Tests of Policy Ineffectiveness in Macroeconometrics*

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June 15, 2014

Abstract

This paper proposes tests of policy ineffectiveness in the context of macroeconometric rational expectations models. It is assumed that there is a policy intervention that takes the form of changes in the parameters of a policy rule, and that there are sufficient observations before and after the intervention. The test is based on the difference between the realisations of the outcome variable of interest and counterfactuals based on no policy intervention, using only the pre-intervention parameter estimates, and in consequence the Lucas Critique does not apply. The paper develops tests of policy ineffectiveness for a full structural model, with and without exogenous, policy or non-policy, variables. Asymptotic distributions of the proposed tests are derived both when the post intervention sample is fixed as the pre-intervention sample expands, and when both samples rise jointly but at different rates. The performance of the test is illustrated by a simulated policy analysis of a three equation New Keynesian Model, which shows that the test size is correct but the power may be low unless the model includes exogenous variables, or if the policy intervention changes the steady states, such as the inflation target.

Keywords: Counterfactuals, policy analysis, policy ineffectiveness test, macroeconomics
JEL classification: C18, C54, E65,

*We are grateful to Karrar Hussain and Alex Chudik for their help with the calibration and simulation exercises reported in the paper. We have also benefited from discussions with Oscar Jorda, Adrian Pagan, Ivan Petrella and Glenn Rudebusch.

1 Introduction

In this paper we propose tests for the effect of a change in policy on some target variable in the context of a macroeconometric dynamic stochastic general equilibrium (DSGE) model. We are concerned with *ex post* evaluation of a policy intervention on a single unit (country), where data are available before and after the intervention.

We consider a policy intervention which takes the form of a change in one or more of the parameters of a policy rule, the non-policy parameters being structural in that they are invariant to the policy intervention. The announcement and implementation of the intervention, at time T_0, are assumed to be understood and to be credible. The null hypothesis of the test is policy ineffectiveness, no change in the parameters, and the tests are based on the mean, over a given policy evaluation horizon, $T_0+1, T_0+2, ..., T_0+H$, of the differences between the post-intervention realizations of the target variable and the associated counterfactual outcomes based on the parameters estimated using data before the policy intervention. The development of the test does not require knowing or estimating the post-intervention parameters, thus it is not a structural change test. We derive the asymptotic distribution of the tests both when the post-intervention sample size, H, is fixed as the pre-intervention sample expands, and when both samples rise jointly but possibly at different rates.

The Lucas Critique is not an issue since the counterfactual, given by the predictions from the model estimated on pre-intervention data, will embody pre-intervention parameters while the actual post-intervention outcomes will embody the effect of any change in the policy parameters and the consequent change in expectations. Different issues are involved in *ex ante* policy formulation where post-intervention data are not available and the Lucas Critique could be an issue since the possible effects of the policy change on parameters and expectations must be taken into account.

We consider both standard DSGE models where all variables, including policy variables, are endogenous and DSGE models augmented by exogenous variables. This also accommodates interventions that change parameters in rules that determine the policy variable exogenously, as in a fixed money supply growth rule or inflation targeting, rather than determine them endogenously, as in a Taylor rule that relates the policy variable to the endogenous variables of the DSGE model. Thus our framework can also accommodate policy changes that alter the steady states of some of the variables, as occurs when the inflation target is changed.

In DSGE models with persistence, we show that the policy intervention will effect the deviations from steady state but these effects are transitory, so the average effect of the intervention on the target variable over the evaluation sample will fall towards zero as the length of the evaluation sample is increased without bound. Thus while the test may have power for short post-intervention evaluation samples, its power does not rise with H. In addition, the power of the test to detect the intervention will depend on the state of the economy at the time of the intervention, T_0. If at T_0 the economy is in steady states the test will have little power to detect the intervention. In practice this may not be a problem, because major policy interventions tend to take place at times when the economy is far from its steady state.

Where there are exogenous variables that have permanent effects or where the policy intervention changes the steady state, such as changing the target rate of inflation, the power of the test increases towards one as the evaluation horizon, H, increases. The most effective policy changes are ones that change the parameters in a way that amplifies the permanent effect of exogenous variables. Policy interventions that change the steady states, such as the target inflation rate, have this property.

We investigate the size and power of the proposed tests by a simulation analysis using a standard three equation New Keynesian DSGE model, where the policy interventions involve changing the parameters of the Taylor rule. The model shows very standard shock impulse response functions which show the time profile of the effects on the variables of monetary policy, demand and supply shocks. We also consider policy impulse response functions which show the time profile of the effects on the variables of a change in the policy rule parameters. The simulations accord with the theoretical results and show that in all applications (where we abstract from parameter uncertainty) the tests have the correct size; but if the intervention does not change the steady state the power of the test can be low. The simulations demonstrate how the power varies with the the magnitude of the policy change, the difference between pre and post-intervention parameter values; the particular parameters of the Taylor rule which are changed; the state of the economy at the time of policy intervention, and the post-intervention evaluation horizon.

We also consider policy changes that aim at increasing or reducing the inflation target, with the latter scenario being currently of interest for Japan. Once again simulation results are in line with the theory, and show that our test has power when it is applied to interest rates and inflation and the power rises with H and eventually appraoches unity. But when the test is applied to output deviations, the power could be low and does not rise with H, since the effect of the policy change on output is only transitory. The simulations also show interesting interactions between the direction of the policy change in the inflation target and the degree of interest rate smoothing. In cases where the policy change aims at increasing the inflation target then smoothing of interest rate changes can have beneficial effects on output, but in cases where the inflation target is reduced further smoothing of interest rate changes can be costly as output losses will be greater.

The rest of the paper is organized as follows: Section 2 develops the counterfactuals and policy impulse response functions and derives the policy ineffectiveness test for a standard DSGE where all the variables, including the policy variable, are endogenous. Section 3 augments the standard DSGE model to allow for exogenous variables including exogenous policy variables. Section 4 provides the simulated policy analysis of the New Keynesian model. Section 5 ends with some concluding remarks. The more technical derivations are given in an Appendix.

2 Policy ineffectiveness tests for a standard RE model

2.1 Derivation of the counterfactuals and policy effects: standard case

Consider a standard rational expectations (RE) model, where all the variables are endogenous. We suppose that the target variable, y_t, is affected directly by a vector of variables, $\mathbf{z}_t$, including the policy variable, and assume that the $(k_z+1)\times 1$ vector $\mathbf{q}_t = (y_t, \mathbf{z}_t')'$ is determined by the RE model (which could result from some well defined decision problem) of the form

$$\mathbf{A}_0\mathbf{q}_t = \mathbf{A}_1 E_t(\mathbf{q}_{t+1}) + \mathbf{A}_2\mathbf{q}_{t-1} + \mathbf{u}_t, \tag{1}$$

where the structural shocks, $\mathbf{u}_t$, have mean zero, $E(\mathbf{u}_t) = 0$, are serially uncorrelated and have the constant variance matrix, $E(\mathbf{u}_t\mathbf{u}_t') = \boldsymbol{\Sigma}_u$, typically a diagonal matrix. $E_t(\mathbf{q}_{t+1}) = E(\mathbf{q}_{t+1} \mid \Im_t)$, $\Im_t$ is the information set that includes $\mathbf{u}_t$, and the lagged values of the variables, $\mathbf{q}_t$. We assume that $\mathbf{q}_t$ are measured as deviations from their steady state values, but discuss policy changes that alter the steady states, such as the inflation target, below.

Initially we abstract from parameter estimation uncertainty and denote the vector of structural and policy parameters by $\boldsymbol{\theta} = vec(\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2)$, and assume that $\boldsymbol{\Sigma}_u$ remains invariant to the policy change. The parameter vector, θ, is composed of a set of policy parameters, $\boldsymbol{\theta}_p$, and a set of structural parameters, $\boldsymbol{\theta}_s$, that are invariant to changes in $\boldsymbol{\theta}_p$. A policy intervention is defined in terms of a change in one or more elements of $\boldsymbol{\theta}_p$. The null hypothesis of our test is policy ineffectiveness to be defined more formally below. We assume that the model is known by economic agents, the announcement and implementation of the intervention are credible, and no further changes are expected.[1] We suppose that the policy intervention occurs at the end of time $t = T_0$, and we have a pre-intervention sample that runs from $t = M, M+1, ..., T_0$, and a post-intervention sample for $t = T_0+1, T_0+2, .., T_0+H$. Therefore, the post-intervention evaluation horizon is H and the sample size for estimation of the pre-intervention parameters is $T = T_0 - M + 1$. This notation allows us to increase the sample size T (by letting $M \to -\infty$), while keeping the time of intervention, T_0, fixed.

[1] Kulish and Pagan (2014) consider solutions of forward looking models in the case of imperfect credibility where policy announcements are not necessarily incorporated into expectations.

A prominent example of a system of this form is the three equation new Keynesian DSGE model, which has an IS curve determining log output-gap, y_t, a Phillips curve determining inflation, π_t, and a Taylor rule determining the short interest rate, R_t. The policy parameters are the parameters of the Taylor rule. We use such a model in the simulation analysis in Section 4. In the literature the effects of policy is usually measured by "shock impulse response functions", SIRFs. For instance in the NK model, this estimates the expected effect over time of a one standard error monetary policy shock to the interest rate equation, assuming that the shock is small enough to leave the parameters unchanged. In contrast, we focus on a "policy impulse response function" (PIRF), that measures the effect over time of a policy intervention that takes the form of a change in the policy parameters, θ_p, such as those of the Taylor rule, rather than a shock to its equation error. In the context of the SIRF, it is often not clear what is the source of this policy implementation error that is shocked to produce the IRF in response to a monetary policy shock.

Under the above set up, the RE model (1) has the unique solution

$$\mathbf{q}_t = \mathbf{\Phi}(\theta)\mathbf{q}_{t-1} + \mathbf{\Gamma}(\boldsymbol{\theta})\mathbf{u}_t, \tag{2}$$

if the quadratic matrix equation

$$\mathbf{A}_1\mathbf{\Phi}^2 - \mathbf{A}_0\mathbf{\Phi} + \mathbf{A}_2 = \mathbf{0}, \tag{3}$$

has a solution, $\mathbf{\Phi}$, with all its eigenvalues inside the unit circle, and $\mathbf{\Gamma}(\theta) = (\mathbf{A_0} - \mathbf{A_1}\mathbf{\Phi})^{\mathbf{-1}}$. Below we shall also use the reduced form shocks, $\boldsymbol{\varepsilon}_t = \mathbf{\Gamma}(\theta)\mathbf{u}_t$, and we note that

$$\mathbf{\Sigma}_\varepsilon(\theta) = E(\varepsilon_t\varepsilon_t') = (\mathbf{A}_0 - \mathbf{A}_1\mathbf{\Phi})^{-1}\mathbf{\Sigma}_u(\mathbf{A}_0 - \mathbf{A}_1\mathbf{\Phi})^{'-1}. \tag{4}$$

Notice that (2) is a vector autoregression and corresponds to the reduced form of a standard simultaneous equations model where there are no exogenous variables.

A policy change, defined as a change in one or more elements of θ_p, will affect the mean outcomes through changes in $\mathbf{\Phi}(\theta)$ and the variance of the outcomes through $\mathbf{\Gamma}(\theta)$. Denote the pre-intervention parameters by $\theta^0 = (\theta_p^{0\prime}, \theta_s')'$, and the post-intervention parameters by $\theta^1 = (\theta_p^{1\prime}, \theta_s')'$, where only one or more elements of the policy parameters are changed. If the intervention at T_0 is transparently and fully communicated, it is understood to be credible, with expectations adjusting immediately, then the process switches from

$$\mathbf{q}_t = \mathbf{\Phi}(\theta^0)\mathbf{q}_{t-1} + \mathbf{\Gamma}(\theta^0)\mathbf{u}_t,\ t = M, M+1, M+2, ..., T_0$$

to

$$\mathbf{q}_t = \mathbf{\Phi}(\theta^1)\mathbf{q}_{t-1} + \mathbf{\Gamma}(\theta^1)\mathbf{u}_t,\ t = T_0+1, T_0+2, ..., T_0+H.$$

The policy impulse response function for y_t is given by the expected difference in the outcomes associated with the two parameter vectors

$$PIRF_y(h, \theta^0, \theta^1, \mathbf{q}_{T_0}) = \mathfrak{s}' \left\{ \left[\mathbf{\Phi}\left(\theta^1\right) \right]^h - \left[\mathbf{\Phi}\left(\theta^0\right) \right]^h \right\} \mathbf{q}_{T_0}, \tag{5}$$

where $\mathfrak{s}$ is a the $(k_z + 1) \times 1$ selection vector with all its elements zero except for its first element which is set to unity. Unlike the shock impulse response functions the PIRF depends on the state of the economy at the time of the intervention, $\mathbf{q}_{T_0}$. To evaluate this PIRF requires knowing, or being able to estimate, the post-intervention parameters θ^1. However, the counterfactual values of the focus variable, y_{T_0+h}, on the assumption of no change in policy, are given by

$$y^0_{T_0+h} = \mathfrak{s}' \left[\mathbf{\Phi}\left(\theta^0\right) \right]^h \mathbf{q}_{T_0}, \tag{6}$$

and only require estimation of θ^0. The effect of policy on the target variable is then the difference between the realised values, y_{T_0+h}, and the counterfactual values, $y^0_{T_0+h}$,

$$d_{T_0+h} = y_{T_0+h} - y^0_{T_0+h},\ h = 1, 2, ..., H. \tag{7}$$

These measured policy effects will be subject to the post intervention random errors, $\varepsilon_{y,T_0+h.}$.

Notice that if there are no dynamics in (1), $\mathbf{A}_2 = 0$, then assuming that all eigenvalues of $\mathbf{A}_0^{-1}\mathbf{A}_1$ lie within the unit circle, the unique solution is:

$$\mathbf{q}_t = \mathbf{A}_0^{-1}\mathbf{u}_t = \mathbf{\Gamma}(\theta)\mathbf{u}_t.$$

Thus in the absence of persistence (dynamics), a policy intervention (defined by a change in some elements of $\mathbf{A}_0$) has no effect on the mean outcomes, $\mathbf{q}_t$, but does change the variance of the outcomes. The variance of $\mathbf{q}_t$ changes from $\mathbf{\Sigma}_\varepsilon(\theta^0) = \mathbf{\Gamma}(\theta^0)\mathbf{\Sigma}_u\mathbf{\Gamma}(\theta^0)'$ to $\mathbf{\Sigma}_\varepsilon(\theta^1) = \mathbf{\Gamma}(\theta^1)\mathbf{\Sigma}_u\mathbf{\Gamma}(\theta^1)'$. Conditional on the structural error variances, $\mathbf{\Sigma}_u$, remaining constant, one could derive a test statistic for a policy induced variance change corresponding to the policy ineffectiveness test for a mean change discussed below. It will be more challenging to develop tests that separate the effects of a policy change from other changes in $\mathbf{\Sigma}_u$ that happen by chance. When there is persistence, policy can effect mean outcomes, but that effect is transitory since the system returns to steady state.

2.2 Derivation of the test statistic: standard case

To derive the distribution of the policy effects and develop a policy ineffectiveness test we require the following assumptions.

Assumption 1: The RE model defined by (1) has a unique solution given by (2), and the structural parameters, $\theta \in \mathbf{\Theta}$, are identified at θ^0 and $\boldsymbol{\theta}^1$ (the pre and post-intervention parameters). The structural errors, $\mathbf{u}_t$, are serially uncorrelated with zero means and a constant covariance matrix, $\mathbf{\Sigma}_u$.

Assumption 2a: The spectral radius of $\mathbf{\Phi}(\theta)$, defined by $|\lambda_{\max}[\mathbf{\Phi}(\theta)]|$, is strictly less than unity for values of $\boldsymbol{\theta} = \theta^0$ and $\theta^1 \in \mathbf{\Theta}$.[2]

Assumption 2b: There exists a matrix norm of $\mathbf{\Phi}(\boldsymbol{\theta})$, denoted by $\|\mathbf{\Phi}(\theta)\|$, such that $\|\mathbf{\Phi}(\theta)\| < 1$, for values of $\boldsymbol{\theta} = \theta^0$ and $\theta^1 \in \mathbf{\Theta}$.

Assumption 3: Standard regularity assumptions on the structural errors, $\mathbf{u}_t$, and the processes generating the exogenous variables (if any) apply such that θ^0 can be consistently estimated by $\hat{\boldsymbol{\theta}}_T^0$ based on the pre-intervention sample, $t = M, M+1, M+2, ..., T_0$, where $T = T_0 - M + 1$, and $\hat{\theta}_T^0 = \theta^0 + O_p\left(T^{-1/2}\right)$. In particular

$$\sqrt{T}\left(\hat{\theta}_T^0 - \theta^0\right) \overset{a}{\sim} N(\mathbf{0}, \mathbf{\Sigma}_{\theta^0}), \tag{8}$$

$$E\left\|\hat{\theta}_T^0 - \theta^0\right\| = O(T^{-1/2}), \tag{9}$$

where $\mathbf{\Sigma}_{\theta^0}$ is a symmetric positive definite matrix.

Assumption 4: $\mathbf{\Phi}(\theta) = (\phi_{ij}(\theta))$, is bounded and continuously differentiable in $\boldsymbol{\theta}$, such that $\left\|\partial\phi_{ij}(\theta)/\partial\boldsymbol{\theta}'\right\|$, for all i and j exist and are bounded.

Assumption 5: The initial values, $\mathbf{q}_{T_0}$, are bounded, namely $\|\mathbf{q}_{T_0}\| < K$, where K is a fixed positive constant.

Assumptions 1, 2a, 3 and 4 are standard in the literature on the econometric analysis of DSGE models. The conditions for identification in Assumption 1 are discussed in Koop, Pesaran and Smith (2013). Assumption 2a ensures that $\|\mathbf{\Phi}(\theta)\| < \lambda$, where λ is a finite positive constant.[3] Assumption 2b is stronger than 2a and further requires that $\lambda < 1$. This latter restriction allows us to simplify the proofs considerably and obtain the main theoretical results without requiring high order differentiability of $\mathbf{\Phi}(\theta)$ which will be needed in the absence of Assumption 2b.

In the cases where both H and T go to infinity we will also use the following joint asymptotic condition:

Condition 1 *The post-intervention sample size, H, rises with the pre-intervention sample size, T, such that $H = \kappa T^{\epsilon}$, where κ is a fixed positive constant, and $\epsilon \leq 1/2$.*

Using (6), estimates of the counterfactuals in the absence of the policy change are given by

$$\hat{y}_{T_0+h}^0 = \mathfrak{s}'\left[\mathbf{\Phi}\left(\hat{\theta}_T^0\right)\right]^h \mathbf{q}_{T_0}, \tag{10}$$

[2] $\lambda_{\max}(\mathbf{A})$ stands for the largest eigenvalue of matrix $\mathbf{A}$.

[3] Note that there exists a matrix norm, $\|\mathbf{A}\|$, such that $|\lambda_{\max}(A)| \leq \|\mathbf{A}\| \leq |\lambda_{\max}(A)| + \epsilon$, where ϵ is a positive constant. See, for example, Lemma 5.10.10 in Horn and Johnson (1985).

where under Assumption 3, $\hat{\theta}_T^0$ is a $\sqrt{T}$- consistent estimator of $\boldsymbol{\theta}$ based on the pre-intervention sample. Therefore, the estimated policy effects are given by

$$\hat{d}_{T_0+h}(\hat{\theta}_T^0) = \mathfrak{s}'\mathbf{q}_{T_0+h} - \mathfrak{s}' \left[\boldsymbol{\Phi}\left(\hat{\theta}_T^0\right)\right]^h \mathbf{q}_{T_0}, \tag{11}$$

for $h = 1, 2, ..., H$. It is clear that estimation of the policy effects only requires estimates of θ^0 that can be obtained using the pre-intervention sample. Also, the sampling distribution of the $\hat{d}_{T_0+h}(\hat{\theta}_T^0)$, depends on post-intervention parameters only under the alternative that the policy is effective, but not under the null hypothesis of no policy effect as defined by

$$H_0 : \theta^1 = \theta^0. \tag{12}$$

To derive the distribution of the policy effects, $\hat{d}_{T_0+h}(\hat{\theta}_T^0)$, first note that post-intervention realized values, $\mathbf{q}_{T_0+h}$, (for $h = 1, 2, ..., H$) are given by

$$\mathbf{q}_{T_0+h} = \left[\boldsymbol{\Phi}\left(\theta^1\right)\right]^h \mathbf{q}_{T_0} + \sum_{j=0}^{h-1} \left[\boldsymbol{\Phi}\left(\theta^1\right)\right]^j \boldsymbol{\Gamma}(\theta^1)\mathbf{u}_{T_0+h-j}. \tag{13}$$

Using (13) and substituting the results for $\mathbf{q}_{T_0+h}$ in (11) we have

$$\hat{d}_{T_0+h}(\hat{\theta}_T^0) = \hat{\mu}_{T_0,h}(\hat{\theta}_T^0) + v_{T_0,h} \tag{14}$$

where

$$\hat{\mu}_{T_0,h}(\hat{\theta}_T^0) = -\mathfrak{s}' \left\{\left[\boldsymbol{\Phi}\left(\hat{\theta}_T^0\right)\right]^h - \left[\boldsymbol{\Phi}\left(\theta^1\right)\right]^h\right\} \mathbf{q}_{T_0}, \tag{15}$$

$$v_{T_0,h} = \sum_{j=0}^{h-1} \mathfrak{s}' \left[\boldsymbol{\Phi}\left(\theta^1\right)\right]^j \boldsymbol{\Gamma}(\theta^1)\mathbf{u}_{T_0+h-j}, \tag{16}$$

In (14) the estimated policy effect, $\hat{d}_{T_0+h}(\hat{\theta}_T^0)$, has a systematic component, $\hat{\mu}_{T_0,h}(\hat{\theta}_T^0)$, and a random component, $v_{T_0,h}$. The random component is a weighted linear combination of serially uncorrelated shocks, $\mathbf{u}_t$ with the weights decaying exponentially under Assumption 2a. A policy ineffectiveness test of H_0 can now be based on the policy effects, $\hat{d}_{T_0+h}(\hat{\theta}_T^0)$, $h = 1, 2, ..., H$. But to develop formal statistical tests of policy ineffectiveness, we also need to make distributional assumptions regarding the shocks, $\mathbf{u}_t$. The role of such assumptions can be minimized by basing the policy ineffectiveness test on a "mean policy effect" computed over the post-intervention horizon $T_0 + h$, for $h = 1, 2, ..., H$, namely

$$\bar{\hat{d}}_H(\hat{\theta}_T^0) = \frac{1}{H}\sum_{h=1}^{H} \hat{d}_{T_0+h}(\hat{\theta}_T^0). \tag{17}$$

For a fixed H, the implicit null hypothesis of no policy effects can now be specified as

$$H_0' : p \lim_{T\to\infty} \left(H^{-1/2}\sum_{h=1}^{H} \hat{\mu}_{T_0,h}(\hat{\theta}_T^0)\right) = 0. \tag{18}$$

As we shall see, this condition is met under Assumptions 1, 2a, 3 and 4 when H is fixed and as $T \to \infty$.

Interestingly enough, H_0' continues to hold even if $H \to \infty$, so long as Assumption 2b holds and the rate of increase of H in relation to T is governed by the joint asymptotic condition 1. If the underlying RE model is correctly specified, then under the null of no policy change, H_0, we have

$$H^{-1/2}\sum_{h=1}^{H}\hat{\mu}_{T_0,h}(\hat{\theta}_T^0) = -\mathfrak{s}'\left\{H^{-1/2}\sum_{h=1}^{H}\left[\mathbf{\Phi}\left(\hat{\theta}_T^0\right)\right]^h - \left[\mathbf{\Phi}\left(\theta^0\right)\right]^h\right\}\mathbf{q}_{T_0}. \tag{19}$$

Now using results in Lemmas 2 and 3, given in the Appendix, we have

$$\begin{aligned}\left\|H^{-1/2}\sum_{h=1}^{H}\hat{\mu}_{T_0,h}(\hat{\theta}_T^0)\right\| &\leq \left\|\mathfrak{s}'\right\|\left\|\mathbf{q}_{T_0}\right\|H^{-1/2}\left\|\sum_{h=1}^{H}\left[\mathbf{\Phi}\left(\hat{\theta}_T^0\right)\right]^h - \left[\mathbf{\Phi}\left(\theta^0\right)\right]^h\right\| \\ &\leq K\left\|\mathfrak{s}'\right\|\left\|\mathbf{q}_{T_0}\right\|H^{-1/2}\left(\sum_{h=1}^{H}h\lambda_T^{h-1}\right)\left\|\hat{\theta}_T^0-\theta^0\right\|,\end{aligned} \tag{20}$$

where K is a fixed constant. Using (70) in Lemma 3, we have

$$\left\|\mathbf{\Phi}(\hat{\theta}_T^0)\right\| \leq \left\|\mathbf{\Phi}(\theta^0)\right\| + a_T\left\|\hat{\theta}_T^0-\theta^0\right\|,$$

where $a_T = \left\|\partial\mathbf{\Phi}\left(\bar{\theta}_T^0\right)/\partial\theta'\right\|$, and elements of $\bar{\theta}_T^0$ lie on the line segment joining θ^0 and $\hat{\theta}_T^0$. Considering that $\bar{\theta}_T^0 \to_p \theta^0$, and by Assumption 4 $\|\partial\phi_{ij}(\theta)/\partial\theta'\|$ for all i and j exist and are bounded, then it must also follow that a_T is bounded in T. Hence, recalling that under Assumption 3, $\sqrt{T}\left\|\hat{\theta}_T^0-\theta^0\right\| = O_p(1)$, then $\lambda_T \leq \lambda + a_T T^{-1/2}$, where $\left\|\mathbf{\Phi}\left(\theta^0\right)\right\| \leq \lambda$, and a_T is bounded in T. In the case where H is fixed and $T \to \infty$,

$$\left|H^{-1/2}\left(\sum_{h=1}^{H}h\lambda_T^{h-1}\right)\right| \leq H^{-1/2}\sum_{h=1}^{H}h\left(\lambda + a_T T^{-1/2}\right)^{h-1} \to H^{-1/2}\sum_{h=1}^{H}h\lambda^{h-1} < K, \text{ as } T \to \infty.$$

Using this result in (20) and noting that under Assumptions 3 and 5, $\|\mathbf{q}_{T_0}\|$ is bounded in T, and $\left\|\hat{\theta}_T^0-\theta^0\right\| = O_p\left(T^{-1/2}\right)$, then under the null of no policy change, H_0, for a fixed H and as $T \to \infty$, we have $\left\|H^{-1/2}\sum_{h=1}^{H}\hat{\mu}_{T_0,h}(\hat{\theta}_T^0)\right\| \to_p 0$, as required.

Consider now the case where H rises with T and the rate of increase of H in relation to T is governed by the joint asymptotic condition 1. Note also that under Assumption 2b, $\lambda < 1$. Then using (71) and (72) in Lemma 4 we have

$$\sum_{h=1}^{H}h\lambda_T^{h-1} = \frac{1}{(1-\lambda)^2} + O_p\left(T^{-1/2}\right) + O_p\left(H\lambda^H\right), \tag{21}$$

$$\sum_{h=1}^{H}\sum_{j=0}^{h-1}j\lambda_T^{j-1} = \frac{1}{(1-\lambda)^2}\left(H - \frac{1+\lambda}{1-\lambda}\right) + O_p\left(T^{-1/2}\right) + O_p\left(H\lambda^H\right). \tag{22}$$

Using (21) in (20), and (22) we obtain

$$H^{-1/2}\sum_{h=1}^{H}\hat{\mu}_{T_0,h}(\hat{\theta}_T^0) = O_p\left(H^{-1/2}T^{-1/2}\right) + O_p\left(\frac{H^{-1/2}\lambda^H}{T^{-1/2}}\right), \text{ under } H_0 \tag{23}$$

Therefore, under H_0, $H^{-1/2}\sum_{h=1}^{H}\hat{\mu}_{T_0,h}(\hat{\boldsymbol{\theta}}_T^0)$ tends to zero in probability if $H = \kappa T^{\epsilon}$, for $\epsilon \leq 1/2$, as H and $T \to \infty$ (the joint asymptotic condition 1).

To derive the distribution of $\bar{\hat{d}}_H(\hat{\theta}_T^0)$, using Lemma 1, in the Appendix, we first note that

$$\frac{1}{H}\sum_{h=1}^{H}\upsilon_{T_0,h} = \frac{1}{H}\sum_{h=1}^{H}\sum_{j=0}^{h-1}\mathfrak{s}'\left[\boldsymbol{\Phi}(\theta^1)\right]^j\boldsymbol{\Gamma}(\theta^1)\mathbf{u}_{T_0+h-j} = \frac{1}{H}\sum_{j=1}^{H}\mathfrak{s}'\mathcal{A}_{H-j}\left(\boldsymbol{\Phi}_1\right)\boldsymbol{\Gamma}(\theta^1)\mathbf{u}_{T_0+h-j}, \tag{24}$$

where

$$\mathcal{A}_{H-j}\left(\boldsymbol{\Phi}_1\right) = \mathbf{I}_{k_z+1} + \boldsymbol{\Phi}_1 + \boldsymbol{\Phi}_1^2 + ... + \boldsymbol{\Phi}_1^{H-j} = \left(\mathbf{I}_{k_z+1} - \boldsymbol{\Phi}_1\right)^{-1}\left(\mathbf{I}_{k_z+1} - \boldsymbol{\Phi}_1^{H-j+1}\right). \tag{25}$$

To simplify notation we have used $\boldsymbol{\Phi}_1$ for $\boldsymbol{\Phi}(\theta^1)$. Considering that under H_0, $\bar{\hat{\mu}}_{T_0,H}(\hat{\theta}_T^0) = O_p(T^{-1/2})$, we have

$$Var\left(\sqrt{H}\bar{\hat{d}}_H(\hat{\theta}_T^0)\right) = \omega_{0q}^2 + o(1),$$

where

$$\omega_{0q}^2 = \mathfrak{s}'\left[H^{-1}\sum_{j=1}^{H}\mathcal{A}_{H-j}\left(\boldsymbol{\Phi}_1\right)\boldsymbol{\Sigma}_{\varepsilon}(\theta^1)\mathcal{A}'_{H-j}\left(\boldsymbol{\Phi}_1\right)\right]\mathfrak{s},$$

$\boldsymbol{\Sigma}_{\varepsilon}(\boldsymbol{\theta}^1) = E(\varepsilon_{T+j}\varepsilon'_{T+j}) = \boldsymbol{\Gamma}(\theta^1)\boldsymbol{\Sigma}_u\boldsymbol{\Gamma}(\theta^1)'$. See (4) for the definition of $\boldsymbol{\Gamma}(\boldsymbol{\theta})$. Therefore, the policy ineffectiveness test statistic can be written as

$$\mathcal{T}_{d,H} = \frac{\sqrt{H}\bar{\hat{d}}_H(\hat{\theta}_T^0)}{\sqrt{\hat{\omega}_{0q}^2}}, \tag{26}$$

where ω_{0q}^2 can be estimated using pre-intervention sample as:

$$\hat{\omega}_{0q}^2 = \mathfrak{s}'\left\{H^{-1}\sum_{j=1}^{H}\mathcal{A}_{H-j}\left(\boldsymbol{\Phi}(\hat{\theta}_T^0)\right)\boldsymbol{\Sigma}_{\varepsilon}\left(\hat{\theta}_T^0\right)\mathcal{A}'_{H-j}\left(\boldsymbol{\Phi}(\hat{\theta}_T^0)\right)\right\}\mathfrak{s}, \tag{27}$$

where

$$\mathcal{A}_{H-j}\left(\boldsymbol{\Phi}(\hat{\theta}_T^0)\right) = \mathbf{I}_{k_z+1} + \boldsymbol{\Phi}(\hat{\theta}_T^0) + \left[\boldsymbol{\Phi}(\hat{\theta}_T^0)\right]^2 + ... + \left[\boldsymbol{\Phi}(\hat{\theta}_T^0)\right]^{H-j} \tag{28}$$

$$\boldsymbol{\Sigma}_{\varepsilon}\left(\hat{\theta}_T^0\right) = T^{-1}\sum_{t=M}^{T_0}\left[\mathbf{q}_t - \boldsymbol{\Phi}(\hat{\theta}_T^0)\mathbf{q}_{t-1}\right]\left[\mathbf{q}_t - \boldsymbol{\Phi}(\hat{\theta}_T^0)\mathbf{q}_{t-1}\right]', \tag{29}$$

Under the null hypothesis of policy ineffectiveness, and assuming that the underlying RE model is correctly specified and the innovations $\varepsilon_{T_0+h} = \boldsymbol{\Gamma}(\theta)\mathbf{u}_{T_0+h}$ for $h = 1, 2, ..., H$ are normally distributed, then for a fixed H and as $T \to \infty$, we have $\mathcal{T}_{d,H} \to_d N(0,1)$. For moderate values of

H, small departures from normality of the innovations over the post-intervention sample might not be that serious for the validity of the test.

Finally, the above derivations abstract from the pre-intervention sampling uncertainty by assuming that T is sufficiently large and H/T sufficiently small. Allowing for the effects of sampling uncertainty on the distribution of $\mathcal{T}_{d,H}$ when dealing with dynamic RE models with complicated non-linear cross-equation restrictions is likely to be challenging and will not be attempted here. Alternatively, one could adopt a Bayesian approach and compute the posterior distribution of $\bar{\hat{d}}_H(\hat{\theta}_T^0)$ using Markov chain Monte Carlo simulations.

2.3 Power of the policy ineffectiveness test: standard case

The power of $\mathcal{T}_{d,H}$ test, defined by (26), depends on the probability limit of $\mathcal{T}_{d,H}$ under the alternative hypothesis that $\theta^1 \neq \theta^0$. In particular, the test is consistent if its power exceeds its size in finite samples, and if the power tends to unity as $H \to \infty$. Using (14) and suppressing the dependence on $(\hat{\theta}_T^0)$ for simplicity, we note that

$$\sqrt{H}\bar{\hat{d}}_H = H^{-1/2}\sum_{h=1}^{H}\hat{\mu}_{T_0,h} + H^{-1/2}\sum_{h=1}^{H}\upsilon_{T_0,h}. \tag{30}$$

It is now easily seen that the purely random component, $H^{-1/2}\sum_{h=1}^{H}\upsilon_{T_0,h}$, has a limiting distribution with mean zero and a finite variance both under the null and the alternative hypotheses. Therefore, for the test to be consistent the mean component of $\sqrt{H}\bar{\hat{d}}_H$ must diverge to infinity with H. We shall consider the limiting behaviour of $H^{-1/2}\sum_{h=1}^{H}\hat{\mu}_{T_0,h}$, which relates to the internal dynamics of the DSGE model. Under $H_1 : \theta^1 \neq \theta^0$, we have

$$\begin{aligned} H^{-1/2}\sum_{h=1}^{H}\hat{\mu}_{T_0,h} &= -\mathfrak{s}'\left\{H^{-1/2}\sum_{h=1}^{H}\left[\boldsymbol{\Phi}^h\left(\hat{\theta}_T^0\right) - \boldsymbol{\Phi}^h\left(\theta^1\right)\right]\right\}\mathbf{q}_{T_0} \\ &= \mathfrak{s}'\left\{H^{-1/2}\sum_{h=1}^{H}\left[\boldsymbol{\Phi}^h\left(\theta^1\right) - \boldsymbol{\Phi}^h\left(\theta^0\right)\right]\right\}\mathbf{q}_{T_0} - \mathfrak{s}'\left\{H^{-1/2}\sum_{h=1}^{H}\left[\boldsymbol{\Phi}^h\left(\hat{\theta}_T^0\right) - \boldsymbol{\Phi}^h\left(\theta^0\right)\right]\right\}\mathbf{q}_{T_0}. \end{aligned} \tag{31}$$

But it has been already established that (see (23))

$$\mathfrak{s}'\left\{H^{-1/2}\sum_{h=1}^{H}\left[\boldsymbol{\Phi}^h\left(\hat{\theta}_T^0\right) - \boldsymbol{\Phi}^h\left(\theta^0\right)\right]\right\}\mathbf{q}_{T_0} = O_p\left(H^{-1/2}T^{-1/2}\right) + O_p\left(\frac{H^{-1/2}\lambda^H}{T^{-1/2}}\right).$$

Hence, under H_1

$$H^{-1/2}\sum_{h=1}^{H}\hat{\mu}_{T_0,h} = \mathfrak{s}'\left\{H^{-1/2}\sum_{h=1}^{H}\left[\boldsymbol{\Phi}^h\left(\theta^1\right) - \boldsymbol{\Phi}^h\left(\theta^0\right)\right]\right\}\mathbf{q}_{T_0} + O_p\left(H^{-1/2}T^{-1/2}\right) + O_p\left(\frac{H^{-1/2}\lambda^H}{T^{-1/2}}\right).$$

Now set $\mathbf{\Phi}_1 = \mathbf{\Phi}\left(\theta^1\right)$ and $\mathbf{\Phi}_0 = \mathbf{\Phi}\left(\theta^0\right)$, and note that $\sum_{h=1}^{H}\mathbf{\Phi}_1^h = \mathbf{\Phi}_1(\mathbf{I}_{k_z+1}-\mathbf{\Phi}_1^H)(\mathbf{I}_{k_z+1}-\mathbf{\Phi}_1)^{-1}$. Under Assumption 2, $(\mathbf{I}_{k_z+1} - \mathbf{\Phi}_1)^{-1}$ exists and is finite and $\mathbf{\Phi}_1^H \to \mathbf{0}$ as $H \to \infty$. Hence,

$$H^{-1/2}\sum_{h=1}^{H}\mathbf{\Phi}^h\left(\theta^1\right) = H^{-1/2}\mathbf{\Phi}_1(\mathbf{I}_{k_z+1} - \mathbf{\Phi}_1^H)(\mathbf{I}_{k_z+1} - \mathbf{\Phi}_1)^{-1} \to \mathbf{0}, \text{ as } H \to \infty.$$

Similarly, $H^{-1/2}\sum_{h=1}^{H}\mathbf{\Phi}^h\left(\theta^0\right) \to \mathbf{0}$, with H, and $H^{-1/2}\sum_{h=1}^{H}\hat{\mu}_{T_0,h} = o_p(1)$, under the alternative hypothesis. Hence, $H^{-1/2}\sum_{h=1}^{H}\hat{\mu}_{T_0,h} \to_p 0$ under both the null and the alternative hypotheses as T and $H \to \infty$, subject to the joint asymptotic condition 1. Therefore, the internal dynamics of the RE model do not contribute to the power of the policy ineffectiveness test for T and H large. Thus tests based on the average policy effects, $\bar{\hat{d}}_H$, will not be consistent in the case of stationary DSGE models. In such cases, the best that can be hoped for is to base the test of the policy ineffectiveness on a short post-intervention sample and accept that the test is likely to lack power and be sensitive to the specifications of the post-intervention error processes, ε_{T_0+h}, $h = 1, 2, ..., H$.

3 Policy ineffectiveness tests for the RE model with exogenous variables

3.1 Derivation of the counterfactuals and policy effects with exogenous variables

We now allow for exogenous policy and non-policy variables. Endogenous policy rules, such as the Taylor rule, follow closed loop control with feedback, but there may be open loop control without feedbacks, such as fixed money supply rules, where the policy variable x_t is exogenous. There may also be non-policy variables, $\mathbf{w}_t$, such as global variables that affect $\mathbf{z}_t$ and/or y_t but are invariant to changes in x_t. This framework also accomodates changes that shift steady states such as target inflation.

As before let $\mathbf{q}_t = (y_t, \mathbf{z}_t')'$, be a $(k_z + 1) \times 1$ vector, but now introduce $\mathbf{s}_t = (x_t, \mathbf{w}_t')'$, a $(1 + k_w) \times 1$ vector. The RE model is now

$$\mathbf{A}_0\mathbf{q}_t = \mathbf{A}_1 E_t(\mathbf{q}_{t+1}) + \mathbf{A}_2\mathbf{q}_{t-1} + \mathbf{A}_3\mathbf{s}_t + \mathbf{u}_t, \tag{32}$$

and suppose that the forcing variables, $\mathbf{s}_t$, follow the VAR(1) specification

$$\mathbf{s}_t = \mathbf{R}\mathbf{s}_{t-1} + \eta_t, \tag{33}$$

where

$$\mathbf{R} = \begin{pmatrix} \rho & 0 \\ 0 & \mathbf{R}_w \end{pmatrix}, \ \eta_t = \begin{pmatrix} \eta_{xt} \\ \eta_{wt} \end{pmatrix},$$

so that $\mathbf{w}_t$ is invariant to changes in x_t. The errors, $\mathbf{u}_t$ and η_t are assumed to be serially and cross sectionally uncorrelated, with zero means and constant variances, $\mathbf{\Sigma}_u$, and $\mathbf{\Sigma}_\eta$, respectively.

Initially, consider the case where there are no dynamics, namely $\mathbf{A}_2 = \mathbf{0}$, and all eigenvalues of $\mathbf{A}_0^{-1}\mathbf{A}_1$ lie within the unit circle. Then the unique solution of (32) is given by

$$\mathbf{A}_0\mathbf{q}_t = \mathbf{G}(\theta)\mathbf{s}_t + \mathbf{u}_t, \tag{34}$$

where $\boldsymbol{\theta}$ includes both the structural coefficients, $\mathbf{a} = vec(\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_3)$, and the parameters of the processes generating the exogenous variables, $\phi = (\rho, vec(\mathbf{R}_w)')'$.

$$vec(\mathbf{G}) = \left[(\mathbf{I}_{k_w+1}\otimes\mathbf{I}_{k_z+1}) - \left(\mathbf{R}'\otimes\mathbf{A}_1\mathbf{A}_0^{-1}\right)\right]^{-1} vec\left(\mathbf{A}_3\right).$$

Equation (34) is the structural form of a standard simultaneous equations model. The reduced form is

$$\begin{aligned} \mathbf{q}_t &= \mathbf{A}_0^{-1}\mathbf{G}(\theta)\mathbf{s}_t + \mathbf{A}_0^{-1}\mathbf{u}_t \\ &= \mathbf{\Pi}\left(\theta\right)\mathbf{s}_t + \mathbf{\Gamma}(\theta)\mathbf{u}_t. \end{aligned} \tag{35}$$

Under the same assumptions as before about the intervention at T_0, then the process switches from

$$\mathbf{q}_t = \mathbf{A}_0^{-1}\mathbf{G}(\theta^0)\mathbf{s}_t + \mathbf{A}_0^{-1}\mathbf{u}_t = \mathbf{\Pi}^0\mathbf{s}_t + \mathbf{\Gamma}(\theta)\mathbf{u}_t,\ t = M, M+1, M+2, ..., T_0,$$

to

$$\mathbf{q}_t = \mathbf{A}_0^{-1}\mathbf{G}(\theta^1)\mathbf{s}_t + \mathbf{A}_0^{-1}\mathbf{u}_t = \mathbf{\Pi}^1\mathbf{s}_t + \mathbf{\Gamma}(\theta)\mathbf{u}_t,\ t = T_0+1, T_0+2, ..., T_0+H.$$

In the general case where $\mathbf{A}_2 \neq \mathbf{0}$, the RE solution is

$$\mathbf{q}_t = \mathbf{\Phi}\left(\theta\right)\mathbf{q}_{t-1} + \mathbf{\Psi}_x\left(\theta\right)\mathbf{x}_t + \mathbf{\Psi}_w\left(\theta\right)\mathbf{w}_t + \mathbf{\Gamma}(\theta)\mathbf{u}_t, \tag{36}$$

where $\boldsymbol{\theta}$ contains $\mathbf{a} = vec(\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3)$ and $\phi = (\rho, vec(\mathbf{R}_w)')'$. The counterfactual values of y_{T_0+h}, are now given by

$$y^0_{T_0+h} = \mathfrak{s}'\left[\mathbf{\Phi}\left(\theta^0\right)\right]^h \mathbf{q}_{T_0} + \mathfrak{s}'\sum_{j=0}^{h-1}\left[\mathbf{\Phi}\left(\theta^0\right)\right]^j\left[\mathbf{\Psi}_x\left(\theta^0\right)\mathbf{x}^0_{T_0+h-j} + \mathbf{\Psi}_w\left(\theta^0\right)\mathbf{w}_{T_0+h-j}\right], \tag{37}$$

where $x^0_{T_0+j}$ for $j = 1, 2, ..., H$ denote the counterfactual values of the policy variable, and $\mathbf{w}_{T_0+j}$, for $j = 1, 2, ..., H$, are the realized values of the policy invariant variables. In the case where there is a single policy variable that follows the AR(1) process, $x_t = \rho x_{t-1} + \eta_{xt}$, we also have $x^0_{T_0+h} = \left(\rho^0\right)^h x_{T_0}$. Notice that the counterfactual outcomes are neither *ex ante* forecasts, since $y^0_{T_0+h}$ is computed conditional on the realizations of $\mathbf{w}_{T_0+h}$ and not their predictions, nor are they *ex post* forecasts since they are based on projected values of the policy variables, $\mathbf{x}_{T_0+h}$, and the initial values of the endogenous variables, $\mathbf{q}_{T_0}$.

3.2 Derivation of the test statistic with exogenous variables

In addition to assumptions 1-3 and the joint asymptotic condition given above we amend assumptions 4 and 5 to allow for the exogenous variables.

Assumption 4a: $\mathbf{\Phi}(\theta) = (\phi_{ij}(\theta))$, and $\mathbf{\Psi}(\boldsymbol{\theta}) = (\psi_{ij}(\theta))$, are bounded and continuously differentiable in $\boldsymbol{\theta}$, such that $\left\|\partial\phi_{ij}(\theta)/\partial\boldsymbol{\theta}'\right\|$, and $\left\|\partial\psi_{ij}(\theta)/\partial\boldsymbol{\theta}'\right\|$ exist and are bounded, for all i and j.

Assumption 5a: The initial values, $\mathbf{q}_{T_0}$, and post policy exogenous variables, $\mathbf{s}_{T_0+j}$, for $j = 1, 2, ..., H$ are bounded, namely $\|\mathbf{q}_{T_0}\| < K$, and $\|\mathbf{s}_{T_0+j}\| < K$ for all T_0 and j, where K is a fixed positive constant.

Using (37), the estimated counterfactuals are

$$\hat{y}^0_{T_0+h} = \mathfrak{s}' \left[\mathbf{\Phi}\left(\hat{\theta}^0_T\right)\right]^h \mathbf{q}_{T_0} + \mathfrak{s}' \sum_{j=0}^{h-1} \left[\mathbf{\Phi}\left(\hat{\theta}^0_T\right)\right]^j \left[\mathbf{\Psi}_x\left(\hat{\theta}^0_T\right)\hat{\mathbf{x}}^0_{T_0+h-j} + \mathbf{\Psi}_w\left(\hat{\theta}^0_T\right)\mathbf{w}_{T_0+h-j}\right]. \quad (38)$$

As before under Assumption 3, $\hat{\theta}^0_T$ is $\sqrt{T}$ consistent estimator of θ based on pre-intervention period, $t = 1, 2, ..., T$. In the case of the AR(1) specification for x_t we also have $x^0_{T_0+h} = (\rho^0)^h x_{T_0}$; $h = 1, 2, ...H$, where ρ^0 is the pre-intervention value of ρ, which can be estimated using the pre-intervention sample, namely

$$\hat{\mathbf{x}}^0_{T_0+h-j} = \left[\hat{\rho}^0_T\right]^{h-j} x_{T_0}.$$

Therefore, the estimated policy effects are

$$\hat{d}_{T_0+h} = \mathfrak{s}'\mathbf{q}_{T_0+h} - \mathfrak{s}' \left[\mathbf{\Phi}\left(\hat{\theta}^0_T\right)\right]^h \mathbf{q}_{T_0} - \mathfrak{s}' \sum_{j=0}^{h-1} \left[\mathbf{\Phi}\left(\hat{\theta}^0_T\right)\right]^j \left[\mathbf{\Psi}_x\left(\hat{\theta}^0_T\right)\left[\hat{\rho}^0_T\right]^{h-j} x_{T_0} + \mathbf{\Psi}_w\left(\hat{\theta}^0_T\right)\mathbf{w}_{T_0+h-j}\right], \quad (39)$$

for $h = 1, 2, ..., H$. The dependence of $\hat{d}_{T_0+h}$ on $\left(\hat{\boldsymbol{\theta}}^0_T\right)$ has not been made explicit for simplicity. The null hypothesis of no policy effect is $H_0 : \theta^1 = \theta^0$ and $\rho^1 = \rho^0$.

The post-intervention realized values, $\mathbf{q}_{T_0+h}$, (for $h = 1, 2, ..., H$) are given by

$$\begin{aligned}\mathbf{q}_{T_0+h} = {} & \left[\mathbf{\Phi}\left(\theta^1\right)\right]^h \mathbf{q}_{T_0} + \sum_{j=0}^{h-1} \left[\mathbf{\Phi}\left(\theta^1\right)\right]^j \left[\mathbf{\Psi}_x\left(\theta^1\right)\left[\rho^1\right]^{h-j} x_{T_0} + \mathbf{\Psi}_w\left(\theta^1\right)\mathbf{w}_{T_0+h-j}\right] \\ & + \sum_{j=0}^{h-1} \left[\mathbf{\Phi}\left(\theta^1\right)\right]^j \mathbf{\Psi}_x\left(\theta^1\right)\xi_{T_0,h-j} + \sum_{j=0}^{h-1} \left[\mathbf{\Phi}\left(\theta^1\right)\right]^j \varepsilon_{T_0+h-j}.\end{aligned}$$

where $\varepsilon_t = \mathbf{A}_0^{-1}\mathbf{u}_t$, and

$$\xi_{T_0,h-j} = \sum_{i=1}^{h-j} \left[\rho^1\right]^{h-j-i} \eta_{x,T_0+i}.$$

But after some algebra it follows that

$$\sum_{j=0}^{h-1}\left[\mathbf{\Phi}\left(\theta^{1}\right)\right]^{j}\mathbf{\Psi}_{x}\left(\theta^{1}\right)\xi_{T_0,h-j}=\sum_{j=0}^{h-1}\mathcal{B}_{j}\left(\theta^{1},\rho^{1}\right)\eta_{x,T_0+h-j}, \tag{40}$$

where

$$\mathcal{B}_{j}\left(\theta^{1},\rho^{1}\right)=\sum_{i=0}^{j}\left[\mathbf{\Phi}\left(\theta^{1}\right)\right]^{i}\mathbf{\Psi}_{x}\left(\theta^{1}\right)\left[\rho^{1}\right]^{j-i}.$$

The estimated policy effects are then

$$\hat{d}_{T_0+h}=\left(\hat{\mu}^{q}_{T_0,h}+\hat{\mu}^{w}_{T_0,h}+\hat{\mu}^{x}_{T_0,h}\right)+\left(v^{q}_{T_0,h}+v^{x}_{T_0,h}\right)=\hat{\mu}_{T_0,h}+v_{T_0,h}.$$

The terms

$$\hat{\mu}^{q}_{T_0,h}=-\mathfrak{s}'\left\{\left[\mathbf{\Phi}\left(\hat{\theta}^{0}_{T}\right)\right]^{h}-\left[\mathbf{\Phi}\left(\theta^{1}\right)\right]^{h}\right\}\mathbf{q}_{T_0},$$

$$v^{q}_{T_0,h}=\sum_{j=0}^{h-1}\mathfrak{s}'\left[\mathbf{\Phi}\left(\theta^{1}\right)\right]^{j}\varepsilon_{T_0+h-j},$$

are the same as $\hat{\mu}_{T,h}$ in (15) and $v_{T,h}$ in (16) in section (2.2), without exogenous variables. The other terms are

$$\hat{\mu}^{w}_{T_0,h}=-\mathfrak{s}'\sum_{j=0}^{h-1}\left\{\left[\mathbf{\Phi}\left(\hat{\theta}^{0}_{T}\right)\right]^{j}\mathbf{\Psi}_{w}\left(\hat{\theta}^{0}_{T}\right)-\left[\mathbf{\Phi}\left(\theta^{1}\right)\right]^{j}\mathbf{\Psi}_{w}\left(\theta^{1}\right)\right\}\mathbf{w}_{T_0+h-j}, \tag{41}$$

$$\hat{\mu}^{x}_{T_0,h}=-\mathfrak{s}'\sum_{j=0}^{h-1}\left\{\left[\mathbf{\Phi}\left(\hat{\theta}^{0}_{T}\right)\right]^{j}\mathbf{\Psi}_{x}\left(\hat{\theta}^{0}_{T}\right)\left[\hat{\rho}^{0}_{T}\right]^{h-j}-\left[\mathbf{\Phi}\left(\theta^{1}\right)\right]^{j}\mathbf{\Psi}_{x}\left(\theta^{1}\right)\left[\rho^{1}\right]^{h-j}\right\}x_{T_0}, \tag{42}$$

and

$$v^{x}_{T_0,h}=\sum_{j=0}^{h-1}\mathfrak{s}'\mathcal{B}_{j}\left(\theta^{1},\rho^{1}\right)\eta_{x,T_0+h-j}. \tag{43}$$

For a fixed H, the implicit null hypothesis of no policy effects can now be specified as

$$H_0':p\lim_{T\to\infty}\left[H^{-1/2}\sum_{h=1}^{H}\left(\hat{\mu}^{q}_{T_0,h}+\hat{\mu}^{w}_{T_0,h}+\hat{\mu}^{x}_{T_0,h}\right)\right]=0, \tag{44}$$

which is a generalization of (18). To establish the above result, we consider each of the three terms in (44), separately. The first term, relates to the internal dynamics of the DSGE model, which we already discussed above in Section 2.2, while the next two terms capture the effects of exogenous variables.

Consider now the second term in (44), and note that (under Assumption 5a)

$$\left\|H^{-1/2}\sum_{h=1}^{H}\hat{\mu}^{w}_{T_0,h}\right\|\leq K\;H^{-1/2}\sum_{h=1}^{H}\sum_{j=0}^{h-1}\left\|\left[\mathbf{\Phi}\left(\hat{\theta}^{0}_{T}\right)\right]^{j}\mathbf{\Psi}_{w}\left(\hat{\theta}^{0}_{T}\right)-\left[\mathbf{\Phi}\left(\theta^{1}\right)\right]^{j}\mathbf{\Psi}_{w}\left(\theta^{1}\right)\right\|. \tag{45}$$

But under H_0

$$\left[\mathbf{\Phi}\left(\hat{\theta}_T^0\right)\right]^j \mathbf{\Psi}_w\left(\hat{\theta}_T^0\right)-\left[\mathbf{\Phi}\left(\theta^1\right)\right]^j \mathbf{\Psi}_w\left(\theta^1\right)=\hat{\mathbf{\Phi}}_0^j\hat{\mathbf{\Psi}}_{0,w}-\mathbf{\Phi}_0^j\mathbf{\Psi}_{0,w}=\left(\hat{\mathbf{\Phi}}_0^j-\mathbf{\Phi}_0^j\right)\hat{\mathbf{\Psi}}_{0,w}+\mathbf{\Phi}_0^j\left(\hat{\mathbf{\Psi}}_{0,w}-\mathbf{\Psi}_{0,w}\right),$$

where $\mathbf{\Phi}_0=\mathbf{\Phi}\left(\theta^0\right)$, $\hat{\mathbf{\Phi}}_0=\mathbf{\Phi}\left(\hat{\theta}_T^0\right)$, $\mathbf{\Psi}_{0,w}=\mathbf{\Psi}_w\left(\theta^0\right)$ and $\hat{\mathbf{\Psi}}_{0,w}=\mathbf{\Psi}_w\left(\hat{\theta}_T^0\right)$. Hence

$$\begin{aligned}&\sum_{h=1}^{H}\sum_{j=0}^{h-1}\left\|\left[\mathbf{\Phi}\left(\hat{\theta}_T^0\right)\right]^j \mathbf{\Psi}_w\left(\hat{\theta}_T^0\right)-\left[\mathbf{\Phi}\left(\theta^1\right)\right]^j \mathbf{\Psi}_w\left(\boldsymbol{\theta}^1\right)\right\| \\ &\leq \sum_{h=1}^{H}\sum_{j=0}^{h-1}\left\|\hat{\mathbf{\Phi}}_0^j-\mathbf{\Phi}_0^j\right\|\left\|\hat{\mathbf{\Psi}}_{0,w}\right\|+\left\|\mathbf{\Phi}_0^j\right\|\left\|\hat{\mathbf{\Psi}}_{0,w}-\mathbf{\Psi}_{0,w}\right\|.\end{aligned} \tag{46}$$

Once again using results in Lemmas 2 and 3 we have

$$\begin{aligned}&\sum_{h=1}^{H}\sum_{j=0}^{h-1}\left\|\hat{\mathbf{\Phi}}_0^j-\mathbf{\Phi}_0^j\right\|\left\|\hat{\mathbf{\Psi}}_{0,w}\right\|+\left\|\mathbf{\Phi}_0^j\right\|\left\|\hat{\mathbf{\Psi}}_{0,w}-\mathbf{\Psi}_{0,w}\right\| \\ &\leq\left(\sum_{h=1}^{H}\sum_{j=0}^{h-1}j\lambda_T^{j-1}\right)\left\|\hat{\mathbf{\Phi}}_0-\mathbf{\Phi}_0\right\|\left\|\hat{\mathbf{\Psi}}_{0,w}\right\|+\left(\sum_{h=1}^{H}\sum_{j=0}^{h-1}\lambda^{j-1}\right)\left\|\hat{\mathbf{\Psi}}_{0,w}-\mathbf{\Psi}_{0,w}\right\|,\end{aligned} \tag{47}$$

where λ is the upper bound of $\|\mathbf{\Phi}_0\|$, and as before $\lambda_T\leq\lambda+a_T T^{-1/2}$. Once again when H is fixed and $T\to\infty$, $\left(\sum_{h=1}^{H}\sum_{j=0}^{h-1}j\lambda_T^{j-1}\right)$ and $\left(\sum_{h=1}^{H}\sum_{j=0}^{h-1}\lambda^{j-1}\right)$ are bounded in T, by Lemma 3 and under Assumption 3, $\left\|\hat{\mathbf{\Phi}}_0-\mathbf{\Phi}_0\right\|$ and $\left\|\hat{\mathbf{\Psi}}_{0,w}-\mathbf{\Psi}_{0,w}\right\|$ both tend to zero in probability and we have $H^{-1/2}\sum_{h=1}^{H}\hat{\mu}_{T,h}^w\to_p 0$, as desired. A similar result also obtains for $H^{-1}\sum_{h=1}^{H}\hat{\mu}_{T,h}^x$. Therefore, for a fixed H and under Assumptions 1-3, 4a, 5a, and the null of no policy change, H_0, we have

$$H^{-1/2}\sum_{h=1}^{H}\left(\hat{\mu}_{T_0,h}^q+\hat{\mu}_{T_0,h}^w+\hat{\mu}_{T_0,h}^x\right)\to_p 0,\text{ as }T\to\infty,\text{ for a fixed }H.$$

In the case where H rises with T and the rate of increase of H in relation to T is governed by the joint asymptotic condition 1, in addition to the results above, following (21), we have

$$H^{-1/2}\sum_{h=1}^{H}\hat{\mu}_{T_0,h}^w=O_p\left(H^{1/2}T^{-1/2}\right)+O_p\left(\frac{H^{-1/2}\lambda^H}{T^{-1/2}}\right),\text{ under }H_0.$$

Therefore, under $H_{0,}$ $H^{-1/2}\sum_{h=1}^{H}\hat{\mu}_{T_0,h}^q$ and $H^{-1/2}\sum_{h=1}^{H}\hat{\mu}_{T_0,h}^w$, both tend to zero in probability if $H=\kappa T^{\epsilon}$, for $\epsilon\leq 1/2$, as H and $T\to\infty$ (the joint asymptotic condition 1). A similar result also holds for $H^{-1/2}\sum_{h=1}^{H}\hat{\mu}_{T_0,h}^x$.

To derive the distribution of the mean effect, $\bar{\hat{d}}_H$, we use (24) and

$$\begin{aligned}\frac{1}{H}\sum_{h=1}^{H}v_{T_0,h}^x&=\frac{1}{H}\sum_{h=1}^{H}\sum_{j=0}^{h-1}\mathfrak{s}'\mathcal{B}_j\left(\theta^1,\rho^1\right)\eta_{x,T_0+h-j}\\&=\frac{1}{H}\sum_{j=1}^{H}\mathfrak{s}'\mathcal{C}_{H-j}\left(\theta^1,\rho^1\right)\eta_{x,T_0+j},\end{aligned}$$

where $\mathcal{C}_{H-j}\left(\theta^1,\rho^1\right) = \sum_{i=0}^{H-j}\mathcal{B}_i\left(\theta^1,\rho^1\right)$. Hence, since by assumption η_{xt} and ε_t are serially uncorrelated and are distributed independently of each other, and considering that under H_0, $\bar{\hat{\mu}}_{T_0,H} = O_p(T^{-1/2})$, we have

$$Var\left(\sqrt{H}\bar{\hat{d}}_H\right) = \omega_{0q}^2 + \omega_{0x}^2 + o(1),$$

where

$$\omega_{0q}^2 = \mathfrak{s}'\left[H^{-1}\sum_{j=1}^{H}\mathcal{A}_{H-j}\left(\mathbf{\Phi}_1\right)\mathbf{\Sigma}_{\varepsilon}(\boldsymbol{\theta}^1)\mathcal{A}'_{H-j}\left(\mathbf{\Phi}_1\right)\right]\mathfrak{s},$$

$\mathbf{\Sigma}_{\varepsilon}(\theta^1) = E(\varepsilon_{T+j}\varepsilon_{T+j})$ and (see also 4))

$$\omega_{0x}^2 = \sigma_{\eta x}^2\mathfrak{s}'\left[H^{-1}\sum_{j=1}^{H}\mathcal{C}_{H-j}\left(\theta^1,\rho^1\right)\mathcal{C}_{H-j}\left(\theta^1,\rho^1\right)'\right]\mathfrak{s}.$$

Therefore, the policy ineffectiveness test statistic is given by

$$\mathcal{T}_{d,H} = \frac{\sqrt{H}\bar{\hat{d}}_H}{\sqrt{\hat{\omega}_{0q}^2 + \hat{\omega}_{0x}^2}}, \tag{48}$$

where ω_{0q}^2 and ω_{0x}^2 are estimated using pre-intervention sample as:

$$\hat{\omega}_{0q}^2 = \mathfrak{s}'\left\{H^{-1}\sum_{j=1}^{H}\mathcal{A}_{H-j}\left(\mathbf{\Phi}(\hat{\theta}_T^0)\right)\mathbf{\Sigma}_{\varepsilon}\left(\hat{\theta}_T^0\right)\mathcal{A}'_{H-j}\left(\mathbf{\Phi}(\hat{\theta}_T^0)\right)\right\}\mathfrak{s},$$

where

$$\mathcal{A}_{H-j}\left(\mathbf{\Phi}(\hat{\theta}_T^0)\right) = \mathbf{I}_{k_z+1} + \mathbf{\Phi}(\hat{\theta}_T^0) + \left[\mathbf{\Phi}(\hat{\theta}_T^0)\right]^2 + ... + \left[\mathbf{\Phi}(\hat{\theta}_T^0)\right]^{H-j}$$

$$\mathbf{\Sigma}_{\varepsilon}\left(\hat{\theta}_T^0\right) = T^{-1}\sum_{t=M}^{T_0}\varepsilon_t\left(\hat{\theta}_T^0\right)\varepsilon'_t\left(\hat{\theta}_T^0\right),$$

$$\varepsilon_t\left(\hat{\theta}_T^0\right) = \mathbf{q}_t - \mathbf{\Phi}(\hat{\theta}_T^0)\mathbf{q}_{t-1} - \mathbf{\Psi}_x\left(\hat{\theta}_T^0\right)\mathbf{x}_t - \mathbf{\Psi}_w\left(\hat{\theta}_T^0\right)\mathbf{w}_t,$$

$$\hat{\omega}_{0x}^2 = \hat{\sigma}_{0,\eta x}^2\mathfrak{s}'\left[H^{-1}\sum_{j=1}^{H}\left(\widehat{\mathcal{C}}_{H-j}\widehat{\mathcal{C}}'_{H-j}\right)\right]\mathfrak{s}, \tag{49}$$

$\widehat{\mathcal{C}}_{H-j} = \sum_{i=0}^{H-j}\mathcal{B}_i\left(\hat{\theta}_T^0,\hat{\rho}_T^0\right)$, and $\hat{\sigma}_{0,\eta x}^2 = T^{-1}\sum_{t=M}^{T_0}\left(x_t - \hat{\rho}_T^0 x_{t-1}\right)^2$. Under the null hypothesis of policy ineffectiveness, and assuming that the underlying RE model is correctly specified and the innovations ε_{T_0+h} and η_{x,T_0+h} for $h = 1, 2, ..., H$ are normally distributed, then for a fixed H and as $T \to \infty$, we have $\mathcal{T}_{d,H} \to_d N(0,1)$. Notice that (48) differs from (26) in the explicit inclusion of the estimated variance of the exogenous policy variable $\hat{\omega}_{0x}^2$. In (26) the variance of the endogenous policy variable was included in $\hat{\omega}_{0q}^2$.

3.3 Power of the policy ineffectiveness test with exogenous variables

In the numerator of (48)

$$\sqrt{H}\hat{\bar{d}}_H = H^{-1/2}\sum_{h=1}^{H}\left(\hat{\mu}^q_{T_0,h} + \hat{\mu}^w_{T_0,h} + \hat{\mu}^x_{T_0,h}\right) + H^{-1/2}\sum_{h=1}^{H}\left(\upsilon^q_{T_0,h} + \upsilon^x_{T_0,h}\right).$$

the purely random component, $H^{-1/2}\sum_{h=1}^{H}\left(\upsilon^q_{T_0,h} + \upsilon^x_{T_0,h}\right)$, has a limiting distribution with mean zero and a finite variance both under the null and the alternative hypotheses. Therefore, for the test to be consistent the mean component of $\sqrt{H}\hat{\bar{d}}_H$ must diverge to infinity with H. The limiting behaviour of $H^{-1/2}\sum_{h=1}^{H}\hat{\mu}^q_{T_0,h}$, which relates to the internal dynamics of the DSGE model is considered above in section (2.3) following (31). The terms $H^{-1/2}\sum_{h=1}^{H}\hat{\mu}^w_{T_0,h}$, and $H^{-1/2}\sum_{h=1}^{H}\hat{\mu}^x_{T_0,h}$, capture the effects of exogenous variables. Under $H_1 : \boldsymbol{\theta}^1 \neq \boldsymbol{\theta}^0$, we have

$$H^{-1/2}\sum_{h=1}^{H}\hat{\mu}^w_{T_0,h} = -H^{-1/2}\sum_{h=1}^{H}\sum_{j=0}^{h-1}\mathfrak{s}'\left\{\hat{\boldsymbol{\Phi}}^j_0\hat{\boldsymbol{\Psi}}_{0,w} - \boldsymbol{\Phi}^j_1\boldsymbol{\Psi}_{1,w}\right\}\mathbf{w}_{T_0+h-j}, \tag{50}$$

where $\hat{\boldsymbol{\Psi}}_{0,w} = \boldsymbol{\Psi}_w\left(\hat{\theta}^0_T\right)$, $\boldsymbol{\Psi}_{1,w} = \boldsymbol{\Psi}_w\left(\boldsymbol{\theta}^1\right)$, $\hat{\boldsymbol{\Phi}}_0 = \boldsymbol{\Phi}\left(\hat{\theta}^0_T\right)$, and, $\boldsymbol{\Phi}_1 = \boldsymbol{\Phi}\left(\theta^1\right)$. Also, setting $\boldsymbol{\Psi}_{0,w} = \boldsymbol{\Psi}_w\left(\theta^0\right)$ we have

$$\hat{\boldsymbol{\Phi}}^j_0\hat{\boldsymbol{\Psi}}_{0,w} - \boldsymbol{\Phi}^j_1\boldsymbol{\Psi}_{1,w} = \left(\hat{\boldsymbol{\Phi}}^j_0 - \boldsymbol{\Phi}^j_0\right)\hat{\boldsymbol{\Psi}}_{0,w} + \boldsymbol{\Phi}^j_0\left(\hat{\boldsymbol{\Psi}}_{0,w} - \boldsymbol{\Psi}_{0,w}\right) + \boldsymbol{\Phi}^j_0\boldsymbol{\Psi}_{0,w} - \boldsymbol{\Phi}^j_1\boldsymbol{\Psi}_{1,w}, \tag{51}$$

and we have

$$\begin{aligned} H^{-1/2}\sum_{h=1}^{H}\hat{\mu}^w_{T_0,h} &= -H^{-1/2}\sum_{h=1}^{H}\sum_{j=0}^{h-1}\mathfrak{s}'\left[\left(\hat{\boldsymbol{\Phi}}^j_0 - \boldsymbol{\Phi}^j_0\right)\hat{\boldsymbol{\Psi}}_{0,w} + \boldsymbol{\Phi}^j_0\left(\hat{\boldsymbol{\Psi}}_{0,w} - \boldsymbol{\Psi}_{0,w}\right)\right]\mathbf{w}_{T_0+h-j} \\ &= -H^{-1/2}\sum_{h=1}^{H}\sum_{j=0}^{h-1}\mathfrak{s}'\left(\boldsymbol{\Phi}^j_0\boldsymbol{\Psi}_{0,w} - \boldsymbol{\Phi}^j_1\boldsymbol{\Psi}_{1,w}\right)\mathbf{w}_{T_0+h-j}. \end{aligned}$$

Noting that under Assumption 5a, $\|\mathbf{w}_{T_0+h-j}\| < K$, the first term of the above is given by

$$\begin{aligned} &- H^{-1/2}\sum_{h=1}^{H}\sum_{j=0}^{h-1}\mathfrak{s}'\left[\left(\hat{\boldsymbol{\Phi}}^j_0 - \boldsymbol{\Phi}^j_0\right)\hat{\boldsymbol{\Psi}}_{0,w} + \boldsymbol{\Phi}^j_0\left(\hat{\boldsymbol{\Psi}}_{0,w} - \boldsymbol{\Psi}_{0,w}\right)\right]\mathbf{w}_{T_0+h-j} \\ &= O_p\left(H^{1/2}T^{-1/2}\right) + O_p\left(\frac{H^{-1/2}\lambda^H}{T^{-1/2}}\right) \end{aligned}$$

and under H_1 and the joint asymptotic condition 1 we have

$$H^{-1/2}\sum_{h=1}^{H}\hat{\mu}^w_{T_0,h} = -H^{-1/2}\sum_{h=1}^{H}\sum_{j=0}^{h-1}\mathfrak{s}'\left\{\boldsymbol{\Phi}^j_0\boldsymbol{\Psi}_{0,w} - \boldsymbol{\Phi}^j_1\boldsymbol{\Psi}_{1,w}\right\}\mathbf{w}_{T_0+h-j} + O_p\left(H^{1/2}T^{-1/2}\right) + O_p\left(\frac{H^{-1/2}\lambda^H}{T^{-1/2}}\right) \tag{52}$$

Now using the result in Lemma 1 we have

$$H^{-1/2}\sum_{h=1}^{H}\sum_{j=0}^{h-1}\mathbf{\Phi}_0^j\mathbf{\Psi}_{0,w}\mathbf{w}_{T_0+h-j} = \left(\mathbf{I}_{k_z+1}-\mathbf{\Phi}_0\right)^{-1}\mathbf{\Psi}_{0,w}\left(H^{-1/2}\sum_{j=1}^{H}\mathbf{w}_{T_0+j}\right) - \left(\mathbf{I}_{k_z+1}-\mathbf{\Phi}_0\right)^{-1}\left(H^{-1/2}\sum_{j=1}^{H}\mathbf{\Phi}_0^{H-j+1}\mathbf{\Psi}_{0,w}\mathbf{w}_{T_0+j}\right).$$

But considering that $\|\mathbf{\Psi}_{0,w}\|$ and $\|\mathbf{w}_{T+j}\|$ are bounded in H, and $\|\mathbf{\Phi}_0\| < 1$ (under Assumption 2b), then we have

$$\left\|H^{-1/2}\sum_{j=1}^{H}\mathbf{\Phi}_0^{H-j+1}\mathbf{\Psi}_{0,w}\mathbf{w}_{T_0+j}\right\| \leq H^{-1/2}\sum_{j=1}^{H}\|\mathbf{\Phi}_0\|^{H-j+1}\|\mathbf{\Psi}_{0,w}\|\|\mathbf{w}_{T_0+j}\| \leq K\ H^{-1/2}\frac{\|\mathbf{\Phi}_0\|\left(1-\|\mathbf{\Phi}_0\|^H\right)}{\left(1-\|\mathbf{\Phi}_0\|\right)} = O\left(H^{-1/2}\right)$$

Hence

$$H^{-1/2}\sum_{h=1}^{H}\hat{\mu}_{T_0,h}^{w} = \sqrt{H}\mathfrak{s}'\left[\left(I_{k_z+1}-\mathbf{\Phi}_1\right)^{-1}\mathbf{\Psi}_{1,w} - \left(\mathbf{I}_{k_z+1}-\mathbf{\Phi}_0\right)^{-1}\mathbf{\Psi}_{0,w}\right]\bar{\mathbf{w}}_{T_0,H} + O_p\left(H^{1/2}T^{-1/2}\right) + O_p\left(\frac{H^{-1/2}\lambda^H}{T^{-1/2}}\right). \tag{53}$$

Therefore, under $H_1: \theta^1 \neq \theta^0$**,** the power of the test rises with $\sqrt{H}$ if $p\lim_{H\to\infty}\bar{\mathbf{w}}_{T_0,H} \neq \mathbf{0}$, and so long as $H = \kappa T^{\epsilon}$, with $\epsilon \leq 1/2$, as $T \to \infty$.[4] The power of the test also depends on the size of the difference between the pre and post-intervention long-run effects of the exogenous variables.

Whereas with standard stationary DSGE models the tests based on the average policy effects, $\bar{\hat{d}}_H$, were not consistent, when there are also exogenous variables the tests are consistent.

4 Simulated policy analysis using a New Keynesian model

To illustrate the issues discussed above we calibrate a standard three equation New Keynesian DSGE model, using parameter estimates from the literature. We assume that there is no parameter or specification uncertainty. We first consider a model where the variables are all measured in deviations from their steady states. These are R_t, the interest rate, y_t, log real output, and π_t, the inflation rate. The policy intervention takes place at time T_0, with a post-intervention sample, $T_0+1, T_0+2, ..., T_0+H$. We set out the model; examine the shock and policy impulse response functions introduced in Section 2.1, and then examine the size and power of the tests discussed in Section 2.3. In 4.4 we consider a second model where the inflation target is changed. The first model, where the variables are deviations from steady state is

[4] The same consideration also apply to the policy variable, x_t. In cases where $\left|\rho^1\right| < 1$, the policy can only have short term effects.

$$R_t = \delta_R R_{t-1} + (1-\delta_R)(\psi_\pi \pi_t + \psi_y y_t) + u_{Rt}, \tag{54}$$

$$y_t = \delta_y y_{t-1} + \kappa E(y_{t+1} \,|\Im_t) - \sigma \left[R_t - E(\pi_{t+1} \,|\Im_t)\right] + u_{yt}, \tag{55}$$

$$\pi_t = \delta_\pi \pi_{t-1} + \beta E(\pi_{t+1} \,|\Im_t) + \gamma y_t + u_{\pi t}, \tag{56}$$

which can be written more compactly as (1), and repeated here for convenience

$$\mathbf{A}_0 \mathbf{q}_t = \mathbf{A}_1 E_t(\mathbf{q}_{t+1} \,|\Im_t) + \mathbf{A}_2 \mathbf{q_{t-1}} + \mathbf{u}_t,$$

where $\mathbf{q}_t = (R_t, y_t, \pi_t)'$, $\mathbf{u}_t = (u_{Rt}, u_{yt}, u_{\pi t})'$, and

$$\mathbf{A}_0 = \begin{pmatrix} 1 & -(1-\delta_r)\psi_y & -(1-\delta_r)\psi_\pi \\ \sigma & 1 & 0 \\ 0 & -\gamma & 1 \end{pmatrix}, \ \mathbf{A}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \kappa & \sigma \\ 0 & 0 & \beta \end{pmatrix}, \ \mathbf{A}_2 = \begin{pmatrix} \delta_R & 0 & 0 \\ 0 & \delta_y & 0 \\ 0 & 0 & \delta_\pi \end{pmatrix}. \tag{57}$$

We also assume that the structural shocks are orthogonal and have the following diagonal covariance matrix

$$\mathbf{\Sigma}_u = \begin{pmatrix} \sigma^2_{uR} & 0 & 0 \\ 0 & \sigma^2_{uy} & 0 \\ 0 & 0 & \sigma^2_{u\pi} \end{pmatrix}. \tag{58}$$

As in Section 2.1, the solution of the model is given by (2):

$$\mathbf{q}_t = \mathbf{\Phi}(\theta)\mathbf{q}_{t-1} + \mathbf{\Gamma}(\theta)\mathbf{u}_t, \tag{59}$$

suppressing the dependence on θ, for the moment, $\mathbf{\Phi}$, is the solution of $\mathbf{A}_1\mathbf{\Phi}^2 - \mathbf{A}_0\mathbf{\Phi} + \mathbf{A}_2 = \mathbf{0}$ and $\mathbf{\Gamma} = (\mathbf{A}_0 - \mathbf{A}_1\mathbf{\Phi})^{-1}$. The value of $\mathbf{\Phi}$ can be solved by iterative back-substitution procedure which involves iterating on an initial arbitrary choice of $\mathbf{\Phi}$ say $\mathbf{\Phi}_{(0)}$ using the recursive relation

$$\mathbf{\Phi}_{(r)} = \left[\mathbf{I}_k - (\mathbf{A}_0^{-1}\mathbf{A}_1)\mathbf{\Phi}_{(r-1)}\right]^{-1}(\mathbf{A}_0^{-1}\mathbf{A}_2). \tag{60}$$

See Binder and Pesaran (1995) for further details. The iterative procedure is continued until convergence using the criteria $\| \mathbf{\Phi}_{(r)} - \mathbf{\Phi}_{(r-1)} \|_{\max} \leq 10^{-6}$.

In the numerical calculations all unknown parameters are replaced by calibrated values from the DSGE literature. Parameters of (56) are calibrated based on average estimates from eight major economies as summarized in Table 5 of Dees et al (2009). The parameters of (55) and the long run parameters of the Taylor rule, (54), are calibrated using the results in Dennis (2009). The calibrated values of θ^0 are summarized in Table 1 below. The standard deviations of the errors were all set equal to 0.005, or half a percent per quarter, which is similar to the US values found in Dees et al. (2009).

Table 1. Pre-intervention parameter values, θ^0, used in the Monte Carlo Analysis

$\sigma = 0.065$	$\kappa = 0.57$	$\beta = 0.65$	$\gamma = 0.045$	$\psi_\pi = 1.5$	$\psi_y = 0.5$
$\delta_y = 0.42$	$\delta_\pi = 0.34$	$\delta_R = 0.7$	$\sigma_{u\pi} = 0.005$	$\sigma_{uy} = 0.005$	$\sigma_{uR} = 0.005$

The solution matrices for the pre-intervention parameters in (59) are given by:

$$\mathbf{\Phi}(\theta^0) = \begin{pmatrix} 0.65 & 0.13 & 0.20 \\ -0.17 & 0.62 & -0.05 \\ -0.06 & 0.08 & 0.47 \end{pmatrix}, \ \mathbf{\Gamma}(\theta^0) = \begin{pmatrix} 0.93 & 0.31 & 0.60 \\ -0.24 & 1.49 & -0.15 \\ -0.08 & 0.19 & 1.39 \end{pmatrix}. \tag{61}$$

The solution is a VAR(1) and in each equation the coefficient of the autoregressive term is the largest in absolute value and the persistence of inflation is lower than the persistence of output and interest rates. We also note that even though the structural shocks are orthogonal the reduced form shocks are correlated.

We consider four separate policy interventions, in which each of the parameters of the Taylor rule are changed one at a time, leaving the other parameters unchanged. Intervention 1_A increases the interest rate persistence in the Taylor Rule, δ_R, from 0.7 to 0.9. Intervention 1_B reduces δ_R from 0.7 to 0.25. Intervention 1_C increases the inflation coefficient in the Taylor rule, ψ_π, from 1.5 to 2.5. Intervention 1_D increases the output coefficient in the Taylor rule, ψ_y, from 0.5 to 1. The values of θ^1 that are changed under alternative policy interventions are given in Table 2.

Table 2: Policy interventions

Interventions*	θ^0	θ^1
1_A	$\delta_R = 0.7$	$\delta_R = 0.9$
1_B	$\delta_R = 0.7$	$\delta_R = 0.25$
1_C	$\psi_\pi = 1.5$	$\psi_\pi = 2.5$
1_D	$\psi_y = 0.5$	$\psi_y = 1.0$

* The other elements of θ^1 are kept at their pre-intervention values.

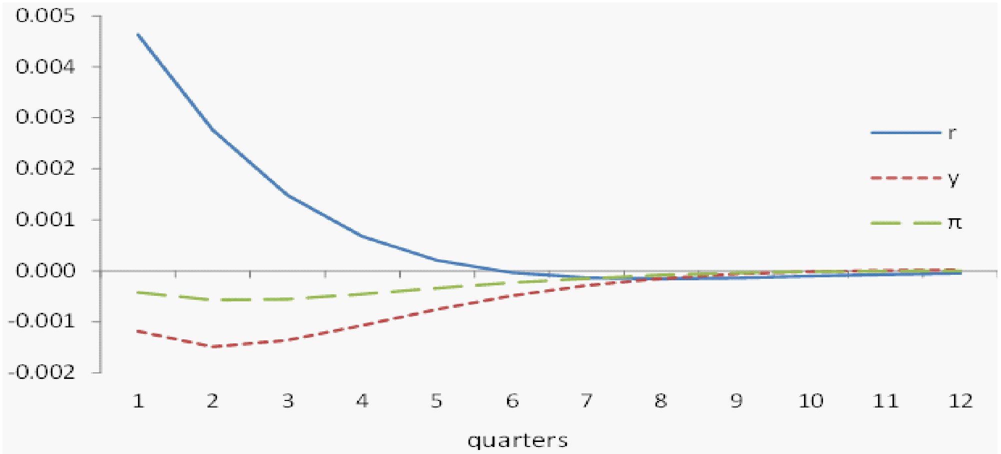
4.1 Shock Impulse Response Functions

As noted above, a shock impulse response function, SIRF, gives the time profile for a shock to one of the structural errors assuming that the parameters are constant. For example the monetary policy shock impulse response function represents the effects of a one standard error shock to u_{Rt}, the error in the Taylor rule, and is given by:

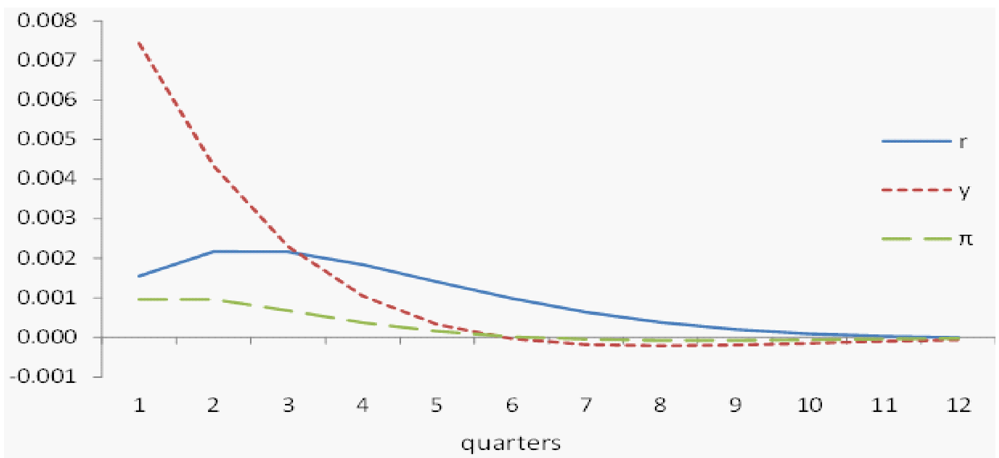
$$SIRF_R(h, \theta^0, \sigma_{uR}) = \sigma_{uR} \left[\mathbf{\Phi}\left(\theta^0\right)\right]^h \mathbf{\Gamma}(\theta^0)\mathbf{e}_R, \tag{62}$$

where $\mathbf{e}_R$= $(1, 0, 0)'$. For a linear model this SIRF is independent of the value of $\mathbf{q}_{T_0}$, the state of the economy at the time of the shock. In terms of the SIRF analysis the behaviour of the model is very standard. As Figure 1 shows a contractionary monetary policy shock raises interest rates and reduces output and inflation, with output falling by more than inflation.

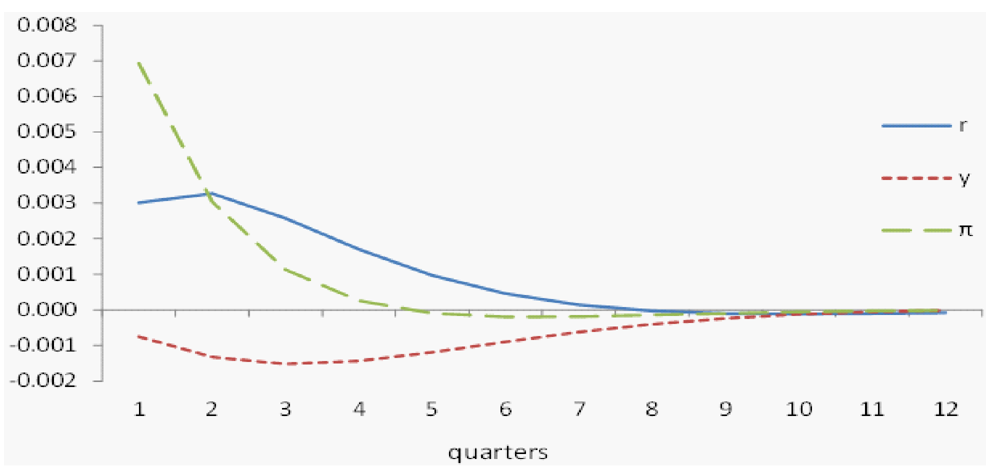
Figure 1: Shock impulse response functions for interest rates, R_t, output, y_t, and inflation π_t



1a. Monetary Policy Shock



1b. Demand Shock



1c. Supply Shock

A positive demand shock increases all three variables; output by the most, then interest rates, and then inflation. A negative supply shock, increases inflation, the interest rate rises to offset the higher inflation, but not by as much as inflation and output falls. The impact effects of the monetary policy shock are given by the first column of $\mathbf{\Gamma}(\theta^0)$ defined by (61), while the impact effects of the demand and supply shocks are given by its second and third columns.

4.2 Policy Impulse Response Functions

The policy impulse response function, PIRF, gives the response of the system over time to a permanent change in the policy parameter(s) and, for a particular variable identified by the selection vector, $\mathfrak{s}$, is given by (5), repeated here for convenience:

$$PIRF(h, \theta^1, \theta^0, \mathbf{q}_{T_0}) = \mathfrak{s}' \left[\left[\mathbf{\Phi}\left(\theta^1\right) \right]^h - \left[\mathbf{\Phi}\left(\theta^0\right) \right]^h \right] \mathbf{q}_{T_0}. \tag{63}$$

As noted above, the PIRF requires knowledge of the parameters before and after the intervention and, as can be seen from (15) and (31) above, it is the PIRF which largely determines the power of the test to determine whether the mean effect of the policy is different from zero.

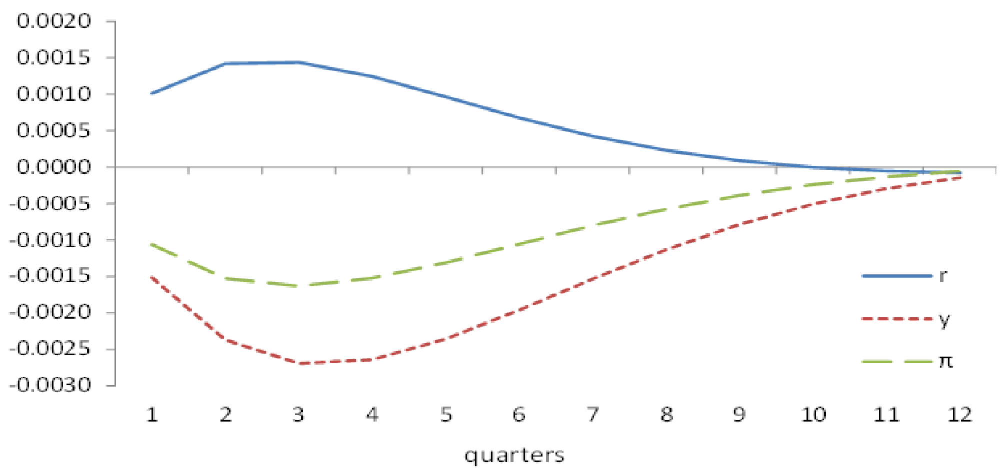
Unlike SIRFs, the PIRFs and the policy ineffectiveness tests depend on the chose of initial states, $\mathbf{q}_{T_0}$, at the time of the policy change. It is therefore important that the choice of $\mathbf{q}_{T_0}$reflects a sensible combination of values of interest rate, inflation and output. One possible approach is to set $\mathbf{q}_{T_0}$ equal to the impact effects of SIRFs. For example, one could set $\mathbf{q}_{T_0}$ to $\mathbf{q}_{R,T_0} = \sigma_{uR}\mathbf{\Gamma}(\theta^0)\mathbf{e}_R$, which is the impact effect of a monetary policy shock as given by (62) for $h = 0$. Similarly, for the demand and supply shocks $\mathbf{q}_{T_0}$ can be set to $\mathbf{q}_{y,T_0} = \sigma_{u\ y}\mathbf{\Gamma}(\theta^0)\mathbf{e}_y$ and $\mathbf{q}_{\pi,T_0} = \sigma_{u\pi}\mathbf{\Gamma}(\theta^0)\mathbf{e}_\pi$, respectively, where $\mathbf{e}_y = (0, 1, 0)'$ and $\mathbf{e}_\pi = (0, 0, 1)'$.[5] Considering values of the initial states, $\mathbf{q}_{T_0}$, that correspond to impact effects of structural shocks seems sensible given the focus of the literature on SIRFs. One could also consider multiples of the effects of such shocks as representing different degrees of deviations from equilibrium. The power of the policy ineffectiveness test will then be an increasing function of the extent to which, at the time of the policy change, the economy has deviated from the equilibrium.

Figure 2 shows PIRFs for the effects of changing the degree of persistence (or the interest rate smoothing) associated with the Taylor rule, Figure 2a shows the effect of intervention 1_A and Figure 2b of 1_B. These are the only policy changes which have much effect. This is consistent with the theoretical results that it is the dynamics that are central to policy having mean effects. We set the initial states at $\mathbf{q}_{R,T_0} = \sigma_{uR}\mathbf{\Gamma}(\theta^0)\mathbf{e}_R$, the values of the variables that result from the monetary policy shock on impact. Intervention 1_A increases the degree of persistence from $\delta_R = 0.7$, to $\delta_R = 0.9$. This causes the interest rate to rise and output and inflation to fall initially, with a maximum effect after about three periods before returning to zero. Intervention 1_B reduces the degree of persistence from $\delta_R = 0.7$, to $\delta_R = 0.25$. This has the opposite effect causing the

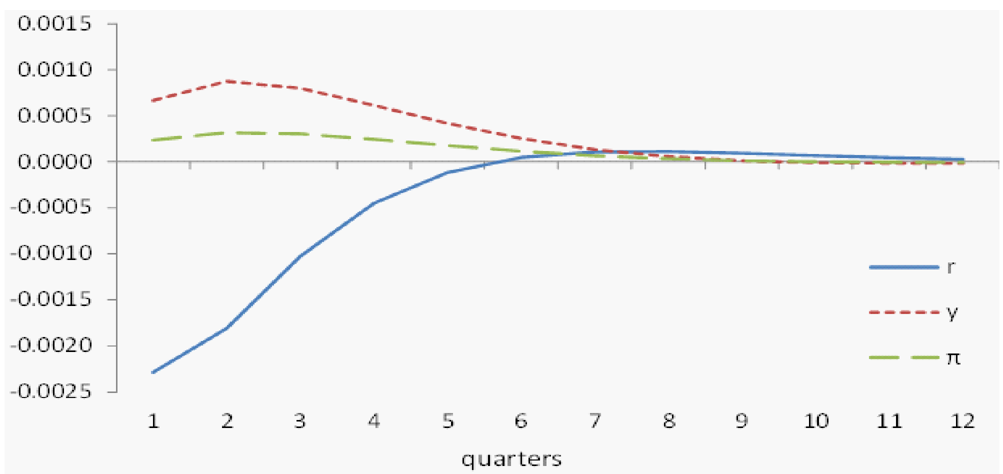
[5] As noted above these values are given by the columns of $\mathbf{\Gamma}(\theta^0)$ defined by (61).

interest rate to fall, by more than it rose in case 1_A, and output and inflation to rise by rather less than they fell under case 1_A. The initial effects are the same as the values of $\left[\mathbf{\Phi}\left(\theta^1\right) - \mathbf{\Phi}\left(\theta^0\right)\right]$ for the two cases. When the degree of persistence is low as in intervention 1_B, the variables return to zero much faster, making the mean effect of policy much smaller. As we shall see, this is reflected in the power of the policy ineffectiveness tests to be discussed below.

Figure 2: Policy Impulse Response Functions: $\mathbf{q}_{R,T_0} = \sigma_{uR}\mathbf{\Gamma}(\theta^0)\mathbf{e}_R$.



2a. Intervention $1_A : \delta_R = 0.7$, to $\delta_R = 0.9$



2b. Intervention $1_B : \delta_R = 0.7$, to $\delta_R = 0.25$

4.3 Policy Ineffectiveness tests

The test performance was evaluated using the calibrated values of θ^0 ignoring estimation error and for various settings of the initial states, $\mathbf{q}_{T_0}$. Values of $\mathbf{q}_{T_0+h}$, $h = 1, 2, ..., H$, for horizons $H = 8$, and $H = 24$. are generated from (59) assuming $\mathbf{u}_t^{(b)} \sim IIDN(\mathbf{0}, \mathbf{\Sigma}_u)$, where $\mathbf{\Sigma}_u$ is given in (58) for $b = 1, 2, .., 2000$, replications.[6] For replication b the policy effects are simulated as

$$\hat{\mathbf{d}}_{T_0+h}^{(b)} = \mathbf{q}_{T_0+h}^{(b)} - \left[\mathbf{\Phi}\left(\theta^0\right)\right]^h \mathbf{q}_{T_0}, \tag{64}$$

for $h = 1, 2, ..., H$. The policy mean effect is calculated as

$$\bar{\hat{\mathbf{d}}}_H^{(b)} = \frac{1}{H}\sum_{h=1}^{H} \hat{\mathbf{d}}_{T_0+h}^{(b)},$$

and the test statistic as

$$\mathcal{T}_{d,H}^{(b)} = \frac{\sqrt{H}\bar{\hat{\mathbf{d}}}_H^{(b)}}{\hat{\omega}_{0q}},$$

where

$$\hat{\omega}_{0q}^2 = \left\{ H^{-1}\sum_{j=1}^{H} \mathcal{A}_{H-j}\left(\mathbf{\Phi}(\theta^0)\right) \mathbf{\Sigma}_\varepsilon\left(\theta^0\right) \mathcal{A}'_{H-j}\left(\mathbf{\Phi}(\theta^0)\right) \right\},$$

$$\mathcal{A}_{H-j}\left(\mathbf{\Phi}(\theta^0)\right) = \mathbf{I}_{k_z+1} + \mathbf{\Phi}(\theta^0) + \left[\mathbf{\Phi}(\theta^0)\right]^2 + ... + \left[\mathbf{\Phi}(\theta^0)\right]^{H-j}.$$

Table 3 shows the size and power of the policy ineffectiveness tests against four alternative policy interventions, two evaluation horizons and three initial states. The size was calculated when $\mathbf{q}_{T_0+h}^{(b)}$ was generated using θ^0, the power was calculated when $\mathbf{q}_{T_0+h}^{(b)}$ was generated using one of the four alternative policy interventions which change θ^0 to $\theta^{1A}, ..., \theta^{1D}$, as set out in Table 2. The initial states are given in different rows of the Table. The rows labelled $\mathbf{q}_{R,T_0}$ give the rejection frequencies for the initial state corresponding to the effects of a one standard deviation monetary policy shock, ; the rows labelled $\mathbf{q}_{y,T_0}$ a demand shock and and the rows labelled $\mathbf{q}_{\pi,T_0}$ a a supply shock.

The size seems very well controlled. The power is highest for intervention, 1_A, where the degree of persistence of the Taylor rule increases from $\delta_R = 0.7$, to $\delta_R = 0.9$, confirming what was apparent from the PIRFs. However, even in this case the power is not high. At $H = 8$ the highest power is 20% for testing the effect on y_t and using the initial state, $\mathbf{q}_{R,T_0}$ or $\mathbf{q}_{\pi,T_0}$. At $H = 24$ the highest power is 25% for testing the effect on y_t. The test has little power against the other three types of interventions.[7] Whereas the test has power against the increase in persistence of the Taylor rule it has less power against the reduction in the persistence of the Taylor rule for output and inflation because the variables return to zero quickly. The test has little power against changes in the coefficients of inflation and output in the Taylor rule because they have relatively little effect on the other variables on impact.

[6] More specifically, $\mathbf{q}_{T_0+h}^{(b)} = \mathbf{\Phi}(\theta)\mathbf{q}_{T_0+h-1}^{(b)} + \mathbf{\Gamma}(\theta)\mathbf{u}_{T_0+h}^{(b)}$, for $h = 1, 2, ..., H$, with $\mathbf{q}_{T_0}^{(b)} = \mathbf{q}_{T_0}$.

[7] Similar outcomes are also reported by Rudebusch (2005) who, in the context of the Lucas Critique, shows that the apparent policy invariance of reduced forms is consistent with the magnitude of historical policy shifts and the relative insensitivity of the reduced forms of plausible forward looking macroeconomic specifications to policy shifts. However, here we use formal tests based on structural models.

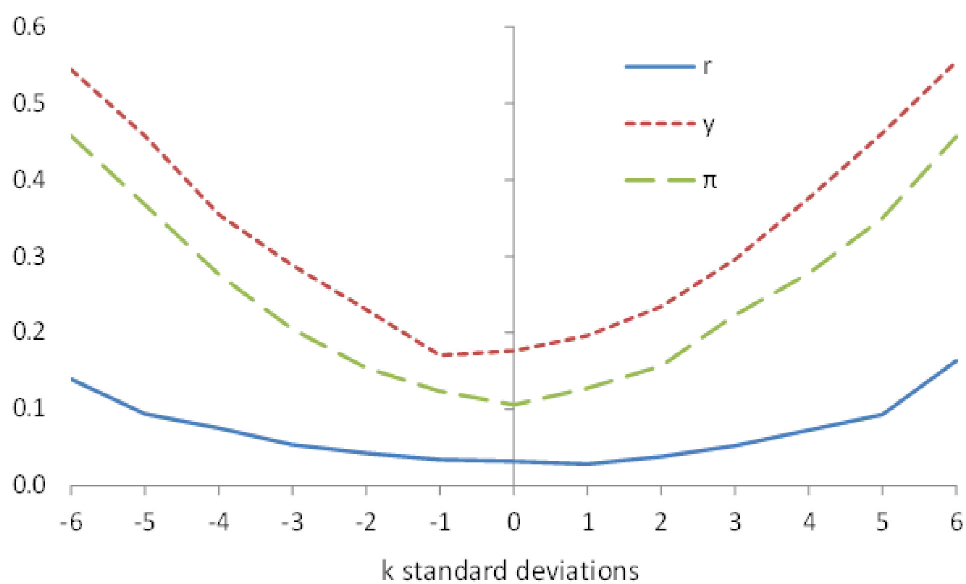
Table 3: Size, θ^0, and power of policy ineffectiveness tests against 4 alternatives $\theta^{1A}, \theta^{1B}, \theta^{1C}, \theta^{1D}$; horizons $H = 8, 24$; 3 initial states

	Size (θ^0)			Power (θ^{1A})			Power (θ^{1B})			Power (θ^{1C})			Power (θ^{1D})		
	R	y	π	R	y	π	R	y	π	R	y	π	R	y	π
	$H = 8$														
$\mathbf{q}_{R,T_0}$	0.05	0.05	0.05	0.03	0.20	0.13	0.13	0.04	0.08	0.11	0.06	0.03	0.07	0.02	0.07
$\mathbf{q}_{y,T_0}$	0.04	0.05	0.05	0.03	0.18	0.12	0.11	0.04	0.07	0.10	0.06	0.03	0.07	0.01	0.06
$\mathbf{q}_{\pi,T_0}$	0.05	0.04	0.05	0.04	0.20	0.12	0.12	0.04	0.08	0.12	0.05	0.03	0.07	0.02	0.06
	$H = 24$														
$\mathbf{q}_{R,T_0}$	0.05	0.05	0.05	0.04	0.25	0.17	0.11	0.04	0.09	0.10	0.06	0.02	0.07	0.02	0.07
$\mathbf{q}_{y,T_0}$	0.05	0.06	0.05	0.04	0.25	0.16	0.11	0.03	0.09	0.10	0.05	0.02	0.07	0.01	0.06
$\mathbf{q}_{\pi,T_0}$	0.05	0.04	0.05	0.04	0.24	0.18	0.12	0.04	0.09	0.10	0.07	0.02	0.07	0.02	0.06

Notes: The rows labelled q_{R,T_0} set the initial state $q_{T_0} = \sigma_{uR}\mathbf{\Gamma}(\theta^0)\mathbf{e}_R$. Similarly for $q_{y,T_0} = \sigma_{uy}\mathbf{\Gamma}(\theta^0)\mathbf{e}_y$, and $q_{\pi,T_0} = \sigma_{u\pi}\mathbf{\Gamma}(\theta^0)\mathbf{e}_\pi$. The alternative hypotheses are set out in Table 2.

Figure 3 shows the rejection frequency for intervention 1_A, increasing the degree of interest rate smoothing, against the impact of a k standard deviation monetary policy shock, $\mathbf{q}_{R,T_0}$. The rejection frequencies increase with the deviation of the initial value from zero and are roughly symmetric for positive and negative values. The rejection frequencies are highest for output, intermediate for inflation and lowest for interest rates. The graphs were similar but with lower rejection frequencies when the initial states are set to multiples of demand and supply shocks.

Figure 3. Rejection frequencies for intervention 1_A (increasing δ_R from 0.7 to 0.9) with the initial states at k standard deviations of $\mathbf{q}_{R,T_0}$, and $H = 8$ quarters



These simulations confirm the theoretical results. The size of the test is correct. The effect of the policy intervention depends on the dynamics, reductions in the degree of persistence reduce the effect of changing the policy parameters. The power of the test depends on the state of the economy at the time of the policy intervention. In our example, the test has some power against increases in the persistence of the Taylor rule, but not against the other policy changes considered. However, the effects of all these policy changes are transitory, none have any effect on the steady states. We now consider interventions that change the steady states.

4.4 Inflation targeting as a policy change

As an example of a policy intervention that changes the steady states, consider an inflation targeting regime when the policy maker changes the target rate of inflation which we denote by π_*. We assume the announcement of the change in the inflation target is credible and fully understood.[8] To represent this intervention in the New Keynesian example, where the variables are measured as deviations from steady state, we need to re-write the inflation and interest rate deviations in terms of their realized values which we denote by $\mathring{\pi}_t$ and $\mathring{R}_t$, namely $\mathring{\pi}_t = \pi_t + \pi_*$ and $\mathring{R}_t = R_t - (r + \pi_*)$, where π_* is the target rate of inflation, and r denotes the steady state value of the real interest rate. In terms of the realized values of inflation and interest rates, $\mathring{\pi}_t$ and $\mathring{R}_t$, and deviations y_t, for the output gap, we have

$$\mathring{R}_t = (1 - \delta_R)\left[r + (1 - \psi_\pi)\pi_*\right] + \delta_R \mathring{R}_{t-1} + (1 - \delta_R)(\psi_\pi \mathring{\pi}_t + \psi_y y_t) + u_{Rt}$$

$$y_t = -\sigma r + \delta_y y_{t-1} + \kappa E(y_{t+1} \,|\mathfrak{I}_t) - \sigma \left[\mathring{R}_t - E(\mathring{\pi}_{t+1} \,|\mathfrak{I}_t)\right] + u_{yt}$$

$$\mathring{\pi}_t = (1 - \delta_\pi - \beta)\,\pi_* + \delta_\pi \mathring{\pi}_{t-1} + \beta E(\mathring{\pi}_{t+1} \,|\mathfrak{I}_t) + \gamma y_t + u_{\pi t}.$$

and setting $\mathring{\mathbf{q}}_t = (\mathring{R}_t, y_t, \mathring{\pi}_t)'$, we obtain

$$\mathbf{A}_0 \mathring{\mathbf{q}}_t = \mathbf{A}_1 E_t(\mathring{\mathbf{q}}_{t+1}) + \mathbf{A}_2 \mathring{\mathbf{q}}_{t-1} + \mathbf{A}_3 \mathbf{s}_t + \mathbf{u}_t,$$

which corresponds to the RE model (32) with exogenous variables, with $\mathbf{s}_t$ replaced by 1, and

$$\mathbf{A}_3 = \begin{pmatrix} (1 - \delta_R)\left[r + (1 - \psi_\pi)\pi_*\right] \\ -\sigma r \\ (1 - \delta_\pi - \beta)\,\pi_* \end{pmatrix}.$$

The other matrices, $\mathbf{A}_0$, $\mathbf{A}_1$, and $\mathbf{A}_2$, are given as before by (57). The solution in terms of $\mathring{\mathbf{q}}_t$ is given by

$$\mathring{\mathbf{q}}_t = \left[\mathbf{I}_3 - \mathbf{\Phi}(\theta)\right] \mathring{\mathbf{q}}_* + \mathbf{\Phi}(\theta) \mathring{\mathbf{q}}_{t-1} + \mathbf{\Gamma}(\theta) \mathbf{u}_t,$$

[8] Kulish and Pagan (2014) consider a change in inflation target when there is both perfect and imperfect credibility.

where $\mathring{\mathbf{q}}_* = (r + \pi_*, 0, \pi_*)'$, and $\mathbf{\Phi}(\theta)$ and $\mathbf{\Gamma}(\theta)$ are defined as before. This solution can be viewed as an example of the model with policy invariant variables discussed in Section 3 where in the solution (36), $\mathbf{\Psi}_x(\theta)\mathbf{x}_t$ is set to zero and $\mathbf{\Psi}_w(\theta)\mathbf{w}_t$ is set to $[\mathbf{I}_3 - \mathbf{\Phi}(\theta)]\mathring{\mathbf{q}}_*$.

Suppose now that the policy intervention at time T_0 took the form of changing the inflation target from π_*^0 to π_*^1. In this case the policy effects are given by (39) with $\mathbf{\Psi}_w(\theta)\mathbf{w}_t$ replaced by $[\mathbf{I}_3 - \mathbf{\Phi}(\theta)]\mathring{\mathbf{q}}_*$, namely

$$\hat{d}_{T_0+h} = \mathfrak{s}'\mathring{\mathbf{q}}_{T_0+h} - \mathfrak{s}'\left[\mathbf{\Phi}\left(\hat{\theta}_T^0\right)\right]^h \mathring{\mathbf{q}}_{T_0} - \mathfrak{s}'\sum_{j=0}^{h-1}\left[\mathbf{\Phi}\left(\hat{\theta}_T^0\right)\right]^j\left[\mathbf{I}_3 - \mathbf{\Phi}\left(\hat{\theta}_T^0\right)\right]\mathring{\mathbf{q}}_*^0,$$

where $\mathring{\mathbf{q}}_*^0 = (r + \pi_*^0, 0, \pi_*^0)'$

$$\hat{d}_{T_0+h} = \mathfrak{s}'\mathring{\mathbf{q}}_{T_0+h} - \mathfrak{s}'\left[\mathbf{\Phi}\left(\hat{\theta}_T^0\right)\right]^h \mathring{\mathbf{q}}_{T_0} - \mathfrak{s}'\left\{\mathbf{I}_3 - \left[\mathbf{\Phi}\left(\hat{\theta}_T^0\right)\right]^h\right\}\mathring{\mathbf{q}}_*^0, \tag{65}$$

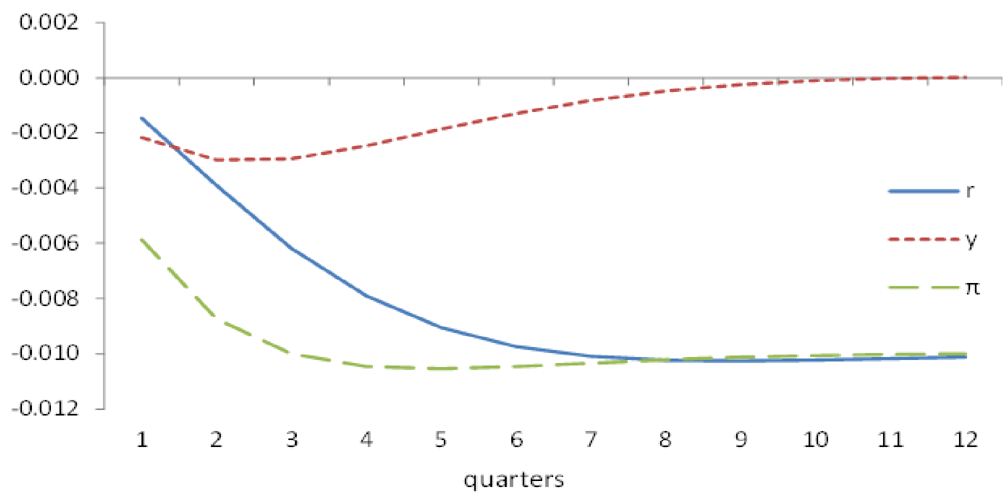
The policy ineffectiveness test is given by (48), noting that there are no policy exogenous variables, $\mathbf{x}_t$, in this example. In the case where only the inflation target is changed the power of the test rises with $\sqrt{H}\mathfrak{s}'\left(\mathring{\mathbf{q}}_*^1 - \mathring{\mathbf{q}}_*^0\right) = \sqrt{H}\left(\pi_*^1 - \pi_*^0\right)(1,0,1)'\mathfrak{s}$, and tends to unity in the case of inflation and the nominal interest rate, as to be expected, and has no power as $H \to \infty$, if real output deviations, y_t, are considered. Nevertheless, the change in the inflation target does have short run effects on real output. This is reflected in the policy impulse response function and the test outcomes. The policy impulse response function when only the inflation taget is changed is given by

$$PIRF(h, \pi_*^1 - \pi_*^0, \theta) = \left(\pi_*^1 - \pi_*^0\right)\left\{\mathbf{I}_3 - \left[\mathbf{\Phi}(\theta)\right]^h\right\}\begin{pmatrix}1\\0\\1\end{pmatrix}, \text{ for } h = 1, 2, ..., H. \tag{66}$$

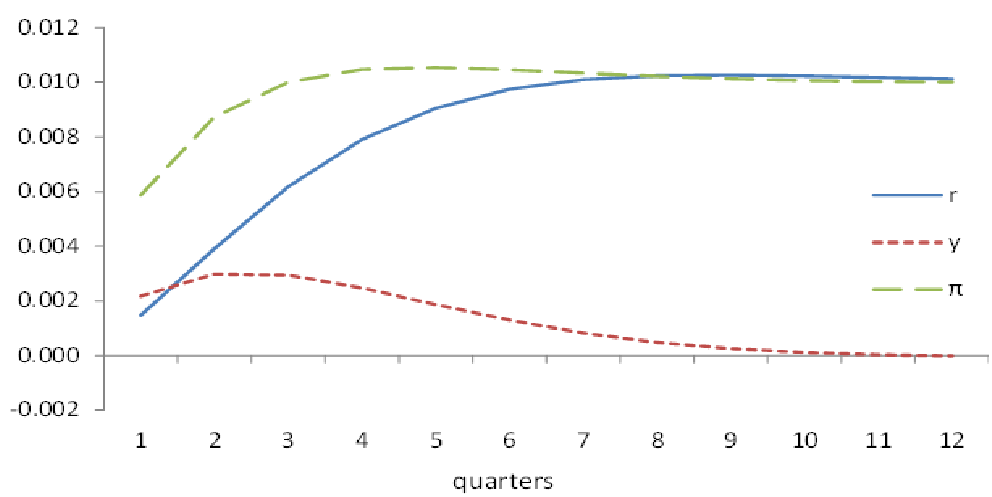
It is clear that in the limit as $H \to \infty$, the PIRF tends to $\left(\pi_*^1 - \pi_*^0\right)(1,0,1)'$, which also confirms that in the NK model only nominal values are affected in the long run by changes in the inflation target.

The short run impacts of changes in the inflation target can be illustrated using the parametrization given above. For this purpose we consider two scenarios, a reduction of π_*^0 from 2% to 1% per quarter and an increase of π_*^0 from 1% to 2% per quarter. The increase in the target inflation is interesting in the context of the Japanese experience. Initially we do not change any of the other policy parameters, which are kept at the baseline values listed in Table 1. Figure 4a gives the responses to the reduction and 4b to the increase in the inflation target. In the case of a reduction, inflation falls more than the interest rate, raising the real interest rate on impact to 0.44% and thus depressing output. The real interest rate and output return to zero, leaving the nominal interest rate and inflation rate at the new target 1% lower after about seven quarters. When the target rate of inflation is increased the effects are reversed: inflation jumps more than interest rates, the real interest rate falls on impact to -0.44%, temporarily raising output. Although the two cases are symmetrical numerically, they are not symmetrical in welfare terms, since the output loss associated with the reduction in inflation is something that one would wish to avoid.

Figure 4: Policy impulse response functions for changes in target rates of inflation



4a. Reduction of $\pi_*^0 = 2\%$ to $\pi_*^1 = 1\%$ per quarter



4b. Increase of $\pi_*^0 = 1\%$ to $\pi_*^1 = 2\%$ per quarter

In the case where there is both a change in the steady state and a change in the policy rule parameters, the policy impulse response functions are given by

$$PIRF(h, \pi_*^1, \theta^1, \pi_*^0, \theta^0) = \left[\left[\mathbf{\Phi}\left(\theta^1\right)\right]^h - \left[\mathbf{\Phi}\left(\theta^0\right)\right]^h \right] \mathring{\mathbf{q}}_{T_0} + \left\{ \mathbf{I}_3 - \left[\mathbf{\Phi}\left(\theta^1\right)\right]^h \right\} \mathring{\mathbf{q}}_*^1 - \left\{ \mathbf{I}_3 - \left[\mathbf{\Phi}\left(\theta^0\right)\right]^h \right\} \mathring{\mathbf{q}}_*^0,$$
$$= \left\{ \left[\mathbf{\Phi}\left(\theta^1\right)\right]^h - \left[\mathbf{\Phi}\left(\theta^0\right)\right]^h \right\} \left(\mathring{\mathbf{q}}_{T_0} - \mathring{\mathbf{q}}_*^0\right) + \left[\mathbf{I}_3 - \mathbf{\Phi}\left(\theta^1\right)\right]^h \left(\mathring{\mathbf{q}}_*^1 - \mathring{\mathbf{q}}_*^0\right)$$

where

$$\mathring{\mathbf{q}}_*^0 = \begin{pmatrix} r + \pi_*^0 \\ 0 \\ \pi_*^0 \end{pmatrix}, \mathring{\mathbf{q}}_*^1 - \mathring{\mathbf{q}}_*^0 = \left(\pi_*^1 - \pi_*^0\right) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

More specifically, for a unit MP shock at the point of intervention, we set $\mathring{\mathbf{q}}_{T_0} = \mathbf{q}_*^0 + \sigma_{uR}\Gamma(\hat{\theta}_T^0)\mathbf{e}_R$, and hence

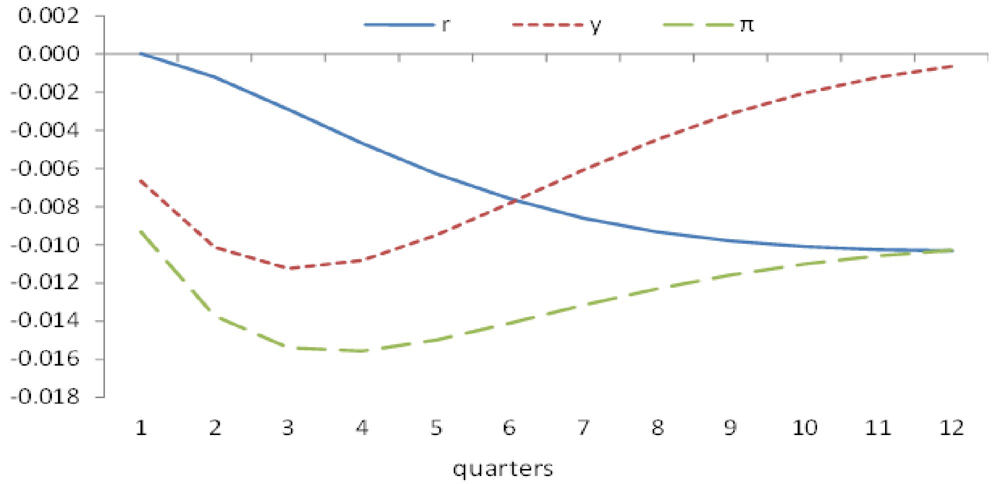
$$PIRF(h, \pi_*^1, \theta^1, \pi_*^0, \theta^0) = \sigma_{uR}\left\{\left[\mathbf{\Phi}\left(\theta^1\right)\right]^h - \left[\mathbf{\Phi}\left(\theta^0\right)\right]^h\right\}\Gamma(\theta^0)\mathbf{e}_R + \left(\pi_*^1 - \pi_*^0\right)\left[\mathbf{I}_3 - \mathbf{\Phi}\left(\theta^1\right)\right]^h \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}. \tag{67}$$

which reduces to (66) when only the inflation target is changed. Similar expressions can be obtained when the initial state is set to values of $\mathring{\mathbf{q}}$ that arise on impact from demand or supply shocks.

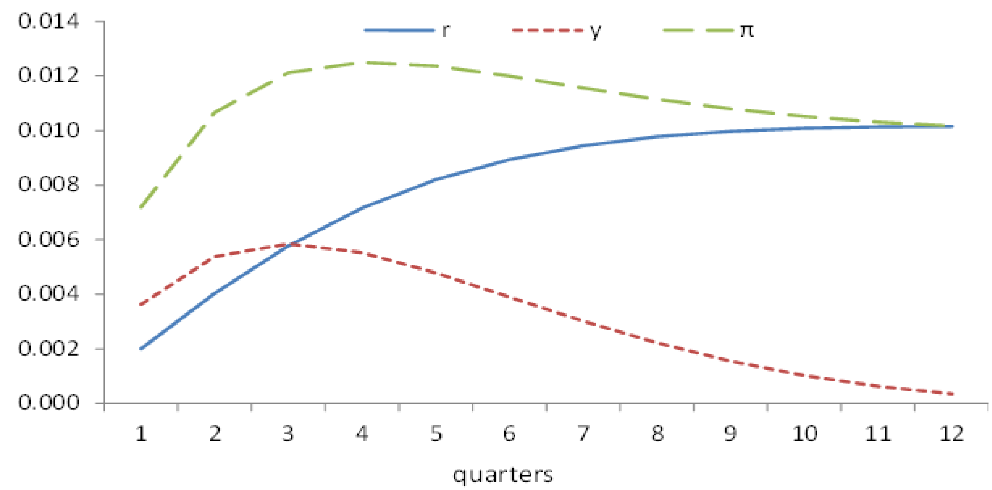
We now consider combining the change in the inflation target with changes in the degree of inflation smoothing. Figure 5a presents the effects of simultaneously reducing the inflation target from 2% to 1% and increasing the inflation smoothing parameter, δ_R, from 0.7 to 0.9**,** intervention 1_A above, with the initial state set to $\mathring{\mathbf{q}}_{R,T_0}$. This intervention causes inflation to drop sharply, overshooting its steady state of 1%, hitting 1.55% after about 4 quarters. The real interest rate rises to 1.25%, depressing output, before the variables return to their steady state. Figure 5b shows that increasing the target rate of inflation has similar but the opposite effects. Comparing the reduction in the target rate of inflation in Figure 5a with that in Figure 4a, the increased interest rate smoothing has resulted in a much larger loss of output. Whereas in Figure 4a the maximum loss of output was 0.3% per quarter, in figure 5a the maximum loss was 1.1%, in both cases around quarter 3.

Figure 5: Policy impulse response functions for changes in target rates of inflation plus increased interest rate smoothing

Intervention 1_A : δ_R from 0.7 to 0.9, initial state $\mathring{\mathbf{q}}_{R,T_0}$



5a. Reduction of $\pi_*^0 = 2\%$ to $\pi_*^1 = 1\%$ per quarter

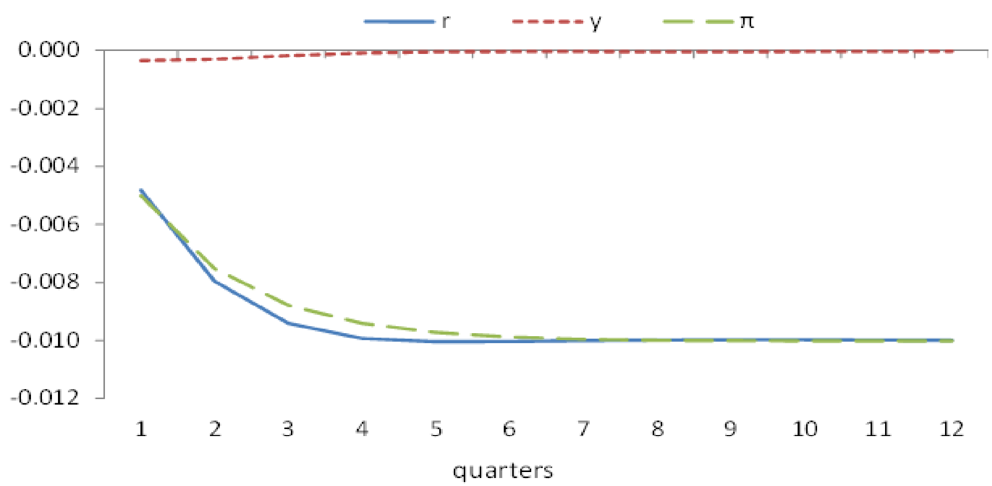


5b. Increase of $\pi_*^0 = 1\%$ to $\pi_*^1 = 2\%$ per quarter

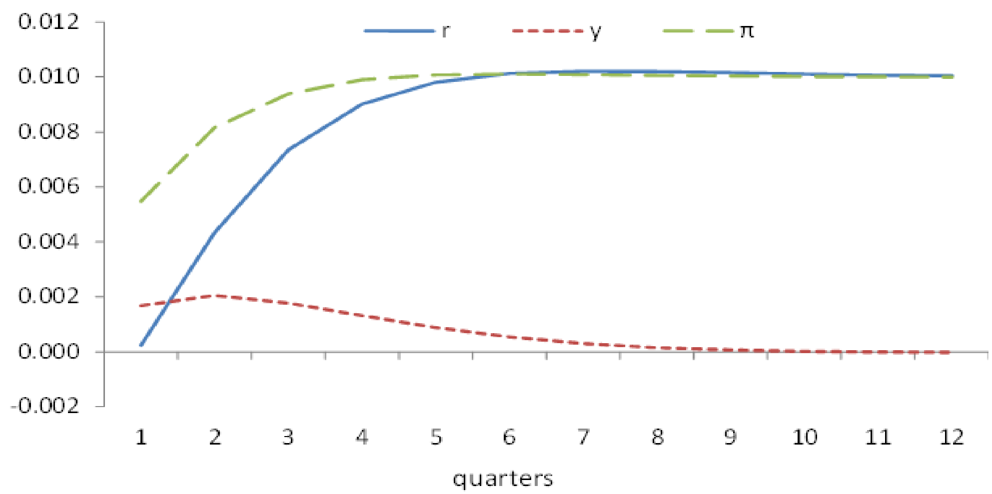
Figure 6 shows the results when the change in inflation target is combined with reduced interest rate smoothing. For a credible reduction in the inflation target and very little interest rate smoothing, the interest rate and the inflation rate reduce by almost exactly the same amount and output hardly falls. With a credible increase in the inflation target and reduced interest rate smoothing, inflation increases more than interest rates and the lower real interest rates provides a boost to output. While the results are specific to this parameterisation and the assumption of credibility, it seems likely that less interest rate smoothing is optimal when reducing the target rate of inflation, as in Figure 6a, since this causes less output loss, and more interest rate smoothing seems more appropriate when increasing the target rate of inflation, as in the Japanese case, since this provides a bigger boost to output.

Figure 6: Policy impulse response functions for changes in target rates of inflation plus reduced interest rate smoothing

Intervention 1_B : δ_R from 0.7 **to** 0.25**, initial state** $\mathring{\mathbf{q}}_{R,T_0}$



6a. Reduction of $\pi_*^0 = 2\%$ to $\pi_*^1 = 1\%$ per quarter



6b. Increase of $\pi_*^0 = 1\%$ to $\pi_*^1 = 2\%$ per quarter

We now consider the effect on size and power of the policy ineffectiveness test in detecting the effects of changes in the target rate of inflation on inflation, output deviations and the interest rate. We only consider the case where the inflation target is reduced from 2% to 1% per quarter, the results for an increase were almost identical. We consider two interventions. In the first, called θ^{1E}, the interest rate smoothing parameter is left unchanged at $\delta_R = 0.7$, in the second, called θ^{1F}, δ_R is increased to 0.9 at the same time as the reduction in the inflation target is announced. If the target rate is reduced without any other policy changes, the power of the tests based on the nominal interest rate and the inflation rate are quite high and rise substantially as the horizon of the test is increased from $H = 8$ to 24 quarters. In contrast, and as to be expected noting the PIRFs depicted in Figure 4, the test has little power for output, since the effect of a change in the inflation target on the real output is small and transitory. Under intervention θ^{1F}, when there is both a change in the inflation target and an increase in interest rate smoothing, the power of the test based on inflation outcomes is increased, but for interest rates the power is reduced relative to the case θ^{1E}, since the increased smoothing means that interest rates do not change as much. The increased smoothing causes a larger movement in real interest rates as noted above and this causes a greater effect on output hence a higher power in detecting the effects of the policy change on realized values of output deviations. Whereas for interest rates and inflation, the power increases as the horizon is extended, for output deviations, which is moving back to its steady state value of zero, the power falls as the horizon is extended.

Table 4: Size and power of policy ineffectiveness tests against reducing the inflation target only (θ^{1E}) and when inflation target reduction is accompanied by a rise in interest rate smoothing (θ^{1F})- Horizons $H = 8, 24$; 3 initial states ($\mathring{\mathbf{q}}_{T_0}$)

	Size (θ^0)			Power (θ^{1E})			Power (θ^{1F})		
	R	y	π	R	y	π	R	y	π
Initial states	$H = 8$								
$\mathring{\mathbf{q}}_{R,T_0}$	0.05	0.05	0.05	0.29	0.07	0.72	0.13	0.39	0.90
$\mathring{\mathbf{q}}_{y,T_0}$	0.06	0.05	0.06	0.26	0.07	0.68	0.17	0.33	0.86
$\mathring{\mathbf{q}}_{\pi,T_0}$	0.05	0.06	0.06	0.28	0.06	0.70	0.16	0.35	0.88
	$H = 24$								
$\mathring{\mathbf{q}}_{R,T_0}$	0.06	0.04	0.05	0.73	0.07	0.99	0.65	0.30	0.98
$\mathring{\mathbf{q}}_{y,T_0}$	0.05	0.06	0.05	0.73	0.05	0.99	0.70	0.28	0.98
$\mathring{\mathbf{q}}_{\pi,T_0}$	0.05	0.05	0.05	0.71	0.05	0.99	0.68	0.29	0.99

Notes: See notes to Table 3. Alternative hypothesis θ^{1E} assumes that the inflation target is reduced from $\pi^0_* = 2\%$ to $\pi^1_* = 1\%$ per quarter. Alternative hypothesis θ^{1F} combines the reduction of the inflation target from $\pi^0_* = 2\%$ to $\pi^1_* = 1\%$ per quarter with a higher degree of interest rate smoothing, raising δ_R from 0.7 to 0.9.

5 Conclusion

In this paper we have derived tests for the null hypothesis of the ineffectiveness of a policy intervention, defined as a change in the parameters of a policy rule. We consider tests conducted using full structural models both of the standard form, where all the variables including the policy variable are endogenous, as well as where the system is augmented with exogenous variables, including, perhaps, exogenous policy variables. The augmented system allows us to consider policy interventions that change steady states, such as changes in the inflation target.

The tests are based on the average differences, over a given policy evaluation horizon, between the post-intervention realizations of the target variable and the associated counterfactual outcomes based on the parameters estimated using data before the policy intervention. The Lucas Critique is not an issue since the counterfactual, given by the predictions from the model estimated on pre-intervention data, will embody pre-intervention parameters, while the actual post-intervention outcomes will embody any effect of the change in policy, the change in parameters and the consequent change in expectations. The tests do not require knowing the post-intervention parameters.

We derive the asymptotic distribution of the policy ineffectiveness tests under alternative assumptions concerning the type of model, the future error processes and the pre and post-intervention sample sizes. The power of the proposed tests depends on the size of the parameter change, the dynamics of the system, the state of the economy at the time of the intervention, the size of the policy evaluation horizon and whether the model contains policy invariant exogenous variables. However, the power of the policy ineffectiveness tests are likely to be low unless the underlying DSGE model contains exogenous variables, or equivalently the policy changes the steady states.

The size and power of the proposed tests are investigated by simulations using a standard three equation New Keynesian DSGE model. These simulations are in accord with the theoretical results. The size of the test is correct, and the tests have power against increases in the persistence of the Taylor rule, but little power against increases in the responses of interest rates to inflation and output. The test does have power against policy interventions that change steady states, such as changes in the target rate of inflation which has a permanent effect on inflation and interest rates but only a transitory effect on output, which eventually returns to its steady state.

The focus of this paper has been on the mean effects of policy changes. But, as mentioned at the end of Section 2.1, the volatility effects of policy change are also of interest. In that simple case, where there are no dynamics and no exogenous variables, the variance of $\mathbf{q}_t$ changes following the policy intervention from $\mathbf{\Sigma}_\varepsilon(\theta^0) = \mathbf{\Gamma}(\theta^0)\mathbf{\Sigma}_u\mathbf{\Gamma}(\theta^0)'$ to $\mathbf{\Sigma}_\varepsilon(\theta^1) = \mathbf{\Gamma}(\theta^1)\mathbf{\Sigma}_u\mathbf{\Gamma}(\theta^1)'$, assuming that $\mathbf{\Sigma}_u$ remains constant. However, in many cases, such as the Great Moderation, the central issue is whether the reduction in the variance of output growth is due to good policy (changes in policy parameters $\boldsymbol{\theta}_p$) or good luck (reductions in $\|\mathbf{\Sigma}_u\|$). The same issues arise when there is dynamics. In the case where the model include exogenous variables, the variance of $\mathbf{q}_t$ can be derived from the RE solution for $\mathbf{q}_t$ given by equation (36). In this case there is an extra contribution to the change in the variance of $\mathbf{q}_t$ after the policy intervention that comes from any change in the variance of the non-policy exogenous variables. An interesting subject for future research is the decomposition of the change in the variance of $\mathbf{q}_t$ into components due to the change in policy parameters, the change in non-policy innovation variances and the change in the variance of the exogenous variables. Unlike the policy ineffectiveness test, such a decomposition requires estimating the parameters of a full structural model before as well as after the intervention.

Appendix: Statement and Proof of Lemmas

Lemma 1 *Let* $\mathbf{A}$ *be a* $k\times k$ *matrix and* $\mathbf{x}_{T+h-j}$ *a* $k\times 1$ *vector, and suppose that* $\mathbf{I}_k - \mathbf{A}$ *is invertible, then*

$$\begin{aligned} H^{-1}\sum_{h=1}^{H}\sum_{j=0}^{h-1}\mathbf{A}^j\mathbf{x}_{T+h-j} &= H^{-1}\sum_{j=1}^{H}\left(\mathbf{I}_k+\mathbf{A}+...+\mathbf{A}^{H-j}\right)\mathbf{x}_{T+j} \\ &= H^{-1}\left(\mathbf{I}_k-\mathbf{A}\right)^{-1}\sum_{j=1}^{H}\left(\mathbf{I}_k-\mathbf{A}^{H-j+1}\right)\mathbf{x}_{T+j} \\ &= \left(\mathbf{I}_k-\mathbf{A}\right)^{-1}\left(H^{-1}\sum_{j=1}^{H}\mathbf{x}_{T+j}\right)-\left(\mathbf{I}_k-\mathbf{A}\right)^{-1}\left(H^{-1}\sum_{j=1}^{H}\mathbf{A}^{H-j+1}\mathbf{x}_{T+j}\right). \end{aligned}$$

Proof. The result follows by direct manipulation of the terms. ■

Lemma 2 *Suppose that the* $k\times k$ *matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *have bounded spectral norms* $\|\mathbf{A}\|\leq\lambda$ *and* $\|\mathbf{B}\|\leq\lambda$*, for some fixed positive constant* λ*. Then*

$$\left\|\mathbf{A}^h-\mathbf{B}^h\right\|\leq h\lambda^{h-1}\left\|\mathbf{A}-\mathbf{B}\right\|, \text{ for } h=1,2,.... \tag{68}$$

Proof. We establish this result by backward induction. It is clear that it holds for $h=1$. For $h=2$, using the identity

$$\mathbf{A}^2-\mathbf{B}^2=\mathbf{A}(\mathbf{A}-\mathbf{B})+(\mathbf{A}-\mathbf{B})\mathbf{B},$$

the result for $h=2$ follows

$$\left\|\mathbf{A}^2-\mathbf{B}^2\right\|\leq\left(\|\mathbf{A}\|+\|\mathbf{B}\|\right)\|\mathbf{A}-\mathbf{B}\|=2\lambda\|\mathbf{A}-\mathbf{B}\|.$$

More generally, we have the identity

$$\mathbf{A}^h-\mathbf{B}^h=\mathbf{A}^h(\mathbf{A}-\mathbf{B})+(\mathbf{A}-\mathbf{B})\mathbf{B}^h+\mathbf{A}(\mathbf{A}^{h-2}-\mathbf{B}^{h-2})\mathbf{B}.$$

Now suppose now that (68) holds for $h-2$, and using the above note that

$$\begin{aligned}
\left\|\mathbf{A}^h-\mathbf{B}^h\right\| &\leq \left\|\mathbf{A}^{h-1}\right\|\|\mathbf{A}-\mathbf{B}\|+\|\mathbf{A}-\mathbf{B}\|\left\|\mathbf{B}^{h-1}\right\|+\|\mathbf{A}\|\left\|\mathbf{A}^{h-2}-\mathbf{B}^{h-2}\right\|\|\mathbf{B}\| \\
&\leq \|\mathbf{A}\|^{h-1}\|\mathbf{A}-\mathbf{B}\|+\|\mathbf{A}-\mathbf{B}\|\|\mathbf{B}\|^{h-1}+\|\mathbf{A}\|\left\|\mathbf{A}^{h-2}-\mathbf{B}^{h-2}\right\|\|\mathbf{B}\| \\
&\leq 2\lambda^{h-1}\|\mathbf{A}-\mathbf{B}\|+\lambda^2\left\|\mathbf{A}^{h-2}-\mathbf{B}^{h-2}\right\| \\
&\leq 2\lambda^{h-1}\|\mathbf{A}-\mathbf{B}\|+\lambda^2\left[(h-2)\lambda^{h-3}\|\mathbf{A}-\mathbf{B}\|\right] \\
&\leq h\lambda^{h-1}\|\mathbf{A}-\mathbf{B}\|.
\end{aligned}$$

Hence, if (68) holds for $h-2$, then it must also hold for h. But since we have established that (68) holds for $h=1$ and $h=2$, then it must hold for any h. ■

Lemma 3 *Consider the $k\times k$ matrix $\mathbf{A}(\boldsymbol{\theta})=(a_{ij}(\boldsymbol{\theta}))$, where k is a finite integer and $a_{ij}(\boldsymbol{\theta})$, for all $i,j=1,2,..,k$, are continuously differentiable real-valued functions of the $p\times 1$ vector of parameters, $\theta\in\mathbf{\Theta}$. Suppose that $a_{ij}(\theta)$ has finite first order derivatives at all points in $\mathbf{\Theta}$, and assume that $\hat{\boldsymbol{\theta}}_T$ is a $\sqrt{T}$ consistent estimator of $\boldsymbol{\theta}^0$. Then*

$$\left\|\mathbf{A}(\hat{\boldsymbol{\theta}}_T)-\mathbf{A}(\boldsymbol{\theta}^0)\right\|\leq a_T\left\|\hat{\boldsymbol{\theta}}_T-\boldsymbol{\theta}^0\right\|, \tag{69}$$

$$\left\|\mathbf{A}(\hat{\boldsymbol{\theta}}_T)\right\|\leq \left\|\mathbf{A}(\boldsymbol{\theta}^0)\right\|+a_T\left\|\hat{\boldsymbol{\theta}}_T-\boldsymbol{\theta}^0\right\|, \tag{70}$$

where $a_T=\left\|\partial\mathbf{A}\left(\bar{\boldsymbol{\theta}}_T\right)/\partial\boldsymbol{\theta}'\right\|$ is bounded in T, and elements of $\bar{\boldsymbol{\theta}}_T\in\mathbf{\Theta}$ lie on the line segment joining $\boldsymbol{\theta}^0$ and $\hat{\boldsymbol{\theta}}_T$

Proof. Consider the mean-value expansions

$$a_{ij}\left(\hat{\theta}_T\right)-a_{ij}\left(\theta^0\right)=\frac{\partial a_{ij}\left(\bar{\theta}_T\right)}{\partial\theta'}\left(\hat{\theta}_T-\theta^0\right),\text{ for } i,j=1,2,...,k,$$

where elements of $\bar{\theta}_T$ lie on the line segment joining θ^0 and $\hat{\theta}_T$. Given that $\hat{\theta}_T$ is consistent for θ^0, it must also be that $\bar{\theta}_T\to_p\theta^0$, as $T\to\infty$. Collecting all the k^2 terms we have

$$\mathbf{A}(\hat{\boldsymbol{\theta}}_T)-\mathbf{A}(\boldsymbol{\theta}^0)=\left(\frac{\partial\mathbf{A}\left(\bar{\boldsymbol{\theta}}_T\right)}{\partial\boldsymbol{\theta}'}\right)\left[\mathbf{I}_k\otimes\left(\hat{\boldsymbol{\theta}}_T-\boldsymbol{\theta}^0\right)\right],$$

where $\otimes$ denotes the Kronecker matrix product. Hence

$$\left\|\mathbf{A}(\hat{\boldsymbol{\theta}}_T)-\mathbf{A}(\boldsymbol{\theta}^0)\right\|\leq\left\|\frac{\partial\mathbf{A}\left(\bar{\boldsymbol{\theta}}_T\right)}{\partial\boldsymbol{\theta}'}\right\|\left\|\hat{\boldsymbol{\theta}}_T-\boldsymbol{\theta}^0\right\|,$$

$$\left\|\mathbf{A}(\hat{\boldsymbol{\theta}}_T)\right\|=\left\|\mathbf{A}(\boldsymbol{\theta}^0)+\left(\frac{\partial\mathbf{A}\left(\bar{\boldsymbol{\theta}}_T\right)}{\partial\boldsymbol{\theta}'}\right)\left[\mathbf{I}_k\otimes\left(\hat{\boldsymbol{\theta}}_T-\boldsymbol{\theta}^0\right)\right]\right\|\leq\left\|\mathbf{A}(\boldsymbol{\theta}^0)\right\|+\left\|\frac{\partial\mathbf{A}\left(\bar{\boldsymbol{\theta}}_T\right)}{\partial\boldsymbol{\theta}'}\right\|\left\|\hat{\boldsymbol{\theta}}_T-\boldsymbol{\theta}^0\right\|.$$

The results (69) and (70) now follow since $\bar{\boldsymbol{\theta}}_T\to_p\boldsymbol{\theta}^0$, and by assumption the derivatives $\partial a_{ij}\left(\boldsymbol{\theta}^0\right)/\partial\boldsymbol{\theta}'$ exist and are bounded in T. ■

Lemma 4 *Suppose that* $\lambda_T = \lambda + T^{-1/2} a_T$, $a_T > 0$ *and bounded in* T, $\lambda_T \neq 1$, $H = \kappa T^{\epsilon}$, *where* $\epsilon \leq 1/2$, $0 < \lambda < 1$, *and* κ *is a positive fixed constant. Then*

$$\sum_{h=1}^{H} h\lambda_T^{h-1} = \frac{1}{(1-\lambda)^2} + O_p\left(T^{-1/2}\right) + O_p\left(H\lambda^H\right), \tag{71}$$

and

$$\sum_{h=1}^{H}\sum_{j=0}^{h-1} j\lambda_T^{j-1} = \frac{1}{(1-\lambda)^2}\left(H - \frac{1+\lambda}{1-\lambda}\right) + O_p\left(T^{-1/2}\right) + O_p\left(H\lambda^H\right). \tag{72}$$

Proof. We first note that

$$\begin{aligned}\sum_{h=1}^{H} h\lambda_T^{h-1} &= \frac{\partial}{\partial\lambda_T}\left(\sum_{h=1}^{H}\lambda_T^h\right) \\ &= \frac{1-\lambda_T^H}{(1-\lambda_T)^2} - \frac{H\lambda_T^H}{(1-\lambda_T)},\end{aligned} \tag{73}$$

Also since $\lambda_T = \lambda + O_p\left(T^{-1/2}\right)$

$$\sum_{h=1}^{H} h\lambda_T^{h-1} = \frac{1}{(1-\lambda)^2} + O_p\left(T^{-1/2}\right) + O_p\left(H\lambda_T^H\right). \tag{74}$$

But,

$$\lambda_T^H = \left(\lambda + T^{-1/2}a_T\right)^H = \lambda^H\left(1 + \frac{T^{-1/2}a_T}{\lambda}\right)^H = O_p\left(\lambda^H e^{d_T H/\sqrt{T}}\right), \tag{75}$$

where $d_T = a_T/\lambda$, which is also bounded in T. Finally, $H/\sqrt{T} = T^{1-\epsilon/2}$ and for $\epsilon \leq 1/2$, $e^{d_T H/\sqrt{T}}$ will be bounded in T. Using this result in (74) yields (71), as desired. Similarly,

$$\begin{aligned}\sum_{h=1}^{H}\sum_{j=0}^{h-1} j\lambda_T^{j-1} &= \sum_{h=1}^{H}\left[\frac{(1-\lambda_T^h) - h(1-\lambda_T)\lambda_T^{h-1}}{(1-\lambda_T)^2}\right] \\ &= \frac{1}{(1-\lambda_T)^2}\left[\sum_{h=1}^{H}\left[\left(1-\lambda_T^h\right) - h(1-\lambda_T)\lambda_T^{h-1}\right]\right] \\ &= \frac{1}{(1-\lambda_T)^2}\left[H - \sum_{h=1}^{H}\lambda_T^h - (1-\lambda_T)\sum_{h=1}^{H} h\lambda_T^{h-1}\right].\end{aligned}$$

Using (73) we have

$$\sum_{h=1}^{H}\sum_{j=0}^{h-1} j\lambda_T^{j-1} = \frac{1}{(1-\lambda_T)^2}\left\{H - \frac{\lambda_T - \lambda_T^{H+1}}{1-\lambda_T} - (1-\lambda_T)\left[\frac{1-\lambda_T^H}{(1-\lambda_T)^2} - \frac{H\lambda_T^H}{(1-\lambda_T)}\right]\right\}.$$

Now using (75) and recalling that $\lambda_T = \lambda + O_p\left(T^{-1/2}\right)$, we obtain (72). ■

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