 - 1)/c, 4)$ on $G \times G$. Similarly, for $p \neq 2$ by Lemma 4.13 there exists a hyperbolic generating triple of type $(p, q + 1, (q^2 - 1)/d)$ on G , which by Lemma 2.8 yields an even triple of type $(p(q+1), p(q+1), (q^2 - 1)/d)$ on $G \times G$.

By Lemma 4.12 there exist hyperbolic generating triples of type (t', t', t') for all p and by Lemma 4.14 we need only show that $\phi(t') > 6e$. For $q = 7, 9$ it can be verified directly that $\phi(t') > 6e$ so we can assume $q \geq 11$. In the case $d = 1$ we have $2^3 < p^e - 1$ so

$$2^f < p^{2f} - p^f + 1 \quad \text{for all } 0 \leq f \leq 6e$$

and we have our inequality. In the case $d = 3$ since $2^3 3 < p(p - 1)$ we have $2^f < t'$ for $1 \leq f \leq 3e$. Similarly, since $3^2 < p - 1$ we have the terms 3^{f-1} for $1 \leq f \leq 4e$ giving us $7e$ terms in total, as was to be shown. \square

Lemma 4.16. *The projective special unitary group $U_3(q)$ and $U_3(q) \times U_3(q)$ admit a mixable Beauville structure for $q \geq 3$.*

Proof. In light of the preceding lemmas in this section it remains only to check the cases $U_3(3)$, $U_3(5)$ and $U_3(q) \times U_3(q)$ for $q = 3, 4, 5$ and 8 . We present words in the standard generators [42, 44] that can be easily checked to give suitable triples for G which, by Lemma 2.8, give mixable Beauville structures for these cases. For $G = U_3(3)$ let

$$\begin{aligned} a_1 &= [a, b^2] \in G, & b_1 &= [a, b^2]^b \in G, \\ a_2 &= (babab^2)^3 \in G, & b_2 &= [a, b^2]^b \in G. \end{aligned}$$

It can be checked that $G = \langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle$ where both hyperbolic generating triples have type $(4, 4, 8)$ but in the former triple both elements of order 4 come from the conjugacy class $4C$, whereas in the latter, $a_2 \in 4AB, b_2 \in 4C$. Since these two triples are then inequivalent under the action of $\text{Aut}(G)$ we have that $(a_1, a_2), (b_1, b_2) \in G \times G$ yields a hyperbolic generating triple of type $(4, 4, 8)$. For the remaining cases we present in Table 1 words in the standard generators for G which, by Lemma 2.8, give a mixable Beauville structure on G and $G \times G$ where necessary. \square

| q | x_1 | y_1 | x_2 | y_2 | Type |
|-----|------------|------------|----------|------------------|----------------------|
| 3 | a_1, a_2 | b_1, b_2 | ab | ba | (4, 4, 8; 7, 7, 3) |
| 4 | a | $(ab)^2$ | b | $[b, a]$ | (2, 13, 13; 3, 5, 3) |
| 5 | a | ab^2 | ab | b^3ab^3 | (3, 8, 8; 7, 7, 5) |
| 8 | a | $(ab)^2$ | $[a, b]$ | $[a, b]^{babab}$ | (2, 19, 19; 9, 9, 7) |

Table 1. Words in the standard generators ([44]) a and b or as otherwise specified in Lemma 4.16 for $G = U_3(q)$.

4.3 The Suzuki groups ${}^2B_2(2^{2n+1}) \cong Sz(2^{2n+1})$

The Suzuki groups ${}^2B_2(q)$ are defined over fields of order $q = 2^{2n+1}$ for $n \geq 0$ and have order $q^2(q - 1)(q^2 + 1)$. They are simple for $q > 2$ and their maximal subgroups can be found in [43].

Lemma 4.17. *Let G be the Suzuki group ${}^2B_2(q)$ for $q > 2$. Then:*

- (1) G admits a mixable Beauville structure of type $(2, 4, 5; q - 1, n, n)$,
- (2) $G \times G$ admits a mixable Beauville structure of type

$$(4, 10, 10; n(q - 1), n(q - 1), n),$$

where $n = q \pm \sqrt{2q} + 1$, whichever is coprime to 5.

Proof. In the proof of [21, Theorem 6.2] Fuertes and Jones prove that there exist hyperbolic generating triples for G of types $(2, 4, 5)$ and $(q - 1, n, n)$. It is clear that $\gcd(10, n) = \gcd(10, q - 1) = 1$. Then, by Lemma 2.8, we need only show that $\gcd(q - 1, n) = 1$. If $q - 1$ and n share a common factor, then so do $q^2 - 1$ and $q^2 + 1$ and similarly their difference. Hence $\gcd(q - 1, n)$ divides 2, but since $q - 1$ is odd we have $\gcd(q - 1, n) = 1$ as was to be shown. \square

4.4 The small Ree groups ${}^2G_2(3^{2n+1}) \cong R(3^{2n+1})$

The small Ree groups are defined over fields of order $q = 3^{2n+1}$ for $n \geq 0$ and have order $q^3(q^3 + 1)(q - 1)$. They are simple for $q > 3$ and their maximal subgroups can be found in [30] or [43].

Lemma 4.18. *Let G be a simple small Ree group ${}^2G_2(q)$ for $q > 3$. Then:*

- (1) G admits a mixable Beauville structure of type

$$\left(\frac{q + 1}{2}, \frac{q + 1}{2}, q + \sqrt{3q} + 1; \frac{q - 1}{2}, \frac{q - 1}{2}, q - \sqrt{3q} + 1 \right),$$

(2) $G \times G$ admits a mixable Beauville structure of type

$$\left(\frac{q+1}{2}, \frac{q+1}{2}n^+, \frac{q+1}{2}n^+; \frac{q-1}{2}, \frac{q-1}{2}n^-, \frac{q-1}{2}n^- \right),$$

where $n^+ = q + \sqrt{3q} + 1$ and $n^- = q - \sqrt{3q} + 1$.

Proof. Let G, q, n^+ and n^- be as in the hypotheses. From the character table of G we see that regular semisimple elements of orders $\frac{q+1}{2}, \frac{q-1}{2}, n^+$ and n^- exist [40]. By Gow's theorem we can find elements $x_1, y_1 \in G$, both of order $\frac{q+1}{2}$, whose product has order n^+ , and elements $x_2, y_2 \in G$, both of order $\frac{q-1}{2}$, whose product has order n^- .

The only maximal subgroups of G containing elements of order n^+ have order $6n^+$. Similarly, the only maximal subgroups of G containing elements of order n^- have order $6n^-$. Since $n^+ - (q+1) = \sqrt{3q}$ we have

$$\gcd\left(\frac{q+1}{2}, n^+\right) = 1$$

as neither is divisible by 3. Then, for $q > 3$ we have $\frac{q+1}{2} > 6$ so (x_1, y_1, x_1y_1) is indeed an even triple for G of type $(\frac{q+1}{2}, \frac{q+1}{2}, q + \sqrt{3q} + 1)$. Similarly, for $q > 3$ we have $\frac{q-1}{2} > 6$ and $n^+n^- + (q-1) = q^2$, hence

$$\gcd\left(\frac{q-1}{2}, n^-\right) = 1$$

as both are coprime to 3. Note this also implies that $\gcd(n^+, \frac{q-1}{2}) = 1$. This gives us an odd triple for G of type $(\frac{q-1}{2}, \frac{q-1}{2}, q - \sqrt{3q} + 1)$.

It is clear that

$$\gcd\left(\frac{q+1}{2}, \frac{q-1}{2}\right) = 1$$

since their difference is q and neither is divisible by 3. Similarly, $\gcd(n^+, n^-) = 1$ since their difference is $2\sqrt{3q}$ and both are clearly coprime to 6. Since we have already shown that $\gcd(n^+, \frac{q-1}{2}) = 1$ it remains to show that $\gcd(n^-, \frac{q+1}{2}) = 1$. We have

$$(q+1) - n^- = \sqrt{3q}$$

and since neither is divisible by 3 we have a mixable Beauville structure on G . Finally, by Lemma 2.8, since $\gcd(\frac{q+1}{2}, n^+) = 1$ and $\gcd(\frac{q-1}{2}, n^-) = 1$, we also have a mixable Beauville structure for $G \times G$. \square

4.5 The large Ree groups ${}^2F_4(2^{2n+1})$

The large Ree groups ${}^2F_4(q)$ are defined over fields of order $q = 2^{2n+1}$ for $n \geq 0$ and have order $q^{12}(q^6 + 1)(q^4 - 1)(q^3 + 1)(q - 1)$. They are simple except for the case $q = 2$ which has simple derived subgroup ${}^2F_4(2)'$, known as the Tits group, which we consider along with the sporadic groups in the next section. The maximal subgroups of the large Ree groups can be found in [33] or [43].

Lemma 4.19. *Let G be a large Ree group ${}^2F_4(q)$ for $q > 2$. Then:*

(1) G admits a mixable Beauville structure of type

$$\left(10, 10, n^+; \frac{q^2 - 1}{3}, n^-, n^-\right),$$

(2) $G \times G$ admits a mixable Beauville structure of type

$$\left(10, 10n^+, 10n^+; \frac{q^2 - 1}{3}n^-, \frac{q^2 - 1}{3}n^-, n^-\right),$$

where $n^+ = q^2 + q + 1 + \sqrt{2q}(q + 1)$ and $n^- = q^2 + q + 1 - \sqrt{2q}(q + 1)$.

Proof. Let G , q , n^+ and n^- be as in the hypotheses. Elements of order 10 exist since G contains maximal subgroups of the form ${}^2B_2(q) \wr 2$, as do elements of order $(q^2 - 1)/3$ since G contains maximal subgroups isomorphic to $SU_3(q) : 2$ and since $\gcd(3, q + 1) = 3$. The only maximal subgroups containing elements of order n^+ have order $12n^+$. Similarly, elements of order n^- are only contained in maximal subgroups of order $12n^-$.

Using the computer program CHEVIE [23] it is possible to determine the structure constant for a pair of elements of order 10 whose product is n^+ and we see that such triples exist. Since $n^+n^- = q^4 - q^2 + 1$ and $q \equiv \pm 2 \pmod{5}$ we have that $\gcd(10, n^-) = \gcd(10, n^+) = 1$. Then, since no maximal subgroup contains both elements of order 10 and of order n^+ this is indeed an even triple for G . Elements of order $(q^2 - 1)/3$ are semisimple and from [37] we see that elements of order n^- are regular semisimple. Then by Gow's theorem there exists a pair of elements of order n^- whose product has order $(q^2 - 1)/3$. Since $(n^+n^-) + (q^2 - 1) = q^4$ any common factor of n^- and $(q^2 - 1)/3$ must be a power of 2, but since n^- and $q^2 - 1$ are both odd we have

$$\gcd\left(\frac{q^2 - 1}{3}, n^-\right) = 1.$$

Note that this also implies

$$\gcd\left(n^+, \frac{q^2 - 1}{3}\right) = 1.$$

Then, since $\frac{q^2-1}{3} > 12$ for $q > 2$, we see that no maximal subgroup contains both elements of order n^- and of $(q^2-1)/3$. Hence odd triples of type $((q^2-1)/3, n^-, n^-)$ exist for G . By Lemma 2.8 are also odd and even triples for $G \times G$.

We have already shown that $\gcd(10, n^-) = 1$, $\gcd(n^+, q^2-1) = 1$ and it is clear that

$$\gcd\left(10, \frac{q^2-1}{3}\right) = 1.$$

Finally, let $c = \gcd(n^+, n^-)$ and note that c is odd. Since $n^+ - n^- = 2\sqrt{2q}(q+1)$, c must divide $q+1$. Also, since $n^+ + n^- = 2(q^2 + q + 1)$, c must also divide $q^2 + q + 1$. Therefore c must divide q^2 and hence $c = 1$ so we have our desired mixable Beauville structures for G and $G \times G$. \square

4.6 The exceptional groups of type $G_2(q)$

The exceptional groups of type $G_2(q)$ are defined over fields of order q and have order $q^6(q^6-1)(q^2-1)$. They are simple for all prime powers, $q \geq 3$, and their maximal subgroups were determined by Cooperstein [13] for q even and by Kleidman [30] for q odd. Their conjugacy classes were determined by Enomoto [14] in the case $p = 2, 3$ and Chang [11] in the case $p \geq 5$.

Lemma 4.20. *Let G be the exceptional group $G_2(q)$ where $q \geq 4$ is even. Then:*

(1) *if $q = 4$, G admits a mixable Beauville structure of type*

$$(8, 8, 7; 5, 5, 13),$$

(2) *if $q \geq 8$ and $q \equiv 1 \pmod{3}$, G admits a mixable Beauville structure of type*

$$(4, 4, q^2 - q + 1; q + 1, q + 1, q^2 + q + 1),$$

(3) *if $q \geq 8$ and $q \equiv -1 \pmod{3}$, G admits a mixable Beauville structure of type*

$$(4, 4, q^2 + q + 1; q - 1, q - 1, q^2 - q + 1).$$

Furthermore, each of these cases yields a mixable Beauville structure on $G \times G$.

Proof. We treat the case $q = 4$ independently, since additional maximal subgroups arise in this case, by presenting words in the standard generators of $G_2(4)$ ([44]) which are easily checked in GAP. For our odd triple $x = b$ and $y = b^a$ gives a triple of type $(5, 5, 13)$ for G and $(5, 65, 65)$ for $G \times G$. For our even triple $x = (ab^2(ab)^2b)^b$ and $y = x^a$ gives a triple of type $(8, 8, 7)$ for G and of type $(8, 56, 56)$ for $G \times G$. These words visibly provide a mixable Beauville structure for G and $G \times G$.

Now suppose that $q \geq 8$. We begin with the odd triples for all $q \geq 8$ and use results from the character tables of $SL_3(q)$ and $SU_3(q)$ ([40]) throughout. From the list of conjugacy classes of G elements of order $q^2 - q + 1$ exist, are regular semisimple and belong only to maximal subgroups isomorphic to $SU_3(q) : 2$. Elements of order $q - 1$ can be chosen so that their centraliser in G has order $(q - 1)^2$ in which case they are regular semisimple. Such elements do not belong to maximal subgroups isomorphic to $SU_3(q) : 2$ since this would force the order of their centraliser in $SU_3(q) : 2$ to divide $(q - 1)^2$. Triples of type $(q - 1, q - 1, q^2 - q + 1)$ exist by Gow's theorem and by the previous discussion generate G . The proof for triples of type $(q + 1, q + 1, q^2 + q + 1)$ is identical with the roles of $SL_3(q) : 2$ and $SU_3(q) : 2$ interchanged. Since

$$\gcd(q - 1, q^2 - q + 1) = 1 \quad \text{and} \quad \gcd(q + 1, q^2 + q + 1) = 1$$

we have our odd triples for G and $G \times G$.

For our even triples we proceed as follows. In the case $q \equiv 1 \pmod 3$, elements of order $q^2 - q + 1$ exist as before. Elements of order 4 are conjugate in $SU_3(q) : 2$ and so also in G , but G contains three conjugacy classes of elements of order 4. We denote them $4A$, $4B$ and $4C$ in decreasing order of their centralisers in G . If we let $n = q^2 - q + 1$ then a calculation in CHEVIE shows that triples of type $(4A, 4C, nA)$ exist. Again, our argument is identical for the case $q \equiv -1 \pmod 3$ with the roles of $SL_3(q) : 2$ and $SU_3(q) : 2$ interchanged yielding an even triple of type $(q + 1, q + 1, q^2 + q + 1)$. Since $q^2 - q + 1$ and $q^2 + q + 1$ are odd we have our even triples for G and for $G \times G$.

It remains to show that $\gcd(q - 1, q^2 + q + 1) = \gcd(q + 1, q^2 - q + 1) = 1$. In the former case any common factor must divide $(q^2 + q + 1) - (q - 1)^2 = 3q$. Since $q - 1$ is odd and since elements of order $q - 1$ are chosen when $q \equiv -1 \pmod 3$ we have that $\gcd(q - 1, q^2 + q + 1) = 1$. The latter case is proven analogously. \square

Remark 4.21. In the case that $q = 2$, there exists an isomorphism

$$G_2(2) \cong \text{Aut}(U_3(3)),$$

but this group is neither simple nor mixable since it contains an index 2 subgroup.

Lemma 4.22. *Let G be the exceptional group $G_2(q)$ where $q \geq 9$ is odd. Then G admits a mixable Beauville structure of type*

$$\left(\frac{q - 1}{2}, \frac{q - 1}{2}, \frac{q^2 - q + 1}{t_1}, \frac{q + 1}{2}, \frac{q + 1}{2}, \frac{q^2 + q + 1}{t_2} \right)$$

where $t_1 = \gcd(3, q + 1)$, $t_2 = \gcd(3, q - 1)$. This also yields a mixable Beauville structure on $G \times G$.

Proof. We follow closely the construction given in [17, Section 5.7] but modify it slightly to ensure we have a mixable Beauville structure. Let

$$k_1 = \frac{q-1}{2}, \quad k_2 = \frac{q+1}{2},$$

$$k_3 = \frac{q^2+q+1}{\gcd(3, q-1)}, \quad k_6 = \frac{q^2-q+1}{\gcd(3, q+1)}.$$

Observe that $\gcd(k_1, k_2) = 1$, so that either k_1 or k_2 is even. It is clear that

$$\gcd(k_3, k_6) = 1, \quad \gcd(k_2, k_3) = 1, \quad \gcd(k_1, k_6) = 1.$$

Both $\gcd(k_1, k_3)$ and $\gcd(k_2, k_6)$ divide 3, but since 9 does not divide k_3 or k_6 , from our choices of k_1, k_2, k_3 and k_6 we have that these four numbers are pairwise coprime. From [11] and [14] we see that there exist conjugacy classes of regular semisimple elements of all of these orders for all odd q . From the character tables of $\mathrm{SL}_3(q)$ and $\mathrm{SU}_3(q)$ elements of orders k_1 and k_2 can be chosen from the conjugacy class denoted C_6 in [38]. By Gow's theorem, there then exists a pair of elements of order k_1 whose product has order k_6 , and a pair of elements of order k_2 whose product has order k_3 . From the list of maximal subgroups, such elements cannot belong to a single maximal subgroup. Then, by the preceding arguments and Lemma 2.8 we have a mixable Beauville structure on G and on $G \times G$. \square

Lemma 4.23. *Let G be the exceptional group $G_2(q)$ where $q \geq 3$. Then G and $G \times G$ admit mixable Beauville structures.*

Proof. In light of Lemmas 4.20 and 4.22 it remains to consider the cases $q = 3, 5$ or 7 .

If $q = 3$ the following explicit words in the standard generators [44] are easily checked in GAP to admit a mixable Beauville structure. Our even triple is given by $x_1 = a, y_1 = aba(bab)^2$, of type $(2, 7, 8)$, and our odd triple is given by $x_2 = b, y_2 = b^a$, of type $(3, 3, 13)$. By Lemma 2.8 these also give mixable Beauville structures on $G_2(3) \times G_2(3)$.

If $q = 5$ we appeal to the character table [12] where it can be verified by direct computation that triples of types $(6C, 6C, 25A)$ and $(7A, 7A, 31A)$ exist. Elements of order 25 must belong to a maximal subgroup of shape $[q^5]:\mathrm{GL}_2(5)$, which does not contain elements from class $6C$, and elements of order 31 must belong to a maximal subgroup conjugate to $L_3(5):2$ which does not contain elements of order 7. Hence we have a mixable Beauville structure on $G_2(5)$ and, by Lemma 2.8, $G_2(5) \times G_2(5)$.

Finally, in the case of $q = 7$, there exist conjugacy classes of regular semisimple elements of orders 8 and 19, as well as elements of orders 49 and 43. By Gow's

theorem there exists a pair of elements of order 8 whose product has order 9, and by computation in CHEVIE there exists a pair of elements of order 49 whose product has order 43. From the list of maximal subgroups and by Lemma 2.8 we have that these triples admit mixable Beauville structures on $G_2(7)$ and $G_2(7) \times G_2(7)$, completing the proof. \square

4.7 The Steinberg triality groups ${}^3D_4(q)$

The Steinberg triality groups ${}^3D_4(q)$ are defined over fields of order q and have order $q^{12}(q^8 + q^4 + 1)(q^6 - 1)(q^2 - 1)$. They are simple for all prime powers, q , and their maximal subgroups can be found in [31] or [43].

Lemma 4.24. *Let G be the Steinberg triality group ${}^3D_4(2)$. Then both G and $G \times G$ admit a mixable Beauville structure.*

Proof. It can be verified using GAP that $(a, (ab)^3b^2; ab, b^{ab^2})$, where a and b are the standard generators as found in [44], is a mixable Beauville structure of type $(2, 7, 28; 13, 9, 13)$ and by Lemma 2.8 this yields a mixable Beauville structure of type $(14, 14, 28; 117, 117, 13)$ on $G \times G$. \square

Lemma 4.25. *Let G be the Steinberg triality group ${}^3D_4(q)$ for $q \geq 3$ and let $d = \gcd(3, q + 1)$. Then:*

(1) *for $p = 2$ there exists a mixable Beauville structure on G of type*

$$(6, 6, \Phi_{12}(q); \Phi_3(q), \Phi_3(q), \Phi_6(q)),$$

(2) *for $p \neq 2$ there exists a mixable Beauville structure on G of type*

$$\left(\frac{q^2 - 1}{d}, \frac{q^2 - 1}{d}, \Phi_{12}(q); \Phi_3(q), \Phi_3(q), \Phi_6(q) \right).$$

Proof. Let G and d be as in the hypothesis. By [17, Lemma 5.24] for $q > 2$ there exists a hyperbolic generating triple of type $(\Phi_3(q), \Phi_3(q), \Phi_6(q))$.

For $p = 2$ one can verify using CHEVIE to compute the structure constants that there exist pairs of elements of order 6 whose product has order $\Phi_{12}(q)$ and it is clear from the list of maximal subgroups that this is indeed an even triple for G .

For $p \neq 2$ elements of order $(q^2 - 1)/d$ exist since G contains subgroups isomorphic to $SU(3, q)$. Using CHEVIE and the list of maximal subgroups it can be shown that an odd triple of type

$$\left(\frac{q^2 - 1}{d}, \frac{q^2 - 1}{d}, \Phi_{12}(q) \right)$$

exists for G . By Zsigmondy's theorem it is also clear that we have coprimeness for both Beauville structures. \square

Lemma 4.26. *Let G be the Steinberg triality group ${}^3D_4(q)$ for $q \geq 3$ and let $d = \gcd(3, q + 1)$. Then:*

(1) *for $p = 2$ there exists a mixable Beauville structure on $G \times G$ of type*

$$(6, 6\Phi_{12}(q), 6\Phi_{12}(q); \Phi_3(q), \Phi_3(q)\Phi_6(q), \Phi_3(q)\Phi_6(q)),$$

(2) *for $p \neq 2$ there exists a mixable Beauville structure on $G \times G$ of type*

$$\left(\frac{q^2 - 1}{d}, \Phi_{12}(q)\frac{q^2 - 1}{d}, \Phi_{12}(q)\frac{q^2 - 1}{d}; \Phi_3(q), \Phi_3(q)\Phi_6(q), \Phi_3(q)\Phi_6(q) \right).$$

Proof. By Lemmas 2.8 and 4.25 we need only verify that $\gcd(6, \Phi_{12}(q)) = 1$ for $p = 2$ as the rest follows by construction. This is clear since $\Phi_{12}(q)$ is both odd and coprime to $q^2 - 1$ which is divisible by 3. \square

5 The sporadic groups

We present in Table 2 explicit mixable Beauville structures for the sporadic groups in terms of words in the standard generators [44], except for the cases of the Baby Monster, \mathbb{B} , and the Monster, \mathbb{M} , for which we simply show existence of such structures in the following lemmas. The types of these structures are given in Table 3.

Lemma 5.1. *There exists a mixable Beauville structure on the Baby Monster, \mathbb{B} , and on $\mathbb{B} \times \mathbb{B}$.*

Proof. From [41] we know there exists a hyperbolic generating triple of type $(2, 3, 8)$. Let

$$x = (ab)^3(ba)^4b(ba)^2b^2, \quad y = x^{ab^2}$$

be words in the standard generators [44]. They both have order 47 and their product has order 55, then from the list of maximal subgroups [43] they will generate \mathbb{B} . This gives a mixable Beauville structure of type $(2, 3, 8; 47, 47, 55)$ on \mathbb{B} and by Lemma 2.8 we have a mixable Beauville structure of type $(6, 6, 8; 47, 2585, 2585)$ on $\mathbb{B} \times \mathbb{B}$. \square

Lemma 5.2. *There exists a mixable Beauville structure on the Monster, \mathbb{M} , and on $\mathbb{M} \times \mathbb{M}$.*

| G | x_1 | y_1 | x_2 | y_2 |
|--------------------|-----------------------|-------------------------------|---------------------|-----------------------------|
| M_{11} | $ab(ab^2)^2$ | $(x_1)^b$ | $(ab)^5$ | $[a, b]^2$ |
| M_{12} | $(ab)^4ba(bab)^2b$ | $x_1^{b^2aba}$ | ab | ba |
| J_1 | $(ab)^2(ba)^3b^2ab^2$ | b^{aba} | ab | $[a, b]$ |
| M_{22} | $(ab)^3b^2ab^2$ | $ba(bab^2)^2a$ | ab | $(ab)^4b^2ab^2$ |
| J_2 | $ab(abab^2)^2ab^2$ | $x_1^{ab^2}, x_1^{(ba)^2b^2}$ | ab | ba |
| M_{23} | $abab^2$ | $babab$ | ab | $(abab)^{ba}$ |
| ${}^2F_4(2)'$ | $(ab)^3bab$ | $(x_1^7)^{baba}$ | ab | ba |
| HS | $abab^3$ | b^3aba | $(ab)^3b$ | $(x_2)^b$ |
| J_3 | $(ab)^2(ba)^3b^2$ | x_1^b | ab | $(ab)^b$ |
| M_{24} | $(ab)^4b$ | $(x_1^3)^b$ | ab | ba |
| McL | ab^2 | bab | ab | $(ab)^b$ |
| He | ab^3 | ab^4 | ab | ba |
| Ru | b | $b^{(ab)^5}$ | $(ab)^2$ | $(ba)^2$ |
| Suz | $ab(abab^2)^2$ | $x_1^{abab^2}$ | ab | ba |
| O'N | $[a, b]$ | $([a, b]^2)^{bab}$ | ab^2 | $(ab^2)^{abab}$ |
| Co ₃ | ab | $(ab)^{b^2}$ | $a(ab)^2b(a^2b)^2b$ | $(x_2^3)^{baba}$ |
| Co ₂ | a | b | $ab(ab^2)^2b$ | $(x_2^2)^{bab^2}$ |
| Fi ₂₂ | $(ab)^3b^3$ | $(x_1)^a$ | b | b^a |
| HN | a | b | $[b, a]$ | $(ab)^2b((ab)^5(ab^2)^2)^2$ |
| Ly | a | b | $abab^3$ | $x_2^{(ab)^7}$ |
| Th | $[a, b]$ | $[a, b]^{(ba)^4b^2}$ | $ababa$ | $(x_2^5)^{bab}$ |
| Fi ₂₃ | a | b | $((ab)^{11}b)^3$ | x_2^a |
| Co ₁ | a | b | $[(ab)^3, aba]$ | $[(ab)^{23}, ab^2]^{babab}$ |
| J_4 | a | b | $a(bab)^3(ab)^2b$ | x_2^a |
| Fi ₂₄ ' | ab | $((ab)^6b)^{15}$ | $b(ba)^3$ | $(x_2^3)^{bab}$ |

Table 2. Mixable structures (x_1, y_1, x_2, y_2) for G in terms of words in the standard generators [44].

Proof. Norton and Wilson [35, Theorem 21] show that the only maximal subgroups of the Monster which contain elements of order 94 are copies of $2 \cdot \mathbb{B}$ which does not contain elements of order 71. Then from a computation of the structure constants an even triple of type $(94, 94, 71)$ can be shown to exist. Finally, in [15] it is shown that there exists a hyperbolic generating triple of type $(21, 39, 55)$ on \mathbb{M} . Therefore we have a mixable Beauville structure of type $(94, 94, 71; 21, 39, 55)$ on \mathbb{M} and of type $(94, 6674, 6674; 21, 2145, 2145)$. This completes the proof. \square

| G | Type of G | Type of $G \times G$ |
|-------------------|--------------------------|----------------------------------|
| M_{11} | (8, 8, 5; 11, 3, 11) | (8, 40, 40; 11, 33, 33) |
| M_{12} | (8, 8, 5; 11, 11, 3) | (8, 40, 40; 11, 33, 33) |
| J_1 | (10, 3, 10; 7, 19, 19) | (10, 30, 30; 133, 133, 19) |
| M_{22} | (8, 8, 5; 11, 7, 7) | (8, 40, 40; 77, 77, 7) |
| J_2 | (10, 10, 10; 7, 7, 3) | (10, 10, 40; 7, 21, 21) |
| M_{23} | (8, 8, 11; 23, 23, 7) | (8, 88, 88; 23, 161, 161) |
| ${}^2F_4(2)'$ | (8, 8, 5; 13, 13, 3) | (8, 40, 40; 13, 39, 39) |
| HS | (8, 8, 15; 7, 7, 11) | (8, 120, 120; 7, 77, 77) |
| J_3 | (8, 8, 5; 19, 19, 3) | (8, 40, 40; 19, 57, 57) |
| M_{24} | (8, 8, 5; 23, 23, 3) | (8, 40, 40; 23, 69, 69) |
| McL | (12, 12, 7; 11, 11, 5) | (84, 84, 12; 55, 55, 11) |
| He | (8, 8, 5; 17, 17, 7) | (8, 40, 40; 17, 119, 119) |
| Ru | (4, 4, 29; 13, 13, 7) | (4, 116, 116; 13, 91, 91) |
| Suz | (8, 8, 7; 13, 13, 3) | (8, 56, 56; 13, 39, 39) |
| O'N | (12, 6, 31; 19, 19, 11) | (12, 186, 186; 209, 209, 19) |
| Co ₃ | (14, 14, 5; 23, 23, 9) | (14, 70, 70; 23, 207, 207) |
| Co ₂ | (2, 5, 28; 23, 23, 9) | (10, 10, 28; 23, 207, 207) |
| Fi ₂₂ | (16, 16, 9; 13, 13, 11) | (144, 144, 16; 143, 143, 13) |
| HN | (2, 3, 22; 5, 19, 19) | (6, 6, 22; 95, 95, 19) |
| Ly | (2, 5, 14; 67, 67, 37) | (10, 10, 14; 2479, 2479, 67) |
| Th | (10, 10, 13; 19, 19, 31) | (130, 130, 10; 589, 589, 19) |
| Fi ₂₃ | (2, 3, 28; 13, 13, 23) | (6, 6, 28; 299, 299, 13) |
| Co ₁ | (2, 3, 40; 11, 13, 23) | (6, 6, 40; 143, 143, 23) |
| J_4 | (2, 4, 37; 43, 43, 23) | (4, 74, 74; 989, 989, 43) |
| Fi' ₂₄ | (29, 4, 20; 33, 33, 23) | (116, 116, 20; 759, 759, 33) |
| \mathbb{B} | (2, 3, 8; 47, 47, 55) | (6, 6, 8; 47, 2585, 2585) |
| \mathbb{M} | (94, 94, 71; 21, 39, 55) | (94, 6674, 6674; 21, 2145, 2145) |

Table 3. Types of the mixable Beauville structures for G and $G \times G$ from the words in Table 2 and Lemmas 5.1 and 5.2.

Finally, we have the following lemma which completes the proof of Theorem 1.6.

Lemma 5.3. *Let G be one of the 26 sporadic groups or the Tits group ${}^2F_4(2)'$. Then there exists a mixable Beauville structure on G and $G \times G$.*

Proof. With the exceptions of \mathbb{B} and \mathbb{M} , the types of all mixable Beauville structures for G as they appear in Table 3 can easily be checked in GAP. In all cases except J_2 it follows from Lemma 2.8 that such structures indeed extend to mixable Beauville structures for $G \times G$. In the case of J_2 we have two even triples,

$$(x_1, x_1^{ab^2}) \quad \text{and} \quad (x_1, x_1^{(ab)^2b^2})$$

of types (10, 10, 10) and (10, 10, 8), which are inequivalent by Lemma 2.6, hence extend to a mixable Beauville structure on $J_2 \times J_2$.

For $G = \mathbb{B}$ or \mathbb{M} we have existence of mixable Beauville structures following from Lemmas 5.1 and 5.2. If G is one of M_{11} , M_{12} , J_1 , M_{22} , J_2 , M_{23} , ${}^2F_4(2)'$, HS, J_3 , M_{24} , McL, He, Ru, Suz, O'N, Co₃, Co₂, Fi₂₂, Fi₂₃ or Fi'₂₄, then it can easily be checked in GAP that $\langle x_1, y_1 \rangle = \langle x_2, y_2 \rangle = G$ for all words appearing in Table 2, including both y_1 for J_2 . The remaining cases are HN, Ly, Th, Co₁ and J_4 . With the exception of Th, in all cases x_1 and y_1 are the standard generators, so there is nothing to check. In the case of Th there are no maximal subgroups containing elements of orders 10 and 13 hence $\langle x_1, y_1 \rangle = \text{Th}$. Now we turn to the generating pairs x_2, y_2 . From the list of maximal subgroups appearing in [12] the following is easily checked. No maximal subgroup of HN contains elements of orders 5 and 19; no maximal subgroup of Ly contains elements of orders 37 and 67; no maximal subgroup of Th contains elements of orders 19 and 31; no maximal subgroup of Co₁ contains elements of orders 11, 13 and 23; and no maximal subgroup of J_4 contains elements of orders 23 and 43. This completes the proof. \square

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