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Hart, Sarah (2017) How many elements of a Coxeter group have a unique reduced expression? Journal of Group Theory 20 (5), pp. 903-910. ISSN 1435-4446.

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How many elements of a Coxeter group have a unique reduced expression?

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Abstract

Let (W, R) be an arbitrary Coxeter system. We determine the number of elements of W that have a unique reduced expression.

1 Introduction

Given a Coxeter group W with distinguished generating set R, every element w of W may be written as a word in R. A reduced expression for w is one of minimal length. There are usually several different reduced expressions for any given element. There are results that enable us, in special cases, to count the number of reduced expressions for elements. For example Stanley [6] gave an algorithm to enumerate the number of reduced expressions for elements of the symmetric group. Eriksson [1] gave a recursive method for elements of affine Weyl groups. Stembridge investigated the reduced expressions for so-called fully commutative elements [7]. It seems fairly natural to ask about elements that have a unique reduced expression. In this short article we show how to determine very quickly from the Coxeter graph of an arbitrary Coxeter group W the number of elements that have a unique reduced expression. Partial results in this direction are known. For the case of finite Coxeter groups these elements form a 2-sided Kazhdan-Lusztig cell, studied in [3]. Enumeration of elements with a unique reduced expression for finite Coxeter groups follows from Proposition 4 of that paper and the examples that follow it.

To state our main result we recall some well-known notation. For more detail on this and other aspects of Coxeter groups see, for example, [2]. A Coxeter system (W, R) is a group W with a generating set R such that $W = \langle R \mid (rs)^{m_{rs}} = 1; r, s \in R\rangle$, where $m_{rr} = 1$ for all $r \in R$, and $m_{rs} = m_{sr}$. That is, m_{rs} is the order of rs. In particular, the elements of R are involutions. We write $m_{rs} = \infty$ where rs has infinite order. A nice way to represent this information is via a Coxeter graph: this is an undirected labelled graph Γ with vertex set R, where distinct r, s in R are joined by an edge labelled m_{rs} whenever $m_{rs} \geq 3$. (Usually by convention the label 3 is omitted.) We say that Γ is *simply laced* if every edge label is 3. Once the generating set R is fixed, then Γ is uniquely determined, and in what follows we

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will assume this has happened. Our technique for counting elements with unique reduced expression relies on an analysis of the Coxeter graph. We remark that other kinds of elements can be counted using the Coxeter graph, such as in the elegant paper by Shi [5] using the Coxeter graph to enumerate Coxeter elements.

Definition 1.1. Let Γ be a Coxeter graph with associated Coxeter group W. We define $U(\Gamma)$ to be the number of *non-identity* elements of W with a unique reduced expression.

It will turn out (Lemma 2.2) that it is quick to reduce the work to the irreducible case (that is, where the Coxeter graph is connected). We will therefore summarise here our results for the irreducible case.

Theorem 1.2. *Suppose Γ is a connected Coxeter graph with n vertices.*

1. *If Γ is a simply laced tree, then $U(\Gamma) = n^2$.*

2. *Suppose that Γ is a tree with no infinite bonds and exactly one edge with a label m greater than three. Let a and b be the orders of the two induced subgraphs obtained by removal of this edge (so $a + b = n$). Then*

$$U(\Gamma) = \begin{cases} \frac{1}{2}mn^2 - 2ab & \text{if } m \text{ even;} \\ \\ \frac{1}{2}(m-1)n^2 & \text{if } m \text{ odd.} \end{cases}$$

3. *If Γ is any other connected Coxeter graph then $U(\Gamma) = \infty$.*

In Section 2 we prove the main results. In Section 3 we give a few example calculations. We finish this section with a final piece of notation. For distinct elements r and s of R, we write $[rs]^n$ for the (not necessarily reduced) expression with n terms beginning with r and alternating $rsrs\cdots$. So, for example $[rs]^5 = rsrsr$. An *elementary operation* on a word consists of replacing $[rs]^{m_{rs}}$ with $[sr]^{m_{rs}}$. It is well known [4] that any two reduced expressions for an element w of a Coxeter group can be obtained from one another by a sequence of elementary operations.

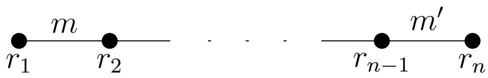
2 Main Results

In this section we first give in Theorem 2.1 necessary conditions for $U(\Gamma)$ to be finite. I am grateful to Nathan Reading for reading an earlier version of this paper and pointing out to me that Theorem 2.1 can be deduced from the first part of the proof of [7, Theorem 5.1]. I have included my proof here for the reader's convenience. These necessary conditions will turn out also to be sufficient conditions. For each case not eliminated by Theorem 2.1 we then find an expression for $U(\Gamma)$, in particular showing that $U(\Gamma)$ is finite.

Theorem 2.1. *Let Γ be the Coxeter graph of W. Suppose W has finitely many elements with a unique reduced expression. Then Γ is finite and each connected component of Γ is a tree with no infinite bonds and at most one edge label greater than three.*

Proof. Clearly Γ is finite, otherwise R would constitute an infinite set of elements of W each having a unique reduced expression. If $m_{rs} = \infty$ for some $r, s \in R$, then $(rs)^k$ has a unique reduced expression for all positive integers k. If Γ contains a cycle then for some n with $n \geq 3$ there are $r_1, \ldots, r_n$ in R for which $m_{r_i r_{i+1}} \geq 3$ when $i < n$ and also $m_{r_n r_1} \geq 3$. Now $(r_1 \cdots r_n)^k$ has a unique reduced expression for all positive integers k. This is because any two reduced expressions for a given element can be obtained from each other by a sequence of elementary operations and clearly no elementary operations are possible in this element. We assume from now on that Γ has no cycles and no infinite bonds.

Suppose Γ contains a subgraph of the following form, where $m \geq m' \geq 4$.



Let $w = r_1 r_2 \cdots r_{n-2} r_{n-1} r_n r_{n-1} r_{n-2} \cdots r_2$. In w^k, for $k \geq 1$, the only expressions $[rs]^i$ for any i greater than 2 are $[r_{n-1} r_n]^3$ and $[r_2 r_1]^3$. However as $m_{r_{n-1} r_n} = m'$ and $m_{r_1 r_2} = m$, and both of these are greater than 3, we see that no elementary operations are possible. Hence the powers of w provide infinitely many elements with a unique reduced expression. We conclude that if W only has finitely many elements with a unique reduced expression, then Γ is a forest each of whose connected components is a tree with no infinite bonds and at most one edge label greater than three. $\square$

The next result allows us to reduce to the case of irreducible Coxeter groups.

Lemma 2.2. *Suppose W is a Coxeter group with Coxeter graph Γ. If $\Gamma = \Gamma_1 \cup \Gamma_2$, where there is no edge connecting any vertex of Γ_1 with any vertex of Γ_2, then $U(\Gamma) = U(\Gamma_1) + U(\Gamma_2)$.*

Proof. We have that W is isomorphic to the direct product $W_1 \times W_2$, where W_1 and W_2 have Coxeter graphs Γ_1 and Γ_2 respectively. Suppose a non-identity element w of W has a unique reduced expression. We can write w canonically as $w = w_1 w_2$, where $w_1 \in W_1$, $w_2 \in W_2$ and at least one of w_1 and w_2 is not the identity. Moreover, $w_1 w_2 = w_2 w_1$. Since there is a unique reduced expression for w, this implies that either $w_1 = 1$ or $w_2 = 1$. If $w_2 = 1$, then $w_1 \neq 1$ and w_1 must have a unique reduced expression in W_1. There are $U(\Gamma_1)$ such elements. Similarly if $w_1 = 1$, then w_2 is one of the $U(\Gamma_2)$ non-identity elements of W_2 with a unique reduced expression. Hence $U(\Gamma) = U(\Gamma_1) + U(\Gamma_2)$. $\square$

Note that in Lemma 2.2 we have not required Γ_1 and Γ_2 to be connected. A simple induction argument therefore shows that when W is a Coxeter group with Coxeter graph Γ having connected components $\Gamma_1, \ldots, \Gamma_n$, then $U(\Gamma) = \sum_{i=1}^{n} U(\Gamma_i)$. We may therefore restrict our attention to the case when W is irreducible, which is equivalent to Γ being connected. By Theorem 2.1 we can assume Γ is a finite tree with no infinite bonds and at most one edge label m being greater than 3.

For the next result, recall that a *path* in a graph is an ordered sequence of vertices connected by edges in which no vertex appears more than once. We use the notation $r_1 \to r_2 \to \cdots \to r_k$ for a path where $r_1, \ldots, r_k$ are the vertices of the path and there is an edge joining r_i and r_{i+1} for each i in $\{1, \ldots, k-1\}$. The *length* of the path is $k-1$.

Lemma 2.3. *A tree of order n contains precisely $n(n-1)$ paths of length at least 1, and n^2 paths in total.*

Proof. In a tree there is a unique path from any vertex to any other vertex (otherwise there would be cycles). Therefore there are precisely $n(n-1)$ paths of length at least 1. Adding the n paths of length 0 (each consisting of a single vertex) we see that there are n^2 paths in total. $\square$

Proposition 2.4. *Suppose Γ is a simply laced tree with n vertices for some positive integer n. Then $U(\Gamma) = n^2$. In particular, $U(\Gamma)$ is finite.*

Proof. Suppose w is a non-identity element with a unique reduced expression $r_1 \cdots r_k$ for some (not necessarily distinct) $r_i \in R$. If any r_i commutes with r_{i+1}, for $1 \leq i < k$, then w would have another reduced expression $r_1 \cdots r_{i-1} r_{i+1} r_i r_{i+2} \cdots r_k$. Hence $m_{r_i r_{i+1}} = 3$ for all i. Suppose $r_i = r_j$ for some $i < j$, and let us assume $|j - i|$ is minimal such that this occurs. That is, $r_i, r_{i+1}, \ldots r_{j-1}$ are all distinct elements of R. Obviously $j = i+1$ is impossible as this is a reduced expression. If $j = i+2$ then we have $r_i r_{i+1} r_i$ as a subexpression of w. But Γ is simply laced, meaning $r_i r_{r+1} r_i = r_{i+1} r_i r_{i+1}$, contradicting the uniqueness of the reduced expression for w. Therefore $j > i+2$. But then the vertices $r_i, r_{i+1}, \ldots, r_{j-1}$ form a cycle of Γ, contradicting the fact that Γ is a tree. Therefore in fact the r_i are all distinct. This means that w corresponds to the unique path from r_1 to r_k in Γ, and Γ_w, the induced subgraph whose vertex set is $\{r_1, \ldots, r_k\}$, is the following.

We have shown that every non-identity element w with a unique expression corresponds to a unique path of Γ. Conversely, suppose $r_1 \to r_2 \to \cdots \to r_k$ is a path of Γ. Let $g = r_1 r_2 \cdots r_k$. Since each vertex in the path is distinct, no elementary operations replacing rsr with srs in g are possible. Moreover, adjacent vertices in the path are by definition joined by an edge. Thus for all i in $\{1, \ldots, k-1\}$ we know that $r_i r_{i+1} \neq r_{i+1} r_i$. These means that no elementary operations replacing rs with sr are possible in g. Hence g has a unique reduced expression. Thus each path gives rise to a non-identity element with a unique reduced expression. Therefore $U(\Gamma)$ is equal to the number of paths in Γ. Hence, by Lemma 2.3, $U(\Gamma) = n^2$. $\square$

Theorem 2.5. *Suppose Γ is a tree with n vertices, no infinite bonds and exactly one edge with a label m greater than three. Let a and b be the orders of the two induced subgraphs obtained by removal of this edge (so $a + b = n$). Then*

$$U(\Gamma) = \begin{cases} \frac{1}{2}mn^2 - 2ab & \text{if } m \text{ even;} \\ \frac{1}{2}(m-1)n^2 & \text{if } m \text{ odd.} \end{cases}$$

Proof. Let r and s be the vertices of Γ which are joined by the edge labelled m. Consider the subgraph induced by removing the edge m. Let Δ be the connected component containing r and Σ be the connected component containing s. Both Δ and Σ are simply laced finite trees. Set $a = |\Delta|$ and $b = |\Sigma|$. Let w be a non-identity element of W that has a unique reduced expression, and let Γ_w be the subgraph of Γ induced by the elements of R contained in the expression for W. If Γ_w does not contain the edge labelled m, then Γ_w is contained in $\Delta \cup \Sigma$. By Proposition 2.4 and Lemma 2.2 there are $a^2 + b^2$ elements of this kind.

We now consider elements w for which Γ_w does contain the edge labelled m. In particular w must contain the subexpression rs or sr at least once. Writing $w = r_1 \cdots r_k$ for some $r_i \in R$, we observe that r_i and r_{i+1} are distinct for each i in $\{1, \ldots, k-1\}$ (otherwise the expression would not be reduced) and moreover there is an edge between r_i and r_{i+1} in Γ_w, otherwise r_i would commute with r_{i+1}, implying the existence of a second reduced expression. Therefore Γ_w is connected. Suppose that there are i, j with $1 \leq i < j \leq k$ such that $\{r_i, r_j\} \subseteq \{r, s\}$, and $\{r_{i+1}, \ldots, r_{j-1}\} \cap \{r, s\} = \emptyset$. Let $u = r_i r_{i+1} \cdots r_{j-1}$. Then u is an element with a unique reduced expression; moreover Γ_u does not contain the edge labelled m. Thus $r_i \to r_{i+1} \to \cdots \to r_{j-1}$ is a path and in particular the elements $r_i, r_{i+1}, \ldots, r_{j-1}$ are all distinct. But r_{j-1} is adjacent in Γ_w to r_j which is either r or s. Either way, it implies that there is a cycle in Γ, a contradiction. Suppose for the moment that r appears before s in w. Then w is of the form $r_1 \cdots r_i [rs]^L s_1 \cdots s_j$ where $\{r_1, \ldots, r_i, s_1, \ldots, s_j\} \subseteq R \setminus \{r, s\}$. To preserve the uniqueness of the expression, we must have $L < m$, and to ensure that Γ_w contains the edge labelled m, we also know that $L \geq 2$. Moreover $r_1 \to \cdots \to r_i \to r$ is a path in Δ. If L is even then $s \to s_1 \to \cdots \to s_j$ is a path in Σ. If L is odd then $r \to s_1 \to \cdots \to s_j$ is a path in Δ.

Suppose that L is even. Then $r_1 \to \cdots \to r_i \to r \to s \to s_1 \to \cdots \to s_j$ is a path between an element of Δ and an element of Σ and each such w (for fixed L) results in exactly one such path. The reverse path will arise from w^{-1}, which is an element where s appears before r. Therefore for each even L lying between 2 and $m - 1$, each of the ab paths from elements of Δ to elements of Σ results in exactly two elements having unique reduced expressions (one where r appears before s, one, for the reverse path, where s appears before r). Therefore there are $2ab\lfloor \frac{m-1}{2} \rfloor$ such elements.

Now suppose that L is odd, and for the moment that r appears before s in the expression for w. This means r also appears before s in the expression for w^{-1}. We have that $r_1 \to \cdots \to r_i \to r$ and $s_j \to \cdots s_1 \to r$ are paths in Δ. Since there is a unique path from any given vertex in Δ to r, the number of such elements w (for fixed odd L) is simply the number of ways of choosing r_1 and s_j (which need not be distinct). Therefore there are a^2 such elements w. Similarly for each odd L there are b^2 elements w where s appears before r. Summing over the odd L between 2 and $m-1$ we get $(a^2+b^2)\lfloor \frac{m-2}{2} \rfloor$ elements.

Combining the calculations for L even and L odd, we see that the total number of elements w with a unique reduced expression such that Γ_w contains the edge labelled m is

$$2ab \left\lfloor \frac{m-1}{2} \right\rfloor + (a^2+b^2) \left\lfloor \frac{m-2}{2} \right\rfloor .$$

To obtain $U(\Gamma)$ we must add to this the a^2+b^2 non-identity elements w for which Γ_w does not contain the edge labelled m. If m is even then, recalling that $a+b=n$, we get

$$U(\Gamma) = a^2+b^2+\tfrac{1}{2}(2ab+a^2+b^2)(m-2) = n^2-2ab+\tfrac{1}{2}n^2(m-2) = \tfrac{1}{2}mn^2-2ab.$$

If m is odd then we get

$$\begin{aligned} U(\Gamma) &= a^2+b^2+\tfrac{1}{2}\left((2ab)(m-1)+(a^2+b^2)(m-3)\right) \\ &= a^2+b^2+2ab+\tfrac{1}{2}(a^2+b^2+2ab)(m-3) \\ &= \tfrac{1}{2}(m-1)n^2. \end{aligned}$$

□

Theorem 2.1, Proposition 2.4 and Theorem 2.5 combine to give Theorem 1.2, along with the following corollary, which classifies the Coxeter groups having finitely many elements with a unique reduced expression.

Corollary 2.6. *Let Γ be the Coxeter graph of W. Then W has finitely many elements with a unique reduced expression if and only if Γ is finite and each connected component of Γ is a tree with no infinite bonds and at most one edge label greater than three.*

3 Examples

In this section we give some example calculations. Proposition 2.4 deals with all Coxeter groups whose graphs are simply laced trees Γ: in each case there are $|\Gamma|^2$ non-identity elements with a unique reduced expression. So for example there are 64 such elements in the Coxeter groups of types A_8, D_8 and E_8. For a group of type B_n we have $m=4$, $a=1$ and $b=n-1$. So by Theorem 2.5 there are $2n^2-2(n-1)$ elements with a unique reduced expression, which is $2(n^2-n+1)$. In B_4 there are 26 such elements, for example. In F_4 we have $a=b=2$ and $U(F_4)=24$. Below is a table listing $U(\Gamma)$ for each irreducible finite and affine Coxeter group.

Γ	$U(\Gamma)$	Γ	$U(\Gamma)$
$A_n(n \geq 1)$	n^2	$\tilde{A}_n(n \geq 1)$	∞
$B_n(n \geq 2)$	$2(n^2 - n + 1)$	$\tilde{B}_n(n \geq 3)$	$2(n^2 + n + 1)$
$D_n(n \geq 4)$	n^2	$\tilde{C}_n(n \geq 2)$	∞
E_6	36	$\tilde{D}_n(n \geq 4)$	$(n+1)^2$
E_7	49	$\tilde{E}_6$	49
E_8	64	$\tilde{E}_7$	64
F_4	24	$\tilde{E}_8$	81
$I_2(m)(m \geq 6)$	$2(m-1)$	$\tilde{F}_4$	38
H_3	18	$\tilde{G}_2$	23
H_4	32		

References

[1] K. Eriksson, Reduced words in affine Coxeter groups, Discrete Math. 157 (1–3) (1996) 127–146.

[2] J.E. Humphreys, Reflection Groups and Coxeter Groups, Cambridge Stud. Adv. Math., vol. 29, 1992.

[3] Tobias Kildetoft, Marco Mackaay, Volodymyr Mazorchuk, Jakob Zimmermann, *Simple transitive 2-representations of small quotients of Soergel bimodules*, arXiv 1605.01373 (2016).

[4] H. Matsumoto, *Générateurs et relations des groupes de Weyl généralisés*, C. R. Acad. Sci. Paris, 258:3419–3422, 1964.

[5] Jian-Yi Shi, *The enumeration of Coxeter elements*, J. Algebraic Combin. 6 (1997), no. 2, 161–171.

[6] R.P. Stanley, *On the number of reduced decompositions of elements of Coxeter groups*, European J. Combin. 5 (4) (1984) 359–372.

[7] J. R. Stembridge, *On the fully commutative elements of Coxeter groups*, J. Algebraic Combin. 5 (1996), 353-385.