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Chimere S. Anabanti and Sarah B. Hart

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Locally maximal product-free sets of size 3

Chimere S. Anabanti* Sarah B. Hart
c.anabanti@mail.bbk.ac.uk s.hart@bbk.ac.uk

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Abstract

Let G be a group, and S a non-empty subset of G. Then S is *product-free* if $ab \notin S$ for all $a, b \in S$. We say S is *locally maximal product-free* if S is product-free and not properly contained in any other product-free set. A natural question is what is the smallest possible size of a locally maximal product-free set in G. The groups containing locally maximal product-free sets of sizes 1 and 2 were classified in [3]. In this paper, we prove a conjecture of Giudici and Hart in [3] by showing that if S is a locally maximal product-free set of size 3 in a group G, then $|G| \leq 24$. This shows that the list of known locally maximal product-free sets given in [3] is complete.

1 Introduction

Let G be a group, and S a non-empty subset of G. Then S is *product-free* if $ab \notin S$ for all $a, b \in S$. For example, if H is a subgroup of G then Hg is a product-free set for any $g \notin H$. Traditionally these sets have been studied in abelian groups, and have therefore been called sum-free sets. Since we are working with arbitrary groups it makes more sense to say 'product-free' in this context. We say S is *locally maximal product-free* if S is product-free and not properly contained in any other product-free set. We use the term *locally maximal* rather than maximal because the majority of the literature in this area uses *maximal* to mean maximal by cardinality (for example [7, 8]).

There are some obvious questions from the definition: given a group G, what is the maximum cardinality of a product-free set in G, and what are the maximal (by cardinality) product-free sets? How many product-free sets are there in G? Given that each product-free set is contained in a locally maximal product-free set, what are the locally maximal product-free sets? What are the possible sizes of locally maximal product-free sets? The question of maximal (by cardinality) product-free sets has been fully solved for abelian groups by Green and Rusza [5]. For the nonabelian case Kedlaya [6] showed that there exists a constant c such

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that the largest product-free set in a group of order n has size at least $cn^{11/14}$. Gowers [4] proved that if the smallest nontrivial representation of G is of dimension k then the largest product-free set in G has size at most $k^{-1/3}n$ (Theorem 3.3 and commentary at the start of Section 5). Much less is known about the minimum sizes of locally maximal product-free sets. This question was first asked in [1] where the authors ask what is the minimum size of a locally maximal product-free set in a group of order n? A good bound is still not known. Small locally-maximal product-free sets when G is an elementary abelian 2-group are of interest in finite geometry, because they correspond to complete caps in $\mathrm{PG}(n-1,2)$. In [3], the groups containing locally maximal product-free sets of sizes 1 and 2 were classified. Some general results were also obtained. Furthermore, there was a classification (Theorem 5.6) of groups containing locally maximal product-free sets S of size 3 for which not every subset of size 2 in S generates $\langle S\rangle$. Each of these groups has order at most 24. Conjecture 5.7 of [3] was that if G is a group of order greater than 24, then G does not contain a locally maximal product-free set of size 3. Table 5 listed all the locally maximal product-free sets in groups of orders up to 24. So the conjecture asserts that this list is the complete list of all such sets. We have reproduced Table 5 as Table 1 in this paper because we need to use it in some of the arguments here. The main result of this paper is the following and its immediate corollary.

Theorem 1.1. *Suppose S is a locally maximal product-free set of size 3 in a group G, such that every two element subset of S generates $\langle S\rangle$. Then $|G| \leq 24$.*

Corollary 1.2. *If a group G contains a locally maximal product-free set S of size 3, then $|G| \leq 24$ and the only possibilities for G and S are listed in Table 1.*

Proof. If not every two-element subset of S generates $\langle S\rangle$, then by Theorem 5.6 of [3], $|G| \leq 24$. We may therefore assume that every two-element subset of S generates $\langle S\rangle$. Then $|G| \leq 24$ by Theorem 1.1. Now Table 1 is just Table 5 of [3]; it is a list of all locally maximal product-free sets of size 3 occurring in groups of order up to 24 (in fact, up to 37 in the original paper). Since we have shown that all locally maximal product-free sets of size 3 occur in groups of order up to 24, this table now constitutes a complete list of possibilities. $\square$

We finish this section by establishing the notation to be used in the rest of the paper, and giving some basic results from [3]. For subsets A, B of a group G, we use the standard notation AB for the product of A and B. That is,

$$AB = \{ab : a \in A, b \in B\}.$$

By definition, a nonempty set $S \subseteq G$ is product-free if and only if $S \cap SS = \varnothing$. In order to investigate locally maximal product-free sets, we introduce some further notations. For a set $S \subseteq G$, we define the following sets:

$$\begin{aligned} S^2 &= \{a^2 : a \in S\}; \\ S^{-1} &= \{a^{-1} : a \in S\}; \\ \sqrt{S} &= \{x \in G : x^2 \in S\}; \\ T(S) &= S \cup SS \cup SS^{-1} \cup S^{-1}S; \\ \hat{S} &= \{s \in S : \sqrt{\{s\}} \not\subseteq \langle S \rangle\}. \end{aligned}$$

For a singleton set $\{a\}$, we usually write $\sqrt{a}$ instead of $\sqrt{\{a\}}$.

For a positive integer n, we will denote by $\mathrm{Alt}(n)$ the alternating group of degree n, by C_n the cyclic group of order n, by D_{2n} the dihedral group of order $2n$, and by Q_{4n} the dicyclic group of order $4n$ given by $Q_{4n} := \langle x, y : x^{2n} = 1, x^n = y^2, yx = x^{-1}y \rangle$.

We finish this section with a few results from [3].

Lemma 1.3. *[3, Lemma 3.1] Suppose S is a product-free set in the group G. Then S is locally maximal product-free if and only if $G = T(S) \cup \sqrt{S}$.*

The next result lists, in order, Proposition 3.2, Theorem 3.4, Propositions 3.6, 3.7, 3.8 and Corollary 3.10 of [3].

Theorem 1.4. *Let S be a locally maximal product-free set in a group G. Then*

(i) $\langle S \rangle$ is a normal subgroup of G and $G/\langle S \rangle$ is either trivial or an elementary abelian 2-group;

(ii) $|G| \leq 2|T(S)| \cdot |\langle S \rangle|$;

(iii) if $\langle S \rangle$ is not an elementary abelian 2-group and $|\hat{S}| = 1$, then $|G| = 2|\langle S \rangle|$;

(iv) every element s of $\hat{S}$ has even order, and all odd powers of s lie in S;

(v) if there exists $s \in S$ and integers $m_1, \ldots, m_t$ such that $\hat{S} = \{s, s^{m_1}, \ldots, s^{m_t}\}$, then $|G|$ divides $4|\langle S \rangle|$;

(vi) if $S \cap S^{-1} = \varnothing$, then $|G| \leq 4|S|^2 + 1$.

We require one final result.

Theorem 1.5. *[3, Theorem 5.1] Up to isomorphism, the only instances of locally maximal product-free sets S of size 3 of a group G where $|G| \leq 37$ are given in Table 1.*

2 Proof of Theorem 1.1

Proposition 2.1. *Suppose S is locally maximal product-free of size 3 in G. If $\langle S\rangle$ is cyclic, then $|G| \leq 24$.*

Proof. Write $S = \{a, b, c\}$. First note that since $\langle S\rangle$ is abelian, $SS^{-1} = S^{-1}S$; moreover $aa^{-1} = bb^{-1} = cc^{-1} = 1$; so $|SS^{-1}| \leq 7$. Also $SS \subseteq \{a^2, b^2, c^2, ab, ac, bc\}$. Thus

$$|T(S)| = |S \cup SS \cup SS^{-1}| \leq 3 + 6 + 7 = 16.$$

By Lemma 1.3, $G = T(S) \cup \sqrt{S}$; so $\langle S\rangle = T(S) \cup (\langle S\rangle \cap \sqrt{S})$. Elements of cyclic groups have at most two square roots. Therefore $|\langle S\rangle| \leq 16 + 6 = 22$. By Table 1, $\langle S\rangle$ must now be one of C_6, C_8, C_9, C_{10}, C_{11}, C_{12}, C_{13} or C_{15}. Theorem 1.4(iv) tells us that every element s of $\hat{S}$ has even order and all odd powers of s lie in S. This means that for C_9, C_{11}, C_{13} or C_{15}, we have $\hat{S} = \varnothing$ and so $G = \langle S\rangle$. In particular, $|G| \leq 24$.

It remains to consider C_6, C_8, C_{10} and C_{12}. For $C_6 = \langle g : g^6 = 1\rangle$, the unique locally maximal product-free set of size 3 is $S = \{g, g^3, g^5\}$. Now if g or g^5 is contained in $\hat{S}$, then $\hat{S}$ consists of powers of a single element; so by Theorem 1.4(v), $|G|$ divides 24. If neither g nor g^5 is in $\hat{S}$, then $|\hat{S}| \leq 1$, and so by Theorem 1.4(iii) therefore, $|G|$ divides 12. In C_8 there is a unique (up to group automorphisms) locally maximal product-free set of size 3, and it is $\{g, g^{-1}, g^4\}$, where g is any element of order 8. If $\hat{S}$ contains g or g^{-1}, then S contains all odd powers of that element by Theorem 1.4(iv), and hence S contains $\{g, g^3, g^5, g^7\}$, a contradiction. Therefore $|\hat{S}| \leq 1$ and so $|G|$ divides 16. Next, we consider $\langle S\rangle = C_{10}$. Recall that elements of $\hat{S}$ must have even order. If $\hat{S}$ contains any element of order 10, then S contains all five odd powers of this element, which is impossible by Theorem 1.4(iv). This leaves only the involution of C_{10} as a possible element of $\hat{S}$. Hence again $|\hat{S}| \leq 1$ and $|G|$ divides 20. Finally we look at C_{12}. If $\hat{S}$ contains any element of order 12, then $|S| \geq 6$, a contradiction. If $\hat{S}$ contains an element x of order 6 then S contains all three of its odd powers, so $S = \{x, x^3, x^5\}$. But then $\langle S\rangle \cong C_6$, contradicting the assumption that $\langle S\rangle = C_{12}$. Therefore, $\hat{S}$ can only contain elements of order 2 or 4. Up to group automorphism, we see from Table 1 that every locally maximal product-free set S of size 3 in C_{12} with $\langle S\rangle = C_{12}$ is one of $\{g, g^6, g^{10}\}$ or $\{g, g^3, g^8\}$ for some generator g of C_{12}. Each of these sets contains exactly one element of order 2 or 4. Therefore in every case, $|\hat{S}| \leq 1$ and so $|G|$ divides 24. This completes the proof. $\square$

Note that the bound on $|G|$ in Proposition 2.1 is attainable. For example in Q_{24} there is a locally maximal product-free set S of size 3, with $\langle S\rangle \cong C_{12}$.

Proposition 2.2. *Suppose S is locally maximal product-free of size 3 in G such that every 2-element subset of S generates $\langle S\rangle$. Then either $|G| \leq 24$ or S contains exactly one involution.*

Proof. First suppose S contains no involutions. If $S \cap S^{-1} = \varnothing$, then Theorem 1.4(vi) tells us that G has order at most 37, and then by Theorem 1.5, (G, S) is one of the possibilities listed in Table 1. In particular $|G| \leq 24$. If $S \cap S^{-1} \neq \varnothing$, then $S = \{a, a^{-1}, b\}$ for some a, b.

But then $\langle S\rangle = \langle a, a^{-1}\rangle = \langle a\rangle$, so $\langle S\rangle$ is cyclic. Now by Proposition 2.1 we get $|G| \leq 24$. Next, suppose that S contains at least two involutions, a and b, with the third element being c. Then, since every 2-element subset of S generates $\langle S\rangle$, we have that $H = \langle S\rangle = \langle a, b\rangle$ is dihedral and S is locally maximal product-free in H. Let $o(ab) = m$, so $H \cong D_{2m}$. The non-trivial coset of the subgroup $\langle ab\rangle$ is product-free of size m. So if c lies in this coset, then we have $m = 3$ and $H \cong D_6$. If c does not lie in this coset then $c = (ab)^i$ for some i, and from the relations in a dihedral group $ac^{-1} = ca$, $c^{-1}a = ac$, $bc^{-1} = cb$ and $c^{-1}b = bc$. The coset $\langle ab\rangle a$ consists of m involutions, which cannot lie in $\sqrt{S}$. Thus $\langle ab\rangle a \subseteq T(S)$ by Lemma 1.3. A straightforward calculation shows that

$$\begin{aligned}\langle ab\rangle a = T(S) \cap \langle ab\rangle a &= \{a, b, ac, ca, bc, cb, ac^{-1}, c^{-1}a, bc^{-1}, c^{-1}b\}\\ &= \{a, b, ac, ca, bc, cb\}\end{aligned}$$

This means $m \leq 6$, and S consists of two generating involutions a, b plus a power of their product ab, with the property that any two-element subset of S generates $\langle a, b\rangle$. A glance at Table 1 shows there are no locally maximal product-free sets of this form in D_{2m} for $m \leq 6$. Therefore the only possibility is that $\langle S\rangle \cong D_6$, with S consisting of the three reflections in $\langle S\rangle$. By Theorem 1.4(i), the index of $\langle S\rangle$ in G is a power of 2. By Theorem 1.4(ii), $|G| \leq 2|T(S)| \cdot |\langle S\rangle|$. Thus $|G| \in \{6, 12, 24, 48\}$. Suppose for contradiction that $|G| = 48$. Now $G = T(S) \cup \sqrt{S}$, and since S consists of involutions, the elements of $\sqrt{S}$ have order 4. So G contains two elements of order 3, three elements of order 2 and the remaining non-identity elements have order 4. Then the 46 elements of G whose order is a power of 2 must lie in three Sylow 2-subgroups of order 16, with trivial pairwise intersection. Each of these groups therefore has a unique involution and 14 elements of order 4, all of which square to the given involution. But no group of order 16 has fourteen elements of order 4. Hence $|G| \neq 48$, and so $|G| \leq 24$. Therefore either $|G| \leq 24$ or G contains exactly one involution. □

Before we establish the next result, we first make a useful observation. Suppose $S = \{a, b, c\}$ where $a, b, c \in G$ and c is an involution. Then a straightforward calculation shows that

$$T(S) \subseteq \left\{ \begin{array}{c} 1, a, b, c, a^2, b^2, ab, ba, ac, ca, bc, cb, \\ ab^{-1}, ba^{-1}, ca^{-1}, cb^{-1}, a^{-1}b, a^{-1}c, b^{-1}a, b^{-1}c \end{array} \right\}. \qquad (1)$$

Lemma 2.3. *Suppose S is a locally maximal product-free set of size 3 in G, every 2-element subset of S generates $\langle S\rangle$, and S contains exactly one involution. Then either $|G| \leq 24$ or $S = \{a, b, c\}$, where a and b have order 3 and c is an involution.*

Proof. Suppose $S = \{a, b, c\}$ where c is an involution and a, b are not. Consider a^{-1}. Recall that $G = T(S) \cup \sqrt{S}$. If $a^{-1} \in \sqrt{S}$ then $a^{-2} \in \{a, b, c\}$ which implies that either a has order 3 or $\langle S\rangle$ is cyclic (because for example if $a^{-2} = b$ then $\langle S\rangle = \langle a, b\rangle = \langle a\rangle$). Thus if $a^{-1} \in \sqrt{S}$ implies that either a has order 3 or (by Lemma 2.1) $|G| \leq 24$. Suppose then that $a^{-1} \in T(S)$. The elements of $T(S)$ are given in Equation 1. If $a^{-1} \in \{b, b^2, ab, ba, ab^{-1}, ba^{-1}, a^{-1}b, b^{-1}a\}$ then by remembering that $\langle S\rangle = \langle a, b\rangle$, we deduce that $\langle S\rangle$ is cyclic, generated by either a or b. For example, $a^{-1} = ba$ implies $b \in \langle a\rangle$. Similarly, if $a^{-1} \in \{c, ac, ca, a^{-1}c, c^{-1}a\}$, then $\langle S\rangle$ is cyclic. Since a has order at least 3, we cannot have $a^{-1} \in \{1, a\}$. If $a^{-1} \in \{bc, cb, b^{-1}c, c^{-1}b\}$, then S would not be product-free. For instance $a^{-1} = b^{-1}c$ implies that $b^{-1}ca = 1$, and hence $ac = b$. The only remaining possibility is $a^{-1} = a^2$, meaning that a has order 3. The same argument with b^{-1} shows that b also has order 3. □

We can now prove Theorem 1.1, which states that if S is a locally maximal product-free set of size 3 in a group G, such that every two element subset of S generates $\langle S\rangle$, then $|G| \leq 24$.

Proof of Theorem 1.1 Suppose S is a locally maximal product-free set of size 3 in G such that every two element subset of S generates $\langle S\rangle$. Then by Lemma 2.3, either $|G| \leq 24$ or $S = \{a, b, c\}$ where a and b have order 3 and c is an involution. In the latter case, we observe that aca^{-1} is an involution, so must be contained in $T(S)$. Using Equation 1 we work through the possibilities. Obviously it is impossible for aca^{-1} to be equal to any of $1, a, b, a^2$ or b^2 because these elements are not of order 2. If any of $ac, ca, a^{-1}c, c^{-1}a, bc, cb, b^{-1}c$ or cb^{-1} were involutions, then it would imply that $\langle S\rangle$ was generated by two involutions whose product has order 3. For example if ac were an involution then $\langle c, ac\rangle = \langle a, c\rangle = \langle S\rangle$. That is, $\langle S\rangle$ would be dihedral of order 6. But there is no product-free set in D_6 containing two elements of order 3, because if x, y are the elements of order 3 in D_6 then $x^2 = y$ and $y^2 = x$. So the remaining possibilities for aca^{-1} are $c, ab, ba, ab^{-1}, ba^{-1}, a^{-1}b$ and $b^{-1}a$. Now $aca^{-1} = ab$ implies $c = ba$, whereas $aca^{-1} = ab^{-1}$ implies $bc = a$ and $aca^{-1} = ba^{-1}$ implies $b = ac$, each of which contradicts the fact that S is product-free. We are now left with the cases $aca^{-1} = c$, $aca^{-1} = ba$ and $aca^{-1} = a^{-1}b$ (which, if it is an involution, equals $b^{-1}a$). If $aca^{-1} = c$, then $\langle S\rangle = \langle a, c\rangle = C_6$, but the only product-free set of size 3 in C_6 contains no elements of order 3, so this is impossible. Therefore $aca^{-1} \in \{ba, a^{-1}b\}$. If $aca^{-1} = ba$, then $a^{-1}ba = ca^{-1}$, so $ac = a^{-1}b^{-1}a$, which has order 3. If $aca^{-1} = a^{-1}b$, then $ac = a^{-1}ba$, again of order 3. So we see that

$$\langle S\rangle = \langle a, c : a^3 = 1, c^2 = 1, (ac)^3 = 1\rangle.$$

This is a well known presentation of the alternating group Alt(4). As c is the only element of S whose order is even, we see that $|\hat{S}| \leq 1$, and hence $|G| \leq 2|\mathrm{Alt}(4)| = 24$. Therefore in all cases $|G| \leq 24$. □

3 Data and Programs

Though Table 1 is essentially just Table 5 from [3], we have taken the opportunity here to correct a typographical error in the entry for the (un-named) group of order 16. We provide below the GAP programs used to obtain the table.

Program 3.1. *A program that tests if a set T is product-free.*

```
## It returns "0" if T is product-free, and "1" if otherwise.
prodtest:= function(T)
local  x, y, prod;
prod:=0;
```

```
for x in T do
    for y in T do
      if x*y in T then
         prod:=1;
      fi;
    od;
od;
return prod;
end;
```

Program 3.2. *A program for finding all locally maximal product-free sets of size* 3 *in* G.

```
##It prints the list of all locally maximal product-free sets of size 3 in G.
LMPFS3:=function(G)
local L, lmpf, combs, x, pf, H, y, z, s, i, q;
L:=AsSortedList(G); lmpf:=[]; combs:=Combinations(L,3);
for i in [1..Binomial(Size(L),3)] do
  pf:=combs[i];
  if prodtest(pf)=0 then
   s:=Size(lmpf); H:=Difference(L,pf);
   for y in [1..3] do
     for z in [1..3] do
       H:=Difference(H, [pf[y]*pf[z], pf[y]*(pf[z])^-1, ((pf[y])^-1)*pf[z]]);
     od;
   od;
   for q in L do
       if q^2 in pf then
          H:=Difference(H, [q]);
       fi;
   od;
   if Size(H) = 0 then
      lmpf:=Union(lmpf, [pf]);
   fi;
  fi;
od;
if Size(lmpf) > 0 then
  Print(G,"\n",L,"\n","Structure Description of G is ",StructureDescription(G),
  "\n", "Gap Id of G is ", IdGroup(G), "\n", "\n", lmpf, "\n", "\n");
fi;
end;
```

G		S	$\langle S\rangle$	# Locally maximal product-free sets of size 3 in G
$\langle g : g^6 = 1\rangle$	$\cong C_6$	$\{g, g^3, g^5\}$	$\cong C_6$	1
$\langle g, h : g^3 = h^2 = 1, hgh = g^{-1}\rangle$	$\cong D_6$	$\{h, gh, g^2h\}$	$\cong D_6$	1
$\langle g : g^8 = 1\rangle$	$\cong C_8$	$\{g, g^{-1}, g^4\}$	$\cong C_8$	2
$\langle g, h : g^4 = h^2 = 1, hgh^{-1} = g^{-1}\rangle$	$\cong D_8$	$\{h, gh, g^2\}$	$\cong D_8$	4
$\langle g : g^9 = 1\rangle$	$\cong C_9$	$\{g, g^3, g^8\}, \{g, g^4, g^7\}$	$\cong C_9$	8
$\langle g, h : g^3 = h^3 = 1, gh = hg\rangle$	$\cong C_3 \times C_3$	$\{g, h, g^2h^2\}$	$\cong C_3 \times C_3$	8
$\langle g : g^{10} = 1\rangle$	$\cong C_{10}$	$\{g^2, g^5, g^8\}, \{g, g^5, g^8\}$	$\cong C_{10}$	6
$\langle g : g^{11} = 1\rangle$	$\cong C_{11}$	$\{g, g^3, g^5\}$	$\cong C_{11}$	10
$\langle g : g^{12} = 1\rangle$	$\cong C_{12}$	$\{g^2, g^6, g^{10}\}$	$\cong C_6$	1
		$\{g, g^6, g^{10}\}, \{g, g^3, g^8\}$	$\cong C_{12}$	8
$\langle g, h : g^6 = 1, g^3 = h^2, hgh^{-1} = g^{-1}\rangle$	$\cong Q_{12}$	$\{g, g^3, g^5\}$	$\cong C_6$	1
Alternating group of degree 4	$=$ Alt(4)	$\{x, y, z : x^2 = y^2 = z^3 = 1\}$	$\cong$ Alt(4)	48
		$\{x, z, xzx : x^2 = z^3 = 1\}$		
		$\{x, z, zxz : x^2 = z^3 = 1\}$		
$\langle g : g^{13} = 1\rangle$	$\cong C_{13}$	$\{g, g^3, g^9\}, \{g, g^6, g^{10}\}$	$\cong C_{13}$	16
$\langle g : g^{15} = 1\rangle$	$\cong C_{15}$	$\{g, g^3, g^{11}\}$	$\cong C_{15}$	4
$\langle g, h : g^4 = h^4 = 1, gh = hg\rangle$	$\cong C_4 \times C_4$	$\{g, h, g^{-1}h^{-1}\}$	$\cong C_4 \times C_4$	16
$\langle g, h : g^8 = 1, g^4 = h^2, hgh^{-1} = g^{-1}\rangle$	$\cong Q_{16}$	$\{g, g^4, g^{-1}\}$	$\cong C_8$	2
$\langle g, h : g^8 = h^2 = 1, hgh^{-1} = g^5\rangle$	(order 16)	$\{g, g^6, g^3h\}$	$\cong G$	8
$\langle g, h : g^{10} = 1, g^5 = h^2, hgh^{-1} = g^{-1}\rangle$	$\cong Q_{20}$	$\{g, g^5, g^8\}, \{g^2, g^5, g^8\}$	$\cong C_{10}$	6
$\langle g, h : g^3 = h^7 = 1, ghg^{-1} = h^2\rangle$	$\cong C_7 \rtimes C_3$	$\{gh, gh^{-1}, g^{-1}\}$	$\cong C_7 \rtimes C_3$	42
$\langle x : x^3 = 1\rangle \times \langle g, h : g^4 = 1, g^2 = h^2, hgh^{-1} = g^{-1}\rangle$	$\cong C_3 \times Q_8$	$\{g^2, xg^2, x^2g^2\}$	$\cong C_6$	1
$\langle g, h : g^{12} = 1, g^6 = h^2, hgh^{-1} = g^{-1}\rangle$	$\cong Q_{24}$	$\{g^2, g^6, g^{10}\}$	$\cong C_6$	1
		$\{g, g^6, g^{10}\}$	$\cong C_{12}$	4

Table 1: Locally maximal product-free sets of size 3 in groups of order up to 24

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