On the Complexity of Generalized Chromatic Polynomials

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Abstract

J. Makowsky and B. Zilber (2004) showed that many variations of graph colorings, called $\mathbf{CP}$-colorings in the sequel, give rise to graph polynomials. This is true in particular for harmonious colorings, convex colorings, mcc_t-colorings, and rainbow colorings, and many more. N. Linial (1986) showed that the chromatic polynomial $\chi(G; X)$ is $\#\mathbf{P}$-hard to evaluate for all but three values $X = 0, 1, 2$, where evaluation is in $\mathbf{P}$. This dichotomy includes evaluation at real or complex values, and has the further property that the set of points for which evaluation is in **P** is finite. We investigate how the complexity of evaluating univariate graph polynomials that arise from $\mathbf{CP}$-colorings varies for different evaluation points. We show that for some $\mathbf{CP}$-colorings (harmonious, convex) the complexity of evaluation follows a similar pattern to the chromatic polynomial. However, in other cases (proper edge colorings, mcc_t-colorings, H-free colorings) we could only obtain a dichotomy for evaluations at non-negative integer points. We also discuss some $\mathbf{CP}$-colorings where we only have very partial results.

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Contents

1. Introduction

By a classical result of R. Ladner, and its generalization by K. Ambos-Spies, [Lad75, AS87], there are infinitely many degrees (via polynomial time reducibility) between $\mathbf{P}$ and $\mathbf{NP}$, and between $\mathbf{P}$ and $\#\mathbf{P}$, provided $\mathbf{P} \neq \mathbf{NP}$. In contrast to this, the complexity of evaluating partition functions or counting graph homomorphisms satisfies a dichotomy theorem: either evaluation is in $\mathbf{P}$ or it is $\#\mathbf{P}$-complete, [DG00, BG05, CCL13]. For the definition of the complexity class $\#\mathbf{P}$, see [GJ79] or [Pap94].

In accordance with the literature in graph theory a finite graph $G = (V(G), E(G))$ with $n(G) = |V(G)|$ and $e(G) = |E(G)|$ has *order* $n(G)$ and *size* $e(G)$. Otherwise, the *size of a finite set* is its cardinality.

In this paper we study the *complexity of the evaluation* of generalized univariate chromatic polynomials, as introduced in [MZ06] and further studied in [KMZ08, KMZ11]. They will be called in the sequel $\mathbf{CP}$-colorings (for **C**ounting **P**olynomials). Among these we find:

Examples 1.1. (i) *Trivial (unrestricted) vertex colorings using at most k colors are just functions $V(G) \to [k]$. We denote by $\chi_{trivial}(G; k)$ the number of trivial colorings of G, hence $\chi_{trivial}(G; k) = k^{|V(G)|} \in \mathbb{Z}[k]$.*

(ii) *Proper vertex colorings using at most k colors, where two neighboring vertices receive different colors, are counted by $\chi(G; k)$, the classical chromatic polynomial .*

(iii) *Proper edge colorings using at most k colors, where two edges with a common vertex receive different colors, are counted by $\chi_{edge}(G; k)$, the edge chromatic polynomial. We note that they are exactly the proper vertex colorings of the line graph $L(G)$ of G.*

(iv) *Convex colorings using at most k colors are vertex colorings, which are not necessarily proper, but where each color class induces a connected subgraph. They are counted by $\chi_{convex}(G; k)$. Convex colorings are first introduced in [MS07].*

(v) *Harmonious colorings using at most k colors are proper vertex colorings such that no two edges have end-vertices receiving the same pair of colors. They were introduced in [HK83, EM95, Edw97]. We denote the number of harmonious colorings using at most k colors by $\chi_{harm}(G; k)$. The graph parameter $\chi_{harm}(G; X)$ is a polynomial in k by [MZ06, KMZ11] which was further studied more recently in [DBG17].*

(vi) *For a fixed connected graph H, $DU(H)$-colorings are vertex colorings, where each color class induces a disjoint collection of copies of H. The graph parameter counting the number of $DU(H)$-colorings with at most k colors is a polynomial in k, and the corresponding graph polynomial is denoted by $\chi_{DU(H)}(G; k)$.*

(vii) *For a fixed $t \in \mathbb{N}^+$, an mcc_t-coloring using at most k colors is a vertex coloring, where the connected components of the subgraphs induced by each color class have at most t vertices. They were previously studied in [ADOV03] and [LMST07]. The graph parameter $\chi_{mcc_t}(G; k)$ counting the number of mcc_t-colorings with at most k colors is also a polynomial in k but not in t.*

(viii) *For a fixed graph H, an H-free coloring using at most k colors is a vertex coloring in which every color class induces an H-free graph. For $H = K_2$ these are* *the proper vertex colorings. The graph parameter $\chi_{H-free}(G;k)$ counting the number of H-free colorings with at most k colors is also a polynomial in k.*

More examples are presented in Section 2, where we also discuss a general theorem which allows us to find infinitely many generalized chromatic polynomials, and in Section 6.

1.1. The complexity spectrum

Let $\mathbf{F}$ be a fixed field that contains $\mathbb{Q}$, the rational numbers, and in which the arithmetic operations are polynomial time computable. For our discussion we use the unit-cost[4] model for the field computations in $\mathbf{F}$.

Given a graph polynomial $P(G;X) \in \mathbf{F}[X]$ and an element $a \in \mathbf{F}$, we view $P_a(G) = P(G;a)$ as a graph parameter. We will look at the complexity of the problem of evaluating $P(G;a)$ for a fixed $a \in \mathbf{F}$ and at the problem of computing all the coefficients of $P(G;X)$.

Problem: $P(G;a)$
Input: Graph G.
Output: The value of $P(G;X)$ for $X = a$.

Problem: $P(G;X)$
Input: Graph G.
Output: All the coefficients of $P(G;X)$ as a vector over $\mathbf{F}$.

We denote by $T_{P_a}(n)$ the time needed to compute $P(G;a)$ on graphs with n vertices in the Turing model of computation. Similarly $T_{P_X}(n)$ denotes the time needed to compute all the coefficients of $P(G;X)$. Clearly, for every $a \in \mathbf{F}$ the problem $P(G;a)$ is reducible to computing the coefficients of $P(G;X)$. The converse is not true in general, but we shall see cases where for certain $a_0 \in \mathbf{F}$ computing the coefficients of $P(G;X)$ is reducible to $P(G;a_0)$. When we speak informally of the *complexity spectrum of* $P(G;X)$ we have in mind the variability of $T_{P_a}(n)$ where $a \in \mathbf{F}$, without giving the term a precise definition. For a graph polynomial $P(G;X)$, we are interested in describing the complexity of $P_a(G)$ for all $a \in \mathbf{F}$. A more modest task would be to describe it only for $a \in \mathbb{N}$. In the case of $a \in \mathbb{N}$ we speak of the *discrete complexity spectrum*, in the case of $a \in \mathbf{F}$ we speak of the *full complexity spectrum*, if the context requires it.

[4]If instead we use the binary cost model for computations in, say, $\mathbb{Q}$, the main results still hold, but have to be formulated more carefully, as $a \in \mathbb{Q}$ could be very large, and the notion of uniformity would be affected.

We define

$$\mathrm{EASY}(P) = \{a \in \mathbf{F} : \text{ there exists } d \in \mathbb{N} \text{ with } T_{P_a}(n) \leq n^d \text{ for all } n \geq 2\}$$

Analogously, we define

$$\#\mathrm{PHARD}(P) = \{a \in \mathbf{F} : P_a(G) \text{ is } \#\mathbf{P}\text{-hard}\}.$$

1.2. Easy computation of the polynomial versus its easy evaluation

In the definition of $\mathrm{EASY}(P)$, we require that the evaluation at $a \in \mathbf{F}$ takes polynomial time, but the exponent d may depend on a, which is to say the polynomial bound for evaluating $P(G; a)$ is *non-uniform*.

If we could compute all of the coefficients of $P(G; X)$ in polynomial time, we could also evaluate $P(G; a)$ for every a in polynomial time $O(n^d)$ where d is independent of a, and $P(G; a)$ can be evaluated in polynomial time uniformly.

However, if $\mathrm{EASY}(P)$ is infinite with a non-uniform polynomial bound, then it does not follow that there exists d such that for every G the coefficients of $P(G; X)$ maybe computed in time $O(n^d)$.

Proposition 1.2. (i) *If the coefficients of the polynomial $P(G; X)$ can be computed in polynomial time from G alone, then* $\mathrm{EASY}(P) = \mathbf{F}$.

(ii) *If, in addition to (i), there is $a \in \mathbf{F}$ such that $a \in \#\mathrm{PHARD}(P)$, then* $\mathbf{P} = \#\mathbf{P}$.

Proof. As all the coefficients of $P(G; X)$ can be computed together in polynomial time, the degree $\delta = \delta(G)$ of $P(G; X)$ is polynomial in the number of vertices of G, and so is the size of the coefficients.

Evaluating such a polynomial can be done in polynomial time in the order of G. Therefore if $a \in \mathrm{EASY}(P) \cap \#\mathrm{PHARD}(P)$ we have $\mathbf{P} = \#\mathbf{P}$. ☐

The characteristic polynomial $p_A(G; X)$ of a graph G is the characteristic polynomial of its adjacency matrix. A variant of this, $p_L(G; X)$, is obtained by replacing the adjacency matrix by the Laplacian of G. Both are obtained by computing a determinant, therefore evaluating both $p_A(G; X)$ and $p_L(G; X)$ can be done in time $O(n^3)$, irrespective of the evaluation point $X = a \in \mathbf{F}$. Hence $\mathrm{EASY}(p_A(G; X)) = \mathbf{F}$ ($\mathrm{EASY}(p_L(G; X)) = \mathbf{F}$) uniformly.

In Section 3.1 we shall see an example where $\mathrm{EASY}(P) = \mathbb{N}$ non-uniformly.

1.3. Linial's Trick

In the case of the chromatic polynomial $\chi(G; X)$, N. Linial set the paradigm in the following theorem:

Theorem 1.3 ([Lin86])**.** $\#\mathrm{PHARD}(\chi) = \mathbf{F} - \{0, 1, 2\}$ *and* $\mathrm{EASY}(\chi) = \{0, 1, 2\}$.

To show this N. Linial observed the following:

Lemma 1.4 (Linial's Trick)**.** *Let $G_1 \bowtie G_2$ be the join of the graphs G_1 and G_2, obtained from the disjoint union $G_1 \sqcup G_2$ by connecting all the vertices of G_1 with all the vertices of G_2.*

(i) *A function* $f : V(G \bowtie K_1) \to [k]$ *is a proper coloring with* k *colors if and only if there is a function* $g : V(G) \to \{1, \ldots, i-1, i+1, \ldots, k\}$ *which is a proper coloring of* G *with* $k-1$ *colors and* $f|_{V(G)} = g$*, i.e.,* g *is the restriction of* f *to* $V(G)$*, and* $f(u) = i$*.*

(ii) $\chi(G \bowtie K_1; k) = k \cdot \chi(G; k-1)$

(iii) $\chi(G \bowtie K_n; k) = k_{(n)} \cdot \chi(G; k-n)$ *where* $k_{(n)}$ *is the falling factorial:* $k_{(n)} = k(k-1)(k-2)\ldots(k-(n-1)) = \frac{k!}{(n-k)!}$.

This allows one for $a \notin \mathbb{N}$ to evaluate $\chi(G, a-n)$ by computing $\chi(G \bowtie K_n, a)$. It also shows that evaluating $\chi(G; k)$ is reducible to evaluating $\chi(G; k+1)$ for $k \geq 3$.

Proof of Theorem 1.3. First one proves that $\chi(G; 3)$ is $\#\mathbf{P}$-complete directly, as in [Lin86]. For the cases $a \in \mathbb{N} - \{0, 1, 2\}$ one uses (ii) of Linial's trick iteratively to reduce the computation of $\chi(G; 3)$ to the computation of $\chi(G; a)$. For the cases $a \in \mathbf{F} - \mathbb{N}$ one uses (iii) of Linial's trick to compute $\chi(G; a-n)$ for $n = 0, 1, \ldots, n(G)$. By Lagrange interpolation the polynomial $\chi(G; X)$ is thereby determined since the degree of $\chi(G; X)$ is $n(G)$. In particular computing $\chi(G; 3)$ is polynomial time reducible to computing $\chi(G; a)$. Hence, Theorem 1.3 follows. $\square$

Similarly, for the generating matching polynomial

$$gm(G; X) = \sum_{M \subseteq E(G)} X^{|M|},$$

we have, see [AM][5].

Proposition 1.5. $\#\mathrm{PHARD}(gm) = \mathbf{F} - \{0\}$ *and* $\mathrm{EASY}(gm) = \{0\}$*,*

G-polynomial	$E = \mathrm{EASY}(P)$	$\#\mathrm{PHARD}(P)$	OTHER	Reference
$\chi_{trivial}(G; X)$	$E_{trivial} = \mathbf{F}$, u	$\emptyset$	$\emptyset$	trivial
$p_A(G; X)$	$E_{char} = \mathbf{F}$, u	$\emptyset$	$\emptyset$	folklore
$gm(G; X)$	$E_{match} = \{0\}$	$\mathbf{F} - E_{match}$	$\emptyset$	folklore
$\chi(G; X)$	$E_{chrom} = \{0, 1, 2\}$	$\mathbf{F} - E_{chrom}$	$\emptyset$	Theorem 1.3
$\chi_{harm}(G; X)$	$E_{harm} = \mathbb{N}$, nu	$\mathbf{F} - E_{harm}$	$\emptyset$	Theorem 3.2
$\chi_{convex}(G; X)$	$E_{convex} = \{0, 1\}$	$\mathbf{F} - E_{convex}$	$\emptyset$	Theorem 3.6
$\chi_{DU(K_\alpha)}(G; X)$	$E_{DU(K_\alpha)} = \{0, 1\}$	$\mathbf{F} - E_{DU(K_\alpha)}$	$\emptyset$	Theorem 3.16
$\alpha \geq 2$				

Table 1: Full complexity spectra, u=uniformly, nu=non-uniformly

[5] It appears that this was known as folklore, but we could not find a suitable reference.

G-polynomial	$E = \text{EASY}(P)$	$\#\text{PHARD}(P)$	OTHER	Reference
$\chi_{edge}(G;X)$	$E_{edge} = \{0,1\}$	$\mathbb{N} - E_{edge}$	$\emptyset$	Theorem 5.2
$\chi_{mcc_t}(G;X)$	$E_{mcc_t} = \{0,1\}$	$\mathbb{N} - E_{mcc_2}$	$\emptyset$	Theorem 5.8
$t \geq 2, k \geq 2$				
$\chi_{H-free}(G;X)$	$E_{H-free} = \{0,1\}$	$\mathbb{N} - \{0,1,2\}$	$\{2\}$ (+)	Theorem 5.10

Table 2: Discrete complexity spectra only, H of size 2, (+) only $\mathbf{NP}$-hard is known

The purpose of this paper is to study the complexity spectrum of generalized univariate chromatic polynomials arising from $\mathbf{CP}$-colorings.

In the examples we study, the complexity spectrum is easily described with the two sets $\text{EASY}(P)$ and $\#\text{PHARD}(P)$. Our results on full complexity spectra are summarized in Table 1. Cases where only the discrete spectrum is understood are given in Table 2.

To get a complete description of the full complexity spectrum of $P(G;a)$, one needs two ingredients:

(i) Enough points $a \in \mathbb{N}$ for which the complexity of $P_a(G) = P(G;a)$ is known, and
(ii) some form of reducibility between $P_a(G)$ and $P_b(G)$ for the remaining values $a, b \in \mathbf{F}$.

From the literature we often, but not always, can get enough information for (i). We give here new results for (i), namely Theorems 3.5, 3.14 and 5.6. For (ii) we try to adapt Linial's Trick, which in some cases is more or less straightforward, while in other cases requires finding a new gadget as in Theorem 3.2. A precise description of what is needed for (ii) is given in [BDM10], which also covers the case for multivariate graph polynomials.

1.4. The Difficult Point Dichotomy

We say that a univariate graph polynomial has the *Difficult Point Dichotomy* if

(i) for every $a \in \mathbf{F}$ either $a \in \text{EASY}(P)$ or $a \in \#\text{PHARD}(P)$, and
(ii) $\text{EASY}(P) = \mathbf{F}$ or $\text{EASY}(P) \subseteq \mathbb{N}$.

In [Mak08a, MKR13] it is conjectured that, for every univariate graph polynomial $P(G;X)$ definable in Second Order Logic SOL, the set $\text{EASY}(P)$ is either finite or $\text{EASY}(P) = \mathbf{F}$ and that $\text{EASY}(P) \cup \#\text{PHARD}(P) = \mathbf{F}$. In [Mak08a] the same was also conjectured for univariate graph polynomials definable in Monadic Second Order Logic MSOL. The example of $\chi_{harm}(G;X)$ is SOL-definable and therefore disproves the conjecture for SOL-definable graph polynomials. However, it was shown in [KM14] that $\chi_{harm}(G;X)$ is not MSOL-definable. Rather than conjecturing frivolously, we state some problems.

Problem 1.6. *Which univariate graph polynomials $P(G; X)$ satisfy the Difficult Point Dichotomy?*

In particular,

(i) *Is it true for every* MSOL*-definable univariate graph polynomial $P(G; X)$?*

(ii) *Can one find a criterion which applies to an infinite family of univariate graph polynomials $P(G; X)$ which are not partition functions, or which do not count homomorphisms, and which implies the Difficult Point Dichotomy.*

Outline of the paper

In Section 2 we give a simplified proof of the result from [MZ06] that not only counting proper graph colorings, but counting many other graph colorings, give rise to infinitely many generalized chromatic graph polynomials. We give many explicit examples, and show that there are infinitely many such graph polynomials which are mutually semantically incomparable, cf. [MRB14, KMR17]. In Sections 3 and 5 we analyze in detail the graph polynomials from Table 1 and 2. In Section 4 we give the proof that counting convex colorings with 2 colors is $\#\mathbf{P}$-complete (Theorem 3.5). In Section 6 we discuss graph polynomials for which we have only partial results. Finally, in Section 7 we summarize our conclusions and list some open problems.

2. One, two, many chromatic polynomials

Let $G = (V(G), E(G))$ be a finite graph and $k \in \mathbb{N}^+$ a positive integer. We denote the set $\{1, \ldots, k\}$ by $[k]$. Unless otherwise stated all graphs are simple, i.e., loop-free and without multiple edges. A *vertex (edge) coloring* f with k colors is a function $f : V(G) \to [k]$ ($f : E(G) \to [k]$). The coloring f is *proper* if no two vertices (edges) with a common edge (vertex) have the same color.

Let $\chi(G; k)$ ($\chi_{edge}(G; k)$) denote the number of *proper* vertex (edge) colorings of G with k colors. In 1912 G. Birkhoff [Bir12] noticed that $\chi(G; k)$ and $\chi_{edge}(G; k)$ are polynomials in $\mathbb{Z}[k]$, and therefore can be extended to polynomials in $\mathbb{C}[X]$, denoted, by abuse of notation, by $\chi(G; X)$ and $\chi_{edge}(G; X)$. Birkhoff's proof for $\chi(G; X)$ was based on a recurrence relation involving deletion and contraction of edges, which was generalized and led, in its most general form, to the Tutte polynomial. For $\chi_{edge}(G; X)$ one simply observes that

$$\chi(L(G); X) = \chi_{edge}(G; X), \qquad (*)$$

where $L(G)$ is the line graph of G. Although proper edge colorings have been studied in the literature, the polynomial $\chi_{edge}(G; X)$ has not received wide attention, probably because of (*).

2.1. Many chromatic polynomials

Let $\mathcal{G}$ denote the class of all finite graphs. We introduce our concepts for vertex colorings, but they can be straightforwardly extended to edge colorings.

Two vertex colorings $f_1, f_2 : V(G) \to [k]$ are isomorphic if there is an automorphism $\alpha : V(G) \to V(G)$ of G and a permutation $\pi : [k] \to [k]$ such that for all $v \in V(G)$

$$\pi(f_1(v)) = f_2(\alpha(v)).$$

Let $\mathrm{COL} = \bigcup_{G \in \mathcal{G}} \bigcup_{k \in \mathbb{N}^+} [k]^{V(G)}$. A *coloring property* Φ is a subset of COL that is closed under isomorphisms of colorings.

For a fixed coloring property Φ, a graph $G \in \mathcal{G}$, $k \in \mathbb{N}^+$ and $I \subseteq [k]$, let

$$\chi_\Phi(G; k) = |\{f \in \Phi : f : V(G) \to [k]\}|$$

and let

$$c_G^\Phi(I, k) = |\{f \in \Phi : f : V(G) \to [k] \text{ with } f(V(G)) = I\}|$$

be the number of colorings $f \in \Phi$ of G which use exactly the colors in I.

We say that Φ is a $\mathbf{CP}$-*Property* if the following two conditions are satisfied:

(A) For all $k \in \mathbb{N}^+$ and $I, J \subseteq [k]$ with $|I| = |J| = i$ we have

$$c_G^\Phi(I, k) = c_G^\Phi(J, k).$$

(B) For all $k, k' \in \mathbb{N}^+$ with $I \subseteq [k] \cap [k']$ we have $c_G^\Phi(I, k) = c_G^\Phi(I, k')$.

If (A) holds we let $c_G^\Phi(i, k)$ denote the common value of $c_G^\Phi(I, k)$, where $|I| = i$, and if both (A) and (B) hold, we let $c_G^\Phi(i)$ denote the common value of $c_G^\Phi(i, k)$, for $k \geq i$.

We now compute $\chi_\Phi(G; k)$ using (A) and (B):

$$\chi_\Phi(G; k) = \sum_{I \subseteq [k]} c_G^\Phi(I, k) = \sum_i c_G^\Phi(i) \binom{k}{i}.$$

This establishes the following result, first shown in [MZ06]:

Theorem 2.1 ([MZ06]). *If Φ is a $\mathbf{CP}$-property, the counting function $\chi_\Phi(G; k)$ is a polynomial in $\mathbb{Z}[k]$.*

Examples 2.2. (i) *In the case of the chromatic polynomial, both (A) and (B) are satisfied. Hence we get a new proof of Birkhoff's Theorem.*

(ii) *Let $\hat{\chi}_\Phi(G; k) = c_G^\Phi(k)$ be the graph parameter that counts the number of colorings in Φ of G which use exactly k colors.*
Note that the function $\hat{\chi}_\Phi(G; k)$ need not be a polynomial in k when $\chi_\Phi(G; k)$ is a polynomial in k.

(iii) *Let Φ_1 be the coloring property which says f is a proper coloring with k colors where all the k colors are used. Here (B) is violated, and indeed, $\chi_{\Phi_1}(G; k)$ is not a polynomial.*

(iv) *Let Φ_2 be the coloring property which says f is a proper coloring with k colors such that $f(v) = i + 1$ if and only if the degree of v is i. Here (A) is violated, but (B) is still true, and $\chi_{\Phi_2}(G; k)$ is still a polynomial.*

(v) *All the examples (i)-(vii) of Section 1 listed in Examples 1.1 satisfy (A) and (B). Hence they are polynomials in k.*

2.2. $\mathcal{P}$-colorings and variations

Two graph polynomials may be compared via their *distinctive power*. Two graphs G_1 and G_2 are *similar* if they have the same number of vertices, edges and connected components. A graph polynomial $Q(G;X)$ is *less distinctive than* $P(G;Y)$, written $Q \preceq P$, if for every two similar graphs G_1 and G_2

$$P(G_1;X) = P(G_2;X) \text{ implies } Q(G_1;Y) = Q(G_2;Y).$$

We also say that $P(G;X)$ *determines* $Q(G;X)$ if $Q \preceq P$. Two graph polynomials $P(G;X)$ and $Q(G;Y)$ are equivalent in distinctive power (d.p.-equivalent) if for every two similar graphs G_1 and G_2

$$P(G_1;X) = P(G_2;X) \text{ iff } Q(G_1;Y) = Q(G_2;Y).$$

Here we show how to obtain infinitely many graph polynomials that are mutually incomparable in distinctive power.

Let $\mathcal{P}$ be any graph property (a class of finite graphs closed under graph isomorphism).

A function $f : V(G) \to [k]$ is a *$\mathcal{P}$-coloring* if for every $i \in [k]$ the set $f^{-1}(i)$ induces a graph $G[f^{-1}(i)] \in \mathcal{P}$. Clearly, this is a $\mathbf{CP}$-coloring for any graph property $\mathcal{P}$. Hence $\chi_{\mathcal{P}}(G;k)$, the number of $\mathcal{P}$-colorings of G with at most k colors, is a polynomial in k.

Theorem 2.3 ([KMR17]). *There are infinitely many graph polynomials of the form $\chi_{\mathcal{P}}(G;k)$ with mutually incomparable distinctive power.*

We can generalize this further. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two graph properties. The $\mathcal{P}_1$-colorings such that the union of any two color classes induces a graph in $\mathcal{P}_2$ form also a $\mathbf{CP}$-property. Let

$$\chi_{\mathcal{P}_1,\mathcal{P}_2}(G;k) =$$
$$|\{f : V(G) \to [k] : \forall_{i\in[k]} G[f^{-1}(\{i\})] \in \mathcal{P}_1, \forall_{i,j\in[k], i\neq j} G[f^{-1}(\{i,j\})] \in \mathcal{P}_2\}|$$

denote the number of such colorings. Then, for $k \in \mathbb{N}$ and a graph G, the graph invariant $\chi_{\mathcal{P}_1,\mathcal{P}_2}(G;k)$ is a polynomial in k.

Problem 2.4. *For which graph properties $\mathcal{P}_1, \mathcal{P}_2$ can we describe the complexity of $\chi_{\mathcal{P}_1,\mathcal{P}_2}(G;k)$? In particular, for which graph properties does the Difficult Point Dichotomy hold?*

Remark 1. *In Problem 2.4 it may be reasonable to impose some complexity restrictions on $\mathcal{P}_1$ and $\mathcal{P}_2$, e.g., we might require them to be in $\mathbf{NP}$.*

Let $\mathcal{AH}$ be any additive induced hereditary property (closed under taking induced subgraphs and disjoint unions). $\mathcal{AH}$-colorings have been studied in [Bro96, Far04], in which the following is shown:

Theorem 2.5. (i) *([Bro96]) There are uncountably many induced hereditary properties $\mathcal{AH}$ of graphs. Therefore, $\chi_{\mathcal{AH}}(G;k)$ may not be computable.*

(ii) *([Far04])* $\chi_{\mathcal{AH}}(G;k)$ *is* $\mathbf{NP}$*-hard, unless* $\mathcal{AH}$ *is the class of empty (=edgeless) graphs.*

Table 3 unifies the $\mathbf{CP}$-colorings considered in Table 1 as $\mathcal{P}_1$-colorings such that the union of any two color classes induces a graph in $\mathcal{P}_2$. Table 3 also contains the definitions of *acyclic colorings, t-improper colorings* and *co-colorings*, which will be discussed in Section 6.

$\mathbf{CP}$-coloring	$\mathcal{P}_1$	$\mathcal{P}_2$
trivial	all graphs	all graphs
proper	edgeless graphs	all graphs
acyclic	edgeless graphs	forests
convex	connected graphs	all graphs
harmonious	edgeless graphs	at most one edge
mcc_t	conn. cpts size $\leq t$	all graphs
$DU(H)$	disjoint union of $\cong H$	all graphs
t-imp	max. degree t	all graphs
co-coloring	clique or edgeless	all graphs
$\mathcal{AH}$-coloring	$\mathcal{AH}$	all graphs

Table 3: $\mathcal{P}_1$-colorings where the union of any two color classes is in $\mathcal{P}_2$. In the last line $\mathcal{P}_1$ is an additive induced hereditary property (closed under taking induced subgraphs and disjoint unions).

3. Detailed case study: Dichotomy theorems

3.1. Harmonious colorings

Recall that a coloring is *harmonious* if it is a proper vertex coloring and every pair of colors occurs along some edge at most once, and $\chi_{harm}(G;X)$ is the corresponding graph polynomial.

Proposition 3.1. *For every* $k \in \mathbb{N}$ *there is a polynomial time Turing machine* $T(k)$ *which computes* $\chi_{harm}(-;k)$*. In other words* $\mathbb{N} \subseteq \mathrm{EASY}(\chi_{harm})$ *non-uniformly.*

Proof. For $X = k$ a positive integer and a graph G on n vertices, $\chi_{harm}(G;k) \neq 0$ implies that G has at most $e(k) = \binom{k}{2}$ edges. Furthermore, there are $k^{2\cdot e(k)}$ colorings of $2 \cdot e(k)$ vertices with k colors. Let $i(G)$ be the number of isolated vertices of G,

$T(k)$ proceeds as follows:

(i) Determine $|E(G)|$. If $|E(G)| \geq e(k)+1$ we have $\chi_{harm}(G;k) = 0$.
(ii) Otherwise, we strip G of all its isolated vertices to obtain G', which has at most $2 \cdot e(k)$ vertices.
(iii) We count the colorings of G' which are harmonious, i.e., $\chi_{harm}(G';k)$, which takes time $t(k)$, independently of the number of vertices of G.
(iv) Therefore $\chi_{harm}(G;k) = k^{i(G)} \cdot \chi_{harm}(G';k)$.

It follows that $T(k)$ runs in time $O(n^2)$ where the constants depend on k. □

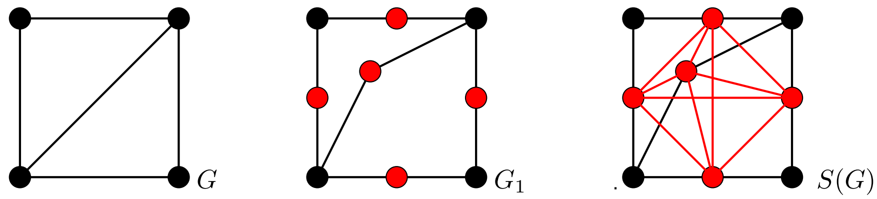


Figure 1: Constructing $S(G)$ from G

Remark 2. *In spite of the low complexity of the above algorithms, we cannot compute all the coefficients of $\chi_{harm}(G;X)$ in polynomial time. To compute $\chi_{harm}(G;X)$ for a graph G on n vertices we would have to compute for $n+1$ values $k_1,\ldots,k_{n+1}$ of X the value of the function $\chi_{harm}(G;X)$ and then use Lagrange interpolation. However, the above algorithm inspects $k^{2\cdot e(k)}$ colorings, which for at least one of the values $\chi_{harm}(G;k_i)$ is bigger than $n^{2\cdot e(n)}$. It follows from Theorem 3.2 below that, indeed, $\chi_{harm}(G;X)$ cannot be computed in polynomial time unless $\mathbf{P} = \#\mathbf{P}$.*

Theorem 3.2. *For each $a \in \mathbf{F} - \mathbb{N}$ the evaluation of $\chi_{harm}(G;a)$ is $\#\mathbf{P}$-hard.*

Proof. Let G be a graph. We form $S(G)$ in the following way (see Figure 1).
We first form G_1 using G by adding a new vertex v_e for each edge $e = (u,v)$ of G. Then replace the edge e by two new edges (u,v_e) and (v_e,v).
Using G_1 we now form $S(G)$: We connect all the new vertices $v_e : e \in E(G)$ of G_1 such that they form a complete graph on $|E(G)|$ vertices.

For a graph G and $k \in \mathbb{N}$,

$$\chi_{harm}(S(G);k+e(G)) = \chi(G;k)\cdot\binom{k+e(G)}{e(G)}e(G)!, \tag{1}$$

where $e(G) = |E(G)|$ and $\chi(G;k)$ is the chromatic polynomial of G evaluated at k. Since Equation (1) holds for every $k \in \mathbb{N}$, it is a polynomial identity, which can be written as follows:

$$\chi_{harm}(S(G);X) = X_{(e(n))}\cdot\chi(G;X-e(G)). \tag{2}$$

Equation (2) provides a polynomial time reduction from the coefficients of $\chi(G;X)$ to the coefficients of $\chi_{harm}(S(G);X)$, and vice versa. In particular, determining $\chi_{harm}(S(G);X)$ is $\#\mathbf{P}$-hard. We also see from Equation (2) that for $a \in \mathbf{F} - \mathbb{N}$, the graph parameter $\chi(G;a-e(G))$ is polynomial time equivalent to the evaluation $\chi_{harm}(S(G);a)$.

Finally, evaluating $\chi(G;a-e(G))$ is $\#\mathbf{P}$-hard for $a \notin \mathbb{N}$. To see this, first observe the following polynomial identity by the multiplicativity of the chromatic polynomial over disjoint unions:

$$\begin{aligned}&\chi(G\sqcup K_{1,n};X-e(G\sqcup K_{1,n})) =\\&\chi(K_{1,n};X-e(G\sqcup K_{1,n}))\cdot\chi(G;X-e(G\sqcup K_{1,n})) =\\&(X-e(G)-n)\cdot(X-e(G)-n-1)^n\cdot\chi(G;X-e(G)-n).\end{aligned} \tag{3}$$

For $a \in \mathbf{F} - \mathbb{N}$, we use Equation (3) to obtain the evaluations $\chi(G; a-e(G)-n)$ for $n = 0, 1, \ldots, |V(G)|$ and we apply Lagrange interpolation to compute $\chi(G; X-e(G))$. By Equation (2) this determines $\chi_{harm}(S(G); X)$, which as we have seen is a $\#\mathbf{P}$-hard problem. Consequently evaluating χ_{harm} at $X = a$ is $\#\mathbf{P}$-hard. □

Remark 3. *From Equation* (1) *we also see that, since evaluating* $\chi(G; 3)$ *is* $\#\mathbf{P}$-*complete, we must indeed have non-uniform polynomial time evaluation of* $\chi_{harm}(G; X)$ *at positive integer points, as stated in Proposition 3.1.*

Hence we have shown:

Theorem 3.3. *The Difficult Point Dichotomy is true for* $\chi_{harm}(G; X)$ *with* $\mathrm{EASY}(\chi_{harm}) = \mathbb{N}$ *non-uniformly, and* $\#\mathrm{PHARD}(\chi_{harm}) = \mathbf{F} - \mathbb{N}$.

3.2. Convex colorings

Recall that a *convex coloring* of a graph G is an assignment of colors to its vertices so that for each color c the subgraph of G induced by the vertices receiving color c is connected[6]. The resulting graph polynomial is $\chi_{convex}(G; X)$.

The following is easily verified:

Proposition 3.4. (i) *For* $X = 1$ *we have*

$$\chi_{convex}(G; 1) = \begin{cases} 1 & \text{if } G \text{ is connected} \\ 0 & \text{else}. \end{cases}$$

(ii) *For* $k \in \mathbb{N}^+$ *we have*

$$\chi_{convex}(G \sqcup K_1; k) = k \cdot \chi_{convex}(G; k-1).$$

In [Mak08b], it was asked whether $\chi_{convex}(-; 2)$ is $\#\mathbf{P}$-hard. The question was answered in the positive and posted in [GN08], but remained unpublished. Note that the number of convex colorings using at most two colors is equal to zero if G has three or more connected components and equal to two if G has exactly two connected components. So we may restrict our attention to connected graphs.

Theorem 3.5 (A. Goodall and S. Noble, [GN08])**.** *Evaluating* $\chi_{convex}(G; X)$ *for* $X = 2$ *on connected graphs is* $\#\mathbf{P}$-*complete.*

The proof is given in Section 4. Combining Proposition 3.4 and Theorem 3.5 we get the following:

Theorem 3.6. $\mathrm{EASY}(\chi_{convex}) = \{0, 1\}$ *and* $\#\mathrm{PHARD}(\chi_{convex}) = \mathbf{F} - \{0, 1\}$.

---

[6] We consider a graph with no vertices to be connected.

3.3. $DU(H)$-colorings

Let H be a fixed connected graph. A coloring $f : V(G) \to [k]$ is an $DU(H)$-*coloring* with k colors, if each color class induces a disjoint union of copies of H. We denote by $\chi_{DU(H)}(G;k)$ the number of $DU(H)$-colorings of G with at most k colors, and by $\hat{\chi}_{DU(H)}(G;k)$ the number of $DU(H)$ colorings of G with exactly k colors.

We easily verify:

Proposition 3.7. (i) *Being a $DU(H)$-coloring with k colors is a* $\mathbf{CP}$*-property, hence $\chi_{DU(H)}(G;k)$ is a polynomial in k. However, $\hat{\chi}_{DU(H)}(G;k)$ is not a* $\mathbf{CP}$*-property, and in general is not a polynomial in k.*

(ii) $\chi_{DU(H)}(G;k) = \sum_{i=1}^{k} \binom{k}{i} \hat{\chi}_{DU(H)}(G;i)$.

(iii) $\chi_{DU(H)}(G;k)$ *is multiplicative over disjoint unions.*

(iv) *For $k = 1$, a graph G is $DU(H)$-colorable iff G is a disjoint union of Hs.*

(v) *For $n(G) \not\equiv 0 \mod n(H)$ the polynomial $\chi_{DU(H)}(G;k)$ vanishes.*

Let $v \in V(H)$. We define $\Box_{H,v}(G)$ to be the graph with vertex set $V(G) \sqcup V(H)$, and edge set $E(G) \sqcup E(H) \sqcup V(G) \times \{v\}$. We can apply Linial's Trick, cf. Lemma 1.4, to analyze the complexity of $\chi_{DU(H)}(G;a)$.

Proposition 3.8. *Let H be a connected graph.*

(i) $\chi_{DU(H)}(\Box_{H,v}(G);k) = k \cdot \chi_{DU(H)}(G;k-1)$.

(ii) *For every $a, b \in \mathbb{N}$ and $b > a$, $\chi_{DU(H)}(G;a)$ is polynomial time reducible to $\chi_{DU(H)}(G;b)$.*

(iii) *For every $a_0 \in \mathbf{F} - \mathbb{N}$, computing the coefficients of $\chi_{DU(H)}(G;X)$ is polynomial time reducible to $\chi_{DU(H)}(G;a_0)$.*

The proof is the same as in [Lin86]. For the convenience of the reader we sketch it here.

Proof. (i) All the vertices of $V(H)$ have to be colored by the same color but differently from the vertices in $V(G)$.
(ii) Apply (i) $b - a$ many times.
(iii) Let $G_0 = G, G_{i+1} = \Box_{H,v}(G_i)$. Using $\chi_{DU(H)}(-;a_0)$ we can compute $\chi_{DU(H)}(G_i;a_0)$ for sufficiently many i's and then use Lagrange Interpolation to compute the coefficients of $\chi_{DU(H)}(G;X)$. $\Box$

Related decision and counting problems have been considered in the literature.

In the following α is a nonnegative integer.

Problem: $\mathrm{CliqueCover}_\alpha$
Input: Graph G. If $\alpha \geq 1$ then $n(G) = \alpha \cdot m$.
Question: Can we partition $V(G)$ into sets V_i such that each V_i induces a clique (for $\alpha = 0$) or induces a copy of K_α (for $\alpha \geq 1$)?

Problem: $\#\text{CliqueCover}_\alpha$
Input: Graph G. If $\alpha \geq 1$ then $n(G) = \alpha \cdot m$.
Output: The number of partitions of $V(G)$ into sets V_i such that each V_i induces a clique (for $\alpha = 0$) or induces a copy of K_α (for $\alpha \geq 1$).

We note for $\alpha = 0$ this is the classical clique cover decision problem of Richard Karp's original 21 problems, see [GJ79].

From the literature we know the following:

Theorem 3.9. (i) CliqueCover_0 *is* $\mathbf{NP}$*-complete and* $\#\text{CliqueCover}_0$ *is* $\#\mathbf{P}$*-complete, [GJ79, HIMRS98].*

(ii) CliqueCover_1 *and* $\#\text{CliqueCover}_1$ *are both trivial.*

(iii) CliqueCover_2 *is the same as finding a perfect matching, and is in* $\mathbf{P}$*, and* $\#\text{CliqueCover}_2$ *is* $\#\mathbf{P}$*-complete by [Val79].*

(iv) CliqueCover_3 *is* $\mathbf{NP}$*-complete and* $\#\text{CliqueCover}_3$ *is* $\#\mathbf{P}$*-complete, [KH83, HIMRS98].*

Problem 3.10. *For which* $\alpha \geq 4$ *is* $\#\text{CliqueCover}_\alpha$ $\#\mathbf{P}$*-complete?*

The connection between $\#\text{CliqueCover}_\alpha$ and $\chi_{DU(K_\alpha)}(G; X)$ is given as follows:

Proposition 3.11. (i) $\chi_{DU(K_1)}(G; X) = \chi(G; X)$*, the chromatic polynomial.*

(ii) $\hat{\chi}_{DU(K_\alpha)}(G; \frac{n}{\alpha}) = \left(\frac{n}{\alpha}\right)! \cdot \#\text{CliqueCover}_\alpha$ *where* $\alpha \neq 0$ *and* $n(G)$ *is divisible by* α.

Proof. If $n = n(G)$ is divisible by α, the maximal value k such that $\hat{\chi}_{DU(K_\alpha)}(G; k) \neq 0$ is $k = \frac{n}{\alpha}$. In this case the subgraph induced by each color class is isomorphic to K_α, and there are $\left(\frac{n}{\alpha}\right)!$ many such colorings. $\square$

We will prove that $\chi_{DU(K_\alpha)}(G; 2)$ is $\#\mathbf{P}$-hard for every $\alpha \geq 2$ in Theorem 3.14 below by describing a polynomial time reduction from the $\#\mathbf{P}$-hard problem $\#\alpha$-of-2α-SAT. From [CH96] we know that $\#\alpha$-of-2α-SAT is $\#\mathbf{P}$-complete.

Problem: $\#\alpha$-of-2α-SAT
Input: 2αCNF formula Θ.
Output: The number of truth assignments such that, in each clause, α literals are true and the other α are false.

Now we proceed to describe the construction of a graph $\overline{G}_\Theta$ for every input Θ to $\#\alpha$-of-2α-SAT. We will prove that $\#\alpha\text{-of-}2\alpha\text{-SAT}(\Theta) = \chi_{DU(K_\alpha)}(\overline{G}_\Theta; 2)$.

Let $\Theta = \bigwedge_{i=1}^{s} C_i$ be a 2αCNF formula with clauses $C_i = l_{i,1} \vee \cdots \vee l_{i,2\alpha}$. Without loss of generality, let the variables which occur in Θ be $x_t : t \in [r]$. The literals $l_{i,j}$ of Θ

are either variables x_t or negation of variables $\neg x_t$. We define $\mathrm{var}(x_t) = \mathrm{var}(\neg x_t) = x_t$.

For every clause C_i of Θ, let G_i be a clique of size 2α whose vertices are labeled by $l_{i,1}, \dots, l_{i,2\alpha}$ respectively. Let G_Θ be the disjoint union of G_i for all clauses C_i of Θ. For every variable $x_t : t \in [r]$ let D_t be a clique of size 2α in which α vertices are labeled x_t and the other α vertices are labeled $\neg x_t$. Let D_Θ be the disjoint union of $D_t : t \in [r]$.

We construct a graph $\overline{G}_\Theta$ as follows. The graph $\overline{G}_\Theta$ is obtained from the disjoint union $G_\Theta \sqcup D_\Theta$ of G_Θ and D_Θ by adding an edge between any vertex of G_Θ and any vertex of D_Θ whose labels are negations of each other (i.e. one is x_t and the other is $\neg x_t$, $t \in [r]$).

Any coloring $c : V(\overline{G}_\Theta) \to [2]$ can be interpreted as assigning the truth values *true* (for the color 1) and *false* (for the color 2) to the literals labeling the vertices. A coloring c of $\overline{G}_\Theta$ is *consistent* if the truth values assigned by c to the literals induce a well-defined truth value assignment as_c to the variables. More precisely, c is consistent if any two vertices labeled the same (both x_t or both $\neg x_t$, $t \in [r]$) have the same color and any two vertices with opposing labels (x_t and $\neg x_t$, $t \in [r]$) have different colors.

Lemma 3.12. *Let $c : V(\overline{G}_\Theta) \to [2]$ be a coloring of $\overline{G}_\Theta$.*

(i) *If c is a $DU(K_\alpha)$-coloring, then c is consistent.*

(ii) *If c is consistent, then the following are equivalent:*

  (a) *c is a $DU(K_\alpha)$-coloring.*

  (b) *as_c satisfies the condition of* $\#\alpha$-of-2α-SAT.

Proof. For (i), we assume c is a $DU(K_\alpha)$-coloring. For all i and t, G_i and D_t are cliques of size 2α. Hence, in each of the cliques G_i and D_t exactly α vertices are colored 1 and the other α vertices are colored 2. As a consequence, no vertex u is adjacent to any vertex v such that $c(u) = c(v)$ and u and v belong to different cliques in $G_\Theta \sqcup D_\Theta$.

Let u be a vertex of G_Θ such that u is labeled by l and $\mathrm{var}(l) = x_t$. Since there are edges between u and the α vertices labeled by the negation of l in D_t, these α vertices cannot have the color $c(u)$. As a consequence, the α vertices labeled by l in D_t must have the color $c(u)$. We get that all the vertices of $\overline{G}_\Theta$ labeled with l receive the same color, which is different from the color of the vertices labeled by the negation of l, and we get (i). Moreover, c assigns exactly α vertices in each G_i to each of the colors, which implies that as_c assigns exactly α of the literals of C_i to *true* and the other α to *false*. Hence as_c is counted by $\#\alpha$-of-2α-SAT, and we get the direction (a)$\Rightarrow$(b) of (ii).

For the direction (b)$\Rightarrow$(a) of (ii), let c be a consistent coloring such that as_c is counted by $\#\alpha$-of-2α-SAT. We will prove that c is a $DU(K_\alpha)$-coloring. Since as_c is counted by $\#\alpha$-of-2α-SAT, it assigns *true* to α literals and *false* to the other α literals of each clause. Hence, c colors every clique G_i so that α vertices receive color 1 and the other α receive color 2. Each of the two colors induces a clique of size α in G_i. Since as_c is consistent and D_t consists of α vertices labeled by some label l and α vertices labeled by the negation of l, each color class of c induces a clique of size α in D_t. It remains to notice that, since c is consistent, any other edge of $\overline{G}_\Theta$ crosses between the color classes, hence does not belong to the induced subgraph of any of the two colors. Consequently, each of the colors induces a disjoint union of cliques of size α in $\overline{G}_\Theta$.

□

As a consequence of Lemma 3.12 we have $\#\alpha\text{-of-}2\alpha\text{-SAT}(\Theta) = \chi_{DU(K_\alpha)}(\overline{G}_\Theta; 2)$. From [CH96] we have:

Theorem 3.13. *For every $\alpha \geq 2$, $\#\alpha\text{-of-}2\alpha\text{-SAT}$ is $\#\mathbf{P}$-hard.*

From Lemma 3.12 and Theorem 3.13, and using the fact that the construction of $\overline{G}_\Theta$ can be done in polynomial time, we get:

Theorem 3.14. *For every $\alpha \geq 2$, $\chi_{DU(K_\alpha)}(G; 2)$ is $\#\mathbf{P}$-hard.*

Proposition 3.15. *$\chi_{DU(K_\alpha)}(G; 0)$ and $\chi_{DU(K_\alpha)}(G; 1)$ are polynomial time computable.*

Proof. For $X = 0$, $\chi_{DU(K_\alpha)}(G; 0)$ is always 0 and hence trivially polynomial time computable. For $X = 1$, there is exactly one coloring $c : V(G) \to [1]$ and c is a $DU(K_\alpha)$-coloring iff G is a disjoint union of copies of K_α, which can be verified in polynomial time. □

Putting all this together we get the full complexity spectrum for $\chi_{DU(K_\alpha)}(G; X)$ for $\alpha \geq 2$. Recall that $\chi_{DU(K_1)}(G; X) = \chi(G; X)$ is the chromatic polynomial.

Theorem 3.16. *For all $\alpha \geq 2$, we have $\mathrm{EASY}(\chi_{DU(K_\alpha)}) = \{0, 1\}$ and $\#\mathrm{PHARD}(\chi_{DU(K_\alpha)}) = \mathbf{F} - \{0, 1\}$.*
Moreover, for $\alpha \geq 1$ the Difficult Point Dichotomy is true for $\chi_{DU(K_\alpha)}(G; X)$, as for $\alpha = 1$ it includes the chromatic polynomial.

4. Counting convex colorings is $\#\mathbf{P}$-complete: the proof

The purpose of this section is to prove Theorem 3.5. This answers a question originally asked in [Mak08b], whether the problem of counting the number of convex colorings using at most two colors, i.e. computing $\chi_{convex}(-; 2)$, is #P-complete on connected graphs.

This section follows almost verbatim the preprint posted as [GN08].

4.1. Cuts, crossing sets, and cocircuits

Let X and Y be disjoint sets of vertices of a graph G. The set of edges of G that have one endpoint in X and the other in Y is denoted by $\delta(X, Y)$. Given a connected graph G, a cut is a partition of $V(G)$ into two (non-empty) sets called its *shores*. The *crossing set* of a cut with shores X and Y is $\delta(X, Y)$. A cut is a *cocircuit* if no proper subset of its crossing set is the crossing set of a cut. Given a set A of edges, we denote by $G \setminus A$ the graph obtained by removing the edges in A from G. We will use the following observation:

Lemma 4.1. *Let G be a connected graph. Then a cut of G with crossing set A is a cocircuit if and only if $G \setminus A$ has exactly two connected components.*

Note that our terminology is slightly at odds with standard usage in the sense that the terms cut and cocircuit usually refer to what we call the crossing set of respectively a cut and a cocircuit. Our usage prevents some cumbersome descriptions in the proofs. We will however abuse our notation by saying that a cut or cocircuit has size k if its crossing set has size k.

4.2. Reductions

We prove Theorem 3.5 by a sequence of reductions involving the following problems:

Problem: #COCIRCUITS
Input: Connected graph G.
Output: The number of cocircuits of G.

Problem: #REQUIRED SIZE COCIRCUITS
Input: Connected graph G, strictly positive integer k.
Output: The number of cocircuits of G of size k.

Problem: #MAX CUT
Input: Connected graph G, strictly positive integer k.
Output: The number of cuts of G of size k.

Problem: #MONOTONE 2-SAT
Input: A Boolean formula in conjuctive normal form in which each clause contains two variables and there are no negated literals.
Output: The number of satisfying assignments.

It is easy to see that each of these problems is a member of #P.

The following result is from Valiant's seminal paper on #P [Val79].

Theorem 4.2. #MONOTONE 2-SAT *is #P-complete.*

We will establish the following reductions.

$$\#\text{MONOTONE 2-SAT} \propto \#\text{MAX CUT} \propto \#\text{REQUIRED SIZE COCIRCUITS}$$
$$\propto \#\text{COCIRCUITS} \propto \chi_{convex}(-;2)$$

Combining Theorem 4.2 with these reductions shows that each of the five problems that we have discussed is #P-complete. As far as we are aware, each of these reductions is new. We have not been able to find a reference showing that #MAX CUT is $\#P$ complete. Perhaps it is correct to describe this result as 'folklore'. In any case we provide a proof below. Some similar problems, but not exactly what we consider here, are shown to be $\#P$ complete in [PB83].

Lemma 4.3. #MONOTONE 2-SAT $\propto$ #MAX CUT.

Proof. Suppose we have an instance I of #MONOTONE 2-SAT with variables $x_1, \ldots, x_n$ and clauses $\mathcal{C} = \{C_1, \ldots, C_m\}$. We construct a corresponding instance $M(I) = (G, k)$ of #MAX CUT by first defining a graph G with vertex set

$$\{x\} \cup \{x_1, \ldots, x_n\} \cup \bigcup \{\{c_{i,1}, \ldots, c_{i,6}\} : 1 \leq i \leq m\}$$

For each clause we add nine edges to G. Suppose C_j is $x_u \vee x_v$. Then we add the edges

$$xc_{j,1}, c_{j,1}c_{j,2}, c_{j,2}x_u, x_uc_{j,3}, c_{j,3}c_{j,4}, c_{j,4}x_v, x_vc_{j,5}, c_{j,5}c_{j,6}, c_{j,6}x.$$

Distinct clauses correspond to pairwise edge-disjoint circuits, each of size 9. Now let $k = 8|\mathcal{C}|$. We claim that the number of solutions of instance $M(I)$ of #MAX CUT is equal to $3^{|\mathcal{C}|}$ times the number of satisfying assignments of I.

Given a solution of I, let L_1 be the set of variables assigned the value true and L_0 the set of variables assigned false together with x. Observe that for each clause $C_j = x_u \vee x_v$ there are three choices of how to add the vertices $c_{j,1}, \ldots, c_{j,6}$ to either L_0 or L_1 so that exactly eight edges of the circuit corresponding to C_j have one endpoint in L_0 and the other in L_1. Clearly the choices for each clause are independent and distinct satisfying assignments result in distinct choices of L_0 and L_1. Any of the choices of L_0 and L_1 constructed in this way may be taken as the shores of a cut of size $8|\mathcal{C}|$. Hence we have constructed $3^{|\mathcal{C}|}$ solutions of $M(I)$ corresponding to each satisfying assignment of I.

In any graph the intersection of a set of edges forming a circuit and a crossing set of a cut must always have even size. So in a solution of $M(I)$ each of the edge-disjoint circuits making up G and corresponding to clauses of I must contribute exactly eight edges to the cut. Suppose U and $V \setminus U$ are the shores of a cut of G of size $8|\mathcal{C}|$. Then it can easily be verified that for any clause $C = x_u \vee x_v$ both U and $V \setminus U$ must contain at least one element from $\{x_u, x_v, x\}$. So it is straightforward to see that this solution of $M(I)$ is one of those constructed above corresponding to the satisfying assignment where a variable is false if and only if the corresponding vertex is in the same set as x. $\square$

Lemma 4.4. #MAX CUT $\propto$ #REQUIRED SIZE COCIRCUITS.

Proof. Suppose (G,k) is an instance of #MAX CUT. We construct an instance (G',k') of #REQUIRED SIZE COCIRCUITS as follows. Suppose G has n vertices. To form G' add new vertices $x, x', x_1, \dots, x_{n^2}$ to G. Now add an edge from x to every other vertex of G' except x' and similarly add an edge from x' to every other vertex of G' except x. Let $k' = n^2 + n + k$. From each solution of the #MAX CUT instance (G,k) we construct 2^{n^2+1} solutions of the #REQUIRED SIZE COCIRCUITS instance (G',k'). Suppose $C = (U, V(G) \setminus U)$ is a solution of (G,k) then we may freely choose to add $x, x', x_1, \dots, x_{n^2}$ to either U or $V(G) \setminus U$, with the sole proviso that x and x' are not both added to the same set, to obtain a cut in G' of size $k' = n^2 + n + k$. Furthermore this cut is a cocircuit because both shores contain exactly one of x and x' and so they induce connected subgraphs.

Conversely suppose $C = (U, V(G') \setminus U)$ is a cocircuit in G' of size k'. Consider the pair of edges incident with x_j. Note that the partition $(x_j, V(G') \setminus x_j)$ is a cocircuit. So if both of the edges incident with x_j are in the crossing set of C then because of its minimality we must have $C = (x_j, V(G') \setminus x_j)$ which is not possible because C would then have size $2 < k'$. Now suppose that neither edge incident with x_j is in the crossing set of C.

Then both x and x', and hence $x_1, \dots, x_{n^2}$, lie in the same block of the partition constituting C. But since G is a simple graph, the maximum possible size of such a cocircuit is at most $2n + \binom{n}{2} < n^2 + n + k$. Hence precisely one of the edges adjacent to x_j is in the crossing set. So x and x' are in different shores of C. Hence the crossing set of C contains: for each j precisely one edge incident to x_j (n^2 edges in total), for each $v \in V(G)$ precisely one of edges vx and vx' (n edges in total) and k other edges with both endpoints in $V(G)$. So the partition $C' = (U \cap V(G), V(G) \setminus U)$ is a cut of G of size k and hence C is one of the cocircuits constructed in the first part of the proof. Consequently the number of solutions of the instance (G',k') of #REQUIRED SIZE COCIRCUITS is 2^{n^2+1} multiplied by the number of solutions of the instance (G,k) of #MAX CUT □

Lemma 4.5. #REQUIRED SIZE COCIRCUITS $\propto$ #COCIRCUITS

Proof. Given a graph G let $N_k(G)$ denote the number of cocircuits of size k and $N(G)$ denote the total number of cocircuits. Let G_l denote the l-stretch of G, that is, the graph formed from G by replacing each edge of G by a path with l edges. Let $m = |E(G)|$. Then we claim that

$$N(G_l) = \sum_{k=1}^{m} l^k N_k(G) + \binom{l}{2} m.$$

To see this, suppose that C is a cocircuit of G_l. If the crossing set of C contains two edges from one of the paths corresponding to an edge of G then by the minimality of the crossing set of C we see that it contains precisely these two edges. The number of such cocircuits is $\binom{l}{2} m$.

Otherwise the crossing set C contains at most one edge from each path in G_l corresponding to an edge of G. Suppose the crossing set of C contains k such edges. Let A denote the corresponding edges in G. Then A is the crossing set of a cocircuit in G of size k. From each such cocircuit we can construct l^k cocircuits of G_l by choosing one edge from each path corresponding to an edge in A. The claim then follows.

If we compute $N(G_1), \ldots, N(G_m)$ then we may retrieve $N_1(G), \ldots, N_m(G)$ by using Gaussian elimination because the matrix of coefficients of the linear equations is an invertible Vandermonde matrix. The fact that the Gaussian elimination may be carried out in polynomial time follows from [Edm67]. □

Lemma 4.6. $\#$COCIRCUITS $\propto \chi_{convex}(-;2)$

Proof. The lemma is easily proved using the following observation. When two colors are available, there are two convex colorings of a connected graph using just one color and the number of convex colorings using both colors is equal to twice the number of cocircuits. □

The preceding lemmas imply our main result, Theorem 3.5, that $\chi_{convex}(-;2)$ is $\#$P-hard.

5. Detailed case study: Discrete spectra

In this section we dicuss cases where we were not able to find a suitable version of Linial's Trick, but where we could determine the complexity of the evaluations for non-negative integers.

5.1. Proper edge colorings

Recall that $\chi_{edge}(G;k)$ counts the number of proper edge colorings of a graph G with k colors. It is a polynomial in k because

$$\chi_{edge}(G;k) = \chi(L(G);k),$$

where $L(G)$ is the line graph of G and $\chi(G;k)$ is the chromatic polynomial. We have $\chi_{edge}(G;0) = 0$ and

$$\chi_{edge}(G;1) = \begin{cases} 1 & \text{if } G \text{ consists of isolated edges and vertices} \\ 0 & \text{otherwise} \end{cases}$$

Although $\chi_{edge}(G;k) = \chi(L(G);k)$, where $L(G)$ is the line graph of G, not every graph G is the line graph of some graph G'.

The class of all finite line graphs, $\mathcal{LG}$, has been completely characterized, [Bei70, BLS99].

Theorem 5.1 (Beineke, 1970, [Bei70])**.** *There are nine graphs $F_i : 1 \leq i \leq 9$, each with at most 6 vertices, such that $G \in \mathcal{LG}$ if and only if no F_i is an induced subgraph of G.*

The complexity spectrum of the chromatic polynomial restricted to the class $\mathcal{LG}$ has, to the best of our knowledge, not been studied.

Surprisingly, the complexity of counting proper edge colorings was proven $\#$**P**-hard only recently, [CGW14]:

Theorem 5.2 (J. Y. Cai, H. Guo, T. Williams, 2014)**.** *For $k \in \mathbb{N}$ we have:*

(i) $\chi_{edge}(G;k)$ *is* $\#\mathbf{P}$*-hard over planar* r*-regular graphs for all* $k \geq r \geq 3$.

(ii) χ_{edge} *is trivially tractable when* $k \geq r \geq 3$ *does not hold.*

The proof given in [CGW14] reduces $\chi_{edge}(G;k)$ to computation of the diagonal of the Tutte polynomial $T(G;X,X)$ using several intermediate steps via Holants[7].

Problem 5.3. *Find an elementary (holant-free) proof of Theorem 5.2.*

Theorem 5.2 gives us the discrete complexity spectrum for $k \in \mathbb{N}$.

We were unable to adapt Linial's Trick to proper edge colorings. Therefore we do not know how to determine the complexity of $\chi_{edge}(G;X)$ for $X = a$ and $a \in \mathbb{C} - \mathbb{N}$ or even $a \in \mathbb{Q} - \mathbb{N}$.

Problem 5.4. *Determine the full complexity spectrum of* $\chi_{edge}(G;X)$ *for* $X = a$ *and* $a \in \mathbb{Q}$ *or* $a \in \mathbb{C}$.

5.2. mcc_t*-colorings*

Let $t \in \mathbb{N}$. Recall that a coloring $f : V(G) \to [k]$ is an mcc_t *-coloring* with k colors, if the connected components of each color class have at most t vertices.

We easily verify:

Proposition 5.5. (i) *For fixed* $t \in \mathbb{N}^+$ *being an* mcc_t *-coloring with* k *colors is a* $\mathbf{CP}$*-property, hence* $\chi_{mcc_t}(G;k)$ *is a polynomial in* k *(but not in* t*).*

(ii) $\chi_{mcc_t}(G;k)$ *is multiplicative over disjoint unions.*

(iii) *For* $t = 1$ *we have* $\chi_{mcc_1}(G;k) = \chi(G;k)$*, i.e., it is the chromatic polynomial.*

(iv) *For* $k = 1$ *a graph* G *is* mcc_t *-colorable iff* G *is a disjoint union of connected graphs with at most* t *vertices.*

We next establish a complexity result.

Theorem 5.6. *Computing* $\chi_{mcc_t}(G;2)$ *is* $\#\mathbf{P}$*-complete for* $t \geq 2$.

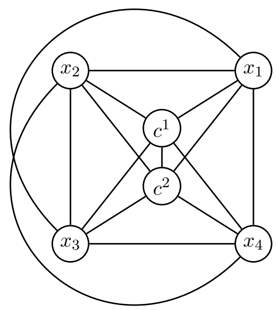
To prove Theorem 5.6 we use a result due to Creignou and Hermann [CH96]. Let NAE_k be the Boolean relation of all k-tuples having at least one 0 and at least one 1, i.e. $\mathrm{NAE}_k = \{0,1\}^k \setminus \{0\cdots 0, 1\cdots 1\}$, commonly called the not-all-equal relation. Let $\mathrm{NAE}_k(x_1,\ldots,x_k)$ be a constraint which is satisfied only by all tuples from the relation NAE_k. A NAE_k formula $\varphi = c_1 \wedge \cdots \wedge c_n$ is satisfied if each clause c_i is not-all-equal satisfied. Given a CNF formula φ, the counting problem $\#\mathrm{NAE}_k\mathrm{SAT}$ asks for the number of not-all-equal satisfying assignments of φ.

Theorem 5.7 ([CH96])**.** $\#\mathrm{NAE}_k\mathrm{SAT}$ *is* $\#\mathbf{P}$*-complete for each* $k \geq 3$.

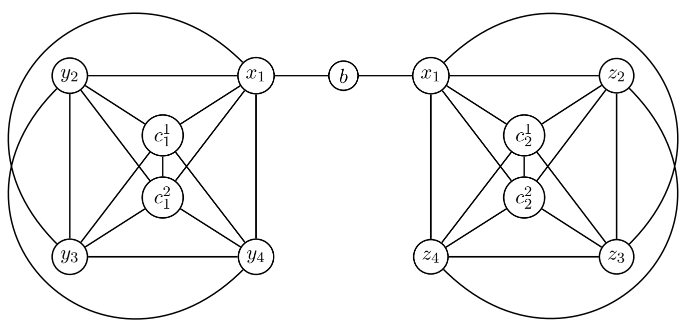
Proof of Theorem 5.6. The membership in $\#\mathbf{P}$ is clear, therefore we focus on the proof of $\#\mathbf{P}$-hardness. We perform a parsimonious reduction from $\#\mathrm{NAE}_{t+1}\mathrm{SAT}$.

Given a NAE_{t+1} formula φ, we associate with it the graph G_φ in the following way. For each clause $c = \mathrm{NAE}_{t+1}(x_1,\ldots,x_{t+1})$ construct the clause gadget (as in

[7]For background on holants, cf. [Val08, CLX11]



(a) Clause gadget



(b) Bridge

Figure 2: Clause gadget (a) for a clause $c = \mathrm{NAE}_4(x_1, x_2, x_3, x_4)$ and a bridge (b) connecting two vertices labeled by the same variable x_1 in two different clauses $c_1 = \mathrm{NAE}_4(x_1, y_2, y_3, y_4)$ and $c_2 = \mathrm{NAE}_4(x_1, z_2, z_3, z_4)$.

Figure 2a for $t = 3$), consisting of a complete graph K_{2t}. Label $t + 1$ vertices of K_{2t} by the variables $x_1, \ldots, x_{t+1}$ and the remaining $t-1$ vertices by $c^1, \ldots, c^{t-1}$, where c is the identifier of the clause.

Connect different copies of the same variable in two clause gadgets by a bridge with a new vertex b. There is a new bridge vertex b for each pair of variable vertices labeled by the same variable x in different clause gadgets. Figure 2b illustrates (for $t = 3$) how two copies of the variable x_1 in clauses $c_1 = \mathrm{NAE}_{t+1}(x_1, y_2, \ldots, y_{t+1})$ and $c_2 = \mathrm{NAE}_{t+1}(x_1, z_2, \ldots, z_{t+1})$ are connected through a bridge with a new node b.

We show that each mcc_t-coloring of the graph G_φ encodes a NAE_{t+1}-satisfiability of the formula φ. A valid mcc_t 2-coloring of each clause gadget forces t variables to be colored by the color 0 and the t others by the color 1. Restricted to the variable nodes $x_1, \ldots, x_{t+1}$ of the clause gadget, this represents a correct assignment of the constraint $\mathrm{NAE}_{t+1}(x_1, \ldots, x_{t+1})$. No other 2-coloring of the clause gadget is a valid mcc_t coloring.

When two occurrences of the same variable x are connected through a bridge with a vertex b, this bridge vertex b must be colored by a different color than the vertices labeled by x. Indeed, in a valid mcc_t coloring, the vertex x in the clause gadget is connected to another $t-1$ vertices colored by the same color. This induces a different color for the vertex b, otherwise there would be a connected component containing $t + 1$ vertices and colored by the same color. This also forces the two copies of the variable x, connected by a bridge, to be colored by the same color.

Hence, the satisfying assignments of a $\mathrm{NAE}_{t+1}\mathrm{SAT}$ formula φ are in one-to-one correspondence to the mcc_t 2-colorings of G_φ, which constitutes a parsimonious reduction from $\#\mathrm{NAE}_{t+1}\mathrm{SAT}$. □

Next we determine the complexity of $\chi_{mcc_t}(G; k)$ for $k, t \in \mathbb{N}$.

Theorem 5.8. *For any integers t and k that are both at least two, $\chi_{mcc_t}(G; k)$ is #**P**-complete.*

Proof. Membership in $\#\mathbf{P}$ is clear. We shall show that if t and k are integers and both at least two, then $\chi_{mcc_t}(G; k)$ is polynomial time reducible to $\chi_{mcc_t}(G; k+1)$. Combining this with Theorem 5.6 gives the result.

Given a graph G, form G' from G by adding a clique on $t(k+1)$ new vertices and joining one of the new vertices to every vertex of G. In any valid coloring of G' with $k+1$ colors, each color class must contain precisely t of the new vertices and these vertices can be colored in $\binom{(k+1)t}{t,t,\ldots,t}$ ways, where $\binom{(k+1)t}{t,t,\ldots,t}$ is the multinomial coefficient counting the number of ways of choosing an ordered collection of $k+1$ subsets each of size t from a set of size $(k+1)t$. Once the new vertices have been colored there are k colors available to color the vertices of G. Thus

$$\chi_{mcc_t}(G'; k+1) = \binom{(k+1)t}{t,t,\ldots,t}\chi_{mcc_t}(G; k).$$

The result follows. □

Remark 4. *The statement*

$$\chi_{mcc_t}(G'; k+1) = \binom{(k+1)t}{t,t,\ldots,t}\chi_{mcc_t}(G; k).$$

appears at first sight to be an instance of Linial's Trick, but there is a subtle difference. In the proof above the choice of $G' = f(G, t, k)$ both depends on t and k.

In applying Linial's Trick we require that the graph $G' = g(G)$ does not depend on k in order to get a polynomial identity of the form

$$\chi_{mcc_t}(g^n(G); k+n) = f(k, n, t) \cdot \chi_{mcc_t}(G; k).$$

Currently we do not know whether there is a version of Linial's Trick which can be used for $\chi_{mcc_t}(G; X)$.

Using the fact that $\chi_{mcc_1}(G; X)$ is the chromatic polynomial and the previous discussion (Proposition 5.5, and Theorems 5.6 and 5.8) we get

Corollary 5. *Evaluating $\chi_{mcc_t}(G; X)$ is in $\mathbf{P}$ for $t = 1$ and $k = 0, 1, 2$, and for $t \geq 2$ and $k = 1$. For all other values of $t, k \in \mathbb{N}$ evaluation is $\#\mathbf{P}$-complete.*

Problem 5.9. *What is the complexity spectrum for $\chi_{mcc_t}(G; a)$ for $t \geq 2$ and $a \in \mathbf{F} - \mathbb{N}$?*

In Theorem 3.16 we have determined completely the complexity spectrum for $\chi_{DU(K_\alpha)}(G; X)$. This was meant to be a warm-up exercise for determining the complexity spectrum for $\chi_{mcc_t}(G; X)$, as each $DU(K_\alpha)$-coloring is also an mcc_α-coloring. However, determining the difficulty of evaluating $\chi_{mcc_t}(G; X)$ seems to be much more demanding.

5.3. H-free-colorings

A function $f : V(G) \to [k]$ is an H-free coloring if no color class induces a graph isomorphic to H. Clearly this is a $\mathbf{CP}$-property, hence $\chi_{H-free}(G; k)$ is a polynomial in k. The discrete complexity spectrum is rather well understood:

Theorem 5.10. (i) *([ABCM98]) $\chi_{H-free}(G; k)$ is $\#\mathbf{P}$-hard for every $k \geq 3$ and H with at least 2 vertices.*

(ii) *([Ach97]) $\chi_{H-free}(G; 2)$ is $\mathbf{NP}$-hard for every H with at most 2 vertices.*

It is easy to see that for $X = 0, 1$ we have the following evaluations:

$$\chi_{H-free}(G; 0) = 0$$

and

$$\chi_{H-free}(G; 1) = \begin{cases} 1 & \text{if } G \text{ is } H - \text{free} \\ 0 & \text{otherwise .} \end{cases}$$

Problem 5.11. *What is the complexity of evaluation of $\chi_{H-free}(G; a)$ for $a \in \mathbf{F} - \mathbb{N}$?*

H-free coloring is another case where Linial's Trick does not seem to work.

6. More graph polynomials

In this section we discuss some graph polynomials for which we have only partial knowledge, if any, about the complexity spectrum. The graph polynomials $\chi_\Phi(G;k)$ we discuss arise from $\mathbf{CP}$-properties Φ defined in Section 6.1. Three of them, (i-iii), belong to the framework of $\mathcal{P}_1 - \mathcal{P}_2$-colorings from Subsection 2.2, listed in Table 3, and the remaining two, (iv,v), are mentioned here because they have a rich literature.

In each case we do not know — but suspect — that evaluation of $\chi_\Phi(G;X)$ is $\#\mathbf{P}$-hard for $X = a$ for at least one $a \in \mathbf{F}$.

Problem 6.1. *Determine the full complexity spectrum of $\chi_\Phi(G;X)$ for each of the graph polynomials in Section 6.1.*

Instead of counting colorings one can look at the corresponding decision problem which asks whether $\chi_\Phi(G;k) > 0$. Clearly, the counting problem is at least as hard as this decision problem. For each of the graph properties Φ above we do know that computing the polynomial $\chi_\Phi(G;X)$ is $\mathbf{NP}$-hard. Furthermore, in two cases we discuss dichotomy theorems showing that an evaluation of $\chi_\Phi(G;X)$ is either in $\mathbf{P}$ or $\mathbf{NP}$-hard.

6.1. Graph polynomials with incomplete complexity spectrum

We consider the following colorings:

(i) Let $t \in \mathbb{N}$. A function $c : V(G) \to [k]$ is a *t-improper coloring* if every color induces a graph of maximal degree t. t-improper colorings were studied in [CGJ97]. They originate in certain network problems.

(ii) A function $c : V(G) \to [k]$ is an *acyclic coloring* if it is proper and there is no two colored cycle in G. Acyclic colorings were introduced in [Gru73] and further studied in [AMR91].

(iii) A function $f : V(G) \to [k]$ is a *co-coloring* if every color class induces a graph which is either a clique or an independent set. Co-colorings were first studied in [HJL77].

(iv) A function $f : E(G) \to [k]$ is a rainbow-path coloring (rainbow coloring for short) if every two vertices are connected by a path in which every two edges are colored differently. Rainbow colorings were first introduced in [CJMZ08].

(v) A function $f : V(G) \to [k]$ is an injective coloring if it is injective on the open neighborhood of every vertex. f does not have to be a proper coloring. In other words, if there is a path of length 2 between v and u then u and v must have different colors.

Let Φ_{t-imp}, Φ_{acyc}, Φ_{co-co}, $\Phi_{rainbow}$, and Φ_{inject} denote respectively the coloring properties of t-improper colorings for fixed $t \in \mathbb{N}$, acyclic colorings, co-colorings, rainbow colorings, and injective colorings. Clearly, each of these coloring properties Φ is a $\mathbf{CP}$-property, and hence the number of colorings in each Φ is a polynomial in k. We denote by $\chi_{t-imp}(G;k)$, $\chi_{acyc}(G;k)$, $\chi_{co-co}(G;k)$, $\chi_{rainbow}(G;k)$, and $\chi_{inject}(G;k)$ the graph polynomials $\chi_\Phi(G;k)$ counting colorings with at most k colors of a graph G in the corresponding Φ.

For acyclic colorings, co-colorings, rainbow colorings and injective colorings, the complexity spectrum is completely unknown. For t-improper colorings, partial results are known. For $t = 0$, the t-improper colorings are exactly the proper colorings, hence $\chi_{0-imp}(G; k) = \chi(G; k)$ and the complexity spectrum is completely understood. For $t = 1$ and $k \in \mathbb{N}$ we have $\chi_{1-imp}(G; k) = \chi_{mcc_2}(G; k)$ and the complexity spectrum is completely understood. For $t = 2$ every color class consists of a disjoint union of paths and cycles. This is the first case where the complexity spectrum of $\chi_{2-imp}(G; k)$ is not known.

6.2. $\mathbf{NP}$*-hardness*

From the literature, we have that each of the graph polynomials defined in Section 6.1 is $\mathbf{NP}$-hard:

Theorem 6.2. (i) *Let $t \in \mathbb{N}$. Let Φ be one of Φ_{t-imp} Φ_{co-co}, $\Phi_{rainbow}$, and Φ_{inject}. Computing the minimal integer $k \in \mathbb{N}$ such that $\chi_\Phi(G; k) > 0$ is $\mathbf{NP}$-hard.*

(ii) *It is NP-hard to decide for a given G and k if the acyclic chromatic number of G is at most k.*

Proof. The case of t-improper colorings in Theorem 6.2(i) follows directly from [CGJ97]. The case of co-colorings is proven in [GKS94]. The case of rainbow colorings is proven in [CFMY11]. The case of injective colorings follows directly from [HKSS02]. Theorem 6.2(ii) is proven in [Kos78]. □

For t-improper colorings we are able to give a dichotomy theorem for graphs with multiple edges:

Proposition 6.3. (i) *For every $a \in \mathbf{F}-\{0, 1\}$ the problem of evaluating $\chi_{t-imp}(G; a)$, where the input runs over the class of graphs with multiple edges allowed, is $\mathbf{NP}$-hard.*

(ii) *The evaluations $\chi_{t-imp}(G; 0)$ and $\chi_{t-imp}(G; 1)$ are in $\mathbf{P}$.*

Proof. Let $G \bowtie_t K_1$ be the graph obtained from G by adding a new vertex v and putting $t + 1$ edges between v and any vertex of G. Clearly, v cannot be colored the same color as any other vertex of G. Therefore, we can use Linial's Trick. □

For acyclic colorings, we have a partial dichotomy. The evaluations of $\chi_{acyc}(G; X)$ with $X = 0$ and $X = 1$ are trivial by definition, because every proper coloring with less than two colors is acyclic.

Proposition 6.4. *For every $a \notin \mathbb{N}$, it is NP-hard to compute $\chi_{acyc}(G; a)$.*

Proof. We use a version of Linial's Trick for the chromatic polynomial. Consider $G \bowtie K_1$. Every acyclic coloring f of $G \bowtie K_1$ with color set $[k]$ must color the vertex K_1 with a unique color. The coloring $f|_G$ induced by f on G is clearly proper and does not contain any two-colored cycles, since f is acyclic. On the other hand, every acyclic coloring g of G with color set $[k - 1]$ can be transformed to an acyclic coloring f of $G \bowtie K_1$ with color set $[k]$ by coloring K_1 with any of the k colors, and then setting

$$f(v) = \begin{cases} g(v) & g(v) < f(K_1) \\ g(v) + 1 & g(v) \geq f(K_1) \end{cases}$$

for any $v \in V(G)$. Hence we get

$$\chi_{acyc}(G \bowtie K_1; x_0) = \chi_{acyc}(G; x_0 - 1) \cdot x_0 .$$

Using Theorem 6.2(ii) we complete the proof. $\square$

A solution of the following problem would complete the dichotomy:

Problem 6.5. *What is the complexity of the evaluation $\chi_{acyc}(G; a)$ for $a \in \mathbb{N} - \{0, 1\}$?*

7. Conclusions and open problems

In the light of the discovery of many univariate graph polynomials in [KMZ11], and the fact that infinitely many of them are incomparable in expressive power, Theorem 2.3, we initiated the systematic study of these graph polynomials. Inspired by N. Linial's work in [Lin86] we have concentrated in this paper on the complexity of evaluating these graph polynomials. We have introduced the full and the discrete complexity spectrum of univariate graph polynomials. In this paper we concentrated our attention on graph polynomials arising from graph colorings previously studied in the literature.

Throughout the paper we have listed[8] open problems we encountered in our explorations. They mostly ask for the complete determination of the full complexity spectrum for specific graph polynomials. Among the more interesting challenges we have the following:

Problem 7.1. *Determine the full complexity spectrum of*

(i) *the edge chromatic polynomial $\chi_{edge}(G; X)$;*
(ii) *the polynomial $\chi_{mcc_t}(G; X)$ for $t \geq 2$;*
(iii) *the polynomials $\chi_{H-free}(G; X)$ for $t \geq 2$;*
(iv) *the generalized chromatic polynomials derived from t-improper colorings, acyclic colorings, co-colorings, rainbow-path colorings and injective colorings of Section 6.*

We have not found a single graph polynomial which does not have a complexity spectrum satisfying the Difficult Point Dichotomy. Our results so far suggest that there might be *meta-theorem* to be formulated, and finally also to be proven, which says that for a large class of graph polynomials Difficult Point Dichotomy holds. The large class in question should be defined by some definability criterion. Examples of such criteria could come from descriptive complexity theory.

Here are some candidates for logically defined classes[9] of univariate graph polynomials

(i) All SOL-definable graph polynomials.

[8]These are Problems 1.6, 2.4, 5.3, 5.4, 3.10, 5.9, 5.11, 6.1 and 6.5.

[9]For a discussion of logically defined classes of graph polynomials, see [Mak08a, MKR13, KMZ11, Kot12].

(ii) All MSOL-definable graph polynomials.

(iii) All $\mathcal{P}$-chromatic polynomials where $\mathcal{P}$ is in $\mathbf{NP}$, or equivalently, where $\mathcal{P}$ is definable in $\exists\mathrm{SOL}$, the existential fragment of SOL.

Previous experience suggests that it is too early to formulate a solid conjecture.

Problem 7.2. *Formulate and prove a meta-theorem for the Difficult Point Dichotomy.*

We hope the search for a meta-theorem will spawn further research and will lead to new insights both in graph theory and in descriptive complexity theory.

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