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On pricing risky loans and collateralized fund obligations

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Loan spreads are analyzed for two types of loans. The first type takes losses at maturity only; the second follows the formulation of collateralized fund obligations, with losses registered over the lifetime of the contract. In both cases, the implementation requires the choice of a process for the underlying asset value and the identification of the parameters. The parameters of the process are inferred from the option volatility surface by treating equity options as compound options with equity itself being viewed as an option on the asset value with a strike set at the debt level following Merton. Using data on the stock of General Motors during 2002–3, we show that the use of spectrally negative Lévy processes is capable of delivering realistic spreads without inflating debt levels, deflating debt maturities or deviating from the estimated probability laws.

1 INTRODUCTION

Credit structuring technology has been very successfully used to develop the gigantic market of collateralized debt obligations (CDOs), where a variety of bonds and debt instruments constitute the underlying assets. Recent times have seen the advent of collateralized fund obligations (CFOs), structures that offer investors exposure to

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funds of funds and, in some cases, private equity funds or managed accounts. From a practical standpoint, portfolio managers benefit from a risk-tranching approach to investing in a diversified pool of funds that is often difficult to access separately. In the case of so-called threshold CFOs, investors receive interest from a fund of funds and the underlying assets are managed according to incremental trigger levels.

In all cases, the obligations are backed by the net asset value of the fund of funds. Typically, this asset value must maintain a specified ratio (termed the advance rate) over the total amount of CFOs issued. When the advance rate is breached, the lowest-priority tranche takes a loss to restore this rate and bring the structure back into compliance. Thereafter, this tranche receives coupons only on the remaining amount loaned. A further reduction of this rate results in further losses until the lowest-priority tranche is exhausted and, hence, takes no further losses and receives no further coupons. At that point, all subsequent losses are absorbed by the next lowest seniority tranche. Traditionally, advance rates are derived from observed asset price volatility: the more homogeneous the collateral, the greater the reliance on historical volatility (Tavakoli (2003); Mahadevan and Schwartz (2002)).

The aim of this paper is to develop in a continuous-time setting the pricing of these loan structures. We obtain closed-form solutions in the case of maturities defined by an exponential distribution independent of the underlying asset value. Geman and Yor (1996) explain that a functional of Brownian motion, eg, the price of an Asian or double barrier option in the Black–Scholes setting, is often easier to compute for such random maturity times, naturally leading to the search for the Laplace transform of the price. Since the advance rate of a CFO is violated when the infimum of the asset value to date breaks a trigger level, we are interested in the law for the lowest asset value to date. When there are jumps in asset value, for spectrally negative Lévy processes (ie, ones exposed only to downward jumps in value), we may easily access the Laplace transform for the infimum to date of the process. The fixed maturity distribution then follows on, employing the efficient inversion procedures described in Abate and Whitt (1995) and Rogers (2000). For related work in jump-diffusion models with exponentially distributed jumps, see Kou and Wang (2003, 2004) and Lipton (2002).

We compare the pricing of such contracts with the more classic and simpler contracts that take loss of coupon and principal just at the final maturity. The classic loans require the computation of deep out-of-the-money option prices, especially for high-priority loans that take loss only after a very substantial loss in asset value. For such options, classic Fourier-inversion methods (Carr and Madan (1999)) for computing option prices from analytical characteristic functions break down, yielding negative prices for call and put options. Here we implement the recently developed saddlepoint methods of Carr and Madan (2009), following the earlier work of Rogers and Zane (1999) among others.

The level of the resulting loan spreads has critical implications for the ratings to be assigned to the various tranches. One may obtain a basic mapping between spreads and ratings from the BondsOnline Quote and Data website[1] by country, sector, date and maturity. For example, for the financial sector of the US on December 29, 2006 the AAA, AA, A, BBB, BB and B five-year spreads were 42, 91, 107, 167, 424 and 375 basis points, respectively. In this regard we note the important differences between classic loan contracts taking losses at the end and those that take them along the way. The analytical methods of this paper illuminate these important issues.

The analysis requires the specification of a risk-neutral process for the underlying asset-value process and we are well aware of the observed limitations of geometric Brownian motion in this regard. For longer-dated contracts like those appearing in structured finance, Eberlein and Madan (2009) observe that spectrally negative Lévy processes are well suited to the option surface at the longer maturities. Hence we restrict attention to this class of processes.

Closed-form results have also been obtained for certain jump-diffusion models (Kou and Wang (2003, 2004)), but we question their relevance. One of the original motivations for the Gaussian distribution is that it is a limiting distribution approximately obtained on summing a large number of independent effects. Especially for longer-maturity contracts, one expects many independent effects on asset values and this suggests that one should use a limit law. Fortunately, the Gaussian distribution is not the only limit law and there are many others. Lévy (1937) and Khintchine (1938) characterized all the limit laws as self-decomposable laws (Carr *et al* (2007)) and showed, in particular, that processes with finitely many jumps or jump diffusions are not limit laws. As a consequence, jump diffusions may not adequately represent the level of activity in the markets. The models we use are associated with self-decomposable laws at unit time and are limit laws.

We perform an analysis of the source of spreads on risky loans and conclude that it is the level of frequent and small price moves in the markets that drives the spreads and not the structure of infrequent large jumps. We focus our attention here on jump processes with finite variation, thereby supposing that the jumps add up to a finite value, and leave for future research the development of results and procedures for processes with infinite variation.

The specific risk-neutral process we employ is the CGMY model with only negative jumps enhanced with a diffusion, which was studied in Eberlein and Madan (2009). The model has four parameters given by $C > 0$, $G > 0$, $0 < Y < 1$ and the diffusion coefficient $\sigma > 0$. We term this model CGYSN for the downsided CGMY with a diffusion component.

[1] URL: www.bondsonlinequotes.com.

Three analyses are conducted: one using stylized parameter values; a second analyzing specifically the effects of activity rates on risky loan spreads; and a third using risk-neutral processes extracted from the calibration of equity option surfaces as a potential source of possibly relevant risk-neutral asset price processes. In this regard we follow the lead of Merton (1974) and Moody's KMV and treat equity prices as call options written on the asset value with a strike set at the debt level and a maturity matching the debt maturity. We then specify the asset-value process to be in the CGYSN class and calibrate the Merton equity model to the surface of quoted option prices. We thereby calibrate, possibly for the first time, the compound-option model to the surface of equity options when the asset-value process is taken to be an infinite activity Lévy process.

We illustrate with an application to data on General Motors for 2002–3 and conclude that the spectrally negative Lévy process calibration to equity or asset values is capable of delivering realistic spread levels without inflating debt levels, deflating debt maturities or deviating from the probabilities embedded in the estimated process.

The outline of the rest of the paper is as follows. The two loan contracts, classic and CFO, are introduced in Section 2 along with the computational details for pricing these contracts. Section 3 presents a report on a stylized study of the effects of different parameters on the resulting loan spreads. Section 4 considers the effects of activity levels or a high frequency of small jumps on risky loan spreads. The details for the calibration of the Merton (1974) compound-option view of equity options are provided in Section 5. An application to options on General Motors is undertaken in Section 6 and Section 7 concludes.

2 ANALYSIS OF THE TWO CONTRACTS

For the classic loan contract, consider a loan of amount L for a maturity T with lower-priority loans in the amount B. Suppose that the asset value of the borrower is A_0 and that there is equity capital of H. The loan of L will suffer a loss of principal if, at the final time T, the asset value A falls short of $(A_0 - (B + H))$ and then the loss of principal will be the smaller of L and $(A_0 - (B + H) - A)$. Hence, the principal returned is given by $(L - (A_0 - (B + H) - A)^+)^+$. Let c be the continuously compounded coupon rate on the loan with a single payment on the outstanding balance at the maturity.

Let the risk-neutral density of the final asset value be $f(A)$. The loan-pricing equation then requires that the expected present value, at a continuously compounded interest rate of r, of the single payment at maturity equals the amount L loaned upfront, or equivalently that:

$$\mathrm{e}^{-rT}\mathrm{e}^{cT}\int_0^\infty (L - (A_0 - (B + H) - A)^+)^+ f(A)\,\mathrm{d}A = L \tag{1}$$

One may solve Equation (1) for the coupon c, noting further that when there is no risk of loss, Equation (1) reduces to:

$$e^{cT}e^{-rT} = 1 \quad \text{and} \quad c = r$$

Let the lower-priority capital be $K = B + H$. The classic coupon equation may be expressed as:

$$e^{cT}[\mathrm{Call}_A(A_0 - K - L) - \mathrm{Call}_A(A_0 - K)] = L$$

where the subscript A denotes the underlying asset and the quantities in parentheses are the strikes of the two call options. We recognize call spreads here, as it is the case of limited insurance protection.

Hence, we obtain that:

$$c = -\log\left(\frac{1}{L}[\mathrm{Call}_A(A_0 - K - L) - \mathrm{Call}_A(A_0 - K)]\right)\frac{1}{T}$$

Typically, the strikes involved in these call options may be deep in the money (with associated put strikes deep out of the money). For example, for highly secured AAA loans, there is a substantial amount of lower-priority capital reducing the strike far below the at-the-money point of A_0. It is precisely for these cases that the Fourier methods of Carr and Madan (1999) break down. This leads us to adopt the saddlepoint pricing methods of Carr and Madan (2009).

Next we consider the continuous-time CFO and first associate with the initial and terminal asset values A_0, A a continuous-time process $(A(t),\ 0 \leqslant t \leqslant T)$ with $A_0 = A(0)$ and $A = A(T)$. We now take the coupon to be paid continuously through time on the outstanding balance. Risk-neutral pricing now requires that the integrated expected discounted coupon payments plus the expected discounted return of principal equal the amount L that was loaned upfront. For an advance rate of η this yields the coupon equation:

$$\left.\begin{aligned} L &= cE\int_0^T e^{-ru}(L - (X(0) - (B + H) - X(u))^+)^+\,du \\ &\qquad\qquad + e^{-rT}(L - (X(0) - (B + H) - X(T))^+)^+ \\ a(t) &= \inf_{0 \leqslant s \leqslant t} A(s) \\ X(t) &= \frac{a(t)}{\eta} \end{aligned}\right\} \tag{2}$$

If there is no chance of loss we observe that Equation (2) reduces to:

$$1 = c\int_0^T e^{-ru}\,du + e^{-rT}$$

or:

$$1 - e^{-rT} = \frac{c}{r}(1 - e^{-rT})$$

and we again have the result that $c = r$.

For the computation of the classic coupon defined in Equation (1) we just need the density of the terminal asset value A, while for the computation of the CFO coupon described in Equation (2), we need the density $g(x, u)$ of x, the infimum to date u of the asset price process deflated by the advance rate. In terms of these densities we may rewrite the equation for the CFO coupon as:

$$c \int_0^T \int_0^\infty e^{-ru}(L - (X(0) - (B + H) - a)^+)^+ g(a, u)\,da\,du \\ + e^{-rT} \int_0^\infty (L - (X(0) - (B + H) - a)^+)^+ g(a, T) = L$$

We may now write this expression in terms of call prices on the infimum process as:

$$c\frac{1}{L} \int_0^T [\mathrm{Call}_{X,u}(X(0) - K - L) - \mathrm{Call}_{X,u}(X(0) - K)]\,du \\ + \frac{1}{L}[\mathrm{Call}_{X,T}(X(0) - K - L) - \mathrm{Call}_{X,T}(X(0) - K)] = 1$$

Hence:

$$c = \frac{1 - (1/L)[\mathrm{Call}_{X,T}(X(0) - K - L) - \mathrm{Call}_{X,T}(X(0) - K)]}{(1/L)\int_0^T [\mathrm{Call}_{X,u}(X(0) - K - L) - \mathrm{Call}_{X,u}(X(0) - K)]\,du}$$

For log asset price processes in the CGYSN class subject to an evolution made up of drift, exposure to a Brownian motion with constant volatility σ and a compensated jump martingale with exposure only to downside or negative jumps, we have access to the Laplace transform of the logarithm of the deflated infimum in terms of the Laplace exponent of the logarithm of $A(1)/\eta$ as follows (Rogers (2000)):

$$\begin{aligned}\psi_\lambda(z) &= \int_0^\infty \lambda e^{-\lambda t} E[e^{zx(t)}]\,dt \\ &= \frac{\lambda}{\lambda - \psi(z)} \frac{\beta - z}{\beta}\end{aligned}$$

where:

$$\psi(z) = \log(E[e^{z \ln(A(1)/\eta)}])$$

and β satisfies:

$$\psi(\beta) = \lambda$$

The specific structure of the Lévy density employed for the down jumps is the negative side of the CGMY model and is given by:

$$k_{\mathrm{CGY}}(x) = C\frac{\mathrm{e}^{-G|x|}}{|x|^{1+Y}}\mathbf{1}_{x<0} \tag{3}$$

The characteristic exponent for the CGYSN process is:

$$\psi_{\mathrm{CGYSN}}(z) = \tfrac{1}{2}\sigma^2 z^2 + C\Gamma(-Y)((G+z)^Y - G^Y)$$

We therefore have access to the characteristic function of the logarithm of the infimum of advance rate deflated asset values taken at an independent exponential time. We may, without loss of generality, absorb the deflation by the advance rate into the underlying process parameters and, henceforth, we work with an advance rate of unity. From the characteristic function of the logarithm at independent exponential times one easily derives the Laplace transform of call prices on the infimum. The actual finite maturity call prices follow on inverting this Laplace transform, which we then integrate to construct the required coupon rates. For the specific Laplace transform inversion algorithm employed we refer to Abate and Whitt (1995). We follow Rogers (2000) to change the contour of integration in λ to avoid having to solve the equation $\psi(\beta) = \lambda$. We find that this method works well for finite variation processes that requires $Y < 1$ for our chosen process. For $Y > 1$ the altered contour remains at distance from the original contour for typical settings advocated in the Laplace inversion as described in Rogers (2000). We leave the case of infinite variation, or $Y > 1$, for future research.

3 STYLIZED INVESTIGATION OF PARAMETERS AND LOAN SPREADS

For a stylized investigation of the effects of varying the parameters of the spectrally negative Lévy process on the structure of loan spreads, we took three settings for each of the parameters that we may regard as low, medium and high. However, instead of choosing the level of C as a parameter we chose instead the aggregate volatility v, where:

$$v^2 = \sigma^2 + \frac{C}{\Gamma(2-Y)G^{2-Y}} \tag{4}$$

We then used three levels for the proportion of total volatility due to the diffusion component that gave us σ in terms of ν and we solved Equation (4) for C given prespecified levels for Y, G. The parameters are then specified on choosing three levels of aggregate volatility, diffusion proportions and G, Y respectively. This gives us 81 cases in all. In addition we have some loan-specific and market-specific variables that influence the loan spread. These are the levels of lower-priority capital, the loan

TABLE 1 Input settings.

Variable	Levels		
Volatility	0.25	0.5	0.75
G	1	5	10
Y	0.25	0.5	0.75
Diffusion proportion	0.25	0.5	0.75
Lower capital	70	80	90
Maturity	1	3	5
Interest rate	0.025	0.05	0.1

maturities for the contract-specific variables and the level of risk-free interest rates as the market-specific variable. Choosing three levels for each of these three variables gives us 27 cases for these variables. In total, we therefore have $2{,}187 = 81 \times 27$ cases. The specific levels of the required variables are presented in Table 1.

For each of these 2,187 cases we computed the classic coupon rate and the CFO coupon rate. We would like to summarize the effects of the various inputs on the coupon rates and recognize that the coupon rate is a deterministic function of these inputs. Recognizing that one may use linear regression to get some idea of the slopes of complicated non-linear functions we set up a fixed-effects regression model, where the constant term reflects the first level for all 7 variables. We then add 14 dummy variables for the second and third levels of the 7 input variables. The regression therefore has 15 explanatory variables inclusive of the constant term. There are two regressions, with the classic coupon and the CFO coupon as the dependent variable in turn. The results of the regression are presented in Table 2 on the facing page. We exclude the presentation of t-statistics as this does not have any real interpretation, since there is no randomness involved but a mere linear projection of a more complicated function. The R^2 is included as a measure of the quality of the linear projection.

We make the following remarks on these results.

- The average CFO coupon exceeds the classic coupon, suggesting that more risk is taken in the CFO structure. This finding is consistent with the relationship between early default models like Longstaff and Schwartz (1995) and the default-at-maturity model of Merton (1974) as observed, for example, in Eom (2004).

- The diffusion component has a positive effect on spreads, suggesting that spreads are responsive to the level of small activity. The role of the diffusion component is analyzed in greater depth in the next section.

TABLE 2 Regressions of classic and CFO coupons in basis points.

Variable	Classic coupon coefficient	CFO coupon coefficient
Constant	99.00	147.1
Vol2	139.9	237.9
Vol3	580.6	959.3
G2	30.19	61.74
G3	31.66	69.10
Y2	2.114	5.105
Y3	4.278	9.620
DP2	0.181	12.53
DP3	4.659	29.40
LC2	−194.6	−365.4
LC3	−345.9	−622.6
T2	136.3	234.4
T3	176.6	301.2
R2	−26.76	−36.41
R3	−72.73	−99.94
RSQUARE	0.767	0.760

- The total volatility has a high, positive and non-linear effect that is more pronounced for the CFO structure.

- Interestingly, the effect of raising G, which increases the relative size of the small activity, has a positive effect that is relatively linear. This also suggests that the cumulated effects of small jumps are important.

- The effect of increasing Y is positive. This again suggests that raising the level of small activity raises spreads. A deeper analysis of activity levels is conducted in the next section.

- The effect of higher priority is negative (as expected), non-linear and more pronounced for the CFO structure.

- The effects of maturity are positive, slightly non-linear and more pronounced for the CFO structures. We do not expect inverted or humped spread curves as the underlying process is one of independent and identically distributed increments with no effects of mean reversion or other Markovian effects in volatility, for example.

TABLE 3 Classic coupons by rate and maturity.

	Rate 1	Rate 2	Rate 3
Maturity 1	160.38	152.92	138.09
Maturity 2	323.88	293.95	241.65
Maturity 3	378.81	336.64	265.14

TABLE 4 CFO coupons by rate and maturity.

	Rate 1	Rate 2	Rate 3
Maturity 1	250.86	237.55	213.15
Maturity 2	519.94	478.64	406.33
Maturity 3	602.71	548.09	454.19

TABLE 5 Classic and CFO loan coupons.

Maturity	Classic	CFO
0.25	49.25	34.34
0.5	160.9	95.95
1	185.1	266.0
2	313.7	546.4
3	405.4	696.4
4	454.5	772.1
5	479.3	810.4

- Lower-interest-rate environments necessitate larger spreads. This finding is consistent with the theoretical observations in Longstaff and Schwartz (1995) and with the findings of Collin-Dufresne *et al* (2001).

In addition, we present, in Tables 3 and 4 respectively, the average levels of spreads for both the classic and CFO coupons in the three interest-rate regimes for the three maturities.

By way of a specific computation for a fixed parameter set we took $G = 5, Y = 0.5$ and, with a diffusion proportion of 25% and a volatility of 50%, the values for σ and C were 0.25 and 2.3654, respectively. For this parameter setting and a lower-priority capital of 70%, a loan amount of 5 with an interest rate of 3%, in Table 5 we present the classic and CFO coupons for maturities of three months, six months and one to five years. We expect that short-maturity spreads, though positive, will be small when priority levels are high, as they are related to the tail of the Lévy measure.

4 ACTIVITY RATES AND RISKY LOAN SPREADS

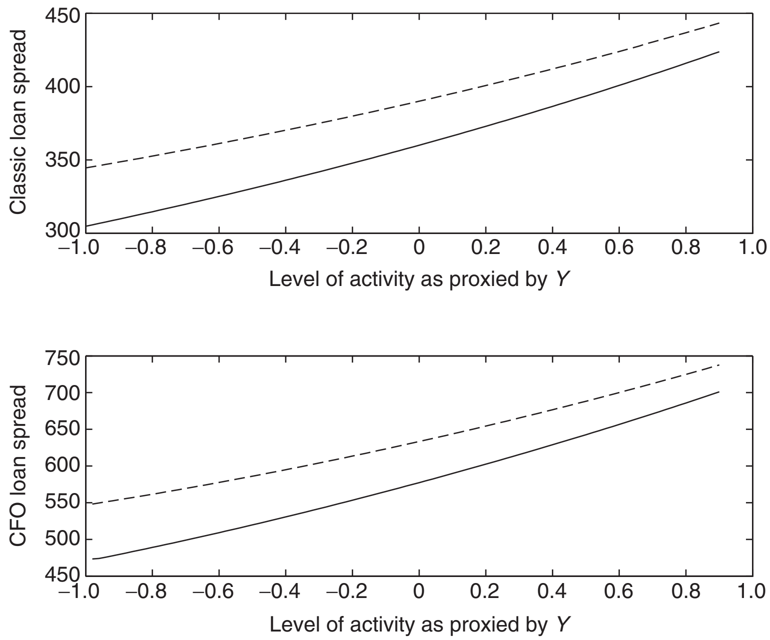
A number of authors have considered, following Merton (1976), the enhancement of diffusion models by the addition of a jump component with either exponential or Gaussian jumps. In the context of structures like the CFO, we have closed-form results for the exponential case (Lipton (2002); Kou and Wang (2004)). We may also apply the methods of this paper for finite jump-activity processes by merely taking $Y < 0$ (Carr *et al* (2002) define the notion of finite and infinite activity for a Lévy process by the integral of the Lévy measure being respectively finite or infinite). The choice $Y = -1$ is the exponential jump case. In this section we analyze the effects of changing Y on the level of loan spreads for risky loans, be they classic or like a CFO.

For this purpose we fix an overall volatility at 50% and view this as the market-calibrated volatility. For the purpose of this study we fix the value of $G = 1$. We consider two levels, 50% and 60%, for the proportion of total volatility attributed to the diffusion component and set the parameter C to take the rest of the volatility. In order to closely study the effects of changing Y from finite to infinite activity we vary the parameter Y from -1 to 0.9 in steps of 0.02. The value of -1 represents the exponential jump case, as then the Lévy measure in Equation (3) is just an exponential function. For other values of $Y > -1$, the activity level or the number of jumps expected in unit time rise until we reach infinite activity when $Y \geqslant 0$. The number of jumps expected in an interval is given by the integral of the Lévy measure over its domain and, for $Y < 0$, this integral is $CG^Y\Gamma(-Y) < \infty$. For $Y > 0$ the integral is infinite and we have infinitely many jumps in any interval, most of which will be small. With $Y < 1$ the sum of all the jumps has a finite expectation given by $CG^Y\Gamma(1-Y)$ obtained on integrating the identity function against the Lévy measure.

We present two graphs in Figure 1 on the next page displaying the loan spread as a solid line for the classic and the CFO loan when the diffusion proportion is 50%, and as a dashed line for the same loans when the diffusion proportion is 60%. We observe that loan spreads rise with an increase in the diffusion component and with an increase in the level of small activity. We conclude that finite activity models may understate the risk of loans, especially when the diffusion component is adjusted downward to create short-maturity spreads. As a consequence, the level of small activity is reduced, because we do not have an infinite level of small jumps in a jump-diffusion model.

With a view to studying the effect of activity levels on the term structure of loan spreads, we construct the term structure for four activity levels represented by $Y = (-0.75, -0.25, 0.25, 0.75)$. We used a volatility of 50% and a diffusion proportion of 50% with $G = 1$ for all the graphs. In Figure 2 on page 49 we present two graphs representing the term structure of spreads for the four activity levels, for the classic loan and the CFO loan. We observe that loan spreads rise substantially with the level

FIGURE 1 Graph displaying risky loan spreads.



The spreads are presented as functions of the level of activity proxied by the parameter Y. Solid line, 50% diffusion component; dashed line, 60% diffusion component on the classic and CFO loan structures.

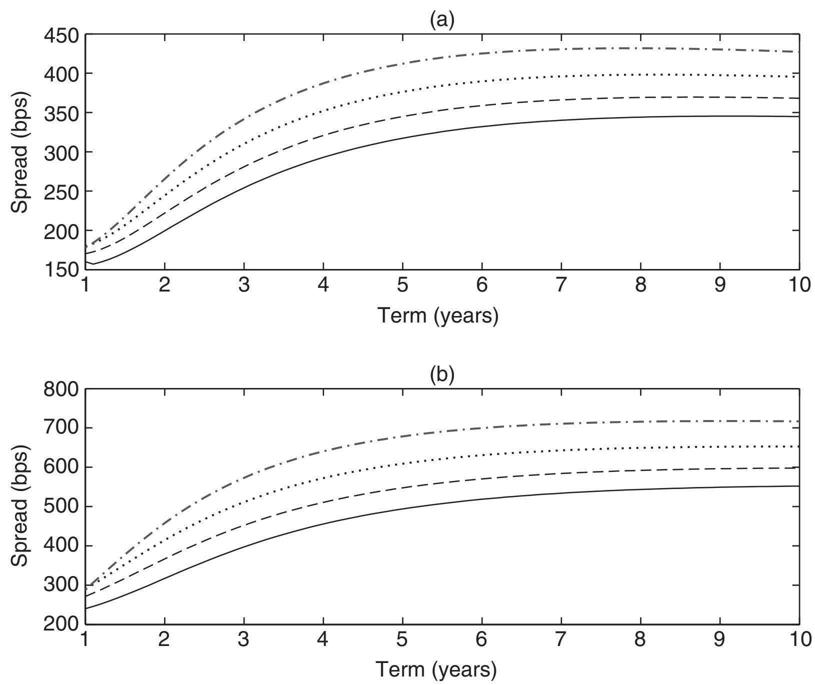
of small activity and longer-maturity loans are likely to be understated using finite activity processes.

We note further that such finite activity models have return distributions that do not correspond to limit laws over any horizon. The class of limit laws includes the Gaussian distribution and is fully characterized by the class of self-decomposable laws (Lévy (1937); Khintchine (1938)). Limit laws are provably associated with Lévy measures displaying infinite activity and furthermore their probability distributions are unimodal (Sato (1999)).

5 CALIBRATION OF THE MERTON (1974) COMPOUND-OPTION INTERPRETATION OF EQUITY OPTIONS TO THE OPTION SURFACE

Merton (1974) introduced the representation of stock prices as call options on the underlying asset value struck at the debt level with a maturity matching the debt maturity. Moody's KMV adopted this representation and used the model to simultaneously

FIGURE 2 Graphs displaying the term structure of loan spreads for (a) the classic loan and (b) the CFO loan at four activity levels.



$Y = -0.75$, solid line; $Y = -0.25$, dashed line; $Y = 0.25$, dotted line; $Y = 0.75$, dash-dotted line.

infer asset values and volatilities from option data to construct their distance-to-default measure. Recently, Bharath and Shumway (2008) reported on an investigation of the performance of this model. We follow this perspective but modify the underlying asset-value process to be in the CGYSN class and employ the full option surface of quoted prices on strikes and maturities to estimate the initial asset value and the model parameters of the underlying Lévy process for asset values. We then use the underlying asset-value process to price the classic loan and the CFO loan on this asset value.

The specific dynamics for the asset-value process $A = (A(t), t \geqslant 0)$ is given by:

$$\begin{aligned} \mathrm{d}A(t) &= (r-q)A(t\_)\,\mathrm{d}t + \sigma A(t\_)\,\mathrm{d}W \\ &\qquad + A(t\_)\int_{-\infty}^{\infty}(\mathrm{e}^{x}-1)(\mu(\mathrm{d}x,\mathrm{d}t) - k_{\mathrm{CGY}}(x)\,\mathrm{d}x\,\mathrm{d}t) \end{aligned}$$

where the Lévy density is as defined in Equation (3). The characteristic function for the logarithm of the asset value at a future date t is given by:

$$E[\mathrm{e}^{\mathrm{i}uA(t)}] = \mathrm{e}^{t\psi_A(u)}$$

where:

$$\psi_A(u) = \psi_{\mathrm{CGYSN}}(\mathrm{i}u) + \mathrm{i}u\omega$$

and:

$$\omega = \ln(A(0)) + (r - q - \psi_{\mathrm{CGYSN}}(1))$$

We use the saddlepoint methods of Carr and Madan (2009) to compute options' prices from this characteristic function.

We take for a prospective company on a particular date an estimate of the debt level D and debt maturity M as the strike and maturity for the call option that represents the equity value. For a prospective set of parameter values σ, C, G and Y we determine the initial asset value A_0 from the observed stock price S_0 on solving the equation:

$$S_0 = C(A_0, D, M; \sigma, C, G, Y)$$

where the function $C(A, K, T; \sigma, C, G, Y)$ is the call-pricing function for the CGYSN model.

To calibrate the parameters of the CGYSN model we then simulate the asset price process starting at A_0 to create an $N \times 10{,}000$ matrix representing asset values on 10,000 paths at N dates matching the equity option maturities. For each equity option maturity we transform the asset values to equity values by:

$$S(t) = C(A(t), D, M; \sigma, C, G, Y)$$

where we take a stationary view of the relationship between asset values and equity values that may come from debt being constantly rolled over to a fixed maturity. Alternatively we could take a non-stationary view and subtract from M the elapsed time. However, these are small values of option maturity relative to the much larger debt maturity and we worked with a stationary view.

Given 10,000 readings of the equity price for each option maturity we may price options at all the traded strikes using a discounted expected cashflow computation. This gives us call prices consistent with the compound-option view of these equity options. We then compute the least-squares criterion to minimize the distance between these model prices and the observed market prices. The underlying asset price process is then estimated by the parameter values that minimize this least-squares criterion. We simulated the asset price process at a weekly time-step and, at this setting, we computed the minimization criterion in 4.5 seconds of CPU time. The seeds of the random number generators for the simulation are frozen to force the objective function to have a non-random output, which is needed for classic optimization algorithms.

6 ILLUSTRATIVE APPLICATION TO THE CASE OF GENERAL MOTORS

We illustrate a first application of the procedure to data on General Motors at the end of years 2002 and 2003. This was a year when the credit default swap rates on General Motors had gone up to some very high levels, reaching 480 basis points in October 2002. From the Compustat database for the year 2003 we took the annual financial statement and obtained a debt level in millions of US dollars of 191,133. The level of equity outstanding in the financial statement was 25,268 and the average stock price for the month of December 2003 was 49.4377. Dividing the equity value by the stock price gives us the number of shares outstanding at 511.1076 million shares. The strike per share was then set at the debt level per share and this number was 373.9584. The period duration reported in Compustat is 12 years and we took this value for M the debt maturity.

We then used the Sato process of the four-parameter VGSSD model reported in Carr *et al* (2007) as an option surface synthesizer. We fitted this model for each trading day in December 2003 and averaged the parameter values over the 22 days. The average parameter values were:

$$\sigma = 0.2803, \qquad \nu = 0.9027, \qquad \theta = -0.1131, \qquad \gamma = 0.5314$$

From these parameter values and the initial stock price of 49.4377 we may construct target strikes and maturities of our own. Essentially we use the Sato process as a surface interpolator to give us option prices at strikes and maturities of our choosing. We used four maturities: three months, six months, nine months and a year. For each of these maturities we used nine strikes struck at 80% of the spot to 120% of the spot at 5% intervals. We obtained a total of 36 target option prices to be calibrated by the Merton compound-option equity pricing model.

We first calibrated the CGYSN Lévy process model to these 36 target option prices at our nine strikes and four maturities. Here we just fitted the equity option data with a CGYSN Lévy model with no Mertonian compound-option input. The estimated parameters were:

$$\sigma = 0.2064, \qquad C = 0.0956, \qquad G = 1.1818, \qquad Y = 0.4953$$

We used these values as the starting values for the estimation of the Merton compound-option model. The calibrated asset-value process was estimated at:

$$\sigma = 0.0189, \qquad C = 0.2297, \qquad G = 1.0991, \qquad Y = 0.3604$$

We then computed the loan spreads for lower-priority capital at 70% for the five-year classic loan and the CFO loan, with both the equity and Merton asset-value calibration.

TABLE 6 General Motors loan spreads 2003.

Loan	Equity calibration	Asset calibration
Classic	110.55	224.94
CFO	192.02	336.52

TABLE 7 General Motors loan spreads 2002 with lower capital: 70%.

Loan	Equity calibration	Asset calibration
Classic	566.82	751.32
CFO	922.98	1,206.04

The average credit default spread for General Motors in the month of December 2003 was 174.50, a number that lies within the range of the equity calibration for a lower-priority capital of 70%. The credit default spread for the month of December 2002 was 345.20. We performed the same exercise for the year 2002 with a level of long-term debt of 134,272 million US dollars, equity of 6,814 and an average share price of 37.14. The equity calibration yielded the parameter values for the model CGYSN of lower-priority capital of 70%, as shown in Table 6:

$$\sigma = 0.005, \qquad C = 3.1340, \qquad G = 3.6533, \qquad Y = 0.2451$$

The asset-value process with a maturity equal to the period duration of 12 years yielded the parameter values of:

$$\sigma = 0.0445, \qquad C = 4.3443, \qquad G = 3.3445, \qquad Y = 0.0086$$

The corresponding loan spreads at lower-priority capital of 70% are presented in Table 7. These spreads are significantly higher than the credit default spread of 345.20. However, if we consider more senior tranches with a higher level of lower-priority capital of, say, 85%, the values are as displayed in Table 8 on the facing page.

We conclude with the observation that the calibration of either the equity value or the asset value to the option surface using the spectrally negative Lévy process model in the CGYSN class is capable of yielding realistic credit spreads without having to deflate maturities to one year and inflate debt levels to total liabilities, as appears to be the case with the diffusion model applications as reported in Bharath and Shumway (2008) and Vassalou and Xing (2003). Furthermore, we do not deviate from the underlying probability laws, as appears to be the case with Moody's KMV.

TABLE 8 General Motors loan spreads 2002 with lower capital: 85%.

Loan	Equity calibration	Asset calibration
Classic	217.59	337.46
CFO	341.19	516.23

7 CONCLUSION

Loan spreads are computed for two types of loans. The first is a classic loan taking loss of coupon and principal at maturity, while the second follows the structure of the more recent CFO contracts. A potential application of the methods presented in this paper relates to the issue of rating counterparties by inferring the level of implied loan spreads. The actual computation of a loan spread requires the choice of the underlying asset-value process and a knowledge of the parameters. We follow Moody's KMV and develop procedures for inferring asset-value parameters from the surface equity option prices when the asset-value process is taken in the CGYSN class of spectrally negative processes. The procedures are illustrated for data on General Motors for 2002–3, and we conclude that the spectrally negative Lévy process calibration in the CGYSN class is capable of delivering realistic spreads without inflating debt levels, deflating debt maturities and deviating from the underlying estimated probability laws.

We also analyze the effects of activity rates on loan spreads to find that these spreads are responsive to a high level of small activity. We note that, although there is some theoretical interest in the dynamics of spreads within the CGYSN model, of greater interest would be the dynamics of spreads as they would be calibrated in an economy with stochastic rates and option surfaces that allow for variations in rates and volatility parameters. The constancy of these factors in the model is just an averaging effect entertained for reasons of tractability in computation, even though one is well aware of their stochasticity.

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