Mathematical Finance, Vol. 0, No. 0 (xxx 2013), 1–22

RISK MEASURES ON $\mathcal{P}(\mathbb{R})$ AND VALUE AT RISK WITH PROBABILITY/LOSS FUNCTION

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We propose a generalization of the classical notion of the $V@R_\lambda$ that takes into account not only the probability of the losses, but the balance between such probability and the amount of the loss. This is obtained by defining a new class of law invariant risk measures based on an appropriate family of acceptance sets. The $V@R_\lambda$ and other known law invariant risk measures turn out to be special cases of our proposal. We further prove the dual representation of Risk Measures on $\mathcal{P}(\mathbb{R})$.

KEY WORDS: Value at Risk, distribution functions, quantiles, law invariant risk measures, quasi-convex functions, dual representation.

1. INTRODUCTION

We introduce a new class of law invariant risk measures $\Phi : \mathcal{P}(\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$ that are directly defined on the set $\mathcal{P}(\mathbb{R})$ of probability measures on $\mathbb{R}$ and are monotone and quasi-convex on $\mathcal{P}(\mathbb{R})$.

As Cherny and Madan (2009) pointed out, for a (*translation invariant*) coherent risk measure defined on random variables, all the positions can be spited in two classes: acceptable and not acceptable; in contrast, for an *acceptability index* there is a whole continuum of degrees of acceptability defined by a system $\{\mathcal{A}^m\}_{m\in\mathbb{R}}$ of sets. This formulation has been further investigated by Drapeau and Kupper (2010) for the quasi-convex case, with emphasis on the notion of an acceptability family and on the robust representation.

We adopt this approach and we build the maps Φ from a family $\{\mathcal{A}^m\}_{m\in\mathbb{R}}$ of acceptance sets of distribution functions by defining:

$$\Phi(P) := -\sup\{m \in \mathbb{R} \mid P \in \mathcal{A}^m\}.$$

The authors would like to thank Prof. Fabio Bellini, University Milano Bicocca, Samuel Drapeau, Humboldt University, as well as an anonymous referee for helpful discussion on this subject. MM acknowledges the financial support provided by the European Social Fund Grant.

Manuscript received February 2012; final revision received October 2012.

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DOI: 10.1111/mafi.12028

In Section 3 we study the properties of such maps, we provide some specific examples, and in particular we propose an interesting generalization of the classical notion of $V@R_\lambda$.

The key idea of our proposal—the definition of the $\Lambda V@R$ in Section 4—arises from the consideration that to assess the risk of a financial position it is necessary to consider not only the probability λ of the loss, as in the case of the $V@R_\lambda$, but the dependence between such *probability* λ and the *amount* of the loss. In other terms, a risk prudent agent is willing to accept greater losses only with smaller probabilities. Hence, we replace the constant λ with a (increasing) function $\Lambda : \mathbb{R} \to [0, 1]$ defined on losses, which we call *Probability/Loss function*. The balance between the probability and the amount of the losses is incorporated in the definition of the family of acceptance sets

$$\mathcal{A}^m := \{Q \in \mathcal{P}(\mathbb{R}) \mid Q(-\infty, x] \leq \Lambda(x),\ \forall x \leq m\}, m \in \mathbb{R}.$$

If P_X is the distribution function of the random variable X, our new measure is defined by:

$$\Lambda V@R(P_X) := -\sup\{m \in \mathbb{R} \mid P(X \leq x) \leq \Lambda(x),\ \forall x \leq m\}.$$

As a consequence, the acceptance sets $\mathcal{A}^m$ are not obtained by the translation of $\mathcal{A}^0$ which implies that the map is not any more translation invariant. However, the similar property

$$\Lambda V@R(P_{X+\alpha}) = \Lambda^\alpha V@R(P_X) - \alpha,$$

where $\Lambda^\alpha(x) = \Lambda(x + \alpha)$, holds true and is discussed in Section 4.

The $V@R_\lambda$ and the worst case risk measure are special cases of the $\Lambda V@R$.

The approach of considering risk measures defined directly on the set of distribution functions is not new and it was already adopted by Weber (2006). However, in this paper we are interested in quasi-convex risk measures based—as the above mentioned map—on families of acceptance sets of distributions and in the analysis of their robust representation. We choose to define the risk measures on the entire set $\mathcal{P}(\mathbb{R})$ and not only on its subset of probabilities having compact support, as it was done by Drapeau and Kupper (2010). For this, we endow $\mathcal{P}(\mathbb{R})$ with the $\sigma(\mathcal{P}(\mathbb{R}), C_b(\mathbb{R}))$ topology. The selection of this topology is also justified by the fact (see Proposition 2.5) that for monotone maps $\sigma(\mathcal{P}(\mathbb{R}), C_b(\mathbb{R})) - lsc$ is equivalent to continuity from above. In Section 5 we briefly compare the robust representation obtained in this paper and those obtained by Cerreia-Vioglio (2009) and Drapeau and Kupper (2010).

Except for $\Phi = +\infty$, we show that there are no *convex*, $\sigma(\mathcal{P}(\mathbb{R}), C_b(\mathbb{R})) - lsc$ translation invariant maps $\Phi : \mathcal{P}(\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$. But there are many quasi-convex and $\sigma(\mathcal{P}(\mathbb{R}), C_b(\mathbb{R})) - lsc$ maps $\Phi : \mathcal{P}(\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$ that in addition are monotone and translation invariant, as for example the $V@R_\lambda$, the entropic risk measure, and the worst case risk measure. This is another good motivation to adopt quasi-convexity versus convexity.

Finally, we provide the dual representation of quasi-convex, monotone, and $\sigma(\mathcal{P}(\mathbb{R}), C_b(\mathbb{R})) - lsc$ maps $\Phi : \mathcal{P}(\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$—*defined on the entire set* $\mathcal{P}(\mathbb{R})$—and compute the dual representation of the risk measures associated to families of acceptance sets and consequently of the $\Lambda V@R$.

2. LAW INVARIANT RISK MEASURES

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $L^0 =: L^0(\Omega, \mathcal{F}, \mathbb{P})$ be the space of $\mathcal{F}$ measurable random variables that are $\mathbb{P}$ almost surely finite. Any random variable $X \in L^0$ induces a probability measure P_X on $(\mathbb{R},\mathcal{B}_{\mathbb{R}})$ by $P_X(B) = \mathbb{P}(X^{-1}(B))$ for every Borel set $B \in \mathcal{B}_{\mathbb{R}}$. We refer to Aliprantis and Border (2005, chapter 15) for a detailed study of the convex set $\mathcal{P} =: \mathcal{P}(\mathbb{R})$ of probability measures on $\mathbb{R}$. Here we just recall some basic notions: for any $X \in L^0$ we have $P_X \in \mathcal{P}$ so that we will associate to any random variable a unique element in $\mathcal{P}$. If $\mathbb{P}(X = x) = 1$ for some $x \in \mathbb{R}$ then P_X is the Dirac distribution δ_x that concentrates the mass in the point x. A map $\rho : L \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$, defined on given subset $L \subseteq L^0$, is law invariant if $X, Y \in L$ and $P_X = P_Y$ implies $\rho(X) = \rho(Y)$.

Therefore, when considering law invariant risk measures $\rho : L^0 \to \overline{\mathbb{R}}$ it is natural to shift the problem to the set $\mathcal{P}$ by defining the new map $\Phi : \mathcal{P} \to \overline{\mathbb{R}}$ as $\Phi(P_X) = \rho(X)$. This map Φ is well defined on the entire $\mathcal{P}$, because there exists a bi-injective relation between $\mathcal{P}$ and the quotient space $\frac{L^0}{\sim}$ (provided that $(\Omega, \mathcal{F}, \mathbb{P})$ supports a random variable with uniform distribution), where the equivalence is given by $X \sim_{\mathcal{D}} Y \Leftrightarrow P_X = P_Y$. However, $\mathcal{P}$ is only a convex set and the usual operations on $\mathcal{P}$ are not induced by those on L^0, namely $(P_X + P_Y)(A) = P_X(A) + P_Y(A) \neq P_{X+Y}(A)$, $A \in \mathcal{B}_{\mathbb{R}}$.

Recall that the first-order stochastic dominance on $\mathcal{P}$ is given by: $Q \preccurlyeq P \Leftrightarrow F_P(x) \leq F_Q(x)$ for all $x \in \mathbb{R}$, where $F_P(x) = P(-\infty, x]$ and $F_Q(x) = Q(-\infty, x]$ are the distribution functions of $P, Q \in \mathcal{P}$. Note that $X \leq Y$ $\mathbb{P}$-a.s. implies $P_X \preccurlyeq P_Y$.

DEFINITION 2.1. A Risk Measure on $\mathcal{P}(\mathbb{R})$ is a map $\Phi : \mathcal{P} \to \mathbb{R} \cup \{+\infty\}$ such that:

(Mon) Φ is monotone decreasing: $P \preccurlyeq Q$ implies $\Phi(P) \geq \Phi(Q)$;
(QCo) Φ is quasi-convex: $\Phi(\lambda P + (1 - \lambda)Q) \leq \Phi(P) \vee \Phi(Q)$, $\lambda \in [0, 1]$.

Quasi-convexity can be equivalently reformulated in terms of sublevel sets: a map Φ is quasi-convex if for every $c \in \mathbb{R}$ the set $\mathcal{A}_c = \{P \in \mathcal{P} \mid \Phi(P) \leq c\}$ is convex. As recalled in Weber (2006), this notion of convexity is different from the one given for random variables (as in Föllmer and Schied 2004) because it does not concern diversification of financial positions. A natural interpretation in terms of compound lotteries is the following: whenever two probability measures P and Q are acceptable at some level c and $\lambda \in [0, 1]$ is a probability, then the compound lottery $\lambda P + (1 - \lambda)Q$, which randomizes over P and Q, is also acceptable at the same level. In terms of random variables (namely X, Y which induce P_X, P_Y), the randomized probability $\lambda P_X + (1 - \lambda)P_Y$ will correspond to some random variable $Z \neq \lambda X + (1 - \lambda)Y$ so that the diversification is realized at the level of distribution and not at the level of portfolio selection.

As suggested by Weber (2006), we define the translation operator T_m on the set $\mathcal{P}(\mathbb{R})$ by: $T_m P(-\infty, x] = P(-\infty, x - m]$, for every $m \in \mathbb{R}$. Equivalently, if P_X is the probability distribution of a random variable X we define the translation operator as $T_m P_X = P_{X+m}$, $m \in \mathbb{R}$. As a consequence we map the distribution $F_X(x)$ into $F_X(x - m)$. Note that $P \preccurlyeq T_m P$ for any $m > 0$.

DEFINITION 2.2. If $\Phi : \mathcal{P} \to \mathbb{R} \cup \{+\infty\}$ is a risk measure on $\mathcal{P}$, we say that (TrI) Φ is translation invariant if $\Phi(T_m P) = \Phi(P) - m$ for any $m \in \mathbb{R}$.

Note that (TrI) corresponds exactly to the notion of cash additivity for risk measures defined on a space of random variables as introduced in Artzner et al. (1999). It is well known (see Cerreia-Vioglio et al. 2011b) that for maps defined on random variables, quasi-convexity and cash additivity imply convexity. However, in the context of distributions (QCo) and (TrI) do not imply convexity of the map Φ, as can be shown with the simple examples of the $V@R$ and the worst case risk measure ρ_w (see the examples in Section 3.1).

The set $\mathcal{P}(\mathbb{R})$ spans the space $ca(\mathbb{R}) := \{\mu \text{ signed measure } \mid V_\mu < +\infty\}$ of all signed measures of bounded variations on $\mathbb{R}$. $ca(\mathbb{R})$ (or simply ca) endowed with the norm $V_\mu = \sup\left\{\sum_{i=1}^n |\mu(A_i)| \text{ s.t. } \{A_1, \ldots, A_n\} \text{ partition of } \mathbb{R}\right\}$ is a norm complete and an Abstract Lebesgue space (see Aliprantis and Border 2005, paragraph 10.11).

Let $C_b(\mathbb{R})$ (or simply C_b) be the space of bounded continuous function $f : \mathbb{R} \to \mathbb{R}$. We endow $ca(\mathbb{R})$ with the weak* topology $\sigma(ca, C_b)$. The dual pairing $\langle \cdot, \cdot \rangle : C_b \times ca \to \mathbb{R}$ is given by $\langle f, \mu \rangle = \int f d\mu$ and the function $\mu \mapsto \int f d\mu$ ($\mu \in ca$) is $\sigma(ca, C_b)$ continuous. Note that $\mathcal{P}$ is a $\sigma(ca, C_b)$-closed convex subset of ca (p. 507 in Aliprantis and Border 2005) so that $\sigma(\mathcal{P}, C_b)$ is the relativization of $\sigma(ca, C_b)$ to $\mathcal{P}$ and any $\sigma(\mathcal{P}, C_b)$-closed subset of $\mathcal{P}$ is also $\sigma(ca, C_b)$-closed.

Even though $(ca, \sigma(ca, C_b))$ is not metrizable in general, its subset $\mathcal{P}$ is separable and metrizable (see Aliprantis and Border 2005, theorem 15.12) and therefore when dealing with convergence in $\mathcal{P}$ we may work with sequences instead of nets.

For every real function F we denote by $\mathcal{C}(F)$ the set of points in which the function F is continuous.

THEOREM 2.3 (Shiryaev 1995, theorem 2, p. 314). *Suppose that P_n, $P \in \mathcal{P}$. Then $P_n \overset{\sigma(\mathcal{P}, C_b)}{\longrightarrow} P$ if and only if $F_{P_n}(x) \to F_P(x)$ for every $x \in \mathcal{C}(F_P)$.*

A sequence of probabilities $\{P_n\} \subset \mathcal{P}$ is decreasing, denoted with $P_n\downarrow$, if $F_{P_n}(x) \leq F_{P_{n+1}}(x)$ for all $x \in \mathbb{R}$ and all n.

DEFINITION 2.4. Suppose that P_n, $P \in \mathcal{P}$. We say that $P_n \downarrow P$ whenever $P_n \downarrow$ and $F_{P_n}(x) \uparrow F_P(x)$ for every $x \in \mathcal{C}(F_P)$. We say that (CfA) Φ *is continuous from above if $P_n \downarrow P$ implies $\Phi(P_n) \uparrow \Phi(P)$.*

PROPOSITION 2.5. *Let $\Phi : \mathcal{P} \to \overline{\mathbb{R}}$ be (Mon). Then the following are equivalent:*

Φ is $\sigma(\mathcal{P}, C_b)$-lower semicontinuous
Φ is continuous from above.

Proof. Let Φ be $\sigma(\mathcal{P}, C_b)$-lower semicontinuous and suppose that $P_n \downarrow P$. Then $F_{P_n}(x) \uparrow F_P(x)$ for every $x \in \mathcal{C}(F_P)$ and we deduce from Theorem 2.3 that $P_n \overset{\sigma(\mathcal{P}, C_b)}{\longrightarrow} P$. (Mon) implies $\Phi(P_n) \uparrow$ and $k := \lim_n \Phi(P_n) \leq \Phi(P)$. The lower level set $A_k = \{Q \in \mathcal{P} \mid \Phi(Q) \leq k\}$ is $\sigma(\mathcal{P}, C_b)$ closed and, because $P_n \in A_k$, we also have $P \in A_k$, i.e., $\Phi(P) = k$, and Φ is continuous from above.

Conversely, suppose that Φ is continuous from above. As $\mathcal{P}$ is metrizable we may work with sequences instead of nets. For $k \in \mathbb{R}$ consider $A_k = \{P \in \mathcal{P} \mid \Phi(P) \leq k\}$ and a sequence $\{P_n\} \subseteq A_k$ such that $P_n \overset{\sigma(\mathcal{P}, C_b)}{\longrightarrow} P \in \mathcal{P}$. We need to show that $P \in A_k$. Lemma 2.6 shows that each $F_{Q_n} := (\inf_{m \geq n} F_{P_m}) \wedge F_P$ is the distribution function of a probability measure and $Q_n \downarrow P$. From (Mon) and $P_n \preccurlyeq Q_n$, we get $\Phi(Q_n) \leq \Phi(P_n)$. From (CfA) then: $\Phi(P) = \lim_n \Phi(Q_n) \leq \liminf_n \Phi(P_n) \leq k$. Thus, $P \in A_k$. $\square$

LEMMA 2.6. *For every* $P_n \overset{\sigma(\mathcal{P},C_p)}{\longrightarrow} P$ *we have that*

$$F_{Q_n} := \inf_{m\geq n} F_{P_m} \wedge F_P, n \in \mathbb{N},$$

is a distribution function associated to a probability measure $Q_n \in \mathcal{P}$ *such that* $Q_n \downarrow P$.

Proof. For each n, F_{Q_n} is increasing and $\lim_{x\to-\infty} F_{Q_n}(x) = 0$. Moreover, for real valued maps right continuity and upper semicontinuity are equivalent. Because the inf-operator preserves upper semicontinuity we can conclude that F_{Q_n} is right continuous for every n. Now we have to show that for each n, $\lim_{x\to+\infty} F_{Q_n}(x) = 1$. By contradiction suppose that, for some n, $\lim_{x\to+\infty} F_{Q_n}(x) = \lambda < 1$. We can choose a sequence $\{x_k\}_k \subseteq \mathbb{R}$ with $x_k \in \mathcal{C}(F_P)$, $x_k\uparrow +\infty$. In particular, $F_{Q_n}(x_k) \leq \lambda$ for all k and $F_P(x_k) > \lambda$ definitively, say for all $k \geq k_0$. We can observe that because $x_k \in \mathcal{C}(F_P)$, we have, for all $k \geq k_0$, $\inf_{m\geq n} F_{P_m}(x_k) < \lim_{m\to+\infty} F_{P_m}(x_k) = F_P(x_k)$. This means that the infimum is attained for some index $m(k) \in \mathbb{N}$, i.e., $\inf_{m\geq n} F_{P_m}(x_k) = F_{P_{m(k)}}(x_k)$, for all $k \geq k_0$. Because $P_{m(k)}(-\infty, x_k] = F_{P_{m(k)}}(x_k) \leq \lambda$ then $P_{m(k)}(x_k, +\infty) \geq 1 - \lambda$ for $k \geq k_0$. We have two possibilities. Either the set $\{m(k)\}_k$ is bounded or $\overline{\lim}_k m(k) = +\infty$. In the first case, we know that the number of those $m(k)$ is finite. Among these $m(k)$ we can find at least one $\overline{m}$ and a subsequence $\{x_h\}_h$ of $\{x_k\}_k$ such that $x_h\uparrow +\infty$ and $P_{\overline{m}}(x_h, +\infty) \geq 1 - \lambda$ for every h. We then conclude that

$$\lim_{h\to+\infty} P_{\overline{m}}(x_h, +\infty) \geq 1 - \lambda$$

and this is a contradiction. If $\overline{\lim}_k m(k) = +\infty$, fix $\overline{k} \geq k_0$ such that $P(x_{\overline{k}}, +\infty) < 1 - \lambda$ and observe that for every $k > \overline{k}$

$$P_{m(k)}(x_{\overline{k}}, +\infty) \geq P_{m(k)}(x_k, +\infty) \geq 1 - \lambda.$$

Take a subsequence $\{m(h)\}_h$ of $\{m(k)\}_k$ such that $m(h)\uparrow +\infty$. Then:

$$\lim_{h\to\infty} \inf P_{m(h)}(x_{\overline{k}}, +\infty) \geq 1 - \lambda > P(x_{\overline{k}}, +\infty),$$

which contradicts the weak convergence $P_n \overset{\sigma(\mathcal{P},C_b)}{\longrightarrow} P$. Finally, note that $F_{Q_n} \leq F_{P_n}$ and $Q_n\downarrow$. From $P_n \overset{\sigma(\mathcal{P},C_b)}{\longrightarrow} P$ and the definition of Q_n, we deduce that $F_{Q_n}(x) \uparrow F_P(x)$ for every $x \in \mathcal{C}(F_P)$ so that $Q_n\downarrow P$. □

EXAMPLE 2.7 [The certainty equivalent]. It is very simple to build risk measures on $\mathcal{P}(\mathbb{R})$. Take any continuous, bounded from below and strictly decreasing function $f : \mathbb{R} \to \mathbb{R}$. Then the map $\Phi_f : \mathcal{P} \to \mathbb{R} \cup \{+\infty\}$ defined by:

$$\Phi_f(P) := -f^{-1}\left(\int f dP\right) \tag{2.1}$$

is a Risk Measure on $\mathcal{P}(\mathbb{R})$. It is also easy to check that Φ_f is (CfA) and therefore $\sigma(\mathcal{P}, C_b)-lsc$. Note that Proposition 5.2 will then imply that Φ_f can not be convex. By selecting the function $f(x) = e^{-x}$ we obtain $\Phi_f(P) = \ln(\int \exp(-x)dF_P(x))$, which is in addition (TrI). Its associated risk measure $\rho : L^0 \to \mathbb{R} \cup \{+\infty\}$ defined on random variables, $\rho(X) = \Phi_f(P_X) = \ln(Ee^{-X})$, is the Entropic (convex) Risk Measure. In Section 5 we will see more examples based on this construction.

3. A REMARKABLE CLASS OF RISK MEASURES ON $\mathcal{P}(\mathbb{R})$

Given a family $\{F_m\}_{m\in\mathbb{R}}$ of functions $F_m : \mathbb{R} \to [0, 1]$, we consider the associated sets of probability measures

$$\mathcal{A}^m := \{Q \in \mathcal{P} \mid F_Q \leq F_m\} \tag{3.1}$$

and the associated map $\Phi : \mathcal{P} \to \overline{\mathbb{R}}$ defined by

$$\Phi(P) := -\sup\left\{m \in \mathbb{R} \mid P \in \mathcal{A}^m\right\}. \tag{3.2}$$

We assume hereafter that for each $P \in \mathcal{P}$ there exists m such that $P \notin \mathcal{A}^m$ so that $\Phi : \mathcal{P} \to \mathbb{R} \cup \{+\infty\}$.

Note that $\Phi(P) := \inf\{m \in \mathbb{R} \mid P \in A_m\}$, where $A_m =: \mathcal{A}^{-m}$ and $\Phi(P)$ can be interpreted as the minimal risk acceptance level under which P is still acceptable. The following discussion will show that under suitable assumption on $\{F_m\}_{m\in\mathbb{R}}$ we have that $\{A_m\}_{m\in\mathbb{R}}$ is a risk acceptance family as defined in Drapeau and Kupper (2010).

We recall from Drapeau and Kupper (2010) the following definition

DEFINITION 3.1. A monotone decreasing family of sets $\{\mathcal{A}^m\}_{m\in\mathbb{R}}$ contained in $\mathcal{P}$ is *left continuous* in m if

$$\mathcal{A}^m =: \bigcap_{\varepsilon>0} \mathcal{A}^{m-\varepsilon}.$$

In particular it is *left continuous* if it is *left continuous* in m for every $m \in \mathbb{R}$.

LEMMA 3.2. *Let $\{F_m\}_{m\in\mathbb{R}}$ be a family of functions $F_m : \mathbb{R} \to [0, 1]$ and $\mathcal{A}^m$ be the set defined in (3.1). Then:*

1. *If, for every $x \in \mathbb{R}$, $F_\cdot(x)$ is decreasing (w.r.t. m) then the family $\{\mathcal{A}^m\}$ is monotone decreasing: $\mathcal{A}^m \subseteq \mathcal{A}^n$ for any level $m \geq n$.*
2. *For any m, $\mathcal{A}^m$ is convex and satisfies: $Q \preceq P \in \mathcal{A}^m \Rightarrow Q \in \mathcal{A}^m$.*
3. *If, for every $m \in \mathbb{R}$, $F_m(x)$ is right continuous w.r.t. x then $\mathcal{A}^m$ is $\sigma(\mathcal{P}, C_b)$−closed.*
4. *Suppose that, for every $x \in \mathbb{R}$, $F_m(x)$ is decreasing w.r.t. m. If $F_m(x)$ is left continuous w.r.t. m, then the family $\{\mathcal{A}^m\}$ is left continuous.*
5. *Suppose that, for every $x \in \mathbb{R}$, $F_m(x)$ is decreasing w.r.t. m and that, for every $m \in \mathbb{R}$, $F_m(x)$ is right continuous and increasing w.r.t. x and* $\lim_{x\to+\infty} F_m(x) = 1$. *If the family $\{\mathcal{A}^m\}$ is left continuous in m* then *$F_m(x)$ is left continuous in m.*

Proof.

1. If $Q \in \mathcal{A}^m$ and $m \geq n$ then $F_Q \leq F_m \leq F_n$, i.e., $Q \in \mathcal{A}^n$.
2. Let $Q, P \in \mathcal{A}^m$ and $\lambda \in [0, 1]$. Consider the convex combination $\lambda Q + (1 - \lambda)P$ and note that

$$F_{\lambda Q+(1-\lambda)P} \leq F_Q \vee F_P \leq F_m,$$

as $F_P \leq F_m$ and $F_Q \leq F_m$. Then $\lambda Q + (1 - \lambda)P \in \mathcal{A}^m$.

3. Let $Q_n \in \mathcal{A}^m$ and $Q \in \mathcal{P}$ satisfy $Q_n \xrightarrow{\sigma(\mathcal{P},C_b)} Q$. By Theorem 2.3 we know that $F_{Q_n}(x) \to F_Q(x)$ for every $x \in \mathcal{C}(F_Q)$. For each n, $F_{Q_n} \leq F_m$ and therefore $F_Q(x) \leq F_m(x)$ for every $x \in \mathcal{C}(F_Q)$. By contradiction, suppose that $Q \notin \mathcal{A}^m$. Then

there exists $\bar{x} \notin \mathcal{C}(F_Q)$ such that $F_Q(\bar{x}) > F_m(\bar{x})$. By right continuity of F_Q for every $\varepsilon > 0$ we can find a right neighborhood $[\bar{x}, \bar{x} + \delta(\varepsilon))$ such that

$$|F_Q(x) - F_Q(\bar{x})| < \varepsilon \quad \forall\, x \in [\bar{x}, \bar{x} + \delta(\varepsilon))$$

and we may require that $\delta(\varepsilon) \downarrow 0$ if $\varepsilon \downarrow 0$. Note that for each $\varepsilon > 0$ we can always choose $x_\varepsilon \in (\bar{x}, \bar{x} + \delta(\varepsilon))$ such that $x_\varepsilon \in \mathcal{C}(F_Q)$. For such x_ε we deduce that

$$F_m(\bar{x}) < F_Q(\bar{x}) < F_Q(x_\varepsilon) + \varepsilon \leq F_m(x_\varepsilon) + \varepsilon.$$

This leads to a contradiction because if $\varepsilon \downarrow 0$ we have that $x_\varepsilon \downarrow \bar{x}$ and thus by right continuity of F_m:

$$F_m(\bar{x}) < F_Q(\bar{x}) \leq F_m(\bar{x}).$$

4. By assumption we know that $F_{m-\varepsilon}(x) \downarrow F_m(x)$ as $\varepsilon \downarrow 0$, for all $x \in \mathbb{R}$. By item 1, we know that $\mathcal{A}^m \subseteq \bigcap_{\varepsilon>0} \mathcal{A}^{m-\varepsilon}$. By contradiction, we suppose that the strict inclusion

$$\mathcal{A}^m \subset \bigcap_{\varepsilon>0} \mathcal{A}^{m-\varepsilon}$$

holds, so that there will exist $Q \in \mathcal{P}$ such that $F_Q \leq F_{m-\varepsilon}$ for every $\varepsilon > 0$ but $F_Q(\overline{x}) > F_m(\overline{x})$ for some $\overline{x} \in \mathbb{R}$. Set $\delta = F_Q(\overline{x}) - F_m(\overline{x})$ so that $F_Q(\overline{x}) > F_m(\overline{x}) + \frac{\delta}{2}$. Because $F_{m-\varepsilon} \downarrow F_m$ we may find $\overline{\varepsilon} > 0$ such that $F_{m-\overline{\varepsilon}}(\overline{x}) - F_m(\overline{x}) < \frac{\delta}{2}$. Thus, $F_Q(\overline{x}) \leq F_{m-\varepsilon}(\overline{x}) < F_m(\overline{x}) + \frac{\delta}{2}$ and this is a contradiction.
5. Assume that $\mathcal{A}^{m-\varepsilon} \downarrow \mathcal{A}^m$. Define $F(x) := \lim_{\varepsilon\downarrow 0} F_{m-\varepsilon}(x) = \inf_{\varepsilon>0} F_{m-\varepsilon}(x)$ for all $x \in \mathbb{R}$. Then $F : \mathbb{R} \to [0, 1]$ is increasing, right continuous (because the inf preserves this property). Note that for every $\varepsilon > 0$ we have $F_{m-\varepsilon} \geq F \geq F_m$ and then $\mathcal{A}^{m-\varepsilon} \supseteq \{Q \in \mathcal{P} \mid F_Q \leq F\} \supseteq \mathcal{A}^m$ and $\lim_{x\to+\infty} F(x) = 1$. Necessarily we conclude $\{Q \in \mathcal{P} \mid F_Q \leq F\} = \mathcal{A}^m$. By contradiction, we suppose that $F(\overline{x}) > F_m(\overline{x})$ for some $\overline{x} \in \mathbb{R}$. Define $F_{\overline{Q}} : \mathbb{R} \to [0, 1]$ by: $F_{\overline{Q}}(x) = F(x)\mathbf{1}_{[\overline{x},+\infty)}(x)$. The above properties of F guarantees that $F_{\overline{Q}}$ is a distribution function of a corresponding probability measure $\overline{Q} \in \mathcal{P}$, and because $F_{\overline{Q}} \leq F$, we deduce $\overline{Q} \in \mathcal{A}^m$, but $F_{\overline{Q}}(\overline{x}) > F_m(\overline{x})$ and this is a contradiction. □

The following Lemma can be deduced directly from the above Lemma 3.2 and from theorem 1.7 in Drapeau and Kupper (2010) (using the risk acceptance family $A_m =: \mathcal{A}^{-m}$, according to definition 1.6 in the aforementioned paper). We provide the proof for sake of completeness.

LEMMA 3.3. *Let $\{F_m\}_{m\in\mathbb{R}}$ be a family of functions $F_m : \mathbb{R} \to [0, 1]$ and Φ be the associated map defined in (3.2). Then:*

1. *The map Φ is (Mon) on $\mathcal{P}$.*
2. *If, for every $x \in \mathbb{R}$, $F_\cdot(x)$ is decreasing (w.r.t. m) then Φ is (QCo) on $\mathcal{P}$.*
3. *If, for every $x \in \mathbb{R}$, $F_\cdot(x)$ is left continuous and decreasing (w.r.t. m) and if, for every $m \in \mathbb{R}$, $F_m(\cdot)$ is right continuous (w.r.t. x) then*

$$A_m := \{Q \in \mathcal{P} \mid \Phi(Q) \leq m\} = \mathcal{A}^{-m}, \forall m, \tag{3.3}$$

and Φ is $\sigma(\mathcal{P}, C_b)$ lower semicontinuous.

Proof.

1. From $P \preccurlyeq Q$ we have $F_Q \leq F_P$ and

$$\{m \in \mathbb{R} \mid F_P \leq F_m\} \subseteq \left\{m \in \mathbb{R} \mid F_Q \leq F_m\right\},$$

which implies $\Phi(Q) \leq \Phi(P)$.

2. We show that $Q_1, Q_2 \in \mathcal{P}$, $\Phi(Q_1) \leq n$, and $\Phi(Q_2) \leq n$ imply that $\Phi(\lambda Q_1 + (1-\lambda)Q_2) \leq n$, that is

$$\sup\left\{m \in \mathbb{R} \mid F_{\lambda Q_1 + (1-\lambda)Q_2} \leq F_m\right\} \geq -n.$$

By definition of the supremum, $\forall \varepsilon > 0$ $\exists m_i$ s.t. $F_{Q_i} \leq F_{m_i}$ and $m_i > -\Phi(Q_i) - \varepsilon \geq -n - \varepsilon$. Then $F_{Q_i} \leq F_{m_i} \leq F_{-n-\varepsilon}$, as $\{F_m\}$ is a decreasing family. Therefore, $\lambda F_{Q_1} + (1-\lambda)F_{Q_2} \leq F_{-n-\varepsilon}$ and $-\Phi(\lambda Q_1 + (1-\lambda)Q_2\lambda) \geq -n-\varepsilon$. As this holds for any $\varepsilon > 0$, we conclude that Φ is quasi-convex.

3. The fact that $\mathcal{A}^{-m} \subseteq A_m$ follows directly from the definition of Φ, as if $Q \in \mathcal{A}^{-m}$

$$\Phi(Q) := -\sup\left\{n \in \mathbb{R} \mid Q \in \mathcal{A}^n\right\} = \inf\left\{n \in \mathbb{R} \mid Q \in \mathcal{A}^{-n}\right\} \leq m.$$

We have to show that $A_m \subseteq \mathcal{A}^{-m}$. Let $Q \in A_m$. Because $\Phi(Q) \leq m$, for all $\varepsilon > 0$ there exists m_0 such that $m + \varepsilon > -m_0$ and $F_Q \leq F_{m_0}$. Because $F_{\cdot}(x)$ is decreasing (w.r.t. m) we have that $F_Q \leq F_{-m-\varepsilon}$, therefore, $Q \in \mathcal{A}^{-m-\varepsilon}$ for any $\varepsilon > 0$. By the left continuity in m of $F_{\cdot}(x)$, we know that$\{\mathcal{A}^m\}$ is left continuous (Lemma 3.2, item 4) and so: $Q \in \bigcap\limits_{\epsilon > 0} \mathcal{A}^{-m-\varepsilon} = \mathcal{A}^{-m}$.

From the assumption that $F_m(\,\cdot\,)$ is right continuous (w.r.t. x) and Lemma 3.2 (item 3), we already know that $\mathcal{A}^m$ is $\sigma(\mathcal{P}, C_b)$-closed, for any $m \in \mathbb{R}$, and therefore the lower level sets $A_m = \mathcal{A}^{-m}$ are $\sigma(\mathcal{P}, C_b)$-closed and Φ is $\sigma(\mathcal{P}, C_b)$ -lower-semicontinuous. □

DEFINITION 3.4. A family $\{F_m\}_{m\in\mathbb{R}}$ of functions $F_m : \mathbb{R} \to [0, 1]$ is *feasible* if

- For any $P \in \mathcal{P}$ there exists m such that $P \notin \mathcal{A}^m$.
- For every $m \in \mathbb{R}$, $F_m(\,\cdot\,)$ is right continuous (w.r.t. x).
- For every $x \in \mathbb{R}$, $F_{\cdot}(x)$ is decreasing and left continuous (w.r.t. m).

From Lemmas 3.2 and 3.3 we immediately deduce:

PROPOSITION 3.5. *Let $\{F_m\}_{m\in\mathbb{R}}$ be a feasible family. Then the associated family $\{\mathcal{A}^m\}_{m\in\mathbb{R}}$ is monotone decreasing and left continuous and each set $\mathcal{A}^m$ is convex and $\sigma(\mathcal{P}, C_b)$-closed. The associated map $\Phi : \mathcal{P} \to \mathbb{R} \cup \{+\infty\}$ is well defined, (Mon), (Qco), and $\sigma(\mathcal{P}, C_b)$-lsc.*

REMARK 3.6. Let $\{F_m\}_{m\in\mathbb{R}}$ be a feasible family. If there exists an $\overline{m}$ such that $\lim_{x\to+\infty} F_{\overline{m}}(x) < 1$ then $\lim_{x\to+\infty} F_m(x) < 1$ for every $m \geq \overline{m}$ and then $\mathcal{A}^m = \varnothing$ for every $m \geq \overline{m}$. Obviously, if an acceptability set is empty then it does not contribute to the computation of the risk measure defined in (3.2). For this reason we will always consider without loss of generality (w.l.o.g.) a class $\{F_m\}_{m\in\mathbb{R}}$ such that $\lim_{x\to+\infty} F_m(x) = 1$ for every m.

3.1. Examples

As explained in the introduction, we define a family of risk measures employing a Probability/Loss function Λ. Fix the *right continuous* function $\Lambda : \mathbb{R} \to [0, 1]$ and define the family $\{F_m\}_{m\in\mathbb{R}}$ of functions $F_m : \mathbb{R} \to [0, 1]$ by

$$F_m(x) := \Lambda(x)\mathbf{1}_{(-\infty,m)}(x) + \mathbf{1}_{[m,+\infty)}(x). \tag{3.4}$$

It is easy to check that if $\sup_{x\in\mathbb{R}} \Lambda(x) < 1$ then the family $\{F_m\}_{m\in\mathbb{R}}$ is feasible and therefore, by Proposition 3.5, the associated map $\Phi : \mathcal{P} \to \mathbb{R} \cup \{+\infty\}$ is well defined, (Mon), (Qco), and $\sigma(\mathcal{P}, C_b)$-*lsc*.

EXAMPLE 3.7. When $\sup_{x\in R} \Lambda(x) = 1$, Φ may take the value $-\infty$. The extreme case is when, in the definition of the family (3.4), the function Λ is equal to the constant one, $\Lambda(x) = 1$, and so: $\mathcal{A}^m = \mathcal{P}$ for all m and $\Phi = -\infty$.

EXAMPLE 3.8. Worst case risk measure: $\Lambda(x) = 0$.
Take in the definition of the family (3.4) the function Λ to be equal to the constant zero: $\Lambda(x) = 0$. Then:

$$\begin{aligned} F_m(x) &:= \mathbf{1}_{[m,+\infty)}(x), \\ \mathcal{A}^m &:= \left\{Q \in \mathcal{P} \mid F_Q \leq F_m\right\} = \{Q \in \mathcal{P} \mid \delta_m \preccurlyeq Q\}, \\ \Phi_w(P) &:= -\sup\{m \mid P \in \mathcal{A}^m\} = -\sup\{m \mid \delta_m \preccurlyeq P\} \\ &= -\sup\{x \in \mathbb{R} \mid F_P(x) = 0\}, \end{aligned}$$

so that, if $X \in L^0$ has distribution function P_X,

$$\Phi_w(P_X) = -\sup\{m \in \mathbb{R} \mid \delta_m \preccurlyeq P_X\} = -ess\inf(X) := \rho_w(X)$$

coincide with the worst case risk measure ρ_w. As the family $\{F_m\}$ is feasible, $\Phi_w : \mathcal{P}(\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$ is (Mon), (Qco), and $\sigma(\mathcal{P}, C_b)$-*lsc*. In addition, it also satisfies (TrI).

Even though ρ_w: $L^0 \to \mathbb{R} \cup \{\infty\}$ is convex, as a map defined on random variables, the corresponding $\Phi_w : \mathcal{P} \to \mathbb{R} \cup \{\infty\}$, as a map defined on distribution functions, is not convex, but it is quasi-convex and quasi-concave. Indeed, let $P \in \mathcal{P}$ and, because $F_P \geq 0$, we set:

$$-\Phi_w(P) = \inf(F_P) := \sup\{x \in \mathbb{R} : F_P(x) = 0\}.$$

If F_1, F_2 are two distribution functions corresponding to P_1, $P_2 \in \mathcal{P}$ then for all $\lambda \in (0, 1)$ we have:

$$\inf(\lambda F_1 + (1-\lambda)F_2) = \min(\inf(F_1), \inf(F_2)) \leq \lambda \inf(F_1) + (1-\lambda)\inf(F_2)$$

and therefore, for all $\lambda \in [0, 1]$

$$\min(\inf(F_1), \inf(F_2)) \leq \inf(\lambda F_1 + (1-\lambda)F_2) \leq \lambda \inf(F_1) + (1-\lambda)\inf(F_2).$$

EXAMPLE 3.9. Value at Risk $V@R_\lambda$: $\Lambda(x)$: $\lambda \in (0, 1)$.

Take in the definition of the family (3.4), the function Λ to be equal to the constant λ, $\Lambda(x) = \lambda \in (0, 1)$. Then

$$\begin{aligned} F_m(x) &:= \lambda \mathbf{1}_{(-\infty,m)}(x) + \mathbf{1}_{[m,+\infty)}(x), \\ \mathcal{A}^m &:= \left\{ Q \in \mathcal{P} \mid F_Q \leq F_m \right\}, \\ \Phi_{V@R_\lambda}(P) &:= -\sup\{m \in \mathbb{R} \mid P \in \mathcal{A}^m\}. \end{aligned}$$

If the random variable $X \in L^0$ has distribution function P_X and $q_X^+(\lambda) = \sup\{x \in \mathbb{R} \mid \mathbb{P}(X \leq x) \leq \lambda\}$ is the right continuous inverse of P_X then

$$\begin{aligned} \Phi_{V@R_\lambda}(P_X) &= -\sup\{m \mid P_X \in \mathcal{A}^m\} \\ &= -\sup\{m \mid \mathbb{P}(X \leq x) \leq \lambda \ \forall x < m\} \\ &= -\sup\{m \mid \mathbb{P}(X \leq m) \leq \lambda\} \\ &= -q_X^+(\lambda) := V@R_\lambda(X) \end{aligned}$$

coincide with the Value at Risk of level $\lambda \in (0, 1)$. As the family $\{F_m\}$ is feasible, $\Phi_{V@R_\lambda} : \mathcal{P} \to \mathbb{R} \cup \{+\infty\}$ is (Mon), (Qco), $\sigma(\mathcal{P}, C_b)$ -*lsc*. In addition, it also satisfies (TrI).

As well known, $V@R_\lambda$: $L^0 \to \mathbb{R} \cup \{\infty\}$ is not quasi-convex, as a map defined on random variables, even though the corresponding $\Phi_{V@R_\lambda} : \mathcal{P} \to \mathbb{R} \cup \{\infty\}$, as a map defined on distribution functions, is quasi-convex (see Drapeau and Kupper 2010 for a discussion on this issue).

EXAMPLE 3.10. Fix the family $\{\Lambda_m\}_{m\in\mathbb{R}}$ of functions $\Lambda_m : \mathbb{R} \to [0, 1]$ such that for every $m \in \mathbb{R}$, $\Lambda_m(\cdot)$ is right continuous (w.r.t. x) and for every $x \in \mathbb{R}$, $\Lambda_\cdot(x)$ is decreasing and left continuous (w.r.t. m). Define the family $\{F_m\}_{m\in\mathbb{R}}$ of functions $F_m : \mathbb{R} \to [0, 1]$ by

$$F_m(x) := \Lambda_m(x)\mathbf{1}_{(-\infty,m)}(x) + \mathbf{1}_{[m,+\infty)}(x). \tag{3.5}$$

It is easy to check that if $\sup_{x\in\mathbb{R}} \Lambda_{m_0}(x) < 1$, for some $m_0 \in \mathbb{R}$, then the family $\{F_m\}_{m\in\mathbb{R}}$ is feasible and therefore the associated map $\Phi : \mathcal{P} \to \mathbb{R} \cup \{+\infty\}$ is well defined, (Mon), (Qco), $\sigma(\mathcal{P}, C_b)$-*lsc*.

4. ON THE $\Lambda V@R$

We now propose a generalization of the $V@R_\lambda$ which appears useful for possible application whenever an agent is facing some ambiguity on the parameter λ, namely λ is given by some uncertain value in a confidence interval $[\lambda^m, \lambda^M]$, with $0 \leq \lambda^m \leq \lambda^M \leq 1$. The $V@R_\lambda$ corresponds to case $\lambda^m = \lambda^M$ and one typical value is $\lambda^M = 0, 05$.

We will distinguish two possible classes of agents:

Risk Prudent Agents: Fix the *increasing* right continuous function $\Lambda : \mathbb{R} \to [0, 1]$, choose as in (3.4)

$$F_m(x) = \Lambda(x)\mathbf{1}_{(-\infty,m)}(x) + \mathbf{1}_{[m,+\infty)}(x)$$

and set $\lambda^m := \inf \Lambda \geq 0$, $\lambda^M := \sup \Lambda \leq 1$. As the function Λ is increasing, we are assigning to a lower loss a lower probability. In particular, given two possible choices Λ_1,

Λ_2 for two different agents, the condition $\Lambda_1 \leq \Lambda_2$ means that the agent 1 is more risk prudent than agent 2. Set, as in (3.1), $\mathcal{A}^m = \{Q \in \mathcal{P} \mid F_Q \leq F_m\}$ and define as in (3.2)

$$\Lambda V@R(P) := -\sup\{m \in \mathbb{R} \mid P \in \mathcal{A}^m\}.$$

Thus, in case of a random variable X

$$\Lambda V@R(P_X) := -\sup\{m \in \mathbb{R} \mid \mathbb{P}(X \leq x) \leq \Lambda(x),\ \forall x \leq m\}.$$

In particular, it can be rewritten as

$$\Lambda V@R(P_X) = -\inf\{x \in \mathbb{R} \mid \mathbb{P}(X \leq x) > \Lambda(x)\}.$$

If both F_X and Λ are continuous $\Lambda V@R$ corresponds to the smallest intersection between the two curves.

In this section, we assume that

$$\lambda^M < 1.$$

Besides its obvious financial motivation, this request implies that the corresponding family F_m is feasible and so $\Lambda V@R(P) > -\infty$ for all $P \in \mathcal{P}$.

The feasibility of the family $\{F_m\}$ implies that the $\Lambda V@R : \mathcal{P} \to \mathbb{R} \cup \{\infty\}$ is well defined, (Mon), (QCo), and (CfA) (or equivalently $\sigma(\mathcal{P}, C_b)$-*lsc*) map.

EXAMPLE 4.1. One possible simple choice of the function Λ is represented by the step function:

$$\Lambda(x) = \lambda^m \mathbf{1}_{(-\infty,\bar{x})}(x) + \lambda^M \mathbf{1}_{[\bar{x},+\infty)}(x).$$

The idea is that with a probability of λ^M we are accepting to loose at most $\bar{x}$. In this case we observe that:

$$\Lambda V@R(P) = \begin{cases} V@R_{\lambda^M}(P) & \text{if } V@R_{\lambda^m}(P) \leq -\bar{x} \\ V@R_{\lambda^m}(P) & \text{if } V@R_{\lambda^m}(P) > -\bar{x}. \end{cases}$$

Even though the $\Lambda V@R$ is continuous from above (Proposition 3.5 and 2.5), it may not be continuous from below, as this example shows. For instance, take $\bar{x} = 0$ and P_{X_n} induced by a sequence of uniformly distributed random variables $X_n \sim U\left[-\lambda^m - \frac{1}{n}, 1 - \lambda^m - \frac{1}{n}\right]$. We have $P_{X_n} \uparrow P_{U[-\lambda^m, 1-\lambda^m]}$ but $\Lambda V@R(P_{X_n}) = -\frac{1}{n}$ for every n and $\Lambda V@R(P_{U[-\lambda^m,1-\lambda^m]}) = \lambda^M - \lambda^m$.

REMARK 4.2.

(i) If $\lambda^m = 0$ the domain of $\Lambda V@R(P)$ is not the entire convex set $\mathcal{P}$. We have two possible cases
- $\text{supp}(\Lambda) = [x^*, +\infty)$: in this case $\Lambda V@R(P) = -\inf \text{supp}(F_P)$ for every $P \in \mathcal{P}$ such that $\text{supp}(F_P) \supseteq \text{supp}(\Lambda)$.
- $\text{supp}(\Lambda) = (-\infty, +\infty)$: in this case

$$\Lambda V@R(P) = +\infty \quad \text{for all } P, \text{ such that } \lim_{x \to -\infty} \frac{F_P(x)}{\Lambda(x)} > 1,$$
$$\Lambda V@R(P) < +\infty \quad \text{for all } P, \text{ such that } \lim_{x \to -\infty} \frac{F_P(x)}{\Lambda(x)} < 1.$$

In the case $\lim_{x\to-\infty}\frac{F_P(x)}{\Lambda(x)}=1$ both the previous behaviors might occur.

(ii) In case that $\lambda^m>0$ then $\Lambda V@R(P)<+\infty$ for all $P\in\mathcal{P}$, so that $\Lambda V@R$ is finite valued.

We can prove a further structural property which is the counterpart of (TrI) for the $\Lambda V@R$. Let $\alpha\in\mathbb{R}$ any cash amount

$$\begin{aligned}\Lambda V@R(P_{X+\alpha}) &= -\sup\{m \mid \mathbb{P}(X+\alpha\le x)\le\Lambda(x),\ \forall x\le m\}\\ &= -\sup\{m \mid \mathbb{P}(X\le x-\alpha)\le\Lambda(x),\ \forall x\le m\}\\ &= -\sup\{m \mid \mathbb{P}(X\le y)\le\Lambda(y+\alpha),\ \forall y\le m-\alpha\}\\ &= -\sup\{m+\alpha \mid \mathbb{P}(X\le y)\le\Lambda(y+\alpha),\ \forall y\le m\}\\ &= \Lambda^\alpha V@R(P_X)-\alpha,\end{aligned}$$

where $\Lambda^\alpha(x)=\Lambda(x+\alpha)$. We may conclude that if we add a sure positive (resp. negative) amount α to a risky position X then the risk decreases (resp. increases) of the value $-\alpha$, constrained to a lower (resp. higher) level of risk prudence described by $\Lambda^\alpha\ge\Lambda$ (resp. $\Lambda^\alpha\le\Lambda$). For an arbitrary $P\in\mathcal{P}$ this property can be written as

$$\Lambda V@R(T_\alpha P)=\Lambda^\alpha V@R(P)-\alpha,\quad \forall\alpha\in\mathbb{R},$$

where $T_\alpha P(-\infty,x]=P(-\infty,x-\alpha]$.

Risk Seeking Agents: Fix the *decreasing* right continuous function $\Lambda:\mathbb{R}\to[0,1]$, with $\inf\Lambda<1$. Similarly as above, we define

$$F_m(x)=\Lambda(x)\mathbf{1}_{(-\infty,m)}(x)+\mathbf{1}_{[m,+\infty)}(x)$$

and the (Mon), (QCo), and (CfA) map

$$\Lambda V@R(P):=-\sup\{m\in\mathbb{R}\mid F_P\le F_m\}=-\sup\{m\in\mathbb{R}\mid \mathbb{P}(X\le m)\le\Lambda(m)\}.$$

In this case, for eventual huge losses we are allowing the highest level of probability. As in the previous example let $\alpha\in\mathbb{R}$ and note that

$$\Lambda V@R(P_{X+\alpha})=\Lambda^\alpha V@R(P_X)-\alpha,$$

where $\Lambda^\alpha(x)=\Lambda(x+\alpha)$. The property is exactly the same as in the former example but here the interpretation is slightly different. If we add a sure positive (resp. negative) amount α to a risky position X then the risk decreases (resp. increases) of the value $-\alpha$, constrained to a lower (resp. higher) level of risk seeking because $\Lambda^\alpha\le\Lambda$ (resp. $\Lambda^\alpha\ge\Lambda$).

REMARK 4.3. For a decreasing Λ, there is a simpler formulation—which will be used in Section 5.3—of the $\Lambda V@R$ that is obtained replacing in F_m the function Λ with the line $\Lambda(m)$ for all $x<m$. Let

$$\tilde F_m(x)=\Lambda(m)\mathbf{1}_{(-\infty,m)}(x)+\mathbf{1}_{[m,+\infty)}(x).$$

This family is of the type (3.5) and is feasible, provided the function Λ is continuous. For a decreasing Λ, it is evident that

$$\Lambda V@R(P)=\Lambda\widetilde{V}@R(P):=-\sup\left\{m\in\mathbb{R}\mid F_P\le\tilde F_m\right\},$$

as the function Λ lies above the line $\Lambda(m)$ for all $x\le m$.

5. QUASI-CONVEX DUALITY

In literature we also find several results about the dual representation of law invariant risk measures. Kusuoka (2001) contributed to the coherent case, although Frittelli and Rosazza Gianin (2005) extended this result to the convex case. Jouini, Schachermayer, and Touzi (2006) and Filipovic and Svindland (2012), in the convex case, and Svindland (2010) in the quasi-convex case, showed that every law invariant risk measure is already weakly lower semicontinuous. Recently, Cerreia-Vioglio et al. (2011b) provided a robust dual representation for law invariant quasi-convex risk measures, which has been extended to the dynamic case by Frittelli and Maggis (2011a, 2011b).

In Sections 5.1 and 5.2 we will treat the general case of maps defined on $\mathcal{P}$, although in Section 5.3 we specialize these results to show the dual representation of maps associated to feasible families.

5.1. Reasons of the Failure of the Convex Duality for Translation Invariant Maps on $\mathcal{P}$

It is well known that the classical convex duality provided by the Fenchel-Moreau theorem (Fenchel 1949) guarantees the representation of convex and lower semicontinuous functions and therefore is very useful for the dual representation of convex risk measures (see Frittelli and Rosazza Gianin 2002). For any map $\Phi : \mathcal{P} \to \mathbb{R} \cup \{\infty\}$ let Φ^* be the convex conjugate:

$$\Phi^*(f) := \sup_{Q \in \mathcal{P}} \left\{ \int f dQ - \Phi(Q) \right\}, \ f \in C_b.$$

Applying the fact that $\mathcal{P}$ is a $\sigma(ca, C_b)$-closed convex subset of ca one can easily check that the following version of Fenchel-Moreau Theorem holds true for maps defined on $\mathcal{P}$.

PROPOSITION 5.1 [Fenchel-Moreau]. *Suppose that $\Phi : \mathcal{P} \to \mathbb{R} \cup \{\infty\}$ is $\sigma(\mathcal{P}, C_b)$-lsc and convex. If $Dom(\Phi) := \{Q \in \mathcal{P} \mid \Phi(Q) < +\infty\} \neq \varnothing$ then $Dom(\Phi^*) \neq \varnothing$ and*

$$\Phi(Q) = \sup_{f \in C_b} \left\{ \int f dQ - \Phi^*(f) \right\}.$$

One trivial example of a proper $\sigma(\mathcal{P}, C_b)$-*lsc* and convex map on $\mathcal{P}$ is given by $Q \to \int f dQ$, for some $f \in C_b$. But this map does not satisfy the (TrI) property. Indeed, we show that in the setting of risk measures defined on $\mathcal{P}$, weakly lower semicontinuity and convexity are incompatible with translation invariance.

PROPOSITION 5.2. *For any map $\Phi : \mathcal{P} \to \mathbb{R} \cup \{\infty\}$, if there exists a sequence $\{Q_n\}_n \subseteq \mathcal{P}$ such that* $\underline{\lim}_n \Phi(Q_n) = -\infty$ then $Dom(\Phi^*) = \varnothing$.

Proof. For any $f \in C_b(\mathbb{R})$

$$\Phi^*(f) = \sup_{Q \in \mathcal{P}} \left\{ \int f dQ - \Phi(Q) \right\} \geq \int f d(Q_n) - \Phi(Q_n) \geq \inf_{x \in \mathbb{R}} f(x) - \Phi(Q_n),$$

which implies $\Phi^* = +\infty$. $\square$

From Propositions (5.1) and (5.2) we immediately obtain:

COROLLARY 5.3. *Let $\Phi : \mathcal{P} \to \mathbb{R} \cup \{\infty\}$ be $\sigma(\mathcal{P}, C_b)$-lsc, convex and not identically equal to $+\infty$. Then Φ is not (TrI), is not cash sup additive (i.e., it does not satisfy: $\Phi(T_m Q) \leq \Phi(Q) - m$) and $\underline{\lim}_n \Phi(\delta_n) \neq -\infty$. In particular, the certainty equivalent maps Φ_f defined in (2.1) can not be convex, as they are $\sigma(\mathcal{P}, C_b)$-lsc and $\Phi_f(\delta_n) = -n$.*

5.2. The Dual Representation

As described in the examples in Section 3, the $\Phi_{V@R_\lambda}$ and Φ_w are proper, $\sigma(ca, C_b)$-*lsc*, quasi-convex, (Mon), and (TrI) maps $\Phi : \mathcal{P} \to \mathbb{R} \cup \{\infty\}$. Therefore, the negative result outlined in Corollary 5.3 for the convex case can not be true in the quasi-convex setting.

We recall that the seminal contribution to quasi-convex duality comes from the dual representation by Volle (1998) and Penot and Volle (1990), which has been sharpened to a complete quasi-convex duality by Cerreia-Vioglio et al. (2011b) (case of M-spaces), Cerreia-Vioglio (2009) (preferences over menus), and Drapeau and Kupper (2010) (for general topological vector spaces).

Here we replicate this result and provide the dual representation of a $\sigma(\mathcal{P}, C_b)$-*lsc* quasi-convex maps defined on the entire set $\mathcal{P}$. The main difference is that our map Φ is defined on a convex subset of ca and not a vector space (a similar result can be found in Drapeau and Kupper 2010 for convex sets). But because $\mathcal{P}$ is $\sigma(ca, C_b)$-closed, the first part of the proof will match very closely the one given by Volle. To achieve the dual representation of $\sigma(\mathcal{P}, C_b)$-*lsc* risk measures $\Phi : \mathcal{P} \to \mathbb{R} \cup \{\infty\}$ we will impose the monotonicity assumption of Φ and deduce that in the dual representation the supremum can be restricted to the set

$$C_b^- = \{f \in C_b \mid f \text{ is decreasing}\}.$$

This is natural as the first-order stochastic dominance implies (see theorem 2.70 in Föllmer and Schied 2004) that

$$C_b^- = \left\{ f \in C_b \mid Q, P \in \mathcal{P} \text{ and } P \preccurlyeq Q \Rightarrow \int f dQ \leq \int f dP \right\}. \tag{5.1}$$

Note that differently from Drapeau and Kupper (2010) the following proposition does not require the extension of the risk map to the entire space $ca(\mathbb{R})$. Once the representation is obtained the uniqueness of the dual function is a direct consequence of theorem 2.19 in Drapeau and Kupper (2010) as explained by Proposition 5.9.

PROPOSITION 5.4.

(i) *Any $\sigma(\mathcal{P}, C_b)$-lsc and quasi-convex functional $\Phi : \mathcal{P} \to \mathbb{R} \cup \{\infty\}$ can be represented as*

$$\Phi(P) = \sup_{f \in C_b} R\left(\int f dP, f \right), \tag{5.2}$$

where $R : \mathbb{R} \times C_b \to \overline{\mathbb{R}}$ is defined by

$$R(t, f) := \inf_{Q \in \mathcal{P}} \left\{ \Phi(Q) \mid \int f dQ \geq t \right\}. \tag{5.3}$$

(ii) *If in addition Φ is monotone then (5.2) holds with C_b replaced by C_b^-.*

Proof. We will use the fact that $\sigma(\mathcal{P}, C_b)$ is the relativization of $\sigma(ca, C_b)$ to the set $\mathcal{P}$. In particular, the lower level sets will be $\sigma(ca, C_b)$-closed.

(i) By definition, for any $f \in C_b(\mathbb{R})$, $R(\int f dP, f) \leq \Phi(P)$ and therefore

$$\sup_{f \in C_b} R\left(\int f dP, f\right) \leq \Phi(P), \quad P \in \mathcal{P}.$$

Fix any $P \in \mathcal{P}$ and take $\varepsilon \in \mathbb{R}$ such that $\varepsilon > 0$. Then P does not belong to the $\sigma(ca, C_b)$-closed convex set

$$\mathcal{C}_\varepsilon := \{Q \in \mathcal{P} : \Phi(Q) \leq \Phi(P) - \varepsilon\}$$

(if $\Phi(P) = +\infty$, replace the set $\mathcal{C}_\varepsilon$ with $\{Q \in \mathcal{P} : \Phi(Q) \leq M\}$, for any M). By the Hahn Banach theorem there exists a continuous linear functional that strongly separates P and $\mathcal{C}_\varepsilon$, i.e., there exists $\alpha \in \mathbb{R}$ and $f_\varepsilon \in C_b$ such that

$$\int f_\varepsilon dP > \alpha > \int f_\varepsilon dQ \quad \text{for all } Q \in \mathcal{C}_\varepsilon. \tag{5.4}$$

Hence:

$$\left\{Q \in \mathcal{P} : \int f_\varepsilon dP \leq \int f_\varepsilon dQ\right\} \subseteq (\mathcal{C}_\varepsilon)^C = \{Q \in \mathcal{P} : \Phi(Q) > \Phi(P) - \varepsilon\} \tag{5.5}$$

and

$$\begin{aligned} \Phi(P) &\geq \sup_{f \in C_b} R\left(\int f dP, f\right) \geq R\left(\textstyle\int f_\varepsilon dP, f_\varepsilon\right) \\ &= \inf\left\{\Phi(Q) \mid Q \in \mathcal{P} \text{ such that } \int f_\varepsilon dP \leq \int f_\varepsilon dQ\right\} \\ &\geq \inf\{\Phi(Q) \mid Q \in \mathcal{P} \text{ satisfying } \Phi(Q) > \Phi(P) - \varepsilon\} \geq \Phi(P) - \varepsilon. \end{aligned} \tag{5.6}$$

(ii) We furthermore assume that Φ is monotone. As shown in (i), for every $\varepsilon > 0$ we find f_ε such that (5.4) holds true. We claim that there exists $g_\varepsilon \in C_b^-$ satisfying:

$$\int g_\varepsilon dP > \alpha > \int g_\varepsilon dQ \quad \text{for all } Q \in \mathcal{C}_\varepsilon, \tag{5.7}$$

and then the above argument (in equations (5.4)–(5.6)) implies the thesis.

We define the decreasing function

$$g_\varepsilon(x) =: \sup_{y \geq x} f_\varepsilon(y) \in C_b^-.$$

First case: Suppose that $g_\varepsilon(x) = \sup_{x \in \mathbb{R}} f_\varepsilon(x) =: s$. In this case there exists a sequence of $\{x_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ such that $x_n \to +\infty$ and $f_\varepsilon(x_n) \to s$, as $n \to \infty$. Define

$$g_n(x) = s\mathbf{1}_{(-\infty, x_n]} + f_\varepsilon(x)\mathbf{1}_{(x_n, +\infty)}$$

and note that $s \geq g_n \geq f_\varepsilon$ and $g_n \uparrow s$. For any $Q \in \mathcal{C}_\varepsilon$ we consider Q_n defined by $F_{Q_n}(x) = F_Q(x)\mathbf{1}_{[x_n,+\infty)}$. Because $Q \preccurlyeq Q_n$, monotonicity of Φ implies $Q_n \in \mathcal{C}_\varepsilon$. Note that

$$\int g_n dQ - \int f_\varepsilon dQ_n = (s - f_\varepsilon(x_n))Q(-\infty, x_n] \overset{n\to+\infty}{\longrightarrow} 0, \text{ as } n \to \infty. \tag{5.8}$$

From equation (5.4) we have

$$s \geq \int f_\varepsilon dP > \alpha > \int f_\varepsilon dQ_n \quad \text{for all } n \in \mathbb{N}. \tag{5.9}$$

Letting $\delta = s - \alpha > 0$ we obtain $s > \int f_\varepsilon dQ_n + \frac{\delta}{2}$. From (5.8), there exists $\overline{n} \in \mathbb{N}$ such that $0 \leq \int g_n dQ - \int f_\varepsilon dQ_n < \frac{\delta}{4}$ for every $n \geq \overline{n}$. Therefore, $\forall\, n \geq \overline{n}$

$$s > \int f_\varepsilon dQ_n + \frac{\delta}{2} > \int g_n dQ - \frac{\delta}{4} + \frac{\delta}{2} = \int g_n dQ + \frac{\delta}{4}$$

and this leads to a contradiction as $g_n \uparrow s$. So the first case is excluded.

Second case: Suppose that $g_\varepsilon(x) < s$ for any $x > \overline{x}$. As the function $g_\varepsilon \in C_b^-$ is decreasing, there will exists at most a countable sequence of intervals $\{A_n\}_{n\geq 0}$ on which g_ε is constant. Set $A_0 = (-\infty, b_0)$, $A_n = [a_n, b_n) \subset \mathbb{R}$ for $n \geq 1$. W.l.o.g. we suppose that $A_n \cap A_m = \emptyset$ for all $n \neq m$ (else, we paste together the sets) and $a_n < a_{n+1}$ for every $n \geq 1$. We stress that $f_\varepsilon(x) = g_\varepsilon(x)$ on $D =: \bigcap_{n\geq 0} A_n^C$. For every $Q \in \mathcal{C}_\varepsilon$ we define the probability $\overline{Q}$ by its distribution function as

$$F_{\overline{Q}}(x) = F_Q(x)\mathbf{1}_D + \sum_{n\geq 1} F_Q(a_n)\mathbf{1}_{[a_n,b_n)}.$$

As before, $Q \preccurlyeq \overline{Q}$ and monotonicity of Φ implies $\overline{Q} \in \mathcal{C}_\varepsilon$. Moreover,

$$\int g_\varepsilon dQ = \int_D f_\varepsilon dQ + f_\varepsilon(b_0)Q(A_0) + \sum_{n\geq 1} f_\varepsilon(a_n)Q(A_n) = \int f_\varepsilon d\overline{Q}.$$

From $g_\varepsilon \geq f_\varepsilon$ and equation (5.4) we deduce

$$\int g_\varepsilon dP \geq \int f_\varepsilon dP > \alpha > \int f_\varepsilon d\overline{Q} = \int g_\varepsilon dQ \quad \text{for all } Q \in \mathcal{C}_\varepsilon.$$

□

We reformulate the Proposition 5.4 and provide two dual representation of $\sigma(\mathcal{P}(\mathbb{R}), C_b)$-*lsc* Risk Measure $\Phi : \mathcal{P}(\mathbb{R}) \to \mathbb{R} \cup \{\infty\}$ in terms of a supremum over a class of probabilistic scenarios. Let

$$\mathcal{P}_c(\mathbb{R}) = \left\{Q \in \mathcal{P}(\mathbb{R}) \mid F_Q \text{ is continuous}\right\}.$$

PROPOSITION 5.5. *Any* $\sigma(\mathcal{P}(\mathbb{R}), C_b)$*-lsc Risk Measure* $\Phi : \mathcal{P}(\mathbb{R}) \to \mathbb{R} \cup \{\infty\}$ *can be represented as*

$$\Phi(P) = \sup_{Q\in\mathcal{P}_c(\mathbb{R})} R\left(-\int F_Q dP, -F_Q\right).$$

Proof. Note that for every $f \in C_b^-$ which is constant we have $R(\int f dP, f) = \inf_{Q\in\mathcal{P}} \Phi(Q)$. Therefore, we may assume w.l.o.g. that $f \in C_b^-$ is not constant. Then $g := \frac{f - f(+\infty)}{f(-\infty) - f(+\infty)} \in C_b^-$, $\inf g = 0$, $\sup g = 1$, and so: $g \in \{-F_Q \mid Q \in \mathcal{P}_c(\mathbb{R})\}$. In addition, because $\int f dQ \geq \int f dP$ i.f.f. $\int g dQ \geq \int g dP$ we obtain from (5.2) and (ii) of Proposition 5.4

$$\Phi(P) = \sup_{f\in C_b^-} R\left(\int f dP, f\right) = \sup_{Q\in\mathcal{P}_c(\mathbb{R})} R\left(-\int F_Q dP, -F_Q\right).$$

□

Finally, we state the dual representations for Risk Measures expressed either in terms of the dual function R as used by Cerreia-Vioglio et al. (2011b), or considering the left continuous version of R (see Lemma 5.7) in the formulation proposed by Drapeau and Kupper (2010). If $R : \mathbb{R} \times C_b(\mathbb{R}) \to \overline{\mathbb{R}}$, the left continuous version of $R(\cdot, f)$ is defined by:

$$R^-(t, f) := \sup\{R(s, f) \mid s < t\}. \tag{5.10}$$

PROPOSITION 5.6. *Any $\sigma(\mathcal{P}(\mathbb{R}), C_b)$-lsc Risk Measure $\Phi : \mathcal{P}(\mathbb{R}) \to \mathbb{R} \cup \{\infty\}$ can be represented as*

$$\Phi(P) = \sup_{f\in C_b^-} R\left(\int f dP, f\right) = \sup_{f\in C_b^-} R^-\left(\int f dP, f\right). \tag{5.11}$$

The function $R^-(t, f)$ defined in (5.10) *can be written as*

$$R^-(t, f) = \inf\{m \in \mathbb{R} \mid \gamma(m, f) \geq t\}, \tag{5.12}$$

where $\gamma : \mathbb{R} \times C_b(\mathbb{R}) \to \overline{\mathbb{R}}$ is given by:

$$\gamma(m, f) := \sup_{Q\in\mathcal{P}} \left\{\int f dQ \mid \Phi(Q) \leq m\right\}, m \in \mathbb{R}. \tag{5.13}$$

Proof. Note that $R(\cdot, f)$ is increasing and $R(t, f) \geq R^-(t, f)$. If $f \in C_b^-$ then $P \preccurlyeq Q \Rightarrow \int f dQ \leq \int f dP$. Therefore,

$$R^-\left(\int f dP, f\right) := \sup_{s<\int f dP} R(s, f) \geq \lim_{P_n \downarrow P} R\left(\int f dP_n, f\right).$$

From Proposition 5.4 (ii) we obtain:

$$\begin{aligned}\Phi(P) &= \sup_{f\in C_b^-} R\left(\int f dP, f\right) \geq \sup_{f\in C_b^-} R^-\left(\int f dP, f\right) \geq \sup_{f\in C_b^-} \lim_{P_n\downarrow P} R\left(\int f dP_n, f\right) \\ &= \lim_{P_n\downarrow P} \sup_{f\in C_b^-} R\left(\int f dP_n, f\right) = \lim_{P_n\downarrow P} \Phi(P_n) = \Phi(P)\end{aligned}$$

by (CfA). This proves (5.11). The second statement follows from Lemma 5.7. □

The following lemma shows that the left continuous version of R is the left inverse of the function γ as defined in 5.13 (for the definition and the properties of the left inverse we refer to Föllmer and Schied 2004, section A.3).

LEMMA 5.7. *Let Φ be any map $\Phi : \mathcal{P}(\mathbb{R}) \to \mathbb{R} \cup \{\infty\}$ and $R : \mathbb{R} \times C_b(\mathbb{R}) \to \overline{\mathbb{R}}$ be defined in (5.3). The left continuous version of $R(\cdot, f)$ can be written as:*

$$R^{-}(t, f) := \sup\{R(s, f) \mid s < t\} = \inf\{m \in \mathbb{R} \mid \gamma(m, f) \geq t\}, \tag{5.14}$$

where $\gamma : \mathbb{R} \times C_b(\mathbb{R}) \to \overline{\mathbb{R}}$ is given in (5.13).

Proof. Let the right-hand side (RHS) of equation (5.14) be denoted by

$$S(t, f) := \inf\{m \in \mathbb{R} \mid \gamma(m, f) \geq t\}, \ (t, f) \in \mathbb{R} \times C_b(\mathbb{R}),$$

and note that $S(\cdot, f)$ is the left inverse of the increasing function $\gamma(\cdot, f)$ and therefore $S(\cdot, f)$ is left continuous.
Step I: To prove that $R^{-}(t, f) \geq S(t, f)$ it is sufficient to show that for all $s < t$ we have:

$$R(s, f) \geq S(s, f). \tag{5.15}$$

Indeed, if (5.15) is true

$$R^{-}(t, f) = \sup_{s<t} R(s, f) \geq \sup_{s<t} S(s, f) = S(t, f),$$

as both R^{-} and S are left continuous in the first argument. Writing explicitly the inequality (5.15)

$$\inf_{Q \in \mathcal{P}} \left\{ \Phi(Q) \mid \int f dQ \geq s \right\} \geq \inf\{m \in \mathbb{R} \mid \gamma(m, f) \geq s\}$$

and letting $Q \in \mathcal{P}$ satisfying $\int f dQ \geq s$, we see that it is sufficient to show the existence of $m \in \mathbb{R}$ such that $\gamma(m, f) \geq s$ and $m \leq \Phi(Q)$. If $\Phi(Q) = -\infty$ then $\gamma(m, f) \geq s$ for any m and therefore $S(s, f) = R(s, f) = -\infty$.

Suppose now that $+\infty > \Phi(Q) > -\infty$ and define $m := \Phi(Q)$. As $\int f dQ \geq s$ we have:

$$\gamma(m, f) := \sup_{Q \in \mathcal{P}} \left\{ \int f dQ \mid \Phi(Q) \leq m \right\} \geq s.$$

Then $m \in \mathbb{R}$ satisfies the required conditions.

Step II: To obtain $R^{-}(t, f) := \sup_{s<t} R(s, f) \leq S(t, f)$ it is sufficient to prove that, for all $s < t$, $R(s, f) \leq S(t, f)$, that is

$$\inf_{Q \in \mathcal{P}} \left\{ \Phi(Q) \mid \int f dQ \geq s \right\} \leq \inf\{m \in \mathbb{R} \mid \gamma(m, f) \geq t\}. \tag{5.16}$$

Fix any $s < t$ and consider any $m \in \mathbb{R}$ such that $\gamma(m, f) \geq t$. By the definition of γ, for all $\varepsilon > 0$ there exists $Q_\varepsilon \in \mathcal{P}$ such that $\Phi(Q_\varepsilon) \leq m$ and $\int f dQ_\varepsilon > t - \varepsilon$. Take ε such that $0 < \varepsilon < t - s$. Then $\int f dQ_\varepsilon \geq s$ and $\Phi(Q_\varepsilon) \leq m$ and (5.16) follows. □

Complete duality: The complete duality in the class of quasi-convex monotone maps on vector spaces was first obtained by Cerreia-Vioglio et al. (2011a). The following proposition is based on the complete duality proved in Drapeau and Kupper (2010) for maps defined on convex sets and therefore the results in Drapeau and Kupper (2010) apply very easily in our setting. To obtain the uniqueness of the dual function in the representation (5.11) we need to introduce the opportune class $\mathcal{R}^{\max}$. Recall that $\mathcal{P}(\mathbb{R})$ spans the space of countably additive signed measures on $\mathbb{R}$, namely $ca(\mathbb{R})$ and that the first stochastic order corresponds to the cone

$$\mathcal{K} = \left\{ \mu \in ca \mid \int f d\mu \geq 0 \; \forall \, f \in \mathcal{K}^\circ \right\} \subseteq ca_+,$$

where $\mathcal{K}^\circ = -C_b^-$ are the nondecreasing functions $f \in C_b$.

Definition 5.8 (Drapeau and Kupper 2010). We denote by $\mathcal{R}^{\max}$ the class of functions $R : \mathbb{R} \times \mathcal{K}^\circ \to \overline{\mathbb{R}}$ such that: (i) R is nondecreasing and left continuous in the first argument,(ii) R is jointly quasi-concave, (iii) $R(s, \lambda f) = R(\frac{s}{\lambda}, f)$ for every $f \in \mathcal{K}^\circ$, $s \in \mathbb{R}$ and $\lambda > 0$, (iv) $\lim_{s \to -\infty} R(s, f) = \lim_{s \to -\infty} R(s, g)$ for every $f, g \in \mathcal{K}^\circ$, (v) $R^+(s, f) = \inf_{s' > s} R(s', f)$, is upper semicontinuous in the second argument.

Proposition 5.9. *Any $\sigma(\mathcal{P}(\mathbb{R}), C_b)$-lsc Risk Measure $\Phi : \mathcal{P}(\mathbb{R}) \to \mathbb{R} \cup \{\infty\}$ can be represented as in 5.11. The function $R^-(t, f)$ given by 5.12 is unique in the class $\mathcal{R}^{\max}$.*

Proof. According to Definition 2.13 in Drapeau and Kupper (2010) a map $\Phi : \mathcal{P} \to \overline{R}$ is continuously extensible to ca if

$$\overline{\mathcal{A}^m + \mathcal{K}} \cap \mathcal{P} = \mathcal{A}^m,$$

where $\mathcal{A}^m$ is acceptance set of level m and $\mathcal{K}$ is the ordering positive cone on ca. Observe that $\mu \in ca_+$ satisfies $\mu(E) \geq 0$ for every $E \in \mathcal{B}_\mathbb{R}$ so that $P + \mu \notin \mathcal{P}$ for $P \in \mathcal{A}^m$ and $\mu \in \mathcal{K}$ except if $\mu = 0$. For this reason the *lsc* map Φ admits a lower semicontinuous extension to ca and then theorem 2.19 in Drapeau and Kupper (2010) applies and we get the uniqueness in the class $\mathcal{R}^{\max}_{\mathcal{P}}$ (see definition 2.17 in Drapeau and Kupper 2010). In addition, $\mathcal{R}^{\max} = \mathcal{R}^{\max}_{\mathcal{P}}$ follows exactly by the same argument at the end of the proof of proposition 3.5 (Drapeau and Kupper 2010). Finally, we note that lemma C.2 in Drapeau and Kupper (2010) implies that $R^- \in \mathcal{R}^{\max}$ as $\gamma(m, f)$ is convex, positively homogeneous, and *lsc* in the second argument. □

5.3. Computation of the Dual Function

The following proposition is useful to compute the dual function $R^-(t, f)$ for the examples considered in this paper.

Proposition 5.10. *Let $\{F_m\}_{m \in \mathbb{R}}$ be a feasible family and suppose in addition that, for every m, $F_m(x)$ is increasing in x and $\lim_{x \to +\infty} F_m(x) = 1$. The associated map $\Phi : \mathcal{P} \to \mathbb{R} \cup \{+\infty\}$ defined in (3.2) is well defined, (Mon), (Qco), and $\sigma(\mathcal{P}, C_b)$-lsc and the representation (5.11) holds true with R^- given in (5.12) and*

$$\gamma(m, f) = \int f dF_{-m} + F_{-m}(-\infty) f(-\infty). \tag{5.17}$$

Proof. From equations (3.1) and (3.3) we obtain:

$$\mathcal{A}^{-m} = \left\{ Q \in \mathcal{P}(\mathbb{R}) \mid F_Q \le F_{-m} \right\} = \{ Q \in \mathcal{P} \mid \Phi(Q) \le m \}$$

so that

$$\gamma(m, f) := \sup_{Q \in \mathcal{P}} \left\{ \int f dQ \mid \Phi(Q) \le m \right\} = \sup_{Q \in \mathcal{P}} \left\{ \int f dQ \mid F_Q \le F_{-m} \right\}.$$

Fix $m \in \mathbb{R}$, $f \in C_b^-$ and define the distribution function $F_{Q_n}(x) = F_{-m}(x)\mathbf{1}_{[-n,+\infty)}$ for every $n \in \mathbb{N}$. Obviously, $F_{Q_n} \le F_{-m}$, $Q_n \downarrow$ and, taking into account (5.1), $\int f dQ_n$ is increasing. For any $\varepsilon > 0$, let $Q^\varepsilon \in \mathcal{P}$ satisfy $F_{Q^\varepsilon} \le F_{-m}$ and $\int f dQ^\varepsilon > \gamma(m, f) - \varepsilon$. Then: $F_{Q_n^\varepsilon}(x) := F_{Q^\varepsilon}(x)\mathbf{1}_{[-n,+\infty)} \uparrow F_{Q^\varepsilon}$, $F_{Q_n^\varepsilon} \le F_{Q_n}$ and

$$\int f dQ_n \ge \int f dQ_n^\varepsilon \uparrow \int f dQ^\varepsilon > \gamma(m, f) - \varepsilon.$$

We deduce that $\int f dQ_n \uparrow \gamma(m, f)$ and, because

$$\int f dQ_n = \int_{-n}^{+\infty} f dF_{-m} + F_{-m}(-n) f(-n),$$

we obtain (5.17). □

EXAMPLE 5.11. Computation of $\gamma(m, f)$ for the $\Lambda V@R$.

Let $m \in \mathbb{R}$ and $f \in C_b^-$. As $F_m(x) = \Lambda(x)\mathbf{1}_{(-\infty,m)}(x) + \mathbf{1}_{[m,+\infty)}(x)$, we compute from (5.17):

$$\gamma(m, f) = \int_{-\infty}^{-m} f d\Lambda + (1 - \Lambda(-m)) f(-m) + \Lambda(-\infty) f(-\infty). \tag{5.18}$$

We apply the integration by parts and deduce

$$\int_{-\infty}^{-m} \Lambda df = \Lambda(-m) f(-m) - \Lambda(-\infty) f(-\infty) - \int_{-\infty}^{-m} f d\Lambda.$$

We can now substitute in equation (5.18) and get:

$$\gamma(m, f) = f(-m) - \int_{-\infty}^{-m} \Lambda df = f(-\infty) + \int_{-\infty}^{-m} (1 - \Lambda) df, \tag{5.19}$$

$$R^-(t, f) = -H_f^l(t - f(-\infty)), \tag{5.20}$$

where H_f^l is the left inverse of the function: $m \to \int_{-\infty}^m (1 - \Lambda) df$.

As a particular case, we match the results obtained in Drapeau and Kupper (2010) for the $V@R$ and the worst case risk measure. Indeed, from (5.19) and (5.20) we get: $R^-(t, f) = -f^l(\frac{t - \lambda f(-\infty)}{1 - \lambda})$, if $\Lambda(x) = \lambda$; $R^-(t, f) = -f^l(t)$, if $\Lambda(x) = 0$, where f^l is the left inverse of f.

If Λ is decreasing we may use Remark 4.3 to derive a simpler formula for γ. Indeed, $\Lambda V@R(P) = \Lambda \widetilde{V}@R(P)$, where $\forall m \in \mathbb{R}$,

$$\widetilde{F}_m(x) = \Lambda(m)\mathbf{1}_{(-\infty,m)}(x) + \mathbf{1}_{[m,+\infty)}(x)$$

and so from (5.19)

$$\gamma(m, f) = f(-\infty) + [1 - \Lambda(-m)] \int_{-\infty}^{-m} df = [1 - \Lambda(-m)] f(-m) + \Lambda(-m) f(-\infty).$$

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