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# Eliciting Ambiguous Beliefs Under α-Maxmin Preference

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# Eliciting Ambiguous Beliefs Under $\alpha$-Maxmin Preference

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**Abstract**

We study the problem of elicitation of subjective beliefs of an agent when the beliefs are ambiguous (the set of beliefs is a non-singleton set) and the agent's preference exhibits ambiguity aversion; in particular, as represented by $\alpha$-maxmin preferences. We construct a direct revelation mechanism such that truthful reporting of beliefs is the agent's unique best response. The mechanism uses knowledge of the preference parameter $\alpha$ and we construct a mechanism that truthfully elicits $\alpha$. Finally, using the two as ingredients, we construct a grand mechanism that elicits ambiguous beliefs and $\alpha$ concurrently.

JEL CLASSIFICATION: D81, D82

KEYWORDS: Ambiguity, $\alpha$-maxmin preferences, maxmin preferences, elicitation of beliefs and $\alpha$

# 1 Introduction

In this paper, we study the problem of elicitation of beliefs of an agent when the beliefs are *ambiguous* and the agent's preferences exhibit *ambiguity-sensitivity*. As an (perhaps the most common) application consider a policy maker (principal) who needs to take an action the outcomes of which depend on the realization of some random event, and an expert (agent) who has *private* information regarding the likelihood of occurrence of the event (henceforth we will simply call this the "belief of the agent"). There are well-known methods for eliciting beliefs when agents have unique priors (in decision theory, such agents are said to have "probabilistically sophisticated preferences," an example being Subjective Expected Utility). In particular, using tools from mechanism design theory, researchers have shown how an agent's beliefs can be elicited through the use of direct revelation mechanisms in which the agent's unique best response is to truthfully report the beliefs.[1] The objective of this research is to extend this to situations when the beliefs are ambiguous (non-unique priors) and the agent's preferences exhibit sensitiveness to ambiguity. In particular, in this paper we consider the case of $\alpha$-maxmin ($\alpha$-MEU) preferences.[2,3]

There are two problems that one encounters when going from eliciting beliefs of subjective expected utility maximizers to eliciting ambiguous beliefs of agents with alpha-maxmin preferences. The first of course is that a set - rather than a singleton - needs to be elicited. Here, we use insights from the mechanism constructed in Karni (2009).[4] In fact for the special case of maxmin (MEU) preferences, our mechanism can

[1]Earliest work in this literature is from the 1950s showing construction of proper scoring rules which are precisely direct revelation mechanisms where truthful reporting is the agent's best response. The earlier works assumed knowledge of the agent's utility function (in fact the earliest assumed risk-neutrality); this was subsequently relaxed through the use of random scoring rules. See the review article by Gneiting and Raftery (2007). The paper by Karni (2009) is the most relevant for our purposes and we discuss the connections shortly.

[2]See Hurwicz (1951), Jaffray (1989), Ghirardato, Maccheroni, and Marinacci (2004). We also show how to elicit ambiguous beliefs when preferences are maxmin (MEU) (Gilboa and Schmeidler (1989)) - a special case of the $\alpha$-maxmin.

[3]The work reported here is part of an ongoing larger project of ours on eliciting of ambiguous beliefs.

[4]Karni's mechanism goes beyond eliciting beliefs of agents with subjective expected utility prefer-

be thought of as a natural extension of the mechanism in Karni.

The second problem is that attempts to elicit beliefs from choice data run into the difficulty that what is recovered from such attempts include not just the agent's beliefs but other aspects of the agent's preference, for example, the agent's attitude towards ambiguity. Specifically, an agent with $\alpha$-maxmin preferences evaluates an act by considering the weighted average of the worst expected payoff and the best expected payoff with $\alpha$ and $1-\alpha$ being the two weights. The parameter $\alpha$ reflects the agent's attitude towards ambiguity (alternatively, degree of optimism and pessimism) and is different from the agent's (ambiguous) beliefs. However, beliefs estimated from choice data will result in obtaining "as if" priors that will incorporate interaction of $\alpha$ with beliefs.[5]

We solve this problem by expanding the scope of the mechanism. Our direct revelation mechanisms ask the agent to report the agent's entire "type" where type includes not just the beliefs but other (privately known) aspects of the agent's preferences. At first blush, it is not clear why that should help; after all the mechanism will now have to satisfy incentive compatibility type constraints not just for reported beliefs but for those other aspects of the preferences as well. Nevertheless, as we show in this paper, this approach does allow us to "disentangle" beliefs from those other aspects of the preference. To help understand how our mechanism works, in our exposition, we break up the "grand" mechanism into smaller parts. First we describe a procedure that truthfully elicits the beliefs if $\alpha$ were known to the designer. Next, we show a procedure that truthfully elicits $\alpha$. This might give the impression that we need to run the mechanisms sequentially, eliciting $\alpha$ before eliciting beliefs, and ensure that the first mechanism does not contaminate the second. However, these concerns do not arise here. After we present the two mechanisms separately, we show how to combine them into a single mechanism that concurrently elicits the beliefs and the parameter $\alpha$ truthfully.

ences and in fact works for all probabilistically sophisticated preferences.

[5]In this respect, $\alpha$ is reminiscent of "nuisance parameters" in the statistical estimation literature.

## 2 Preliminaries

The objective of the mechanism designer is to elicit the agent's ambiguous beliefs regarding an event $E$. Let $[\underline{\mu}, \overline{\mu}]$, with $0 \leq \underline{\mu} < \overline{\mu} \leq 1$ represent the ambiguous beliefs about $E$. The convention is that $\mu \in [\underline{\mu}, \overline{\mu}]$ is the probability that $E$ happens; therefore $1 - \mu$ is the probability that event $E^c$, the complement of $E$, happens.

The agent's risk preference is reflected in the vN-M utility function $u(\cdot)$. Note that in the mechanisms we consider, the mechanism designer does not need to know $u(\cdot)$.

We assume, as is common in most of the literature, that *all* departures from expected utility is due to ambiguity aversion. In particular, this means that the agent's payoffs from *objective compound lotteries* is exactly the same as an expected utility maximizer's with same risk preferences.

We will consider direct revelation mechanisms. By eliciting we mean that truthful reporting is the agent's unique best response when facing the mechanism.

### 2.1 Some further notation

In the following, $x$, $y$, $z$, etc. will denote monetary payoffs. We assume the agent strictly prefers more money: for $x > y$, we have $u(x) > u(y)$.

$x_E y$ denotes the (subjective) act that pays $x$ if event $E$ happens and $y$ if event $E$ does not happen (i.e., event $E^c$ happens).

For $p \in [0,1]$, $\ell(p; x, y)$ denotes the objective lottery that pays $x$ with probability $p$ and $y$ with probability $1 - p$. For $p \in [0,1]$ $p' \in [0,1]$, $q \in [0,1]$, and prizes $x, y, w, z$, let $\mathcal{L}(q; \ell(p; x, y), \ell(p'; w, z))$ denote the objective compound lottery that "pays" the lottery $\ell(p; x, y)$ with probability $q$ and the lottery $\ell(p'; w, z)$ with probability $1 - q$.

## 3 Belief elicitation

For prizes $w$ and $z$, and given that the agent's set of beliefs regarding the event $E$ is the interval $[\underline{\mu}, \overline{\mu}]$, payoff of the agent from the act $w_E z$ is

$$[\alpha]\left[\min_{\mu\in[\underline{\mu},\overline{\mu}]}[\mu][u(w)]+[1-\mu][u(z)]\right]+[1-\alpha]\left[\max_{\mu\in[\underline{\mu},\overline{\mu}]}[\mu][u(w)]+[1-\mu][u(z)]\right]$$

One can readily see the problem of belief elicitation with such preferences. For prizes $x > y$, the agent's payoff from the act $x_E y$ is given by

$$\alpha\left[\underline{\mu}u(x)+(1-\underline{\mu})u(y)\right]+(1-\alpha)\left[\overline{\mu}u(x)+(1-\overline{\mu})u(y)\right]$$

which is equal to

$$[u(x)][\alpha\underline{\mu}+(1-\alpha)\overline{\mu}]+[u(y)][\alpha(1-\underline{\mu})+(1-\alpha)(1-\overline{\mu})]$$

The agent's observed choices therefore will be indistinguishable from those of an agent with subjective expected utility (SEU) preferences with a unique prior $\alpha\underline{\mu} + (1-\alpha)\overline{\mu}$ for the event $E$.

Of course the agent does not truly have a single prior belief and that can be captured by giving the agent the act $y_E x$. However even in that case the agent's elicited belief will conform to the prior $\alpha(1-\overline{\mu})+(1-\alpha)(1-\underline{\mu})$.

Essentially, the problem for belief elicitation is that (irrespective of the monetary prizes chosen,) the agent's "as if" prior incorporates interaction of the preference parameter $\alpha$ with the true set of beliefs.

The way we overcome the problem of *disentangling* the preference parameter $\alpha$ from the set of beliefs is to employ procedures for eliciting beliefs *and* the parameter $\alpha$. To ease exposition we first describe them as two completely separate procedures: one procedure for eliciting beliefs assuming the value of $\alpha$ is known to the designer (see section 3.1) and another to elicit the preference parameter $\alpha$ (as shown in section 3.2). We then show in section 3.3 how the two can be combined as a single mechanism that elicits the set of beliefs while also eliciting the true value of the preference parameter $\alpha$.

### 3.1 Eliciting beliefs; $\alpha$ known

In this subsection, we assume that the mechanism designer knows the value of $\alpha$. The main result is that except for $\alpha = 1/2$, it is possible to elicit agent's beliefs through the use of a direct revelation mechanism in which truth-telling is the agent's unique best response.[6]

**Theorem 1** *Let $[\underline{\mu}, \overline{\mu}]$ be the agent's ambiguous beliefs for the event E and the agent has $\alpha$ maxmin preferences. Suppose $\alpha$ is known. Then, for all $\alpha \neq 1/2$, there exists a direct revelation mechanism such that the agent's unique best response is to report truthfully the belief set $[\underline{\mu}, \overline{\mu}]$.*

The proof is constructive: we describe the mechanism that elicits beliefs truthfully.

**$\Gamma_{\text{belief}}(\alpha)$: Mechanism to elicit $\alpha$-MEU beliefs**

1. The agent is asked to report the belief set. Let $[\underline{r}, \overline{r}]$ denote the reported belief set. (Of course it suffices to ask the agent to report the two numbers - the minimum and the maximum - of the set.)

2. The mechanism chooses, with (objective) probability 1/2, either scheme A or scheme B. For either scheme, two numbers, drawn independently according to the uniform distribution over $[0, 1]$ are used; let the smaller (larger) of the two numbers be $\underline{p}$ ($\overline{p}$).

   - **Scheme A:** The mechanism calculates the two terms $\alpha\underline{p} + (1-\alpha)\overline{p}$ and $\alpha\underline{r} + (1-\alpha)\overline{r}$.
     - If
     $$\alpha\underline{p} + (1-\alpha)\overline{p} \geq \alpha\underline{r} + (1-\alpha)\overline{r}$$
     the agent is given the lottery $\mathcal{L}(\alpha; \ell(\underline{p}; x, y), \ell(\overline{p}; x, y))$.
     - If on the other hand
     $$\alpha\underline{p} + (1-\alpha)\overline{p} < \alpha\underline{r} + (1-\alpha)\overline{r}$$
     the agent is given the act $x_E y$.

[6]However, see the discussion toward the end of this subsection for the case when $\alpha = 1/2$.

- **Scheme B:** The mechanism calculates the two terms $[\alpha][1-\overline{p}\,]+[1-\alpha][1-\underline{p}\,]$ and $[\alpha][1-\overline{r}]+[1-\alpha\,][1-\underline{r}\,]$
  - If
  $$[\alpha][1-\overline{p}\,]+[1-\alpha][1-\underline{p}\,] \geq [\alpha][1-\overline{r}]+[1-\alpha\,][1-\underline{r}\,]$$
  the agent is given the lottery $\mathcal{L}(\,\alpha;\ell(\,1-\overline{p};x,y),\ell(\,1-\underline{p};x,y)\,)$.
  - If, on the other hand
  $$[\alpha][1-\overline{p}\,]+[1-\alpha][1-\underline{p}\,] < [\alpha][1-\overline{r}]+[1-\alpha\,][1-\underline{r}\,]$$
  the agent is given the act $y_E x$

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A few remarks before we continue. First, in both schemes "tie-breaking" rule is to allocate the lottery in the case of equality; it can be checked that nothing would change if the tie is broken the other way and the agent is awarded the act. A second, and more important, issue is the timing of the randomization. While it does not matter if the agent submits report before or concurrently with the choice of scheme by the mechanism, an important assumption is that the resolution of uncertainty regarding the event $E$ happens *after* the randomization to choose the scheme.[7]

[7] Let $\pi_A(\underline{r},\overline{r},\mu)$ and $\pi_B(\underline{r},\overline{r},\mu)$ be the payoffs from scheme A and B when the prior is $\mu$. As noted, it is important that the randomization to select the scheme is done prior to the resolution of uncertainty regarding the event $E$. In other words, when faced with the mechanism, the agent chooses $\underline{r}$ and $\overline{r}$ to maximize

$$\left[\frac{1}{2}\right]\left[(\alpha)\left(\min_{\mu\in[\underline{\mu},\overline{\mu}]}\pi_A(\underline{r},\overline{r},\mu)\right)+(1-\alpha)\left(\max_{\mu\in[\underline{\mu},\overline{\mu}]}\pi_A(\underline{r},\overline{r},\mu)\right)\right]$$
$$+\left[\frac{1}{2}\right]\left[(\alpha)\left(\min_{\mu\in[\underline{\mu},\overline{\mu}]}\pi_B(\underline{r},\overline{r},\mu)\right)+(1-\alpha)\left(\max_{\mu\in[\underline{\mu},\overline{\mu}]}\pi_B(\underline{r},\overline{r},\mu)\right)\right]$$

If, on the other hand, the randomization to select the schemes are done *after* the uncertainty regarding the event $E$ is resolved then the problem faced by the agent would be to choose $\underline{r}$ and $\overline{r}$ to maximize

$$(\alpha)\left(\min_{\mu\in[\underline{\mu},\overline{\mu}]}\left[\left\{\frac{1}{2}\right\}\{\pi_A(\underline{r},\overline{r},\mu)\}+\left\{\frac{1}{2}\right\}\{\pi_B(\underline{r},\overline{r},\mu)\}\right]\right)$$
$$+(1-\alpha)\left(\max_{\mu\in[\underline{\mu},\overline{\mu}]}\left[\left\{\frac{1}{2}\right\}\{\pi_A(\underline{r},\overline{r},\mu)\}+\left\{\frac{1}{2}\right\}\{\pi_B(\underline{r},\overline{r},\mu)\}\right]\right)$$

and in that case the mechanism may cease to be incentive compatible.

We prove the result in two steps. We first show in the following lemma that the agent's best responses $\underline{r}$ and $\overline{r}$ must satisfy a set of linear equations. Using this result, we then prove Theorem 1 by showing that for $\alpha \neq 1/2$ the unique best response must be to choose $\underline{r} = \underline{\mu}$ and $\overline{r} = \overline{\mu}$.

**Lemma 1** *For an agent with the α-maxmin preferences, the best response when facing the mechanism* $\Gamma_{belief}(\alpha)$ *above is to submit reports* $\underline{r}$ *and* $\overline{r}$ *that satisfy the two equations*

$$\alpha\underline{r} + (1-\alpha)\overline{r} = \alpha\underline{\mu} + (1-\alpha)\overline{\mu} \tag{3.1}$$

*and*

$$(1-\alpha)\underline{r} + \alpha\overline{r} = (1-\alpha)\underline{\mu} + \alpha\overline{\mu} \tag{3.2}$$

**Proof of Lemma 1:** The arguments are essentially the same as those used to show that truthfully reporting one's true valuation is the weakly dominant action in a Vickrey (second price) auction.

The agent's payoff from the act $x_E y$ is

$$[u(x)][\alpha\underline{\mu} + (1-\alpha)\overline{\mu}] + [u(y)][\alpha(1-\underline{\mu}) + (1-\alpha)(1-\overline{\mu})] \tag{3.3}$$

whereas payoff from the lottery $\mathcal{L}(\alpha; \ell(\underline{p}; x, y), \ell(\overline{p}; x, y))$ is

$$[u(x)][\alpha\underline{p} + (1-\alpha)\overline{p}] + [u(y)][\alpha(1-\underline{p}) + (1-\alpha)(1-\overline{p})] \tag{3.4}$$

Similarly, the agent's payoff from the act $y_E x$ is

$$[u(x)][\alpha(1-\overline{\mu}) + (1-\alpha)(1-\underline{\mu})] + [u(y)][\alpha\overline{\mu} + (1-\alpha)\underline{\mu}] \tag{3.5}$$

and payoff from the lottery $\mathcal{L}(\alpha; \ell(1-\overline{p}; x, y), \ell(1-\underline{p}; x, y))$ is

$$[u(x)][\alpha(1-\overline{p}) + (1-\alpha)(1-\underline{p})] + [u(y)][\alpha\overline{p} + (1-\alpha)\underline{p}] \tag{3.6}$$

Now suppose the agent reports $\underline{r}$ and $\overline{r}$ such that equation (3.1) is violated; in particular, $\underline{r}$ and $\overline{r}$ such that $\alpha\underline{r} + (1-\alpha)\overline{r} > \alpha\underline{\mu} + (1-\alpha)\overline{\mu}$. (The arguments for the case when $\alpha\underline{r} + (1-\alpha)\overline{r} < \alpha\underline{\mu} + (1-\alpha)\overline{\mu}$ are similar.)

Suppose Scheme A is chosen. If $\underline{p}$ and $\overline{p}$ are such that $\alpha\underline{p} + (1-\alpha)\overline{p} \geq \alpha\underline{r} + (1-\alpha)\overline{r}$, the agent is awarded the lottery $\mathcal{L}(\alpha; \ell(\underline{p}; x, y), \ell(\overline{p}; x, y))$, resulting in payoff $[u(x)][\alpha\underline{p} + (1-\alpha)\overline{p}] + [u(y)][(\alpha)(1-\underline{p}) + (1-\alpha)(1-\overline{p})]$, which is exactly the same payoff that would result if the agent reported $\alpha\underline{r} + (1-\alpha)\overline{r} = \alpha\underline{\mu} + (1-\alpha)\overline{\mu}$. Similarly, if $\alpha\underline{p} + (1-\alpha)\overline{p} < \alpha\underline{\mu} + (1-\alpha)\overline{\mu}$, the agent is awarded the act $x_E y$ resulting in payoff equal to $[u(x)][\alpha\underline{\mu} + (1-\alpha)\overline{\mu}] + [u(y)][(\alpha)(1-\underline{\mu}) + (1-\alpha)(1-\overline{\mu})]$, again the same payoff the agent would have received if $\alpha\underline{r} + (1-\alpha)\overline{r} = \alpha\underline{\mu} + (1-\alpha)\overline{\mu}$. However, if $\alpha\underline{p} + (1-\alpha)\overline{p} \in (\alpha\underline{\mu} + (1-\alpha)\overline{\mu}, \alpha\underline{r} + (1-\alpha)\overline{r})$, then the agent is awarded the act $x_E y$. A report that did not violate equation (3.1) would have resulted in being awarded the lottery $\mathcal{L}(\alpha; \ell(\underline{p}; x, y), \ell(\overline{p}; x, y))$ giving the agent a higher payoff than what the agent receives.

The arguments are similar if the chosen $\underline{r}$ and $\overline{r}$ are such that equation (3.2) is violated. Suppose in particular that $[\alpha][1-\overline{r}] + [1-\alpha][1-\underline{r}] > [\alpha][1-\overline{\mu}] + [1-\alpha][1-\underline{\mu}]$. (Again, the arguments for the case when $[\alpha][1-\overline{r}] + [1-\alpha][1-\underline{r}] < [\alpha][1-\overline{\mu}] + [1-\alpha][1-\underline{\mu}]$ are similar and hence skipped.) Suppose Scheme B is chosen. If $[\alpha][1-\overline{p}] + [1-\alpha][1-\underline{p}] \geq [\alpha][1-\overline{r}] + [1-\alpha][1-\underline{r}]$ or if $[\alpha][1-\overline{p}] + [1-\alpha][1-\underline{p}] < [\alpha][1-\overline{\mu}] + [1-\alpha][1-\underline{\mu}]$, the agent's payoff is the same as the payoff the agent would have received if equation (3.2) was not violated. However, when $[\alpha][1-\overline{p}] + [1-\alpha][1-\underline{p}] \in ([\alpha][1-\overline{\mu}] + [1-\alpha][1-\underline{\mu}], [\alpha][1-\overline{r}] + [1-\alpha][1-\underline{r}])$, the agent receives the act $y_E x$ but would have been strictly better off if the report had not violated equation (3.2) since in that case the agent would have been awarded the lottery $\mathcal{L}(\alpha; \ell(1-\overline{p}; x, y), \ell(1-\underline{p}; x, y))$.

Collecting the arguments from above together, any report of $\underline{r}$ and $\overline{r}$ such that equation (3.1) is violated gives strictly lower expected payoff than reports that satisfy equation (3.1) when Scheme A is chosen. And any report of $\underline{r}$ and $\overline{r}$ such that equation (3.2) is violated gives strictly lower expected payoff than reports that satisfy equation (3.2) when Scheme B is chosen. Since each scheme is chosen with probability $1/2$ the result follows. $\|$

We are now ready to prove Theorem 1.

**Proof of Theorem 1:** From Lemma 1 we know that the agent's best response is to submit reports $\underline{r}$ and $\overline{r}$ such that equation (3.1) and equation (3.2) are satisfied.

For $\alpha \neq 1/2$ the matrix $\begin{bmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{bmatrix}$ has full rank. Hence the system of linear equations (3.1) and (3.2) (in $\overline{r}$ and $\underline{r}$) has a *unique* solution. Finally, since $\overline{r} = \overline{\mu}$ and $\underline{r} = \underline{\mu}$ solve the equations (3.1) and (3.2) it then follows the truthful reporting is the unique best response. ‖

When $\alpha = 1/2$ the matrix above drops rank and the solution is no longer unique. However, the *real* problem is that when $\alpha = 1/2$ for any act, *multiple values* of $\underline{\mu}$ and $\overline{\mu}$ gives rise to *exactly* the same level of payoff. In other words, for $\alpha = 1/2$, the agent's belief ( as can be deduced from observed behaviour) is not uniquely defined. In that case it is no surprise that the elicitation procedure is unable to elicit *unique* beliefs.

## 3.2 Eliciting $\alpha$

This subsection describes a mechanism to elicit the preference parameter $\alpha$. We remind the reader that we assume that all departures from expected utility are due to ambiguity aversion only. In particular, the agent evaluates objective compound lotteries the same way as someone with Expected Utility preferences would.

The mechanism is as follows.

**$\Gamma_\alpha$: Mechanism to elicit $\alpha$**

1. The agent is asked to report $\alpha$. Let the reported value be denoted as $\hat{\alpha}$.
2. Two numbers are drawn independently from the uniform distribution. Let the larger (smaller) number be called $\overline{p}$ ($\underline{p}$).
3. A third number, $\beta$ is also drawn from the uniform distribution.
4. The mechanism calculates the values $[\,\hat{\alpha}\underline{p} + (1-\hat{\alpha})\overline{p}\,]$ and $[\,\beta\underline{p} + (1-\beta)\overline{p}\,]$.
5. The allocation rule is as follows.
   - If
   $$[\,\beta\underline{p} + (1-\beta)\overline{p}\,] \geq [\,\hat{\alpha}\underline{p} + (1-\hat{\alpha})\overline{p}\,]$$
   the agent is awarded the lottery $\mathcal{L}(\,\beta; \ell(\,\underline{p}; x, y), \ell(\,\overline{p}; x, y)\,)$.

- If

  $$[\,\beta \underline{p} + (1-\beta)\overline{p}\,] < [\,\hat{\alpha}\underline{p} + (1-\hat{\alpha})\overline{p}\,]$$

  the agent is given an ambiguous act defined below.

  **Ambiguous Act:**

  – A number $p \in [\,\underline{p},\, \overline{p}\,]$ is chosen.

  – The agent is given no further information regarding how (i.e., the distribution from which) it is chosen. In other words, all that the agent knows is that $p$ is determined randomly and it is at least $\underline{p}$ and at most $\overline{p}$.

  – The act pays $x$ with probability $p$ and $y$ with probability $1-p$. (Note that this is a sort of "reduced form" description of the ambiguous act. A more detailed description would involve describing the physical action/experiment for creation of some event, say D with - crucially - the probability $p$ of the event D being some number in $[\,\underline{p}, \overline{p}\,]$. The act pays $x$ if D happens and $y$ if D does not happen.)

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It is straightforward to check that this mechanism elicits $\alpha$. Notice first that the agent's payoff from the (engineered) ambiguous act is

$$[u(x)][\alpha \underline{p} + (1-\alpha)\overline{p}] + [u(y)][\alpha(1-\underline{p}) + (1-\alpha)(1-\overline{p})]$$

whereas payoff from the compound lottery $\mathcal{L}(\,\beta; \ell(\,\underline{p}; x, y), \ell(\,\overline{p}; x, y)\,)$ is

$$[u(x)][\beta \underline{p} + (1-\beta)\overline{p}] + [u(y)][\beta(1-\underline{p}) + (1-\beta)(1-\overline{p})]$$

We now employ the same type of reasoning we have used in the proof of Lemma 1. So, assume that $\hat{\alpha} > \alpha$. (The case when $\hat{\alpha} < \alpha$ is analogous and left as an exercise). If either $\beta > \hat{\alpha}$ or $\beta < \alpha$, truth-telling and misreporting gives the same payoff [8] but if $\beta \in (\alpha,\, \hat{\alpha})$, the agent receives the act whereas if the agent had reported truthfully,

[8]In the former scenario, the agent receives the lottery $\mathcal{L}(\,\beta; \ell(\,\underline{p}; x, y), \ell(\,\overline{p}; x, y)\,)$ which is what the agent would have received under truth-telling and in the latter the agent receives the ambiguous act which is what the agent would receive if the agent had reported truthfully.

the agent would have received the (compound) lottery. Since $\beta > \alpha$, and $x > y$, misreporting results is lower payoff than truth-telling.

From the discussions above, we have the following result.

**Proposition 1** *It is possible to elicit the preference parameter $\alpha$ by using a direct revelation mechanism in which truthfully reporting the true value of $\alpha$ is the agent's unique best response.*

### 3.3 Eliciting Beliefs and $\alpha$ concurrently

We have described elicitation of beliefs in section 3.1 under the assumption that the true value of $\alpha$ is known to the designer and then shown in section 3.2 the process to elicit $\alpha$. While this was useful for purposes of exposition, one potential drawback is that this might create the impression that when the designer does not know the value of $\alpha$, the task of elicitation of beliefs requires the second process to be conducted *prior* to running the process for belief elicitation. However, we now show that the two processes can be combined into a single mechanism that elicits beliefs and $\alpha$ concurrently.

Section 3.1 uses the mechanism $\Gamma_{\text{belief}}(\alpha)$ to elicit the beliefs. Let $\Gamma_{\text{belief}}(\widehat{\alpha})$ be a modification of this mechanism where, instead of the true $\alpha$, a reported value $\widehat{\alpha}$ is used everywhere. Recall that $\Gamma_{\alpha}$ is the mechanism used to elicit $\alpha$ in section 3.2. Now consider the following.

The agent is asked to report $\underline{\mu}, \overline{\mu}, \alpha$; let $\underline{r}, \overline{r}, \widehat{\alpha}$ denote the report. Consider the following mechanism.

**The mechanism for eliciting beliefs and $\alpha$ concurrently:** With probability 1/2, the mechanism chooses $\mathbf{\Gamma}_{\text{belief}}(\widehat{\alpha})$ and with probability 1/2 it chooses $\mathbf{\Gamma}_{\alpha}$.

We now show that the agent's best response is to report beliefs and $\alpha$ truthfully. Suppose $\mathbf{\Gamma}_{\text{belief}}(\widehat{\alpha})$ is chosen. It is easy to see that the agent's best response is to choose $\underline{r}$, $\overline{r}$ and $\widehat{\alpha}$ such that

$$\widehat{\alpha}\underline{r} + (1 - \widehat{\alpha})\overline{r} = \alpha\underline{\mu} + (1 - \alpha)\overline{\mu}$$

and

$$(1-\widehat{\alpha})\underline{r}+\widehat{\alpha}\overline{r}=(1-\alpha)\underline{\mu}+\alpha\overline{\mu}$$

This is two equations in three unknowns and while $\underline{r}=\underline{\mu}$, $\overline{r}=\overline{\mu}$ and $\hat{\alpha}=\alpha$ is a solution, it is not the only solution. Put differently, if $\mathbf{\Gamma}_{\text{belief}}(\widehat{\alpha})$ were chosen with certainty, then while truth-telling is still a best response it is no longer the unique best response. However, the mechanism also chooses $\mathbf{\Gamma}_\alpha$ with strictly positive probability. And the crucial observation is that if $\mathbf{\Gamma}_\alpha$ is chosen then the reported values of $\underline{r}$ and $\overline{r}$ are completely irrelevant and further, any $\hat{\alpha}\neq\alpha$ gives strictly lower expected payoff than $\hat{\alpha}=\alpha$. Hence, if $\mathbf{\Gamma}_\alpha$ is chosen then the agent is better off having reported $\hat{\alpha}=\alpha$. But then, if the agent reports $\hat{\alpha}=\alpha$, simply repeating the arguments from section 3.1 shows that if $\Gamma_{\text{belief}}(\widehat{\alpha})$ is chosen then reporting $\underline{r}=\underline{\mu}$ and $\overline{r}=\overline{\mu}$ gives strictly higher expected payoff than any other reporting strategy. Therefore truthful reporting of both beliefs and $\alpha$ is the agent's unique best response under the mechanism described.

It is not difficult to see why combining the two separate procedures into one single procedure works. In an informal, sense there is no "contamination" from one to the other. In both procedures truth-telling is a best response. Put differently, the complication that one faces in other situations (when trying to combine two separate mechanisms into a single mechanism) whereby the agent may lie and "sacrifice" payoff in one case to obtain a higher payoff in the other case does not arise here. In the mechanism $\mathbf{\Gamma}_\alpha$, reporting $\hat{\alpha}=\alpha$ is the agent's unique best response irrespective of the values of $\underline{r}$ and $\overline{r}$ . And when facing the mechanism $\mathbf{\Gamma}_{\text{belief}}(\widehat{\alpha})$, conditional on having truthfully reported $\hat{\alpha}=\alpha$, the agent has strictly higher expected payoff from truthfully reporting $\underline{r}=\underline{\mu}$ and $\overline{r}=\overline{\mu}$ than from any false reporting.

## 3.4 Maxmin Expected Utility (MEU) as a Special Case

The maxmin model of Gilboa and Schmeidler (1989), also known in the literature as the MEU model, represents preference of an ambiguity averse agent (with ambiguous beliefs $[\underline{\mu},\overline{\mu}]$ regarding the event $E$) who evaluates the act $x_E y$ according to

$$\min_{\mu\in[\underline{\mu},\overline{\mu}]}\mu u(x)+[1-\mu]u(y)$$

This can be thought of as a special case of $\alpha$-maxmin with $\alpha = 1$. It is easy to check that the mechanism $\mathbf{\Gamma}_{\text{belief}}(\boldsymbol{\alpha} = \mathbf{1})$ elicits the agent's beliefs. (Actually, the mechanism $\mathbf{\Gamma}_{\text{belief}}(\boldsymbol{\alpha} = \mathbf{1})$ can be simplified: there is no need to draw two numbers $\overline{p}$ and $\underline{p}$. A *single* number $p$ can be drawn which will play the role of $\underline{p}$ if scheme A is chosen and $\overline{p}$, if the scheme chosen is B.)

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