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Intraday Pairs Trading Strategies on High Frequency Data: The Case of Oil Companies

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This paper introduces novel 'doubly mean-reverting' processes based on conditional modeling to model spreads between pairs of stocks. Intraday trading strategies using high frequency data are proposed based on the model. This model framework and the strategies are designed to capture 'local' market inefficiencies that are elusive for traditional pairs trading strategies with daily data. Results from real data back-testing for two periods show remarkable returns, even accounting for transaction costs, with annualized Sharpe ratios of 3.9 and 7.2 over the periods June 2013–April 2015 and 2008 respectively. By choosing the particular sector of oil companies, we also confirm the observation that the commodity price is the main driver of the share prices of commodity-producing companies at times of spikes in the related commodity market.

Keywords: Pairs trading; Quantitative trading strategies; Conditional modeling; Doubly mean-reverting model; High frequency data; Transaction costs

JEL Classification: C1, C3, C6, C61, C63

1. Introduction

The idea of pairs trading is quite popular across various asset classes and based on the property that, since companies within a sector are high correlated, some pairs of price returns exhibit strong similarity. We can model the return differences of these pairs as mean-reverting processes. If they deviate too far from the mean, we short/long the pair by simultaneously buying one and short selling another. We keep the position until it reverts back to the mean level.

Let A and B be a pair of closely related stocks, $S_A(t)$ and $S_B(t)$ their prices at time t. The cumulative log return difference—or spread—for this pair is

$$Y(t) = \log\left(\frac{S_A(t)}{S_A(0)}\right) - \log\left(\frac{S_B(t)}{S_B(0)}\right) \tag{1}$$

In pairs trading literature, the spread $Y(t)$ or its variation has been modeled as a mean-reverting process, oscillating either around zero or around a linear function of time. For example, Avellaneda and Lee (2010) define the spread as the difference between $\log\left(\frac{S_A(t)}{S_A(0)}\right)$ and $\beta \cdot \log\left(\frac{S_B(t)}{S_B(0)}\right)$, where

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β is calculated by regressing the cumulative log return of one stock on another in a certain period. The best pairs are commonly identified as either the ones with smallest distance measures defined as the sum of squared deviations (Gatev *et al.* 2006, Bowen *et al.* 2010) or using cointegration relationships (Vidyamurthy 2004, Lin *et al.* 2006).

Both the distance method and the cointegration method have their limitations. Consider two stocks A and B with cumulative log returns both being 0 at the beginning. Suppose $\log\left(\frac{S_A(t)}{S_A(0)}\right)$ goes to a large positive value α in a short amount of time then stays around that level while $\log\left(\frac{S_B(t)}{S_B(0)}\right)$ remains around 0. Then no matter how synchronized they move afterwards, this pair would not likely be identified by the simple distance measure as they have a large average distance of α. Now, further assume A and B co-move only in a subperiod during the whole analysis period, then the pair is not likely to be selected according to cointegration method. Yet there are clearly profits to be made in this scenario. The problem is that both those methods imply a *static* relationship between two stocks during the training period whereas the relationship may very well be changing from one day to the next.

The strategy we propose in this paper is designed exactly to capture this kind of 'local' statistical arbitrage opportunities, by searching for temporary market mispricing inefficiencies. The idea is to seek the pairs of which $Y(t)$ can be characterized by the following modeling procedure: model the long term trend of $Y(t)$, denoted as $L(t)$, as a stochastic process, and then model $Y(t)$ via a mean-reverting process around this long term stochastic trend $L(t)$ using the conditional modeling technique. If the mean reversion speed of $Y(t)$ is fast enough, we can make profit by making intraday pairs trades.

The utilization of two mean-reverting stochastic processes on the same series is partly inspired by Fourier series expansion. $L(t)$ can be regarded as the first term of a Fourier series with the largest period and lowest frequency; imagine that $Y(t)$ is approximated by the Fourier series, with higher-frequency local oscillation being added on top of the lower-frequency waves. To visualize this, imagine a long rope lying on the ground straightened out. If we hold onto one end of the rope and shake it horizontally, then it will display a wavy pattern. Now, pick two points on the rope that are close to each other, pin their locations, then shake the segment between them. The shaken part will likely become a more pronounced local wave. Repeat this to the whole rope segment-by-segment. The resulting rope would look like the $Y(t)$ process while all the pinned positions make up $L(t)$. The rational for this novel doubly mean-reverting model is that, if we can identify pairs with relative stable $L(t)$ and volatile $Y(t)$, then intraday pairs trading should perform well a priori. $Y(t)$ in these cases would return at the end of a trading day—hopefully after wild swings—to more or less the same level as daily open.

The key to *add up* local oscillation is through a framework of 'conditional modeling and conditional inference' (see the overview in Chang 2010). This technique has been applied to analyze the dynamics of financial time series (e.g. the waiting time invariance of return sequences in Chang *et al.* 2013, aggregation theorem in Chang and Geman 2013) as well as to other research fields (e.g. Amarasingham *et al.* 2012, Chang *et al.* 2015). Nevertheless, the technique has not been utilized for designing trading strategies in the literature.

In this study, we focus on the oil sector and look at times when the underlying commodity price is experiencing sharp moves, making it the major factor driving the share prices. Geman and Vergel (2013) showed, in the case of the fertilizer commodity, that shares of fertilizer-mining companies are very sensitive to the commodity price at times of high moves of this price. Geman (2015) extended this property to other types of agriculture-related companies. The literature on the subject counts as a founding paper Tufano (1998) who analyzes the price sensitivities of gold mining companies' shares. Our results confirmed the observation that crude oil price is the main driver of oil company stock prices during market turbulence.

In the literature, most pairs trading studies use daily data and a daily trading frequency (Gatev *et al.* 2006, Lin *et al.* 2006, Avellaneda and Lee 2010, Cummins and Bucca 2012, Bogomolov 2013, Zeng and Lee 2014). Bowen *et al.* (2010), which use 60 minute return series, is one of the very few that uses data with frequency higher than daily. With the availability of tick data, we are able to use five-minute series for $Y(t)$ in the case of some highly liquid oil company stocks. To our best knowledge, we are the first academic study on pairs trading to use such high frequency data, and the first one on intraday pairs trading strategies.

The rest of the paper is organized as follows. Section 2 details the model and its calibration. Trading rules are discussed in section 3. We perform simulations in section 4 in order to validate the model and the strategies. Results from real data back-testing over two periods are presented and analyzed in section 5. Section 6 concludes the paper and provides directions for future research.

2. The model

2.1. *Model specification*

In order to exploit intraday pairs trading profits, we use high frequency data with interval length of five minutes to model $Y(t)$ defined in equation (1). There are 78 five-minute intervals every day during trading hours from 9:30 AM to 4:00 PM, hence 79 $Y(t)$'s. We denote the 79 observed values of $Y(t)$ in day i as

$$Y_{79(i-1)+1}, Y_{79(i-1)+2}, \ldots, Y_{79i} \qquad i = 1, 2, \ldots, N$$

where N is the number of days. The subscript in this paper refers to discretized observations of stochastic processes. Moreover, we assume that the long term trend $L(t)$ is identified by the daily opening and closing values of the process $Y(t)$, namely for day i,

$$L_{2i-1} = Y_{79(i-1)+1} \quad \text{and} \quad L_{2i} = Y_{79i}. \tag{2}$$

The stochastic process $L(t)$, with two observed data points per day, is preferred to have a small variance. In this study, we model $L(t)$ as an Ornstein–Uhlenbeck (OU) process, with mean 0

$$\mathrm{d}L(t) = -\theta_L L(t)\mathrm{d}t + \sigma_L \mathrm{d}W_t^L. \tag{3}$$

Next by the definition of conditional distribution, the joint distribution of $Y_1, Y_2, \ldots, Y_{79N}$ can be written as the product of the distribution of $Y_{79(i-1)+1}$'s and Y_{79i}'s and the conditional distribution of Y_i's given $Y_{79(i-1)+1}$'s and Y_{79i}'s

$$\begin{aligned} & f(Y_1, Y_2, \ldots, Y_{79N}) \\ =& f(Y_{79(i-1)+1}, Y_{79i}, i = 1, \cdots, N) f(Y_1, Y_2, \ldots, Y_{79N} | Y_{79(i-1)+1}, Y_{79i}, i = 1, \cdots, N) \\ =& f(L_1, L_2, \ldots, L_{2N}) f(Y_1, Y_2, \ldots, Y_{79N} | L_1, L_2, \ldots, L_{2N}) \end{aligned} \tag{4}$$

Note that due to equation (2), the last equality is valid, and the joint distribution of L_i's can be obtained by discretizing the process (3).

Now to model the conditional distribution of Y_i's given L_i's, we use the conditional modeling technique by introducing an auxiliary process $\tilde{Y}(t)$ that follows:

$$\mathrm{d}\tilde{Y}(t) = \theta\left(\tilde{L}(t) - \tilde{Y}(t)\right)\mathrm{d}t + \sigma \mathrm{d}W_t^{\tilde{Y}} \tag{5}$$

where the mean process is $\tilde{L}(t) = \frac{L_{2i-2}+L_{2i-1}}{2}$ and $i = i(t)$ refers to the day of time t ($2i-2$ refers to the closing of day $i-1$ and $2i-1$ refers to the opening of day i). We assume the conditional distribution

$$f(Y_1, Y_2, \ldots, Y_{79N} | L_1, L_2, \ldots, L_{2N})$$

is the same as the conditional distribution of $\tilde{Y}_i$'s given $\tilde{Y}_{79(i-1)+1} = L_{2i-1}$ and $\tilde{Y}_{79i} = L_{2i}$, $i = 1, \cdots, N$, where $\tilde{Y}_i$'s are the corresponding discretization of process (5). In other words, conditional on given daily opening and closing values of the process, $Y(t)$ is the same as $\tilde{Y}(t)$ in distribution. Hence, to simplify the notation, we use $Y(t)$ and Y_i in place of $\tilde{Y}(t)$ and $\tilde{Y}_i$ in the rest of the paper.

By defining the mean process $\tilde{L}(t)$ as the average of L_{2i-2} and L_{2i-1}, we assume in any trading day, the mean level that the spread process reverts to is the average of the opening value of the current day and the closing value of the previous trading day.

To recap, the distribution of the $Y(t)$ process is defined by first specifying the dynamics of $L(t)$ from equation (3), then the distributions of the in-between points are given indirectly by equation (5), via the conditional relationship (4), hence the name conditional modeling.

For $Y(t)$, there are 79 observations per day thus 79 time intervals, the lengths of which are not equal: 78 short five minute periods and a long overnight period. The similar problem exists for $L(t)$: the time span between a day's open and close is different from between the day's close and next day's open.

The lengths in real time of the trading hours per day are 6.5 hours while the overnight periods are at least 17.5 hours (from 4 pm market close to next day's market open 9:30 am, or longer in the case of weekends and holidays). However, the amount of information and market movements during daytime trading hours are much richer than that during overnight periods. Therefore, we estimate the lengths of both periods in *effective* time instead of real time in the following way: since the relative (effective) lengths of the trading day and the overnight period are unknown and unequal, we need two time steps. δ_1 is the length of trading hours 9:30 AM and 4:00 PM; δ_2 is the length between market close and next day's open. Hence,

$$\delta_1 + \delta_2 = 1 \text{ day} = \frac{1}{250} \tag{6}$$

and the algorithm to estimate δ_1 and δ_2 is based on the ratio of variances of intraday and overnight changes as detailed in section 2.2.1.

All the model parameters $\theta_L, \sigma_L, \theta, \sigma, \delta_1$ and δ_2 from equation (3)–(6) are calibrated daily using maximum likelihood estimation (MLE).

2.2. *Model calibration*

For better parameter estimation, the calibration is updated every day using a moving window, also called the pairs formation period (see Gatev *et al.* 2006), the length of which is chosen properly. If the duration is too short, the calibration is unreliable due to the lack of training data; if the duration is too long, estimated parameters do not accurately reflect the present situation because of non-stationarity of market dynamics. In our model, the frequencies of $L(t)$ and $Y(t)$ are different, thus requiring training periods of different lengths. We use the past 100 days' daily open and close prices to calibrate the process $L(t)$ and use the past 30 days' five-minute prices to calibrate the process $Y(t)$. Both lengths were decided at the beginning of the study and has not been tuned based on data, to avoid data-snooping biases (Lo and MacKinlay 1990).

2.2.1. Calibration for $L(t)$.

For a general OU process with constant mean

$$\mathrm{d}S(t) = \theta \left(\mu - S(t)\right) \mathrm{d}t + \sigma \mathrm{d}W_t,$$

the discretization $\{S_i\}$ satisfies

$$S_{i+1} = S_i e^{-\theta\delta} + \mu(1 - e^{-\theta\delta}) + \sigma\sqrt{\frac{1 - e^{-2\theta\delta}}{2\theta}} Z_i, \tag{7}$$

for all i, where δ is the time step in discretization and Z_i's are i.i.d. $N(0, 1)$.

For our $L(t)$, the discretized series is the combined daily opening and closing cumulative return differences. μ is assumed to be 0. Equation (7) leads to two equations.

The intraday changes

$$L_{2i} = L_{2i-1} e^{-\theta_L \delta_1} + \sigma_L \sqrt{\frac{1 - e^{-2\theta_L \delta_1}}{2\theta_L}} Z_{2i} \qquad i = 1, \ldots, N; \tag{8}$$

and the overnight changes

$$L_{2i+1} = L_{2i} e^{-\theta_L \delta_2} + \sigma_L \sqrt{\frac{1 - e^{-2\theta_L \delta_2}}{2\theta_L}} Z_{2i+1}, \qquad i = 1, \ldots, N-1 \tag{9}$$

where Z_i's are i.i.d. $N(0, 1)$.

To estimate δ_1 and δ_2, We use the following equation obtained from equations (8) and (9) with the variances approximated by empirical variances :

$$\frac{\mathrm{Var}(L_{2i} - L_{2i-1} e^{-\theta_L \delta_1})}{\mathrm{Var}(L_{2i+1} - L_{2i} e^{-\theta_L \delta_2})} = \frac{1 - e^{-2\theta_L \delta_1}}{1 - e^{-2\theta_L \delta_2}} \tag{10}$$

Notice that solving this equation for δ_1 and δ_2 requires θ_L. Therefore, we develop the following algorithm to iteratively calibrate $\delta_1, \delta_2, \theta_L$ and σ_L together:

1. Initialize δ_1 and δ_2.
2. Using δ_1 and δ_2 values, calibrate θ_L and σ_L using MLE.
3. Plug θ_L into equation (10). Then δ_1 and δ_2 can be solved together with equation (6).
4. Repeat steps 2 and 3 until $\delta_1, \delta_2, \theta_L$ and σ_L all converge.

Step 1:

Since both δ_1 and δ_2 are small, from equation (10),

$$\frac{\mathrm{Var}(L_{2i} - L_{2i-1})}{\mathrm{Var}(L_{2i+1} - L_{2i})} \approx \frac{1 - e^{-2\theta_L \delta_1}}{1 - e^{-2\theta_L \delta_2}} \approx \frac{\delta_1}{\delta_2} \tag{11}$$

where the variance of intraday return $\mathrm{Var}(L_{2i} - L_{2i-1})$ and the variance of overnight return $\mathrm{Var}(L_{2i+1} - L_{2i})$ are estimated empirically. Then initial δ_1 and δ_2 [1] can be obtained by solving equation (6) and (11).

Step 2:

The conditional densities of $L(t)$ are

$$f(L_{2i} | L_{2i-1}; \theta_L, \hat{\sigma}_1) = \frac{1}{\hat{\sigma}_1 \sqrt{2\pi}} \exp\left(-\frac{(L_{2i} - L_{2i-1} e^{-\theta_L \delta_1})^2}{2\hat{\sigma}_1^2} \right)$$

[1] This initial approximation turns out to be pretty close. On average, the approximates are only off by 0.3% from the final converged values.

$$f(L_{2i+1}|L_{2i};\theta_L,\hat{\sigma}_2) = \frac{1}{\hat{\sigma}_2\sqrt{2\pi}}\exp\left(-\frac{(L_{2i+1}-L_{2i}e^{-\theta_L\delta_2})^2}{2\hat{\sigma}_2^2}\right)$$

where

$$\hat{\sigma}_1 = \sigma_L\sqrt{\frac{1-e^{-2\theta_L\delta_1}}{2\theta_L}}, \qquad \hat{\sigma}_2 = \sigma_L\sqrt{\frac{1-e^{-2\theta_L\delta_2}}{2\theta_L}}$$

The log-likelihood function of $(L_1,\ldots,L_{2N})$ is then

$$\begin{aligned}\mathcal{L}(\theta_L,\sigma_L) =&\mathcal{L}(\theta_L,\hat{\sigma}_1,\hat{\sigma}_2) = \sum_{i=1}^{N}\ln f(L_{2i}|L_{2i-1};\theta_L,\hat{\sigma}_1) + \sum_{i=1}^{N-1}\ln f(L_{2i+1}|L_{2i};\theta_L,\hat{\sigma}_2)\\ =& -\frac{N}{2}\ln(2\pi) - N\ln(\hat{\sigma}_1) - \frac{1}{2\hat{\sigma}_1^2}\sum_{i=1}^{N}(L_{2i}-L_{2i-1}e^{-\theta_L\delta_1})^2\\ &-\frac{N-1}{2}\ln(2\pi) - (N-1)\ln(\hat{\sigma}_2) - \frac{1}{2\hat{\sigma}_2^2}\sum_{i=1}^{N-1}(L_{2i+1}-L_{2i}e^{-\theta_L\delta_2})^2\end{aligned}$$

The MLE for θ_L and σ_L are solved numerically from this equation using a quasi-Newton optimization algorithm called limited memory BFGS (Byrd *et al.* 1995).

Step 3 and 4 are straightforward, and from our experiments, the algorithm converges fast (generally only 2 to 4 iterations are needed to reach a tolerance of 10^{-6}).

2.2.2. **Calibration for $Y(t)$.** For $Y(t)$, the constant mean μ in the OU process is replaced by $\tilde{L}_t = \frac{L_{2i-2}+L_{2i-1}}{2}$. Equation (7) becomes

$$Y_{79(i-1)+j} = Y_{79(i-1)+j-1}e^{-\theta\delta} + \frac{L_{2i-2}+L_{2i-1}}{2}(1-e^{-\theta\delta}) + \sigma\sqrt{\frac{1-e^{-2\theta\delta}}{2\theta}}Z_{79(i-1)+j}, \quad \forall i,j$$

where $i = 1,2,\ldots,30$ denotes days; $j = 2,\ldots,79$ denotes 5-minute periods; $\delta = \frac{\delta_1}{78}$ is the effective length of a five-minute interval; $Z_{79(i-1)+j}$'s are i.i.d. $N(0,1)$. Let

$$\begin{aligned}a &= e^{-\theta\delta}\\ b_i &= \frac{L_{2i-2}+L_{2i-1}}{2}(1-e^{-\theta\delta}) = \frac{L_{2i-2}+L_{2i-1}}{2}(1-a), \qquad i = 1,\ldots,30\\ \hat{\sigma} &= \sigma\sqrt{\frac{1-e^{-2\theta\delta}}{2\theta}} = \sigma\sqrt{\frac{1-a^2}{2\theta}}\end{aligned}$$

where L_0 is defined to be 0.

Then

$$Y_{79(i-1)+j} - aY_{79(i-1)+j-1} = b_i + \hat{\sigma}Z_{79(i-1)+j}, \qquad \forall i = 1,\ldots,30, \forall j = 2,\ldots,79$$

Since $\{L_i|i = 1,\ldots,2N\}$ is a subsequence of $\{Y_i|i = 1,\ldots,79N\}$, we have

$$f(\vec{Y};\theta,\sigma) = f(\vec{L},\vec{Y}) = f(\vec{L})f(\vec{Y}|\vec{L})$$

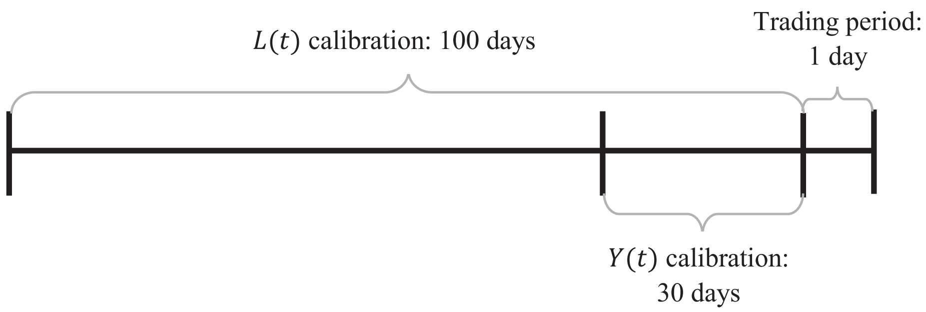


Figure 1. Illustration of formation period and trading period

The log-likelihood for $Y(t)$ is $\ln f(\vec{L}) + \ln f(\vec{Y}|\vec{L})$. The first term $\ln f(\vec{L})$ does not depend on θ and σ. We only focus on the second term

$$
\begin{aligned}
&\ln f(\vec{Y}|\vec{L}) \\
=&\ln f(\vec{Y}|Y_{79(i-1)+1} = L_{2i-1}, Y_{79i} = L_{2i}, \forall i = 1, \dots, 30) \\
=&\ln f(Y_1, Y_2, \dots, Y_{79}|Y_1 = L_1, Y_{79} = L_2) \\
&+ \ln f(Y_{80}, Y_{81}, \dots, Y_{79\times 2}|Y_{79} = L_2, Y_{80} = L_3, Y_{79\times 2} = L_4) \\
&+ \cdots\cdots \\
&+ \ln f(Y_{79\times 29+1}, \dots, Y_{79\times 30}|Y_{79\times 29} = L_{58}, Y_{79\times 29+1} = L_{59}, Y_{79\times 30} = L_{60})
\end{aligned}
\tag{12}
$$

The last equality is due to the definition of $Y_i's$. The remaining derivation of MLE formula is rather cumbersome, thus given in appendix A.

3. Trading rules

As mentioned before, training periods of 100 days and 30 days are fixed for the calibration of $L(t)$ and $Y(t)$ respectively. For an intraday trading strategy, the trading period is one day. The three periods are illustrated in figure 1.

After getting the parameters and consequently the variance estimations of both $L(t)$ and $Y(t)$, we select a set of 'best' pairs to be trading candidates for that day. Our ideal trading candidate pair will have a large $Y(t)$ variance and a small $L(t)$ variance. A large $Y(t)$ variance is preferred because more volatile intraday movements lead to more trading opportunities. The preference of small $L(t)$ variances is to ensure that the long-term value of the spread is not volatile. The most desirable situation would be $L(t)$ remaining constant over time while $Y(t)$ fluctuating a lot during the day but always coming back to the constant level.

The procedure to select the 'best' pairs is: first remove all the pairs with negative θ_L, then rank all remaining pairs by $L(t)$'s short term variance

$$\frac{\sigma_L}{2\theta_L}\left(1 - e^{-2\theta_L(\delta_1+\delta_2)}\right)$$

in *ascending* order, record the ranking r_L, then rank them again by $Y(t)$'s short term variance

$$\frac{\sigma}{2\theta}\left(1 - e^{-\frac{2\theta\delta_1}{78}}\right)$$

in *descending* order, record the ranking r_Y and finally select the top 25 or 50 or 100 pairs with smallest $r_L + r_Y$. In section 5, we test different pair selection criteria on real data by varying the number of pairs selected.

During day i, for each candidate pair stock A and stock B, we make a trade immediately when $Y(t)$, the cumulative return difference between stock A and B, goes out of a 'confidence band'. More specifically, we

(i) short the pair (simultaneously short A and long B) if $Y(t)$ exceeds $\frac{L_{2i-2}+L_{2i-1}}{2} + \epsilon$;
(ii) long the pair (simultaneously long A and short B) if $Y(t)$ drops below $\frac{L_{2i-2}+L_{2i-1}}{2} - \epsilon$.

The value ϵ is the 98% percentile of the absolute daily change in $L(t)$ values in the past 100 days. If ϵ is too large, we miss out trading opportunities by executing only few trades; if ϵ is too small, excessive trading leads to the profits of many trades being dwarfed by transaction costs. In simulations, we also used two other ϵ levels (95% and 90% percentiles) for comparison. In real data back-testing, we stick with the 98% percentile for better performances[1].

For each pair-trade, we buy \$1 worth of one stock and short \$1 worth of the other stock. For example, when we long the pair, we buy $\frac{\$1}{P_A}$ shares of stock A and simultaneously short $\frac{\$1}{P_B}$ shares of stock B, where P_A and P_B are their respective prices. Our net position at the outset of each pair-trade is zero.

The open position is closed by making the opposite trades (selling the stock bought, buying back and returning the stock shorted) when either (a) $Y(t)$ reverts back to $\frac{L_{2i-2}+L_{2i-1}}{2}$, or (b) the market closes for the day at 4pm, whichever happens first.

4. Simulation

In this section, we demonstrate the validity of our doubly mean-reverting model and the proposed trading strategy using simulation. The goal is to simulate L_i's and Y_i's for a whole year using a set of parameters $\theta_L, \sigma_L, \delta_1, \theta, \sigma$, then apply the strategy on the simulated data. For simplicity, we assume the parameters remain constant in the simulation, although we update them daily for trading on real data. If the model and strategy are well designed, the profit should be robust for a 'good' set of parameters but not for a 'bad' one.

4.1. *Simulating L_i's and Y_i's*

First we simulate L_k, $\forall k = 1, \ldots, 2N$ by equation (8) and (9) in section 2.2.1.

For each day i, define $X_{78(i-1)+j} = Y_{79(i-1)+j+1} - aY_{79(i-1)+j}$ for all $j = 1, \cdots, 78$, where $a = e^{-\theta\delta}$. Now given L_k's, we first generate $X_{78(i-1)+j}$, $j = 1, \cdots 78$ using the conditional distribution [derived in appendix equation (A1)–(A3)]:

$$f(X_{78(i-1)+1}, \ldots, Y_{78i} | Y_{79(i-1)} = L_{2(i-1)}, Y_{79(i-1)+1} = L_{2i-1}, Y_{79i} = L_{2i})$$
$$= \frac{\left(\frac{1}{\hat{\sigma}\sqrt{2\pi}}\right)^{77} \exp\left[-\frac{1}{2\hat{\sigma}^2}\left(\sum_{j=1}^{78}(x_{79(i-1)+j} - b_i)^2 + (L_{2i} - a^{78}L_{2i-1} - \sum_{j=1}^{78} a^{78-j} x_{78(i-1)+j} - b_i)^2\right)\right]}{\sqrt{\frac{1-a^2}{1-a^{156}}} \exp\left[-\frac{1}{2\hat{\sigma}^2}\frac{1-a^2}{1-a^{156}}\left(L_{2i} - \frac{1+a^{78}}{2}L_{2i-1} - \frac{1-a^{78}}{2}L_{2i-2}\right)^2\right]}$$

where b_i and $\hat{\sigma}$ are given in section 2.2.2. In fact, the above conditional density is a multivariate

[1]It is possible that the optimal threshold is not 98%. We did not experiment on real data to find out the exact optimal value. Zeng and Lee (2014) derived optimal thresholds for maximum profitability per unit of time in a single OU process model framework.

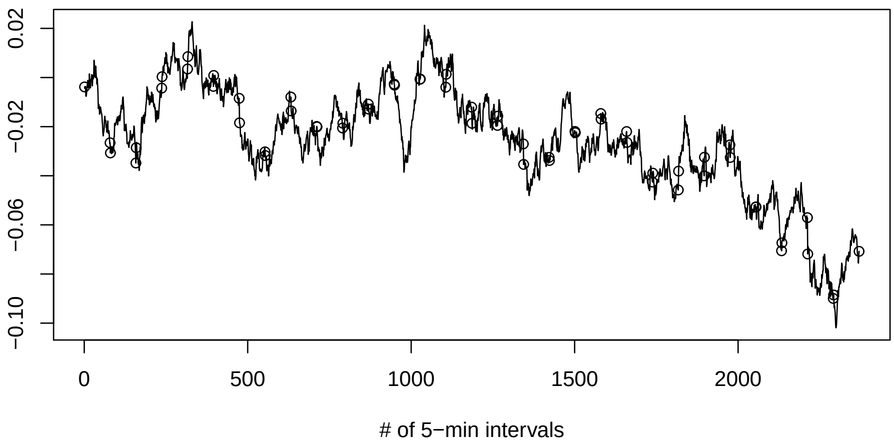


Figure 2. Simulated 30 days' Y_i's and L_i's (circles) for the good pair.

normal density with mean

$$\boldsymbol{\mu} = \begin{bmatrix} 1+a^{154} & a^{153} & \dots & a^{78} \\ a^{153} & 1+a^{152} & \dots & a^{77} \\ \vdots & \vdots & \ddots & \vdots \\ a^{78} & a^{77} & \dots & 1+a^{2} \end{bmatrix}^{-1} \begin{pmatrix} b_i + a^{77}(L_{2i} - a^{78}L_{2i-1} - b_i) \\ b_i + a^{76}(L_{2i} - a^{78}L_{2i-1} - b_i) \\ \vdots \\ b_i + a(L_{2i} - a^{78}L_{2i-1} - b_i) \end{pmatrix}$$

and variance

$$\boldsymbol{\Sigma} = \hat{\sigma}^2 = \begin{bmatrix} 1+a^{154} & a^{153} & \dots & a^{78} \\ a^{153} & 1+a^{152} & \dots & a^{77} \\ \vdots & \vdots & \ddots & \vdots \\ a^{78} & a^{77} & \dots & 1+a^{2} \end{bmatrix}^{-1}$$

Hence, we can generate multivariate normal random variables X_i's, from which Y_i's can be computed straightforwardly.

The simulated L_i's and Y_i's of one sample of 30 days are shown in figure 2 with L_i's indicated by circles.

4.2. *Choosing parameters*

Choosing particular parameters to make the variances of $L(t)$ and $Y(t)$ small and large respectively can easily results in astronomically high profits. However, the simulated spread process trajectory may simply not be achievable by real stock pairs. Therefore, to be more realistic, we use calibrated parameters $\theta_L, \sigma_L, \delta_1, \theta, \sigma$ from real data for simulation.

The dataset is described in detail in section 5. We used the ranking method described in the trading rules to select a 'good pair' and a 'bad pair' as the ones with best and worst parameters respectively in the first trading day.

The good pair's parameters are

$$\theta_L = 1.626237, \sigma_L = 0.229202, \delta_1 = 0.0032902, \theta = 123.22211, \sigma = 0.341101;$$

Table 1. Simulation Trading Results (Top half: the good pair; bottom half: the bad pair)

ϵ level	avg num of trades in 250 days	avg num/ percentage of profitable trades		avg num/ percentage of trades that reverts back to mean		avg pnl	max pnl in 400 simulations	min pnl in 400 simulations	annual Sharpe ratio	annual return	profit per trade in bp (breakeven transac. cost)
98%	29.4	22.8	77.5%	1.4	4.8%	0.230	0.610	0.028	3.325	80.2%	**78**
95%	54.5	41.4	75.9%	4.6	8.4%	0.404	0.800	0.052	4.447	180.9%	74
90%	91.9	68.1	74.1%	13.1	14.3%	0.644	1.111	0.320	**5.668**	**409.5%**	70
98%	27.5	13.4	48.8%	0.1	0.5%	-0.003	0.095	-0.063	-0.165	-0.7%	-1.0
95%	48.2	23.7	49.3%	0.6	1.3%	0.000	0.107	-0.112	-0.039	0.0%	0.0
90%	77.7	38.6	49.7%	2.6	3.4%	0.005	0.134	-0.136	0.074	1.2%	0.6

and the bad pair's parameters are

$$\theta_L = 9.328820, \sigma_L = 0.198623, \delta_1 = 0.0014689, \theta = 482.18402, \sigma = 0.205307.$$

We simulate separately using both sets of parameters and compare results.

4.3. *Simulation trading results*

Following the procedure described in section 4.1, we simulate the spread series of one year (250 days) for a large number of simulations. Each time, we apply the same strategy and calculate the profit and loss (PNL). In table 1, the trading results are shown for both the good pair (top half) and the bad pair (bottom half), each pair for three different threshold ϵ levels, each level for 400 simulations.

For the good pair, in every one of the 1200 simulations, the whole year's profit is positive. On average, there are 29/55/92 trades in the whole 250-day period, for the three ϵ levels respectively; 23/41/68 of them being profitable. The annual Sharpe ratios and annual returns are 3.325/4.447/5.668 and 80.2%/180.9%/409.5%.

Meanwhile for the bad pair, although the number of trades triggered are comparable to the good pair, the profitability is much worse. Slightly less than half the trades are winning ones compared with well over 70% for the good pair. As a result, the Sharpe ratios and annual returns are close to zero.

These simulation results validated our model by showing that a good pair identified by the model can indeed provide stable profits while a bad pair cannot.

For the good pair, as the threshold ϵ is lowered from 98% to 90% percentile, the average number of trades per simulation is more than tripled from 29 to 92. Annual Sharpe ratio and annual return increased significantly due to the larger number of trades triggered. However, both the winning percentage and the profit per trade dropped slightly, from 77.5% and 78 bps to 74.1% and 70 bps respectively. This is expected since lowering the threshold means lowering the 'standard' in identifying trading opportunities.

The results in this section are presented without transaction costs. But as will be discussed and analyzed in detail in section 5, the profits for the good pair are large enough to cover any reasonable transaction costs estimation. Note that there is only one pair in the simulation trading for simplicity, but in real data trading strategy we have at least 25 potential pairs every day.

5. Back-testing on real data

5.1. *The data*

The data for this study are from the NYSE Trade and Quote database on Wharton Research Data Services (WRDS) platform. Tick data for 26 oil company stocks during trading hours 9:30 AM to

Table 2. Oil Company Stocks Descriptions and Statistics (as of December 2014)

	NYSE ticker	mkt cap in Billion	annual revenue	avg daily volume from Google Finance (in M)	avg daily # of trades from tick data	min daily # of trades from tick data
Exxon Mobil Corporation	XOM	393.87	420,836	12.71	62977	30546
Royal Dutch Shell plc (ADR)	RDSA	215.73	451,235	2.5	10478	4281
Chevron Corporation	CVX	207.98	220,264	7.29	42279	20506
Total SA (ADR)	TOT	132.18	227,969	1.4	5773	2749
BP plc (ADR)	BP	119.95	379,136	6.78	22895	8966
ConocoPhillips	COP	82.85	56,185	8.07	34616	15317
Occidental Petroleum Corporation	OXY	61.82	24,561	6.6	28982	14690
Statoil ASA(ADR)	STO	57.53	87,781	2.85	7061	3272
Petroleo Brasileiro Petrobras SA (ADR)	PBR	57.49	141,462	48.82	65939	27431
EOG Resources Inc	EOG	49.35	14,290	6.21	24409	11296
Suncor Energy Inc. (USA)	SU	44.34	35,398	4.56	19876	10121
Anadarko Petroleum Corporation	APC	39.74	14,581	6.05	30750	10493
Phillips 66	PSX	38.83	171,596	4.71	23828	10169
Canadian Natural Resource Ltd (USA)	CNQ	35.85	14,182	4.73	16446	5345
Valero Energy Corporation	VLO	25.47	138,074	7.05	46740	20537
Marathon Petroleum Corp	MPC	25.19	100,248	3.62	26820	13113
Devon Energy Corp	DVN	24.16	10,397	4.34	22804	8650
Apache Corporation	APA	22.31	16,054	4.53	22623	7876
Hess Corp.	HES	21.87	22,247	3.75	18808	7921
Pioneer Natural Resources	PXD	20.69	3,506	2.63	14786	7100
Marathon Oil Corporation	MRO	18.88	14,959	7.75	32701	16308
Plains All American Pipeline, L.P.	PAA	18.27	42,249	1.66	6058	3627
Cenovus Energy Inc (USA)	CVE	15.60	16,389	1.85	7051	3114
Continental Resources, Inc.	CLR	13.18	3,455	3.72	10450	4899
EQT Corporation	EQT	12.91	1,862	1.82	12835	5003
Cabot Oil & Gas Corporation	COG	12.83	1,746	6.11	35182	16146

4:00 PM are downloaded, then processed to be 5-minutes time series by extracting the first tick price right after each 5-minute mark (i.e., 9:30:00, 9:35:00 etc.).

Initially, 31 stocks with the largest market capital in the Oil Refining & Marketing industry group traded on NYSE and/or NASDAQ were downloaded. After processing tick prices, we removed five stocks with too many missing data[1], all of them non-US companies. The information of the 26 stocks are shown in table 2. Notice that our trading universe comprises stocks with extremely large market caps and liquidity, compared with most studies in the pairs trading literature. All the companies have market caps over $12 billion. On average, there are 68.0 tick prices per minute for each stock. The availability of such a high frequency database is critical for this study.

We first did the back-testing on the period of January 2, 2013 to April 29, 2015 (579 business days). Then, in order to examine the performance of the strategy during market turmoil, we back-tested on an earlier period of July 2, 2007 to December 31, 2008 (374 business days). It is worth pointing out that although the data periods in this study may seem short compared with prior literature, the high-frequency nature of the data set makes it actually larger, in terms of numbers of data points per stock. 579 days with 79 data points per day are equivalent to $579 \times 79/252 = 181$ years of daily data.

For the more recent 2013–15 period, we used the 26 stocks described before. For the earlier period however, five stocks (CLR, CVE, MPC, PAA, PSX) either had not started trading or did not have enough liquidity. We used the other 21 stocks for the 2007–08 period.

5.2. *Transaction costs and return calculation*

Pairs trading strategies aim to capture stable and modest profits from market mispricing. As a result, transaction costs can have a major impact on the profitability. Bowen *et al.* (2010) found

[1]A missing data refers to the situation where there is no trade in a five-minute interval. There are only 11 total missing data points among the remaining 26 stocks in the whole analysis period. We interpolate these 11 missing points.

that a moderate level of 15 basis points (bps) transaction costs[2] would reduce the excess returns by more than 50% on one year's data of 100 UK stocks. Testing on the US equity market in the period 1963–2009, Do and Faff (2012) found that profitability of the simple algorithm from the original Gatev *et al.* (2006) paper was largely diminished after various transaction costs.

The magnitude of transaction costs depends on many factors such as the type and size of the investor (institutional vs. retail), liquidity of the particular security and the size and timing of the order. Gatev *et al.* (2006) estimated a large transaction cost of 162 bps per pair per round-trip for the period 1962–2002. However, the figure has been vastly reduced in recent years due to technology advances. Avellaneda and Lee (2010) chose a transaction cost of 10 bps per round-trip pair-trade. Bogomolov (2013) used a more conservative estimation of 40 bps per round-trip pair-trade.

Transaction costs mainly consist of commissions, bid–ask spreads, and short selling costs (Do and Faff 2012). In this paper where we consider trading a pool of highly liquid large cap US stocks from the perspective of hedge funds, commissions and short selling costs are negligible. The bid ask spread—also known as bid ask bounce, slippage, or market impact—can be estimated both directly and indirectly. Since all stocks in our investment universe are highly liquid, we used one of the lower estimates in literature as our baseline number, 10 bps per round-trip per pair-trade as in Avellaneda and Lee (2010). An alternative way to estimate bid–ask spreads is to use delayed trading as a proxy. As argued by Gatev *et al.* (2006), if a trade is made one period (one day in their case) after the divergence signal is identified, instead of immediately, the drop in return would be a rough estimate of half the round-trip transaction cost. As will be seen in detail in the next subsection, the proxy result is consistent with our selection of 10 bps.

Another tricky issue in comparing pairs trading studies is the return calculation, which warrants two considerations. The first is the leverage ratio. Pair trading, as a market neutral strategy, has a zero net initial investment (long \$1 and short \$1 for example) in theory. But it is not zero in practice. In order to short stocks, we need to put margin deposits in the brokerage account. The number of dollars of market exposure allowed for every dollar in the margin account is called the margin leverage ratio. We compute our returns as profit or loss divided by the margin, as does the literature.

Avellaneda and Lee (2010) used a 4:1 leverage, which means \$2 long and \$2 short is permitted for every dollar deposited. Gatev *et al.* (2006) defined excess return as the profit/loss for each \$1 long–\$1 short pair trade. This implied a 2:1 leverage. Large institutional investors can generally get large leverages. In this paper, we assumed a 5:1 leverage. Note that although return numbers largely depend on the leverage selection, Sharpe ratios do not, hence are more suited to be compared across studies. Annualized Sharpe ratio is calculated as

$$\text{Annualized Sharpe ratio} = \frac{\text{Annualized return}}{\text{Annualized volatility}} = \frac{E(\text{daily return}) \times 252}{sd(\text{daily return}) \times \sqrt{252}}$$

The second consideration in return calculation is return on *committed* capital versus return on *actual employed* capital (see Gatev *et al.* 2006). The former uses capital related to all the selected pairs for a day; the latter only uses capital related to those pairs that are traded in the day. Gatev *et al.* argue that '... to the extent that hedge funds are flexible in their sources and uses of funds, computing excess return relative to the actual capital employed may give a more realistic measure of the trading profits.' When discussing results in the next subsection, we refer to the return on actual employed capital (but we displayed both types of returns and corresponding Sharpe ratios in tables 3–6).

To illustrate all the above points, consider an example where we selected 20 pairs each trading

[2]It was unclear to us whether these transaction costs were per round trip or per trade.

Table 3. Jun'13–Apr'15 Trading Results for Different Pair Selections and Transaction Costs

Pair selection criteria	Trans-action cost	# of total trades	avg # of trades per day	% of winning trades	Total PNL($) after TC	Ann. Sharpe based on employed capital	Ann. return on employed capital	Ann. Sharpe based on committed capital	Ann. return on commited capital	PNL per trade in bp (breakeven TC)
top 25	10 bp	1050	2.20	53.9%	1.946	3.353	148.1%	1.256	10.4%	17.6
top 50	10 bp	1897	3.97	54.2%	3.710	3.885	187.8%	1.738	10.1%	19.6
top 75	10 bp	2694	5.64	53.2%	4.371	4.138	192.7%	1.694	7.9%	16.2
top 100	10 bp	3507	7.34	52.9%	4.879	4.078	190.8%	1.592	6.6%	13.9
top 50	**0 bp**	1897	3.97	59.7%	5.607	5.386	346.8%	2.609	15.7%	29.6
top 50	**10 bp**	1897	3.97	54.2%	3.710	3.885	187.8%	1.738	10.1%	19.6
top 50	**20 bp**	1897	3.97	49.0%	1.813	2.338	85.2%	0.850	4.7%	9.6
top 50 **wait one period**	10 bp	1897	3.97	53.1%	2.481	3.129	125.3%	1.244	6.6%	13.1

day based on calibration results. The margin deposit required for each \$1 long–\$1 short trade is

$$\frac{\text{Gross market exposure}}{\text{leverage}} = \frac{\$2}{5} = \$0.4$$

Assuming a 10 bp transaction cost per round trip pair trade, then

(a) daily return on ***committed*** capital

$$= \frac{\text{daily net PNL}}{\text{margin position}} = \frac{(\text{daily PNL}) - \$0.001 \times (\#\text{ of trades})}{\$0.4 \times (\#\text{ of } \boldsymbol{pairs})}$$

(b) daily return on ***actual employed*** capital

$$= \frac{\text{daily net PNL}}{\text{margin position}} = \frac{(\text{daily PNL}) - \$0.001 \times (\#\text{ of trades})}{\$0.4 \times (\#\text{ of } \boldsymbol{trades})}$$

Consider the following three scenarios:

(i) If in one day the daily profit is \$0.3 with 2 trades, return (a) is $\frac{0.3-0.002}{0.4\times 20} = 3.725\%$; return (b) is $\frac{0.3-0.002}{0.4\times 2} = 37.25\%$

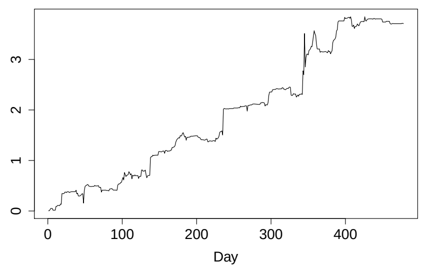
(ii) If the daily loss is -\$1.2 with 5 trades, return (a) is $\frac{-1.2-0.005}{0.4\times 20} = -15.06\%$; return (b) is $\frac{-1.2-.005}{0.4\times 5} = -60.25\%$

(iii) If no trade in one day, the daily return is 0 for both returns (a) and (b).

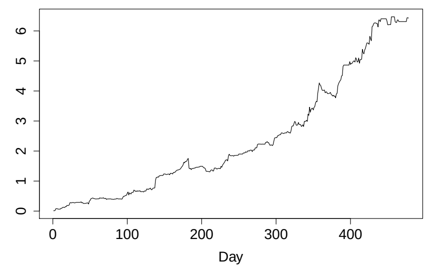
5.3. *Empirical results*

5.3.1. **June 2013–April 2015.** As described in section 3, for each day, we use the previous 100 days' data to select best pairs (i.e. the formation period is 100 days, trading period is one day). Therefore the first five months (January to May 2013) in the data is left out for calibration and the trading starts from June 2013. We report the trading results for different pair selection criteria and transaction cost levels in table 3. As in the simulation, among three threshold ϵ levels (98%, 95% and 90% percentiles), the 98% threshold yields the highest profit per trade. Thus, the results reported in this section are based on threshold $\epsilon = 98\%$.

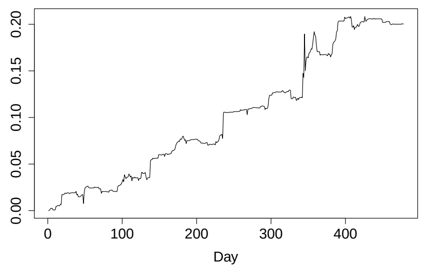
The top part of table 3 shows the results for four different pair selection criteria. As discussed, for each trading day we skipped the pairs with negative estimated θ_L and then selected top



(a) Cumulative Daily PNL



(b) Cumulative Daily Return on Actual Employed Capital



(c) Cumulative Daily Return on Committed Capital

Figure 3. Back-testing performance for period Jun'13–Apr'15 (top 50 pairs, threshold=98%, transaction costs=10 bps)

25/50/75/100 pairs as trading candidates according to their rankings.

As expected, the average number of trades per day depends on the number of pairs selected: the more pairs we select, the higher number of trades per day. The profit per trade ranges from 14 to 20 bps after deducting the 10 bps transaction costs. We selected the 'top 50' as our baseline strategy since it has the best overall performance metrics.

The middle part of table 3 shows the impact of transaction costs for the baseline 'top 50' selection method. Without transaction costs, the profit per trade is 30 bps. In other words, the break-even transaction cost is 30 bps on this dataset period. Even if we relax the estimate to a more conservative 20 bps per trade, we still have a 10 bps per trade profit and a 2.338 annualized Sharpe ratio, compared with the Sharpe ratio of 1.51 from 2003 to 2007 by Avellaneda and Lee (2010).

Finally, we rerun the baseline strategy but with the wait-one-period constraint mentioned before. Gatev *et al.* (2006) argued that when a spread is identified, it is more likely that the winner stock price is an ask price and the loser stock price is a bid price. After waiting a period, five minutes in our strategies, the prices are presumably equally likely to be bid or ask prices. Therefore the drop in PNL after waiting for five minutes as opposed to making the trade at the moment of divergence signal, would be a proxy of half the bid ask bounce—the other half happening at convergence, in the same vein. Of course, part of this drop could also be attributed to the natural mean-reversion in prices. Comparing the second and the bottom lines in table 3, the drop in profit per trade is 6.5 bps. If the drop was exclusively due to the bid–ask bounce, the proxy would be $6.5 \times 2 = 13$ bps[1], which is consistent with our direct estimation of 10 bps.

[1]Out of the 3.97 average trades per day, only 0.21 or about 5% trades converged, i.e., reverting back to $\frac{L_{2i-2}+L_{2i-1}}{2}$ level before day's close. Most other spreads were on their way toward $\frac{L_{2i-2}+L_{2i-1}}{2}$ when market closed. As stated in the trading rules, we close all pairs at market's close. Hence, the waiting-one-period proxy apply to most pairs for only the opening half of the trade.

Table 4. Jun'13-Apr'15 Trading Results by Quarter (Top 50 pairs, transaction costs = 10 bps)

Quarter	# of total trades	% of winning trades	Total PNL ($)	Annl Sharpe based on employed capital	Annl return on employed capital	Annl Sharpe based on committed capital	Annl return on commited capital	PNL per trade in bp
2013 Q3	209	53.1%	0.070	1.96	52.4%	0.39	1.3%	3
2013 Q4	277	49.5%	0.661	4.48	475.6%	2.82	14.3%	24
2014 Q1	212	55.2%	0.304	0.96	27.1%	3.06	6.5%	14
2014 Q2	188	54.8%	0.690	5.56	222.8%	2.50	14.6%	37
2014 Q3	230	53.0%	0.174	4.43	132.8%	1.34	3.5%	8
2014 Q4	500	56.8%	1.489	5.41	459.9%	2.43	34.3%	30
2015 Q1	204	50.0%	-0.007	3.91	172.0%	-0.06	-0.2%	0

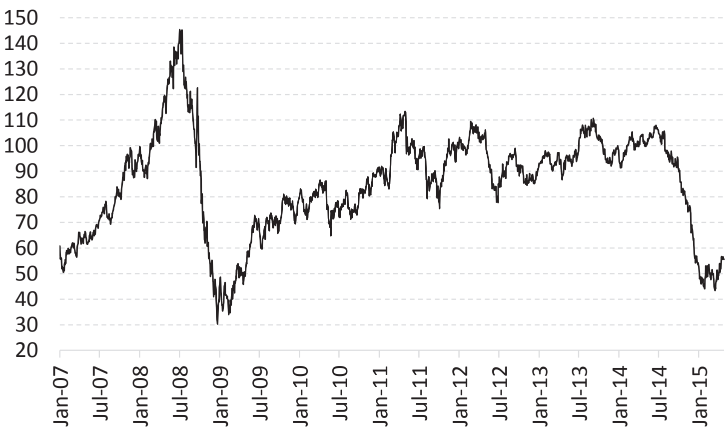


Figure 4. WTI Spot Price 2007–2015

To see the stability of the strategies over time, we plotted the PNL and returns over the trading period of almost two years, and reported the results by quarter. As seen from figure 3, both the PNL and returns increase quite stably over the period.

In table 4, quarterly results are reported for the baseline 'top 50' strategy with 10 bps transaction costs. Out of the seven quarters in the period, six are winning quarters and one breaks even.

5.3.2. **The year 2008.** As a contrarian strategy, pairs trading tends to perform better during markets downturns (Do and Faff 2010). The results during the months of recent oil market crash (Jul'14–Jan'15) show a promising performance: the average monthly PNL without transaction costs during this seven-month span is 43% higher than the whole period. To further test this hypothesis and verify our strategies, we repeated the analysis on the whole year 2008, during which the oil market spiked to an all-time high of $145 per barrel in July then crashed to $30 in December amid global financial crisis, as shown in the US crude oil benchmark index West Texas Intermediate (WTI) history price chart (figure 4).

The original 26 stocks we selected were not all available for the period July 2007–Dec 2008 (the last five months of 2007 were needed for calibration). Three (CVE, MPC, PSX) had not started trading; two (CLR, PAA) had too many missing data due to low liquidity. Therefore we used 21 stocks and $\binom{21}{2} = 210$ total pairs for this period.

The results for 2008 are presented in table 5. The average number of trades per day for the baseline 'top 50' strategy remarkably increases to 6.26, from 3.97 in the 2013–15 period. The

Table 5. 2008 Trading Results for Different Pair Selections and Transaction Costs

Pair selection criteria	Trans-action cost	# of total trades	avg # of trades per day	% of winning trades	Total PNL($) after TC	Ann. Sharpe based on employed capital	Ann. return on employed capital	Ann. Sharpe based on committed capital	Ann. return on commited capital	PNL per trade in bp (breakeven TC)
top 25	10 bp	794	3.18	60.2%	6.260	6.103	1571.5%	4.370	85.9%	78.8
top 50	10 bp	1566	6.26	60.2%	12.133	7.169	1787.6%	4.333	82.4%	77.5
top 75	10 bp	2279	9.12	60.0%	17.406	7.679	1922.5%	4.164	77.7%	76.4
top 100	10 bp	2980	11.92	60.1%	23.405	7.311	1975.3%	4.005	78.4%	78.5
top 50	**0 bp**	1566	6.26	62.7%	13.699	8.223	2938.9%	4.763	97.3%	87.5
top 50	**10 bp**	1566	6.26	60.2%	12.133	7.169	1787.6%	4.333	82.4%	77.5
top 50	**20 bp**	1566	6.26	57.0%	10.567	6.084	1071.2%	3.874	68.7%	67.5
top 50	**40 bp**	1566	6.26	51.6%	7.435	3.834	349.2%	2.867	44.2%	47.5
top 50 **wait one period**	10 bp	1566	6.26	58.0%	8.415	6.476	754.6%	4.002	51.9%	53.7

Table 6. 2008 Trading Results by Quarter (Top 50 pairs, transaction costs = 10 bps)

Quarter	# of total trades	% of winning trades	Total PNL ($)	Annl Sharpe based on employed capital	Annl return on employed capital	Annl Sharpe based on committed capital	Annl return on commited capital	PNL per trade in bp	WTI quarterly return
2008 Q1	302	62.3%	1.021	8.05	1380.8%	4.43	23.3%	34	5.2%
2008 Q2	193	51.8%	0.179	3.17	100.9%	1.56	3.6%	9	35.6%
2008 Q3	511	61.1%	4.286	8.36	3541.4%	5.25	132.3%	84	-29.5%
2008 Q4	560	61.3%	6.648	8.83	12314.0%	6.23	276.3%	119	-62.2%

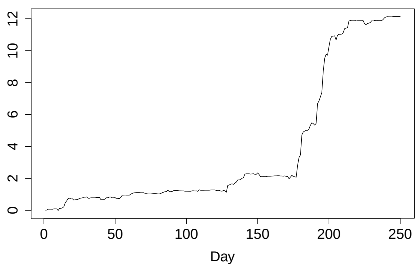
higher numbers of trading opportunities were driven by higher volatilities in stock prices[1]. The annual volatilities of all stocks in the 2013–15 period range from 11.1% to 37.7% with mean 23.1%. In 2008, they range from 31.2% to 63.0% with mean 50.3%. The higher volatilities in stock prices were driven by crude oil's volatile movements (see Geman 2015). WTI's volatilities were 27% in Jun'13–Apr'15 and 55% in 2008.

Furthermore, the strategies' returns are much larger in 2008. For the baseline 'top 50' strategy with a 10 bps transaction cost, the annualized return is 187.8% for the recent period and 1787.6% for 2008. The 7.17 Sharpe ratio of 2008 also dominates the 3.89 in 2013–15. The breakeven transaction cost is 88 bps for 2008 compared with 30 bps for 2013–15. More impressively, the quality performance in 2008 is consistent throughout the year as shown in the quarterly breakdown in table 6. In fact, the returns before transaction costs are positive in every *month*.

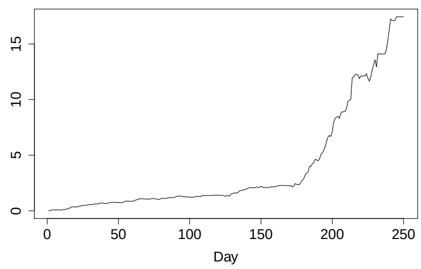
Lastly in table 5, the drop in profit per trade when we apply the wait-one-period constraint is $78-54 = 24$ bps. As in section 5.3.1, less than 13 bps of the drop is likely due to the bid–ask bounce, while the rest is presumably caused by the convergence of the spread, which is more prominent in the more volatile 2008. The fact that our strategy is still profitable in both periods after posing the wait-one-period constraint shows its robustness to the speed of execution (see Bowen *et al.* 2010).

Within 2008, the trading strategy performed extremely well in the second half of year (figure 5), coinciding with the nosedive of WTI price. This is clear from table 6: the order of the strategy performance of the four quarters is the exact reverse order of WTI's quarterly performance. The monthly returns of the baseline trading strategy (not reported) and WTI index are strongly negatively correlated, with a correlation coefficient -0.78. This figure is only -0.04 in the 2013–15 period. We identified two reasons for this significant difference. The first is again volatility: higher WTI and stock volatilities can much better translate the plummeting prices into trading profits

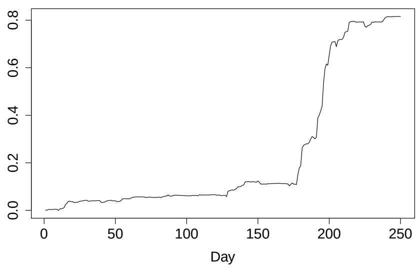
[1]We recorded the number of times each stock is selected and traded over the whole period, to see if there are any discrepancies. Not surprisingly, they have a strong relationship with stocks volatility. The correlation between a stock's volatility and number of times it being selected and traded are 88.2% and 86.6% respectively.



(a) Cumulative Daily PNL



(b) Cumulative Daily Return on Actual Employed Capital



(c) Cumulative Daily Return on Committed Capital

Figure 5. Back-testing performance for period 2008 (top 50 pairs, threshold=98%, transaction costs=10 bps)

through more trading opportunities. After all, pairs trading fundamentally relies on temporary relative mispricing of two stocks. The second reason is that the oil market crash in 2008 was more dramatic and more impactful. In 2008, the WTI index plunged almost 80% in less than six months, compared with a drop of almost 60% in seven months from mid-2014 to early 2015. The stocks in our trading universe lost 49% on average during the 2008 crash and only 29% in the recent market turmoil.

6. Conclusion

This paper introduces a doubly mean-reverting process to model stock price spreads. We developed intraday pairs trading strategies using high frequency data with five-minute intervals on oil company stocks. Results from both simulations and real data back-testing display significant realized profits. In particular, we are able to achieve a 3.9 annualized Sharpe ratio and a 188% annualized return after transaction costs for the period June 2013 to April 2015. We also tested the hypothesis that pairs trading strategies perform better in market turmoil by back-testing on 2008 data. The impressive Sharpe ratio and annualized return of 7.2 and 1788% respectively in that year underpin this theory as well as the fundamental relationship that oil company stocks are driven by crude oil price. We also showed that the strategy is robust to both speed of execution and reasonable transaction costs.

There are several possible directions for future research. First, the frequencies of the two processes may be changed. To utilize stock price data with high liquidity, intervals smaller than five minutes could be used as the frequency for $Y(t)$. On the other hand, we can increase the interval length of $L(t)$, to make the holding period longer, enabling overnight positions. Second, some details in the strategy implementation may be refined to achieve higher returns, such as the optimal thresholds to enter and exit a trade, and the training windows of 100 and 30 days. This has to be done in a careful manner to avoid over-fitting and data-snooping biases. Third, other types of data available on the NYSE Trade and Quote database can be included in the model. In particular, volume data may be used to adjust for different trading intensities throughout the day. Lastly, the model can be extended (a) from pairs trading to groups trading (also known as generalized pairs trading) with the simultaneous buying and selling of more than two stocks which co-move in some pattern and (b) from stock pairs within the Oil Refining industry group to cross-industry pairs, e.g., those in the highly related Oil Services & Equipment industry group.

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Appendix A: Derivation of likelihood function for $\boldsymbol{Y}(t)$ and the maximum likelihood estimation

The 30 summands in equation (12) are similar. Define $Y_0 = L_0 = 0$ so that all 30 terms have the form $\ln f(Y_{79(i-1)+1}, Y_{79(i-1)+2}, \ldots, Y_{79i} | Y_{79(i-1)} = L_{2(i-1)}, Y_{79(i-1)+1} = L_{2i-1}, Y_{79i} = L_{2i})$. We work on the first term for now. Since $(Y_1, Y_2, \ldots, Y_{79})$ are not jointly normal, we perform change of variables. Let

$$
\begin{aligned}
X_1 &= Y_2 - aY_1 \\
X_2 &= Y_3 - aY_2 \\
&\cdots \\
X_{77} &= Y_{78} - aY_{77} \\
X_{78} &= Y_{79} - aY_{78}
\end{aligned}
$$

Then $X_1, \ldots, X_{78}$ are i.i.d. $N(b_1, \hat{\sigma}^2)$, where

$$
\begin{aligned}
a &= e^{-\theta\delta} \\
b_i &= \frac{L_{2i-2} + L_{2i-1}}{2}(1 - e^{-\theta\delta}) = \frac{L_{2i-2} + L_{2i-1}}{2}(1 - a), \qquad i = 1, \ldots, 30 \\
\hat{\sigma} &= \sigma\sqrt{\frac{1 - e^{-2\theta\delta}}{2\theta}} = \sigma\sqrt{\frac{1 - a^2}{2\theta}}
\end{aligned}
$$

Use these 78 equations to recursively cancel out Y_2 to Y_{78}, and express Y_{79} using X's

$$
\begin{aligned}
L_2 = Y_{79} &= X_{78} + aX_{77} + a^2 X_{76} + \cdots + a^{77} X_1 + a^{78} Y_1 \\
&= X_{78} + aX_{77} + a^2 X_{76} + \cdots + a^{77} X_1 + a^{78} L_1
\end{aligned}
$$

Then, the conditional likelihood

$$
\begin{aligned}
&\ln f(X_1, X_2, \ldots, X_{78} | Y_0 = L_0, Y_1 = L_1, Y_{79} = L_2) \\
=&\ln \frac{f(x_1) \cdots f(x_{77}) f(L_2 - a^{78} L_1 - a^{77} x_1 - \cdots - a x^{77})}{f_U(L_2 - a^{78} L_1)} \\
=&\ln(*) \qquad \text{(A1)}
\end{aligned}
$$

where

$$
\begin{aligned}
U &= X_{78} + aX_{77} + a^2 X_{76} + \cdots + a^{77} X_1 \\
&\sim N\Big(b_1(1 + a + \cdots + a^{77}), \hat{\sigma}^2(1 + a^2 + \cdots + a^{154})\Big) = N\left(b_1 \frac{1 - a^{78}}{1 - a}, \hat{\sigma}^2 \frac{1 - a^{156}}{1 - a^2}\right)
\end{aligned}
$$

The f without subscript is the density for $N(b_1, \hat{\sigma}^2)$.

The numerator in $(*)$ is

$$
\left(\frac{1}{\hat{\sigma}\sqrt{2\pi}}\right)^{78} \exp\left[-\frac{1}{2\hat{\sigma}^2}\Big((x_1 - b_1)^2 + \cdots + (x_{77} - b_1)^2 + (L_2 - a^{78} L_1 - a^{77} x_1 - \cdots - a x^{77} - b_1)^2\Big)\right] \qquad \text{(A2)}
$$

The denominator in $(*)$ is

$$\begin{aligned}
&\frac{1}{\hat{\sigma}\sqrt{2\pi}}\sqrt{\frac{1-a^2}{1-a^{156}}}\exp\left[-\frac{1}{2\hat{\sigma}^2}\frac{1-a^2}{1-a^{156}}\left(L_2-a^{78}L_1-b_1\frac{1-a^{78}}{1-a}\right)^2\right]\\
=&\frac{1}{\hat{\sigma}\sqrt{2\pi}}\sqrt{\frac{1-a^2}{1-a^{156}}}\exp\left[-\frac{1}{2\hat{\sigma}^2}\frac{1-a^2}{1-a^{156}}\left(L_2-\frac{1+a^{78}}{2}L_1-\frac{1-a^{78}}{2}L_0\right)^2\right] \qquad \text{(A3)}
\end{aligned}$$

Plug equation (A2) and (A3) into (A1). Then plug (A1) and 29 other similar terms into equation (12),

$$\begin{aligned}
\mathcal{L}&=\ln f(\vec{Y}|\vec{L})=\sum_{i=1}^{30}\ln f(Y_{79(i-1)+1},\ldots,Y_{79i}|Y_{79(i-1)}=L_{2(i-1)},Y_{79(i-1)+1}=L_{2i-1},Y_{79i}=L_{2i})\\
&=\sum_{i=1}^{30}\left\{\begin{array}{c}-77\ln(\hat{\sigma})-\frac{77}{2}\ln(2\pi)-\frac{1}{2\hat{\sigma}^2}\Big[(x_{78(i-1)+1}-b_i)^2+\cdots\\
+(x_{78i-1}-b_i)^2+(L_{2i}-a^{78}L_{2i-1}-a^{77}x_{79(i-1)+1}-\cdots-ax_{79i-1}-b_i)^2\Big]\\
+\frac{1}{2}\ln\left(\frac{1-a^{156}}{1-a^2}\right)+\frac{1}{2\hat{\sigma}^2}\frac{1-a^2}{1-a^{156}}\left(L_{2i}-\frac{1+a^{78}}{2}L_{2i-1}-\frac{1-a^{78}}{2}L_{2i-2}\right)^2\end{array}\right\}\\
&\text{(plug in } \hat{\sigma})\\
&=\sum_{i=1}^{30}\left\{\begin{array}{c}-77\ln(\sigma)-\frac{77}{2}\ln\left(\frac{1-a^2}{\theta}\right)-\frac{77}{2}\ln(\pi)-\frac{\theta}{\sigma^2(1-a^2)}\left[\sum_{j=1}^{78}(x_{78(i-1)+j}-b_i)^2\right]\\
+\frac{1}{2}\ln\left(\frac{1-a^{156}}{1-a^2}\right)+\frac{\theta}{\sigma^2(1-a^{156})}\left(L_{2i}-\frac{1+a^{78}}{2}L_{2i-1}-\frac{1-a^{78}}{2}L_{2i-2}\right)^2\end{array}\right\}\\
&\text{(plug in } b_{i-1} \text{ and } x)\\
&=\sum_{i=1}^{30}\left\{\begin{array}{c}-77\ln(\sigma)-\frac{77}{2}\ln\left(\frac{1-a^2}{\theta}\right)-\frac{77}{2}\ln(\pi)-\frac{\theta}{\sigma^2(1-a^2)}(**)\\
+\frac{1}{2}\ln\left(\frac{1-a^{156}}{1-a^2}\right)+\frac{\theta}{\sigma^2(1-a^{156})}\left(L_{2i}-\frac{1+a^{78}}{2}L_{2i-1}-\frac{1-a^{78}}{2}L_{2i-2}\right)^2\end{array}\right\} \qquad \text{(A4)}
\end{aligned}$$

where

$$\begin{aligned}
(**)=&\sum_{j=1}^{78}\left(Y_{79(i-1)+j+1}-aY_{79(i-1)+j}+(a-1)\frac{L_{2i-2}+L_{2i-1}}{2}\right)^2\\
=&\sum_{j=1}^{78}Y^2_{79(i-1)+j+1}+a^2\sum_{j=1}^{78}Y^2_{79(i-1)+j}+78(a-1)^2\frac{(L_{2i-2}+L_{2i-1})^2}{4}\\
&+(a-1)(L_{2i-2}+L_{2i-1})\sum_{j=1}^{78}Y_{79(i-1)+j+1}-a(a-1)(L_{2i-2}+L_{2i-1})\sum_{j=1}^{78}Y_{79(i-1)+j}\\
&-2a\sum_{j=1}^{78}Y_{79(i-1)+j+1}Y_{79(i-1)+j} \qquad \text{(A5)}
\end{aligned}$$

Plug (A5) into (A4) and expand the summations,

$$\begin{aligned}\mathcal{L}(\theta,\sigma) = & -77\times 30\ln(\sigma) - \frac{77\times 30}{2}\ln\left(\frac{1-a^2}{\theta}\right) - \frac{77\times 30}{2}\ln(\pi) + \frac{30}{2}\ln\left(\frac{1-a^{156}}{1-a^2}\right) \\ & - \frac{\theta}{\sigma^2(1-a^2)}(Aa^2+Ba+C) + \frac{\theta}{\sigma^2(1-a^{156})}(Da^{156}+Ea^{78}+F)\end{aligned} \tag{A6}$$

where

$$A = \sum_{i=1}^{30}\left[\sum_{j=1}^{78} Y_{79(i-1)+j}^2 + 78\frac{(L_{2i-2}+L_{2i-1})^2}{4} - (L_{2i-2}+L_{2i-1})\sum_{j=1}^{78} Y_{79(i-1)+j}\right]$$

$$\begin{aligned}B = \sum_{i=1}^{30}\Bigg[& -156\frac{(L_{2i-2}+L_{2i-1})^2}{4} + (L_{2i-2}+L_{2i-1})\sum_{j=1}^{78} Y_{79(i-1)+j+1} \\ & +(L_{2i-2}+L_{2i-1})\sum_{j=1}^{78} Y_{79(i-1)+j} - 2\sum_{j=1}^{78} Y_{79(i-1)+j+1}Y_{79(i-1)+j}\Bigg]\end{aligned}$$

$$C = \sum_{i=1}^{30}\left[\sum_{j=1}^{78} Y_{79(i-1)+j+1}^2 + 78\frac{(L_{2i-2}+L_{2i-1})^2}{4} - (L_{2i-2}+L_{2i-1})\sum_{j=1}^{78} Y_{79(i-1)+j+1}\right]$$

$$D = \sum_{i=1}^{30}\left[\frac{1}{4}L_{2i-1}^2 + \frac{1}{4}L_{2i-2}^2 - \frac{1}{2}L_{2i-1}L_{2i-2}\right]$$

$$E = \sum_{i=1}^{30}\left[\frac{1}{2}L_{2i-1}^2 - \frac{1}{2}L_{2i-2}^2 - L_{2i}L_{2i-1} + L_{2i}L_{2i-2}\right]$$

$$F = \sum_{i=1}^{30}\left[L_{2i}^2 + \frac{1}{4}L_{2i-1}^2 + \frac{1}{4}L_{2i-2}^2 - L_{2i}L_{2i-1} - L_{2i}L_{2i-2} + \frac{1}{2}L_{2i-1}L_{2i-2}\right]$$

In the expression of $\mathcal{L}$ in (A6), the two parameters are θ and σ. A, B, C, D, E and F are functions of L_i and Y_i; $a = e^{-\theta\delta}$ contains parameter θ; $\delta = \frac{\delta_1}{78}$ is the discretization step size.

Setting first order derivatives of $\mathcal{L}$ with respect to σ to zero

$$\frac{\partial\mathcal{L}}{\partial\sigma} = -77\times 30\frac{1}{\sigma} - \frac{2}{\sigma^3}\left[\frac{\theta}{1-a^{156}}(Da^{156}+Ea^{78}+F) - \frac{\theta}{1-a^2}(Aa^2+Ba+C)\right] = 0 \tag{A7}$$

The optimal pair (θ,σ) that maximizes $\mathcal{L}$ satisfies $\frac{\partial\mathcal{L}}{\partial\sigma} = 0$. From (A7), we can express σ using θ

$$\sigma(\theta) = \sqrt{\frac{2}{77\times 30}\left[\frac{\theta}{1-a^2}(Aa^2+Ba+C) - \frac{\theta}{1-a^{156}}(Da^{156}+Ea^{78}+F)\right]} \tag{A8}$$

Plug (A8) into (A6), $\mathcal{L}(\theta,\sigma)$ becomes $\mathcal{L}^*(\theta)$:

$$\mathcal{L}^*(\theta) = -77\times 30\ln\big(\sigma(\theta)\big) - \frac{77\times 30}{2}\ln\left(\frac{1-a^2}{\theta}\right) - \frac{77\times 30}{2}\ln(\pi) + \frac{30}{2}\ln\left(\frac{1-a^{156}}{1-a^2}\right) - \frac{77\times 30}{2}$$

The maximal solution for $\mathcal{L}(\theta,\sigma)$ is the maximal solution for $\mathcal{L}^*(\theta)$, which is solved numerically.