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On the growth of Kronecker coefficients (extended abstract)

Emmanuel Briand[*1], Amarpreet Rattan[†2], and Mercedes Rosas[‡3§]

[1]*Departamento Matemática Aplicada I, Universidad de Sevilla, Spain,*
[2]*Department of Economics, Mathematics and Statistics, Birkbeck, University of London, UK,*
[3]*Departamento de Álgebra, Universidad de Sevilla, Spain.*

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Abstract. We present a new stability phenomenon for Kronecker coefficients, that we call *hook stability*: the Kronecker coefficients stabilize if we add cells to the first row and first column of each of the indexing partitions, simultaneously. We also show that when we increase the sizes of the first two rows of their three indexing partitions, in some appropriate way, the Kronecker coefficients grow linearly, and we are able to give asymptotic estimates.

Résumé. Nous présentons une nouvelle propriété de stabilité des coefficients de Kronecker, que nous appelons *Stabilité équerre*: les coefficients de Kronecker se stabilisent lorsqu'on ajoute des cases dans la première ligne et première colonne des diagrammes de chacune de leurs trois partitions, en même temps. Nous montrons aussi que lorsqu'on augmente les deux premières parts des trois partitions, d'une façon contrôlée, alors les coefficients de Kronecker croissent linéairement. Nous réussissons à donner des estimations asymptotiques dans ce cas.

Keywords: Kronecker coefficients, Representation Stability.

1 Introduction.

The Kronecker coefficients $g_{\lambda,\mu,\nu}$ are fundamental constants in representation theory. They describe how irreducible representations of $GL(V \otimes W)$ split when viewed as representations of $GL(V) \times GL(W)$. They are also the structural constants for the tensor products of irreducible representations of the symmetric groups.

In spite of their importance, little is known about the Kronecker coefficients, leaving some fundamental questions unanswered. For example, no combinatorial description

[*] `ebriand@us.es`
[†] `a.rattan@bbk.ac.uk`
[‡] `mrosas@us.es`
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akin to the Littlewood–Richardson rule is known for the Kronecker coefficients. Another important question is to determine how difficult it is, algorithmically, to compute Kronecker coefficients, or to merely determine which are nonzero.

A feature of the Kronecker coefficients that has been studied recently is the *stability* phenomena: the fact that some sequences of Kronecker coefficients are eventually constant. The first example of such a behavior was observed by Murnaghan [13] in 1938. The Kronecker coefficients $g_{\lambda,\mu,\nu}$ are indexed by triples of partitions (λ, μ, ν), and Murnaghan's stable sequences are obtained by incrementing the first part of all three partitions at each step. Their limit values (the *reduced*, or *stable* Kronecker coefficients) are interesting objects in their own right.

Many more sequences of Kronecker coefficients are stable. Large families have been produced by means of methods from geometry [10, 11], enumerative combinatorics [19, 20], or recently by symmetric functions calculations [14]. These stable sequences of Kronecker coefficients have general terms of the form $g_{\lambda+n\alpha,\mu+n\beta,\nu+n\gamma}$, where Murnaghan's case corresponds to $\alpha = \beta = \gamma = (1)$. The sequences of Kronecker coefficients of the form $g_{\lambda+n\alpha,\mu+n\beta,\nu+n\gamma}$ that are stable have been completely characterized recently in [16, 17].

In this paper, we present two new results related to the stability of Kronecker coefficients.

The first one (Section 3) is indeed a result of stability, but the sequences that we consider are not of the type $g_{\lambda+n\alpha,\mu+n\beta,\nu+n\gamma}$. At each step, we simultaneously increase the first row and the first column of the Young diagrams of all three indexing partitions. We call this phenomenon *hook stability*. This hook stability does not seem to fit straightforwardly in the representation theory of fixed general linear groups, since it involves sequences of Kronecker coefficients indexed by partitions with unbounded lengths.

The second result (Section 4) concerns the *asymptotics* of some sequences of Kronecker coefficients of type $g_{\lambda+n\alpha,\mu+n\beta,\nu+n\gamma}$ that do not stabilize, but grow linearly.

We describe the relevant coefficients (the limits for hook stability, and the coefficients appearing in quasi-polynomial formulas for the asymptotic estimates, for the result on linear growth) by means of generating series (Section 6).

All these results are derived from a simple factorization of a formal series of symmetric functions (Section 5), obtained by computations in the framework of the λ–ring formalism for symmetric functions, and involving vertex operators. The two sets of results in this paper, hook stability and linear growth, are obtained by first considering these properties for families of reduced Kronecker coefficients, and then translating them to Kronecker coefficients.

Most of the proofs of the results in this extended abstract are skipped or merely sketched; the full proofs are given in the complete version [2]. Full bibliographical references can also be found there.

2 Preliminaries.

Partitions. We will use the following notation for integer partitions. The weight $|\lambda|$ of the partition λ is the sum of its parts. The conjugate of a partition λ is denoted by λ'. Let $\cup$ and $+$ be the standard operations on partitions, as defined in [9, I.§1]. If n is a nonnegative integer and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a partition, then $n\lambda$ is the dilation of λ by a factor n and has parts $(n\lambda_1, n\lambda_2, \ldots, n\lambda_k)$. Let $\overline{\lambda}$ be the partition obtained by removing the first term of λ. Let $\widehat{\lambda}$ be the partition obtained after removing the first row and the first column in the diagram of λ. The sequence defined by prepending a first term a to the partition λ will be denoted (a, λ). The resulting sequence $(a, \lambda_1, \lambda_2, \ldots)$ is not necessarily a partition since we may have that $a < \lambda_1$. Finally, for any non–empty partition λ, we will write $\lambda \oplus (a|b)$ for the partition $\lambda + (a) \cup (1^b)$.

For example, if $\lambda = (8, 3, 3, 1)$, then we have that $\overline{\lambda} = (3, 3, 1)$, $\widehat{\lambda} = (2, 2)$, $(5, \lambda) = (5, 8, 3, 3, 1)$ (not a partition), and $\lambda \oplus (7|4) = (15, 3, 3, 1, 1, 1, 1, 1)$.

Reduced Kronecker coefficients and Murhaghan's stability. As mentioned in the introduction, Murnaghan observed in [13] that for any triple of partitions λ, μ, ν of some positive integer n, the sequence of Kronecker coefficients $g_{\lambda+(m),\mu+(m),\nu+(m)}$ stabilizes (*i.e.* is eventually constant). Several proof have been given of this fact. The original one due to Littlewood [8].

The stable value of the sequence $g_{\lambda+(m),\mu+(m),\nu+(m)}$ does not depend on the first part of λ, μ and ν. Accordingly, it will be denoted $\overline{g}_{\overline{\lambda},\overline{\mu},\overline{\nu}}$, and called the *reduced Kronecker coefficient* (some authors called it also *stable Kronecker coefficient*).

3 Reduced Kronecker coefficients with first column increment, and hook stability for Kronecker coefficients.

In what follows, *"if $x \gg y$"* means *"there exists k such that, if $x \geq y + k \ldots$"*.

The following theorem is a new *column* stability property for reduced Kronecker coefficients.

Theorem 1. *For any triple of partitions α, β, γ, there exists an integer $\overline{\overline{g}}_{\alpha,\beta,\gamma}$ such that whenever $a \geq \ell(\alpha)$, $b \geq \ell(\beta)$, $c \geq \ell(\gamma)$, and $b + c \gg a$, $a + c \gg b$ and $a + b \gg c$, we have*

$$\overline{g}_{\alpha+(1^a),\beta+(1^b),\gamma+(1^c)} = \overline{\overline{g}}_{\alpha,\beta,\gamma}.$$

Combining the result in Theorem 1 with Murnaghan stability, we obtain that the Kronecker coefficients are stable when we increase the first row and first column of the three indexing partitions *simultaneously*. We call this property *hook stability* for Kronecker coefficients, since it appears when increasing, in some balanced way, the main hook in the three indexing partitions.

Corollary 1. *For any triple of non–empty partitions λ, μ, ν of the same weight, for all $(a,b,c,m) \in \mathbb{N}^4$ with $m \geq a,b,c$, $m \gg (a+b+c)/2$, $a+b \gg c$, $a+c \gg b$ and $b+c \gg a$, we have,*

$$g_{\lambda\oplus(m-a|a),\mu\oplus(m-b|b),\nu\oplus(m-c|c)} = \overline{g}_{\overline{\lambda}\cup(1^a),\overline{\mu}\cup(1^b),\overline{\nu}\cup(1^c)} = \overline{\overline{g}}_{\widehat{\lambda},\widehat{\mu},\widehat{\nu}}. \tag{3.1}$$

Example 1. Table 1 presents the Kronecker coefficients $g_{\lambda\oplus(i|j),\lambda\oplus(i|j),\lambda\oplus(i|j)}$ for $\lambda = (3,3)$ and i and j between 0 and 9. Each column of the table is stable because of Murnaghan's stability, Each row is eventually zero by a classical result on the vanishing of the Kronecker coefficients. The region in gray with value 145 corresponds to a "hook stable" region.

i \ j	0	1	2	3	4	5	6	7	8	9
0	0	1	5	5	1	0	0	0	0	0
1	1	8	27	40	30	11	1	0	0	0
2	1	15	53	89	91	64	33	11	1	0
3	2	19	62	108	129	122	97	64	33	11
4	2	19	63	112	138	141	135	122	97	64
5	2	19	63	112	139	145	144	141	135	122
6	2	19	63	112	139	145	145	145	144	141
7	2	19	63	112	139	145	145	145	145	145
8	2	19	63	112	139	145	145	145	145	145
9	2	19	63	112	139	145	145	145	145	145

Table 1: The Kronecker coefficients $g_{(3,3)\oplus(i|j),(3,3)\oplus(i|j),(3,3)\oplus(i|j)}$.

As a very particular case, we obtain the following corollary.

Corollary 2. *Let λ, μ and ν be non–empty partitions of the same weight. The sequence of Kronecker coefficients $g_{\lambda\oplus(n|n),\mu\oplus(n|n),\nu\oplus(n|n)}$ stabilizes to $\overline{\overline{g}}_{\widehat{\lambda},\widehat{\mu},\widehat{\nu}}$.*

4 Reduced Kronecker coefficients with first rows increment.

When increasing the first row of the three indexing partitions α, β, γ of a reduced Kronecker coefficient $\overline{g}_{\alpha,\beta,\gamma}$, *i.e.* when considering a sequence of coefficients $\overline{g}_{\alpha+(m),\beta+(m),\gamma+(m)}$, the sequence does not stabilize. But the same techniques as those used for hook stability provide a rather precise description of the asymptotic behavior of the sequence: it grows linearly with coefficients that are quasipolynomials of period at most 2.

We in fact obtain the following more general theorem, where the first parts of the three partitions do not have to be increased by exactly the same amount.

Theorem 2. *Let α, β and γ be three partitions. There exists integers $A_{\alpha,\beta,\gamma}$, $B_{\alpha,\beta,\gamma}$ and $C_{\alpha,\beta,\gamma}$, such that whenever $a \geq \alpha_1$, $b \geq \beta_1$, $c \geq \gamma_1$, $a \gg b$, $a \gg c$ and $b + c \gg a$, we have*

$$\overline{g}_{(a,\alpha),(b,\beta),(c,\gamma)} = \frac{1}{2} A_{\alpha,\beta,\gamma} \cdot (b + c - a) + B_{\alpha,\beta,\gamma} + \begin{cases} 0 & \text{if } b + c - a \text{ even,} \\ C_{\alpha,\beta,\gamma}/2 & \text{if } b + c - a \text{ odd.} \end{cases}$$

We get another related result. Let $\mathcal{C}$ be the cone of all $(a, b, c) \in \mathbb{R}^3$ such that $a + b \geq c$, $a + c \geq b$ and $b + c \geq a$.

Theorem 3. *Let λ, μ and ν be three partitions and $(a, b, c) \in \mathbb{N}^3$. Without loss of generality, we may assume that* $\max(a, b, c) = a$. *Suppose that there exists n such that $\overline{g}_{\lambda+n(a),\mu+n(b),\nu+n(c)}$ is non–zero. Then $A_{\overline{\lambda},\overline{\mu},\overline{\nu}}$ is nonzero, and if (a, b, c) is in the interior of $\mathcal{C}$, then, as $n \to \infty$,*

$$\overline{g}_{\lambda+n(a),\mu+n(b),\nu+n(c)} \sim \frac{A_{\overline{\lambda},\overline{\mu},\overline{\nu}} \cdot (b + c - a)}{2} \cdot n.$$

From known properties of Kronecker coefficients, one shows easily that if (a, b, c) is on the border of $\mathcal{C}$, then $\overline{g}_{\lambda+n(a),\mu+n(b),\nu+n(c)}$ is eventually constant. And if $(a, b, c) \notin \mathcal{C}$, then $\overline{g}_{\lambda+n(a),\mu+n(b),\nu+n(c)} = 0$ for $n \gg 0$.

From Theorem 2 and Theorem 3 are derived similar statements for Kronecker coefficients whose indexing partitions fulfill linear inequalities guaranteeing that they are equal to the corresponding reduced Kronecker coefficients; see [2].

5 Sketch of the proofs.

The proofs of the properties presented in Sections 3 and 4 are based on a factorization lemma for some formal series of symmetric functions. We introduce below some material on symmetric functions useful for stating this lemma. In particular, the generating series σ of the complete symmetric functions h_n, and the λ–ring formalism, will be useful as well in Section 6 to describe the coefficients $\overline{\overline{g}}_{\alpha,\beta,\gamma}$, $A_{\alpha,\beta,\gamma}$, $B_{\alpha,\beta,\gamma}$ and $C_{\alpha,\beta,\gamma}$ of Sections 3 and 4.

5.1 Preliminaries on symmetric functions

Symmetric functions. Let $Sym_{\mathbb{Q}} = Sym_{\mathbb{Q}}(X)$ be the algebra of symmetric functions with rational coefficients, with underlying alphabet $X = \{x_1, x_2, \ldots\}$. We denote by $\langle \, | \, \rangle_X$, or $\langle \, | \, \rangle$ when there is no ambiguity, the scalar product on $Sym_{\mathbb{Q}}$ where the Schur functions form an orthonormal basis. The scalar product is conveniently extended to symmetric series whenever it makes sense. We also consider symmetric functions in different alphabets (sets of variables) X, Y, Z. The scalar product is canonically extended to the algebras $Sym_{\mathbb{Q}}(X) \otimes_{\mathbb{Q}} Sym_{\mathbb{Q}}(Y)$ or $Sym_{\mathbb{Q}}(X) \otimes_{\mathbb{Q}} Sym_{\mathbb{Q}}(Y) \otimes_{\mathbb{Q}} Sym_{\mathbb{Q}}(Z)$, for example, and denoted by $\langle \, | \, \rangle_{X,Y}$ and $\langle \, | \, \rangle_{X,Y,Z}$.

The λ–ring formalism for symmetric functions, and specializations. Let $\mathcal{A}$ be any commutative algebra over a field $\mathcal{K}$ of characteristic zero. Given a morphism of algebras A from $Sym_{\mathbb{Q}}$ to $\mathcal{A}$, the image of a symmetric function f under A will be denoted with $f[A]$ rather than $A(f)$, and is called the "specialization of f at A".

Since the power sum symmetric functions p_k ($k \geq 1$) generate $Sym_{\mathbb{Q}}$ and are algebraically independent, the map $A \mapsto (p_1[A], p_2[A], \ldots)$ is a bijection from the set of all morphisms of algebras $Sym_{\mathbb{Q}} \to \mathcal{A}$ to the set of infinite sequences of elements from $\mathcal{A}$. This set of sequences is endowed with its operations of component-wise sum, product, and product by a scalar. The above bijection is used to lift these operations to the set of morphisms $Sym_{\mathbb{Q}} \to \mathcal{A}$. This defines expressions like $f[A+B]$, $f[-A]$, $f[AB]$, $f[A/B]$ *etc.* where f is a symmetric function and A and B are two specializations, and more general expressions $f[P(A, B, \ldots)]$, where $P(A, B, \ldots)$ is a rational function in several specializations A, B ... with coefficients in $\mathcal{K}$.

Here are some important specializations. The specialization at -1 is defined on power sums by $p_k[-1] = -1$ for all k. The specialization ε is defined by $p_k[\varepsilon] = (-1)^k$ for all k. The product of the two previous specializations is $-\varepsilon$ and fulfills $p_k[-\varepsilon X] = (-1)^{k+1} p_k[X]$ for all k. As a consequence, the transformation $f[X] \mapsto f[-\varepsilon X]$ coincides with the standard involution defined by $s_\lambda \mapsto s_{\lambda'}$ for all partitions λ. There is also the specialization $X^\perp$ such that for any symmetric function f, $f[X^\perp] = f^\perp$, the adjoint of the multiplication by f with respect to $\langle\,|\,\rangle_X$.

The series σ. Let σ be the generating function for the complete homogeneous symmetric functions in X:

$$\sigma[X] = \prod_i \frac{1}{1 - x_i} = \sum_{n \geq 0} h_n[X].$$

Let Y and Z be additional alphabets. The following identities are well known, except for the last one.

- Cauchy's identity:
$$\sigma[XY] = \prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_\lambda s_\lambda[X] s_\lambda[Y].$$
- A generating series for the Kronecker coefficients:
$$\sigma[XYZ] = \prod_{i,j,k} \frac{1}{1 - x_i y_j z_k} = \sum_{\lambda,\mu,\nu} g_{\lambda,\mu,\nu} s_\lambda[X] s_\mu[Y] s_\nu[Z]. \tag{5.1}$$
- A similar generating series for the Littlewood–Richardson coefficients:
$$\sigma[XY + XZ] = \prod_{i,j} \frac{1}{1 - x_i y_j} \frac{1}{1 - x_i z_j} = \sum_{\lambda,\mu,\nu} c^\lambda_{\mu,\nu} s_\lambda[X] s_\mu[Y] s_\nu[Z].$$

- A generating series for the reduced Kronecker coefficients, that can be derived (see [2]) from a formula due to M. Brion [3, §3.4, Corollary 1]:

$$\sigma[XYZ + XY + XZ + YZ] = \sum_{\lambda,\mu,\nu} \overline{g}_{\lambda,\mu,\nu} s_\lambda[X] s_\mu[Y] s_\nu[Z]. \tag{5.2}$$

5.2 The Factorization Lemma.

The proofs of the results presented in the paper are based on a simple factorization of a formal series. Consider six alphabets X, Y, Z, X', Y', Z', and associate to any polynomial F in three variables and to any three partitions α, β, γ the formal series

$$\Phi_{\alpha,\beta,\gamma} = \left\langle \sigma[F(X,Y,Z)] \,\middle|\, \Gamma_{(X'|X)} s_\alpha[X] \Gamma_{(Y'|Y)} s_\beta[Y] \Gamma_{(Z'|Z)} s_\gamma[Z] \right\rangle_{X,Y,Z} \tag{5.3}$$

where $\Gamma_{(X'|X)} = \sigma[X'X]\sigma\left[-\frac{1}{X'}X^\perp\right]$. The operator $\Gamma_{(X'|X)}$ is a convenient generalization of the vertex operator $\Gamma_{(t|X)} = \sigma[tX]\sigma\left[-\frac{1}{t}X^\perp\right]$(where t is a variable) that sends any Schur function s_α to the series $\sum_{n\in\mathbb{Z}} s_{(n,\alpha)} t^n$. When $\lambda = (n,\alpha)$ is not a partition, *i.e.* when n is less than the first part of α, s_λ should be interpreted as the Jacobi–Trudi determinant $\det(h_{\lambda_j+i-j})_{i,j=1\ldots n}$. The vertex operator $\Gamma_{(t|X)}$ is a classical tool in the theory of symmetric functions used, in particular, by Jing (see, for instance, [7]), and, for the study of various phenomena of stability by Thibon and his collaborators (see for instance [4, 18]). It is also the generating series for Bernstein's creation operators introduced by Zelevinsky.

The series $\Phi_{\alpha,\beta,\gamma}$ is a formal sum of products of symmetric functions in X, Y, and Z and elements of $\mathbb{L}(X') \otimes_{\mathbb{Q}} \mathbb{L}(Y') \otimes_{\mathbb{Q}} \mathbb{L}(Z')$, where $\mathbb{L}$ is the $\mathbb{Q}$-algebra obtained from $Sym_{\mathbb{Q}}$ by adjoining an inverse to each power sum p_k.

The factorization property is the following.

Lemma 1. *For any partitions α, β and γ, there exists an element $Q_{\alpha,\beta,\gamma}$ of $\mathbb{L}(X') \otimes_{\mathbb{Q}} \mathbb{L}(Y') \otimes_{\mathbb{Q}} \mathbb{L}(Z')$ such that*

$$\Phi_{\alpha,\beta,\gamma} = \sigma[F(X',Y',Z')] \cdot Q_{\alpha,\beta,\gamma}.$$

Thus, $Q_{\alpha,\beta,\gamma}$ is the coefficient of $s_\alpha[X]s_\beta[Y]s_\gamma[Z]$ in the expansion in the Schur basis of $\sigma[H]$ (as a symmetric series in X, Y and Z), where

$$H = F(X+X', Y+Y', Z+Z') - F(X',Y',Z') - X/X' - Y/Y' - Z/Z'.$$

The main point of this lemma is that $Q_{\alpha,\beta,\gamma}$ has only finitely many non–zero homogeneous components.

5.3 Specializations.

The main results of this paper are obtained by the following convenient specializations of $\Phi_{\alpha,\beta,\gamma}$. Murnaghan's stability can also be recovered in this way.

Murnaghan stability. Applying Lemma 1 with $F(X,Y,Z) = XYZ$, and specializing X' at t and Y' and Z' at 1, we get that

$$\sigma[t] \cdot P(t) = \frac{1}{1-t} \cdot P(t) = \sum_n g_{(n-|\alpha|,\alpha),(n-|\beta|,\beta),(n-|\gamma|,\gamma)} t^n + \text{ terms } t^n \text{ with small } n.$$

with $P(t)$ a Laurent polynomial. This is Murnaghan's stability. Indeed, a sequence whose generating series is a polynomial $P(t)$ times $1/(1-t)$ is eventually constant with limit value $P(1)$. Setting $t=1$, we get the limit value (which is the reduced Kronecker coefficient $\overline{g}_{\alpha,\beta,\gamma}$) as the coefficient of $s_\alpha[X]s_\beta[Y]s_\gamma[Z]$, in the expansion in the Schur basis, of $\sigma[(X+1)(Y+1)(Z+1)-1-X-Y-Z]$, which simplifies to $\sigma[XYZ+XY+XZ+YZ]$; this is the symmetric form (5.2) of Brion's Formula.

Column stability for reduced Kronecker coefficients. If we apply Lemma 1 with $F(X,Y,Z) = XYZ+XY+YZ+XZ$, and specialize all three alphabets X', Y' and Z' to $-\varepsilon x$, $-\varepsilon y$ and $-\varepsilon z$ respectively, (with x, y and z single variables, and $-\varepsilon$ the main involution on symmetric functions, that sends s_λ to $s_{\lambda'}$), we get

$$\frac{1+xyz}{(1-x)(1-y)(1-z)} Q^-_{\alpha,\beta,\gamma}(x,y,z) = \sum_{a,b,c} \overline{g}_{\alpha+(1^a),\beta+(1^b),\gamma+(1^c)} x^a y^b z^c + \text{ terms } x^a y^b z^c \text{ with small } a,\, b,\, c. \tag{5.4}$$

with $Q^-_{\alpha,\beta,\gamma}$ a Laurent polynomial. Then the column stability for reduced Kronecker coefficients is a consequence of this factorization and the fact that

$$\frac{1+xyz}{(1-x)(1-y)(1-z)} = \sum_{(a,b,c)\in C\cap\mathbb{N}^3} x^a y^b z^c$$

where $\mathcal{C}$ is the cone of $\mathbb{R}^3$ defined by $a+b \geq c$, $a+c \geq b$, $b+c \geq a$. The contribution of each monomial $m_{a,b,c}x^a y^b z^c$ of $Q^-_{\alpha,\beta,\gamma}$, to the generating series (5.4), is $m_{a,b,c} \cdot \sum x^p y^q z^r$ where the sum is over all (p,q,r) is in the translation of $\mathcal{C}$ by vector (a,b,c). For all (p,q,r) in the cone intersection of all these translated cones (there are finitely many of them, since $Q^-_{\alpha,\beta,\gamma}$ is a polynomial), the coefficient of $x^p y^q z^r$ in the generating series is $\sum m_{a,b,c}$, which is $Q^-_{\alpha,\beta,\gamma}(1,1,1)$, and does not depend on p, q, r.

Linear growth for reduced Kronecker coefficients, under first part increment. Last, if we apply Lemma 1 with $F(X,Y,Z) = XYZ+XY+YZ+XZ$, and specialize all three alphabets X', Y' and Z' to a single variable x, y and z respectively, we get a factorization

$$\frac{1}{(1-xyz)(1-x)(1-y)(1-z)} Q^+_{\alpha,\beta,\gamma}(x,y,z) = \sum_{a,b,c} \overline{g}_{\alpha+(a),\beta+(b),\gamma+(c)} x^a y^b z^c + \text{ terms } x^a y^b z^c \text{ with small } a,\, b \text{ and } c.$$

with $Q^+_{\alpha,\beta,\gamma}$ a Laurent polynomial. Row stability for reduced Kronecker coefficients is then derived in a similar way to column stability, but now the factor of the Laurent polynomial is

$$\frac{1}{(1-xyz)(1-x)(1-y)(1-z)} = \sum_{(a,b,c)\in\mathcal{C}} \left(1+\left\lfloor \frac{\min(a+b-c,a+c-b,b+c-a)}{2}\right\rfloor\right) x^a y^b z^c,$$

which is at the origin of the asymptotic quasipolynomiality and linear growth.

6 Generating series.

Four families of constants were defined in the previous sections: the limits $\overline{\overline{g}}_{\alpha,\beta,\gamma}$ under hook stability (Section 3) and the coefficients $A_{\alpha,\beta,\gamma}$, $B_{\alpha,\beta,\gamma}$ and $C_{\alpha,\beta,\gamma}$ appearing in the quasi-polynomial formulas of Section 4. We get, as a byproduct of our proofs, generating series for these constants, in the style of those of Section 5.1.

Theorem 4. *Let α, β, γ be three partitions. Let X, Y and Z be three alphabets. Set $W = XY + XZ + YZ + X + Y + Z$. Denote $\chi = \sum_{n=1}^{\infty} p_n$, the formal sum of all power sum symmetric functions. Then, for all triples of partitions α, β and γ, the constants $\overline{\overline{g}}_{\alpha,\beta,\gamma}$ (in Theorem 1) and $A_{\alpha,\beta,\gamma}$, $C_{\alpha,\beta,\gamma}$, and $B_{\alpha,\beta,\gamma}$ (in Theorem 2), are the coefficients of $s_\alpha[X]s_\beta[Y]s_\gamma[Z]$ in the expansion, in the Schur basis, of, respectively,*

$$\sigma\left[XYZ+(1-\varepsilon)W\right], \qquad \sigma[XYZ+2W], \qquad \sigma[XYZ+(1+\varepsilon)W],$$
$$\textit{and} \quad \sigma[XYZ+2W]\cdot\left(\frac{3}{4}+\frac{1}{4}\sigma[(\varepsilon-1)W]-\frac{1}{2}\chi[W]+\chi[YZ-X]\right).$$

Example 2. One can derive from Theorem 4 the following formulas for the coefficients in the paper, when two of the three indices are the empty partition $\varnothing$.

$$A_{(\alpha_1,\alpha_2),\varnothing,\varnothing} = \alpha_1-\alpha_2+1,$$
$$C_{(\alpha_1,\alpha_2),\varnothing,\varnothing} = \begin{cases}(-1)^{\alpha_2} & \text{if } \alpha_1\equiv\alpha_2 \pmod 2,\\ 0 & \text{otherwise,}\end{cases}$$
$$B_{(\alpha_1,\alpha_2),\varnothing,\varnothing} \text{ is the nearest integer to } -3\cdot\frac{(\alpha_1)^2-(\alpha_2-1)^2}{4},$$

and, when (α_1,α_2) is not the empty partition,

$$B_{\varnothing,(\alpha_1,\alpha_2),\varnothing} \text{ is the nearest integer to } -3\cdot\frac{(\alpha_1-1)^2-(\alpha_2-2)^2}{4}.$$

Similarly, one derives that

$$\overline{\overline{g}}_{\alpha,\emptyset,\emptyset} = \begin{cases} 2 & \text{if } \alpha \text{ is a hook,} \\ 1 & \text{if } \alpha = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

7 Final remarks.

Geometric interpretation of the generating series. The Schur generating series in the paper have straightforward geometric interpretations. Take any three finite–dimensional complex vector spaces X, Y and Z. Then (5.1) means that, when the lengths of α, β and γ are at most the dimension of X, Y and Z respectively, the Kronecker coefficient $g_{\alpha,\beta,\gamma}$ is the multiplicity of the irreducible representation $S_\alpha(X) \otimes S_\beta(Y) \otimes S_\gamma(Z)$, of the group $GL(X) \times GL(Y) \times GL(Z)$, in the symmetric algebra over $X \otimes Y \otimes Z$. Similarly, (5.2) means that the reduced Kronecker coefficient $\overline{g}_{\alpha,\beta,\gamma}$ is the multiplicity of the same irreducible representation in the symmetric algebra over

$$(X \otimes Y \otimes Z) \oplus (X \otimes Y) \oplus (Y \otimes Z) \oplus (X \otimes Z).$$

The other generating series in Theorem 4 have similar interpretations.

Alternative approach to hook stability, through Murnaghan's stability and symmetries of Kronecker coefficients. Is there a simpler approach to hook stability (Corollary 1) based only a clever use of the well–known symmetry properties of the Kronecker coefficients,

$$g_{\lambda,\mu,\nu} = g_{\lambda',\mu',\nu} = g_{\lambda',\mu,\nu'} = g_{\lambda,\mu',\nu'}$$

and of Murnaghan's stability? It seems that this leads only to a weaker result of hook stability "modulo 2", see [2, Section 5.3] and the appendices. This approach also suggests the following interesting conjecture:

Conjecture. *For any three partitions λ, μ and ν of the same weight,*

$$g_{\lambda,\mu,\nu} \leq g_{\lambda\oplus(1|1),\mu\oplus(1|1),\nu\oplus(1|1)}.$$

Bounds for stability. Our methods also provide explicit bounds for when the stability of Section 3, and the quasi–polynomial formulas of Section 4, hold. See [2].

Alternative approach through Hilbert series. S. Sam has recently announced [15] that he can derive some of our main results by interpreting the generating series of Section 5.2 as Hilbert series of suitable finitely-generated modules, from which the Hilbert series of the base ring can be factored out.

Adding cells to other rows. The rate of growth experienced by the Kronecker coefficients and the reduced Kronecker coefficients as we add cells to remaining rows is difficult to understand. Some particular instances of this problem have been studied in the literature.

For any fixed partitions λ, μ, ν, the *stretched Kronecker coefficients* $g_{k\lambda,k\mu,k\nu}$ are known to be quasi-polynomial in k. Results on this can be found in the work of Manivel [11], Mulmuley [12], and Baldoni and Vergne [1]. This property is inherited by the reduced Kronecker coefficients.

When $|\lambda| = |\mu| + |\nu|$, the reduced Kronecker coefficient $\overline{g}_{\lambda,,\mu,\nu}$ coincides with the Littlewood-Richardson coefficient $c^{\lambda}_{\mu,\nu}$. The corresponding stretching function $\overline{g}_{k\lambda,k\mu,k\nu} = c^{k\lambda}_{k\mu,k\nu}$ is well understood: it has been shown by Rassart, and Derksen and Weyman that this stretching function is polynomial in k, and its degree has been studied by King et al.

The families $\overline{g}_{(k),(k^a),(k^a)}$ and $\overline{g}_{(k),(2k-j,k^{a-1}),(k^a)}$, with $k \geq 2j$, with a and j fixed, are considered by Colmenarejo and Rosas [5, 6]: they are quasi polynomials in k of degree respectively $2a-1$ and $3a-2$.

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