

## BIROn - Birkbeck Institutional Research Online

Bresolin, D. and Kurucz, A. and Muñoz-Velasco, E. and Ryzhikov, Vladislav and Sciavicco, G. and Zakharyashev, Michael (2017) Horn fragments of the Halpern-Shoham Interval Temporal Logic. *ACM Transactions on Computational Logic* 18 (3), pp. 1-39. ISSN 1529-3785.

Downloaded from: <https://eprints.bbk.ac.uk/id/eprint/20550/>

*Usage Guidelines:*

Please refer to usage guidelines at <https://eprints.bbk.ac.uk/policies.html> or alternatively contact [lib-eprints@bbk.ac.uk](mailto:lib-eprints@bbk.ac.uk).

# Horn Fragments of the Halpern-Shoham Interval Temporal Logic

DAVIDE BRESOLIN, University of Padova, Italy  
 AGI KURUCZ, King's College London, UK  
 EMILIO MUÑOZ-VELASCO, University of Malaga, Spain  
 VLADISLAV RYZHIKOV, Free University of Bozen-Bolzano, Italy  
 GUIDO SCIAVICCO, University of Ferrara, Italy  
 MICHAEL ZAKHARYASCHEV, Birkbeck, University of London, UK

We investigate the satisfiability problem for Horn fragments of the Halpern-Shoham interval temporal logic depending on the type (box or diamond) of the interval modal operators, the type of the underlying linear order (discrete or dense), and the type of semantics for the interval relations (reflexive or irreflexive). For example, we show that satisfiability of Horn formulas with diamonds is undecidable for any type of linear orders and semantics. On the contrary, satisfiability of Horn formulas with boxes is tractable over both discrete and dense orders under the reflexive semantics and over dense orders under the irreflexive semantics, but becomes undecidable over discrete orders under the irreflexive semantics. Satisfiability of binary Horn formulas with both boxes and diamonds is always undecidable under the irreflexive semantics.

Categories and Subject Descriptors: I.2.4 [Knowledge Representation Formalisms and Methods]: representation languages; F.4.1 [Mathematical Logic]: temporal logic; F.2.2 [Nonnumerical Algorithms and Problems]: complexity of proof procedures.

General Terms: languages, theory.

Additional Key Words and Phrases: temporal logic, modal logic, computational complexity.

## ACM Reference Format:

Davide Bresolin, Agi Kurucz, Emilio Muñoz-Velasco, Vladislav Ryzhikov, Guido Sciacicco, and Michael Zakharyashev. 2017. Horn Fragments of the Halpern-Shoham Interval Temporal Logic. *ACM Trans. Comput. Logic* 0, 0, Article 0 (0), 39 pages.

DOI: 0

## 1. INTRODUCTION

Our concern in this paper is the satisfiability problem for Horn fragments of the interval temporal (or modal) logic introduced by Halpern and Shoham [1991] and known since then under the moniker  $\mathcal{HS}$ . Syntactically,  $\mathcal{HS}$  is a classical propositional logic with modal diamond operators of the form  $\langle R \rangle$ , where  $R$  is one of Allen's [1983] twelve interval relations: *After*, *Begins*, *Ends*, *During*, *Later*, *Overlaps* and their inverses. The propositional variables of  $\mathcal{HS}$  are interpreted by sets of closed intervals  $[i, j]$  of some flow of time (such as  $\mathbb{Z}$ ,  $\mathbb{R}$ , etc.), and a formula  $\langle R \rangle \varphi$  is regarded to be true in  $[i, j]$  if and only if  $\varphi$  is true in some interval  $[i', j']$  such that  $[i, j]R[i', j']$  in Allen's interval algebra.

---

The authors acknowledge the support from the Italian INDAM-GNCS project 2017 'Logics and Automata for Interval Model Checking' (D. Bresolin, G. Sciacicco), the Spanish project TIN15-70266-C2-P-1 (E. Muñoz-Velasco), and the EPSRC UK project EP/M012670 'iTract: Islands of Tractability in Ontology-Based Data Access' (M. Zakharyashev).

DOI: 0

The elegance and expressive power of  $\mathcal{HS}$  have attracted attention of the temporal and modal communities, as well as many other areas of computer science, AI, philosophy and linguistics, e.g., [Allen 1984; Cau et al. 2002; Zhou and Hansen 2004; Cimatti et al. 2015; Della Monica et al. 2011; Pratt-Hartmann 2005]. However, promising applications have been hampered by the fact, already discovered by Halpern and Shoham [1991], that  $\mathcal{HS}$  is highly undecidable (for example, validity over  $\mathbb{Z}$  and  $\mathbb{R}$  is  $\Pi_1^1$ -hard).

A quest for ‘tame’ fragments of  $\mathcal{HS}$  began in the 2000s, and has resulted in a substantial body of literature that identified a number of ways of reducing the expressive power of  $\mathcal{HS}$ :

- *Constraining the underlying temporal structures.* Montanari et al. [2002] interpreted their Split Logic  $\mathcal{SL}$  over structures where every interval can be chopped into at most a constant number of subintervals.  $\mathcal{SL}$  shares the syntax with  $\mathcal{HS}$  and  $\mathcal{CDT}$  [Venema 1991] and can be seen as their decidable variant.
- *Restricting the set of modal operators.* Complete classifications of decidable and undecidable fragments of  $\mathcal{HS}$  have been obtained for finite linear orders (62 decidable fragments), discrete linear orders (44),  $\mathbb{N}$  (47),  $\mathbb{Z}$  (44), and dense linear orders (130). For example, over finite linear orders, there are two maximal decidable fragments with the relations  $A, \bar{A}, B, \bar{B}$  and  $A, \bar{A}, E, \bar{E}$ , both of which are non-primitive recursive. Smaller fragments may have lower complexity: for example, the  $B, \bar{B}, L, \bar{L}$  fragment is NP-complete,  $A, \bar{A}$  is NEXPTIME-complete, while  $A, B, \bar{B}, L$  is EXPSPACE-complete. For more details, we refer the reader to [Lodaya 2000; Montanari et al. 2010b; Bresolin et al. 2012a; Bresolin et al. 2012b; Bresolin et al. 2015] and references therein.
- *Softening semantics.* Allen [1983] and Halpern and Shoham [1991] defined the semantics of interval relations using the irreflexive  $<$ : for example,  $[x, y]L[x', y']$  if and only if  $y < x'$ . By ‘softening’  $<$  to reflexive  $\leq$  one can make the undecidable D fragment of  $\mathcal{HS}$  [Marcinkowski and Michaliszyn 2014] decidable and PSPACE-complete [Montanari et al. 2010a].
- *Relativisations.* The results of Schwentick and Zeume [2010] imply that some undecidable fragments of  $\mathcal{HS}$  become decidable if one allows models in which not all the possible intervals of the underlying linear order are present.
- *Restricting the nesting of modal operators.* Bresolin et al. [2014a] defined a decidable fragment of  $\mathcal{CDT}$  that mimics the behaviour of the (NP-complete) Bernays-Schöenfinkel fragment of first-order logic, and one can define a similar fragment of  $\mathcal{HS}$ .
- *Coarsening relations.* Inspired by Golumbic and Shamir’s [1993] coarser interval algebra, Muñoz-Velasco et al. [2015] reduce the expressive power of  $\mathcal{HS}$  by defining interval relations that correspond to (relational) unions of Allen’s relations. They proposed two coarsening schemata, one of which turned out to be PSPACE-complete.

In this article, we analyse a different way of taming the expressive power of logic formalisms while retaining their usefulness for applications, viz., taking Horn fragments. Universal first-order Horn sentences  $\forall x(A_1 \wedge \dots \wedge A_n \rightarrow A_0)$  with atomic  $A_i$  are rules (or clauses) of the programming language Prolog. Although Prolog itself is undecidable due to the availability of functional symbols, its function-free subset Datalog, designed for interacting with databases, is EXPTIME-complete for combined complexity, even PSPACE-complete when restricted to predicates of bounded arity, and P-complete in the propositional case [Dantsin et al. 2001]. Horn fragments of the Web Ontology Language  $OWL2$  [W3C OWL Working Group 2012] such as the tractable profiles  $OWL2QL$  and  $OWL2EL$  were designed for ontology-based data access via query rewriting and applications that require ontologies with

very large numbers of properties and classes (e.g., SNOMED CT). More expressive decidable Horn knowledge representation formalisms have been designed in Description Logic [Hustadt et al. 2007; Krötzsch et al. 2013], in particular, temporal description logics; see [Lutz et al. 2008; Artale et al. 2014] and references therein. Horn fragments of modal and (metric) temporal logics have also been considered [Fariñas Del Cerro and Penttonen 1987; Chen and Lin 1993; Chen and Lin 1994; Nguyen 2005; Artale et al. 2013; Brandt et al. 2017].

In the context of the Halpern-Shoham logic, we observe first that any  $\mathcal{HS}$ -formula can be transformed to an equisatisfiable formula in *clausal normal form*:

$$\varphi ::= \lambda \mid \neg\lambda \mid [\text{U}](\neg\lambda_1 \vee \dots \vee \neg\lambda_n \vee \lambda_{n+1} \vee \dots \vee \lambda_{n+m}) \mid \varphi_1 \wedge \varphi_2, \quad (1)$$

where  $\text{U}$  is the *universal relation* (which can be expressed via the interval relations as  $[\text{U}]\psi = \bigwedge_{\text{R}}(\psi \wedge [\text{R}]\psi \wedge [\bar{\text{R}}]\psi)$ ), and  $\lambda$  and the  $\lambda_i$  are (*positive temporal*) *literals* given by

$$\lambda ::= \top \mid \perp \mid p \mid \langle \text{R} \rangle \lambda \mid [\text{R}]\lambda, \quad (2)$$

with  $\text{R}$  being one of the interval relations and  $p$  a *propositional variable* and  $[\text{R}]$  the dual of  $\langle \text{R} \rangle$ . We now define the *Horn fragment*  $\mathcal{HS}_{\text{horn}}$  of  $\mathcal{HS}$  as comprising the formulas given by the grammar

$$\varphi ::= \lambda \mid [\text{U}](\lambda_1 \wedge \dots \wedge \lambda_k \rightarrow \lambda) \mid \varphi_1 \wedge \varphi_2. \quad (3)$$

The conjuncts of the form  $\lambda$  are called the *initial conditions* of  $\varphi$ , and those of the form  $[\text{U}](\lambda_1 \wedge \dots \wedge \lambda_k \rightarrow \lambda)$  the *clauses* of  $\varphi$ . We also consider the  $\mathcal{HS}_{\text{horn}}^{\square}$  fragment of  $\mathcal{HS}_{\text{horn}}$ , whose formulas do not contain occurrences of diamond operators  $\langle \text{R} \rangle$ , and the  $\mathcal{HS}_{\text{horn}}^{\diamond}$  fragment whose formulas do not contain box operators  $[\text{R}]$ . We denote by  $\mathcal{HS}_{\text{core}}$  ( $\mathcal{HS}_{\text{core}}^{\square}$  or  $\mathcal{HS}_{\text{core}}^{\diamond}$ ) the fragment of  $\mathcal{HS}_{\text{horn}}$  (respectively,  $\mathcal{HS}_{\text{horn}}^{\square}$  or  $\mathcal{HS}_{\text{horn}}^{\diamond}$ ) with only clauses of the form  $[\text{U}](\lambda_1 \rightarrow \lambda_2)$  and  $[\text{U}](\lambda_1 \wedge \lambda_2 \rightarrow \perp)$ . We remind the reader that propositional Horn logic is P-complete, while the (core) logic of binary Horn clauses is NLOGSPACE-complete.

We illustrate the expressive power of the Horn fragments introduced above by a few examples describing constraints on a summer school timetable. The clause

$$[\text{U}](\langle \bar{\text{D}} \rangle \text{MorningSession} \wedge \text{AdvancedCourse} \rightarrow \perp)$$

says that advanced courses cannot be given during the morning sessions defined by

$$[\text{U}](\langle \bar{\text{B}} \rangle \text{LectureDay} \wedge \langle \text{A} \rangle \text{Lunch} \leftrightarrow \text{MorningSession}).$$

The clause

$$[\text{U}](\text{teaches} \rightarrow [\text{D}]\text{teaches})$$

claims that *teaches* is *downward hereditary* (or *stative*) in the sense that if it holds in some interval, then it also holds in all of its sub-intervals. If, instead, we want to state that *teaches* is *upward hereditary* (or *coalesced*) in the sense that *teaches* holds in any interval covered by sub-intervals where it holds, then we can use the clause<sup>1</sup>

$$[\text{U}](([\text{D}](\langle \text{O} \rangle \text{teaches} \vee \langle \bar{\text{D}} \rangle \text{teaches}) \wedge \langle \text{B} \rangle \text{teaches} \wedge \langle \text{E} \rangle \text{teaches} \rightarrow \text{teaches}).$$

By removing the last two conjuncts on the left-hand side of this clause, we make sure that *teaches* is both upward and downward hereditary. For a discussion of these notions in temporal databases, consult [Böhlen et al. 1996; Terenziani and Snodgrass 2004]. Note also that all of the above example clauses—apart from the implication  $\leftarrow$  in the second one—are equisatisfiable to  $\mathcal{HS}_{\text{horn}}^{\square}$ -formulas (see Section 2 for details).

<sup>1</sup>Here we assume that the interval relations are reflexive; see Section 2.

Table I. Horn and core  $\mathcal{HS}$ -satisfiability over various linear orders.

	Irreflexive semantics	Reflexive semantics
$\mathcal{HS}_{horn}$	undecidable* (Thm. 4.3)	
$\mathcal{HS}_{core}$	undecidable* (Thm. 4.4)	PSPACE-hard* (Thm. 4.1) <span style="border: 1px solid black; padding: 2px;">decidable?</span>
$\mathcal{HS}_{horn}^{\diamond}$	undecidable* (Thm. 4.3)	
$\mathcal{HS}_{core}^{\diamond}$	<span style="border: 1px solid black; padding: 2px;">decidable?</span>	
$\mathcal{HS}_{horn}^{\square}$	discrete: undecidable (Thm. 4.5) dense: P-complete (Thm. 3.5)	P-complete (Thm. 3.5)
$\mathcal{HS}_{core}^{\square}$	discrete: PSPACE-hard (Thm. 4.2) <span style="border: 1px solid black; padding: 2px;">decidable?</span> dense: in P (Thm. 3.5)	in P (Thm. 3.5)

\*actually holds for any class of linear orders containing unbounded orders.

*Our contribution.* In this article, we investigate the satisfiability problem for the Horn fragments of  $\mathcal{HS}$  along two main axes. We consider:

- both the standard ‘irreflexive’ semantics for  $\mathcal{HS}$ -formulas given by Halpern and Shoham [1991] and its reflexive variant
- over classes of discrete and dense linear orders (such as  $(\mathbb{Z}, \leq)$  and  $(\mathbb{R}, \leq)$ ), and general linear orders.

The obtained results are summarised in Table I. Most surprising is the computational behaviour of  $\mathcal{HS}_{horn}^{\square}$ , which turns out to be undecidable over discrete orders under the irreflexive semantics (Theorem 4.5), but becomes tractable under all other choices of semantics (Theorem 3.5). The tractability result, coupled with the ability of  $\mathcal{HS}_{horn}^{\square}$ -formulas to express interesting temporal constraints, suggests that  $\mathcal{HS}_{horn}^{\square}$  can form a basis for tractable interval temporal ontology languages that can be used for ontology-based data access over temporal databases or streamed data. Some preliminary steps in this direction have been made by Artale et al. [2015b] and Kontchakov et al. [2016]. We briefly discuss applications of  $\mathcal{HS}_{horn}^{\square}$  for temporal ontology-based data access in Section 3.1.

On the other hand, the undecidability of  $\mathcal{HS}_{horn}^{\square}$  over discrete orders with the irreflexive semantics prompted us to investigate possible sources of high complexity.

- What is the crucial difference between the irreflexive discrete and other semantic choices? In discrete models, there is a natural notion of ‘interval length’. With the irreflexive semantics, one can single out intervals of any ‘fixed’ length using very simple ( $\mathcal{HS}_{core}^{\square}$ ) formulas: for example,  $[R]_{\perp}$ , where  $R$  is either  $E$  or  $B$ , defines either intervals of length 0 (punctual intervals) or of length 1 (depending on whether one allows punctual intervals or not). Looking at  $\mathcal{HS}$ -models from the 2D perspective as in Fig. 1, we see that intervals of the same fixed length form a *diagonal*. Such a ‘definable’ diagonal might provide us with some kind of ‘horizontal’ and ‘vertical’ counting

capabilities along the 2D grid, even though the horizontal and vertical ‘next-time operators’ are not available in  $\mathcal{HS}$ . It is a well-known fact about 2D *modal product logics* that, if such a ‘unique controllable diagonal’ is expressible in a logic, then the satisfiability problem for the logic is of high complexity [Gabbay et al. 2003]. Here we show that  $\mathcal{HS}_{horn}^{\square}$  has sufficient counting power to make it undecidable (Theorem 4.5), and that even the seemingly very limited expressiveness of  $\mathcal{HS}_{core}^{\square}$  is still enough to make it PSPACE-hard (Theorem 4.2).

- When  $\diamond$ -operators are available, even if the models are reflexive and/or dense, one can generate a unique sequence of ‘diagonal-squares’ (like on a chessboard) and perform some horizontal and vertical counting on it. In particular, bimodal logics over products of (reflexive/irreflexive) linear orders [Marx and Reynolds 1999; Reynolds and Zakharyashev 2001] and also over products of various transitive (not necessarily linear) relations [Gabelaia et al. 2005b] are all shown to be undecidable in this way. It follows that full Boolean  $\mathcal{HS}$ -satisfiability with the reflexive semantics over any unbounded timelines is undecidable. Here we generalise this methodology and show that undecidability still holds even within the  $\mathcal{HS}_{horn}^{\diamond}$ -fragment (Theorem 4.3).
- We also analyse to what extent the above techniques can be applied within the *core* fragments having  $\diamond$ -operators. We develop a few new ‘tricks’ that encode a certain degree of ‘Horn-ness’ to prove intractable lower bounds for  $\mathcal{HS}_{core}$ -satisfiability: undecidability with the irreflexive semantics (Theorem 4.4) and PSPACE-hardness with the reflexive one (Theorem 4.1).

The undecidability of  $\mathcal{HS}_{horn}$  under the irreflexive semantics was established in a conference paper by Bresolin et al. [2014b], and the tractability of  $\mathcal{HS}_{horn}^{\square}$  over  $(\mathbb{Z}, \leq)$  under the reflexive semantics by Artale et al. [2015b].

## 2. SEMANTICS AND NOTATION

$\mathcal{HS}$ -formulas are interpreted over the set of intervals of any linear order<sup>2</sup>  $\mathfrak{T} = (T, \leq)$  (where  $\leq$  is a reflexive, transitive, antisymmetric and connected binary relation on  $T$ ). As usual, we use  $x < y$  as a shortcut for ‘ $x \leq y$  and  $x \neq y$ ’. The linear order  $\mathfrak{T}$  is

- *dense* if, for any  $x, y \in T$  with  $x < y$ , there exists  $z$  such that  $x < z < y$ ;
- *discrete* if every non-maximal  $x \in T$  has an immediate  $<$ -successor, and every non-minimal  $x \in T$  has an immediate  $<$ -predecessor.

Thus, the rationals  $(\mathbb{Q}, \leq)$  and reals  $(\mathbb{R}, \leq)$  are dense orders, while the integers  $(\mathbb{Z}, \leq)$  and the natural numbers  $(\mathbb{N}, \leq)$  are discrete. Any finite linear order is obviously discrete. We denote by  $\text{Lin}$  the class of all linear orders, by  $\text{Fin}$  the class of all finite linear orders, by  $\text{Dis}$  the class of all discrete linear orders, and by  $\text{Den}$  the class of all dense linear orders. We say that a linear order *contains an infinite ascending (descending) chain* if it has a sequence of points  $x_n, n < \omega$ , such that  $x_0 < x_1 < \dots < x_n < \dots$  (respectively,  $x_0 > x_1 > \dots > x_n > \dots$ ). Clearly, any infinite linear order contains an infinite ascending or an infinite descending chain.

Following Halpern and Shoham [1991], by an *interval in  $\mathfrak{T}$*  we mean any ordered pair  $\langle x, y \rangle$  such that  $x \leq y$ , and denote by  $\text{int}(\mathfrak{T})$  the set of all intervals in  $\mathfrak{T}$ . Note that  $\text{int}(\mathfrak{T})$  contains all the *punctual intervals* of the form  $\langle x, x \rangle$ , which is often referred to as the *non-strict* semantics. Under the *strict* semantics adopted by Allen [1983], punctual intervals are disallowed. All of our results hold for both semantics, with slight adjust-

<sup>2</sup>Originally, Halpern and Shoham [1991] also consider more complex temporal structures based on partial orders with *linear intervals* such that, whenever  $x \leq y$ , the closed interval  $\{z \in T \mid x \leq z \leq y\}$  is linearly ordered by  $\leq$ . In particular, trees are temporal structures in this sense.

ments in the proofs in case of the strict semantics. We define the interval relations over  $\text{int}(\mathfrak{T})$  in the same way as Halpern and Shoham [1991] by taking (see Fig. 1):

- $\langle x_1, y_1 \rangle \mathbf{A} \langle x_2, y_2 \rangle$  iff<sup>3</sup>  $y_1 = x_2$  and  $x_2 < y_2$ ; (After)
- $\langle x_1, y_1 \rangle \mathbf{B} \langle x_2, y_2 \rangle$  iff  $x_1 = x_2$  and  $y_2 < y_1$ ; (Begins)
- $\langle x_1, y_1 \rangle \mathbf{E} \langle x_2, y_2 \rangle$  iff  $x_1 < x_2$  and  $y_1 = y_2$ ; (Ends)
- $\langle x_1, y_1 \rangle \mathbf{D} \langle x_2, y_2 \rangle$  iff  $x_1 < x_2$  and  $y_2 < y_1$ ; (During)
- $\langle x_1, y_1 \rangle \mathbf{L} \langle x_2, y_2 \rangle$  iff  $y_1 < x_2$ ; (Later)
- $\langle x_1, y_1 \rangle \mathbf{O} \langle x_2, y_2 \rangle$  iff  $x_1 < x_2 < y_1 < y_2$ ; (Overlaps)
- $\langle x_1, y_1 \rangle \bar{\mathbf{A}} \langle x_2, y_2 \rangle$  iff  $y_2 = x_1$  and  $x_2 < y_2$ ;
- $\langle x_1, y_1 \rangle \bar{\mathbf{B}} \langle x_2, y_2 \rangle$  iff  $x_1 = x_2$  and  $y_1 < y_2$ ;
- $\langle x_1, y_1 \rangle \bar{\mathbf{E}} \langle x_2, y_2 \rangle$  iff  $x_2 < x_1$  and  $y_1 = y_2$ ;
- $\langle x_1, y_1 \rangle \bar{\mathbf{D}} \langle x_2, y_2 \rangle$  iff  $x_2 < x_1$  and  $y_1 < y_2$ ;
- $\langle x_1, y_1 \rangle \bar{\mathbf{L}} \langle x_2, y_2 \rangle$  iff  $y_2 < x_1$ ;
- $\langle x_1, y_1 \rangle \bar{\mathbf{O}} \langle x_2, y_2 \rangle$  iff  $x_2 < x_1 < y_2 < y_1$ .

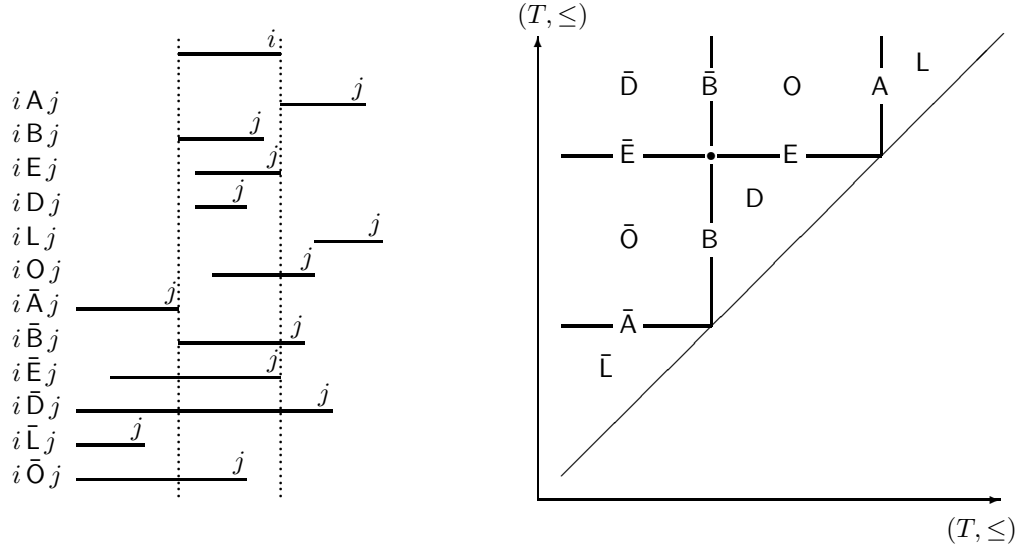


Fig. 1. The interval relations and their 2D representation.

Observe that all of these relations are irreflexive, so we refer to the definition above as the *irreflexive semantics*. As an alternative, we also consider the *reflexive semantics*, which is obtained by replacing each  $<$  with  $\leq$ . We write  $\mathfrak{T}(\leq)$  or  $\mathfrak{T}(<)$  to indicate that the semantics is reflexive or, respectively, irreflexive. When formulating results where the choice of semantics for each interval relation does not matter, we use the term *arbitrary semantics*<sup>4</sup>.

As observed by Venema [1990], if we represent intervals  $\langle x, y \rangle \in \text{int}(\mathfrak{T})$  by points  $(x, y)$  of the ‘north-western’ subset of the two-dimensional Cartesian product  $T \times T$ ,

<sup>3</sup>It is to be noted that there exist slightly different versions of  $\mathbf{A}$  and  $\bar{\mathbf{A}}$  in the literature. All of our results hold with those versions as well, with slight adjustments in the proofs.

<sup>4</sup>It may be of interest to note that the query language SQL:2011 has seven interval temporal operators three of which are under the reflexive semantics and four under the irreflexive one [Kulkarni and Michels 2012].

then  $\text{int}(\mathfrak{T})$  together with the interval relations (under any semantics) forms a multi-modal Kripke frame (see Fig. 1). We denote it by  $\mathfrak{F}_{\mathfrak{T}}$  and call an  $\mathcal{HS}$ -frame.<sup>5</sup> Given a linear order  $\mathfrak{T}$ , an  $\mathcal{HS}$ -model based on  $\mathfrak{T}$  is a pair  $\mathfrak{M} = (\mathfrak{F}_{\mathfrak{T}}, \nu)$ , where  $\mathfrak{F}_{\mathfrak{T}}$  is an  $\mathcal{HS}$ -frame and  $\nu$  a function from the set  $\mathcal{P}$  of propositional variables to subsets of  $\text{int}(\mathfrak{T})$ . The *truth-relation*  $\mathfrak{M}, \langle x, y \rangle \models \varphi$ , for an  $\mathcal{HS}_{\text{horn}}$ -formula  $\varphi$  (read:  $\varphi$  holds at  $\langle x, y \rangle$  in  $\mathfrak{M}$ ), is defined inductively as follows, where  $R$  is any interval relation:

- $\mathfrak{M}, \langle x, y \rangle \models \top$  and  $\mathfrak{M}, \langle x, y \rangle \not\models \perp$ , for any  $\langle x, y \rangle \in \text{int}(\mathfrak{T})$ ;
- $\mathfrak{M}, \langle x, y \rangle \models p$  iff  $\langle x, y \rangle \in \nu(p)$ , for any  $p \in \mathcal{P}$ ;
- $\mathfrak{M}, \langle x, y \rangle \models \langle R \rangle \lambda$  iff there exists  $\langle x', y' \rangle$  such that  $\langle x, y \rangle R \langle x', y' \rangle$  and  $\mathfrak{M}, \langle x', y' \rangle \models \lambda$ ;
- $\mathfrak{M}, \langle x, y \rangle \models [R] \lambda$  iff for every  $\langle x', y' \rangle$  with  $\langle x, y \rangle R \langle x', y' \rangle$ , we have  $\mathfrak{M}, \langle x', y' \rangle \models \lambda$ ;
- $\mathfrak{M}, \langle x, y \rangle \models [U](\lambda_1 \wedge \dots \wedge \lambda_k \rightarrow \lambda)$  iff, for every  $\langle x', y' \rangle \in \text{int}(\mathfrak{T})$  with  $\mathfrak{M}, \langle x', y' \rangle \models \lambda_i$  for  $i = 1, \dots, k$ , we have  $\mathfrak{M}, \langle x', y' \rangle \models \lambda$ ;
- $\mathfrak{M}, \langle x, y \rangle \models \varphi_1 \wedge \varphi_2$  iff  $\mathfrak{M}, \langle x, y \rangle \models \varphi_1$  and  $\mathfrak{M}, \langle x, y \rangle \models \varphi_2$ .

A model  $\mathfrak{M}$  based on  $\mathfrak{T}$  satisfies  $\varphi$  if  $\mathfrak{M}, \langle x, y \rangle \models \varphi$ , for some  $\langle x, y \rangle \in \text{int}(\mathfrak{T})$ . Given a class  $\mathcal{C}$  of linear orders, we say that a formula  $\varphi$  is  $\mathcal{C}$ -satisfiable (respectively,  $\mathcal{C}(\leq)$ - or  $\mathcal{C}(<)$ -satisfiable) if it is satisfiable in an  $\mathcal{HS}$ -model based on some order from  $\mathcal{C}$  under the arbitrary (respectively, reflexive or irreflexive) semantics.

To facilitate readability, we use the following *syntactic sugar*, where  $\psi = \lambda_1 \wedge \dots \wedge \lambda_k$ :

- $[U](\psi \rightarrow \neg \lambda)$  as an abbreviation for  $[U](\psi \wedge \lambda \rightarrow \perp)$ ;
- $[U](\psi \rightarrow \lambda'_1 \wedge \dots \wedge \lambda'_n)$  as an abbreviation for

$$\bigwedge_{i=1}^n [U](\psi \rightarrow \lambda'_i);$$

- $[U](\psi \rightarrow [R](\lambda'_1 \wedge \dots \wedge \lambda'_n \rightarrow \lambda))$  as an abbreviation for

$$[U](\psi \rightarrow [R]p) \wedge [U](p \wedge \lambda'_1 \wedge \dots \wedge \lambda'_n \rightarrow \lambda),$$

where  $p$  is a fresh variable, and similarly for  $\langle R \rangle$  in place of  $[R]$ .

Note also that  $[U](\langle R \rangle \lambda \wedge \psi \rightarrow \lambda')$  is equivalent to  $[U](\lambda \rightarrow [\bar{R}](\psi \rightarrow \lambda'))$ . This allows us to use  $\langle R \rangle$  on the left-hand side of the clauses in  $\mathcal{HS}_{\text{horn}}^{\square}$ -formulas, and  $[R]$  on the right-hand side of the clauses in  $\mathcal{HS}_{\text{horn}}^{\diamond}$ -formulas.

### 3. TRACTABILITY OF $\mathcal{HS}_{\text{HORN}}^{\square}$

Let  $\mathfrak{T} = (T, \leq)$  be a linear order,  $\langle a, b \rangle \in \text{int}(\mathfrak{T})$ , and let  $\varphi$  be an  $\mathcal{HS}_{\text{horn}}^{\square}$ -formula. Suppose we want to check whether there exists a model  $\mathfrak{M}$  based on  $\mathfrak{T}$  such that  $\mathfrak{M}, \langle a, b \rangle \models \varphi$  under the reflexive (or irreflexive) semantics, in which case we will say that  $\varphi$  is  $\langle a, b \rangle$ -satisfiable in  $\mathfrak{T}(\leq)$  (respectively,  $\mathfrak{T}(<)$ ). Let  $\triangleleft \in \{\leq, <\}$ . We set

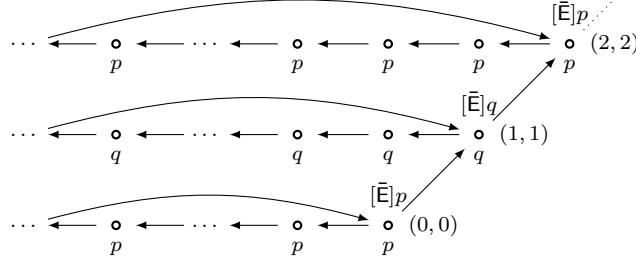
$$\mathfrak{V}_{\varphi} = \{\lambda @ \langle a, b \rangle \mid \lambda \text{ an initial condition of } \varphi\} \cup \{\top @ \langle x, y \rangle \mid \langle x, y \rangle \in \text{int}(\mathfrak{T})\}$$

and denote by  $\text{cl}(\mathfrak{V}_{\varphi})$  the result of applying non-recursively the following rules to  $\mathfrak{V}_{\varphi}$ , where  $R$  is any interval relation in  $\mathfrak{T}(\triangleleft)$ :

- (cl1) if  $[R] \lambda @ \langle x, y \rangle \in \mathfrak{V}_{\varphi}$ , then we add to  $\mathfrak{V}_{\varphi}$  all  $\lambda @ \langle x', y' \rangle$  such that  $\langle x', y' \rangle \in \text{int}(\mathfrak{T})$  and  $\langle x, y \rangle R \langle x', y' \rangle$ ;
- (cl2) if  $\lambda @ \langle x', y' \rangle \in \mathfrak{V}_{\varphi}$  for all  $\langle x', y' \rangle \in \text{int}(\mathfrak{T})$  such that  $\langle x, y \rangle R \langle x', y' \rangle$  and  $[R] \lambda$  occurs in  $\varphi$ , then we add  $[R] \lambda @ \langle x, y \rangle$  to  $\mathfrak{V}_{\varphi}$ ;

<sup>5</sup>Note that if we consider  $\mathfrak{T} = (T, \leq)$  as a unimodal Kripke frame, then  $(\text{int}(\mathfrak{T}), E, \bar{B})$  with the reflexive semantics is an expanding subframe of the modal product frame  $\mathfrak{T} \times \mathfrak{T}$ ; see [Gabbay et al. 2003, Section 3.9].



Fig. 2. The sequence of the canonical model construction for  $(\mathbb{Z}, \leq)$ .

(cl3) if  $[\mathbf{U}](\lambda_1 \wedge \dots \wedge \lambda_k \rightarrow \lambda)$  is a clause of  $\varphi$  and  $\lambda_i @ \langle x, y \rangle \in \mathfrak{V}_\varphi$ , for  $1 \leq i \leq k$ , then we add  $\lambda @ \langle x, y \rangle$  to  $\mathfrak{V}_\varphi$ .

Now, we set  $\text{cl}^0(\mathfrak{V}_\varphi) = \mathfrak{V}_\varphi$  and, for any successor ordinal  $\alpha + 1$  and limit ordinal  $\beta$ ,

$$\text{cl}^{\alpha+1}(\mathfrak{V}_\varphi) = \text{cl}(\text{cl}^\alpha(\mathfrak{V}_\varphi)), \quad \text{cl}^\beta(\mathfrak{V}_\varphi) = \bigcup_{\alpha < \beta} \text{cl}^\alpha(\mathfrak{V}_\varphi) \quad \text{and} \quad \text{cl}^*(\mathfrak{V}_\varphi) = \bigcup_{\gamma \text{ an ordinal}} \text{cl}^\gamma(\mathfrak{V}_\varphi).$$

Define an  $\mathcal{HS}$ -model  $\mathfrak{R}_\varphi^{(a,b)} = (\mathfrak{T}_\varphi, \nu)$  based on  $\mathfrak{T}(\triangleleft)$  by taking, for every variable  $p$ ,

$$\nu(p) = \{\langle x, y \rangle \mid p @ \langle x, y \rangle \in \text{cl}^*(\mathfrak{V}_\varphi)\}.$$

*Example 3.1.* Let  $\mathfrak{T} = (\mathbb{Z}, \leq)$ . The model  $\mathfrak{R}_\varphi^{(0,0)}$  based on  $\mathfrak{T}(\triangleleft)$  for the  $\mathcal{HS}_{horn}^\square$ -formula

$$\varphi = p \wedge [\mathbf{U}]([\mathbf{E}]p \wedge \langle \mathbf{E} \rangle \top \rightarrow p) \wedge [\mathbf{U}]([\mathbf{E}]q \wedge \langle \mathbf{E} \rangle \top \rightarrow q) \wedge [\mathbf{U}](\langle \bar{\mathbf{E}} \rangle [\mathbf{B}] [\bar{\mathbf{E}}] p \rightarrow q) \wedge [\mathbf{U}](\langle \bar{\mathbf{E}} \rangle [\mathbf{B}] [\bar{\mathbf{E}}] q \rightarrow p)$$

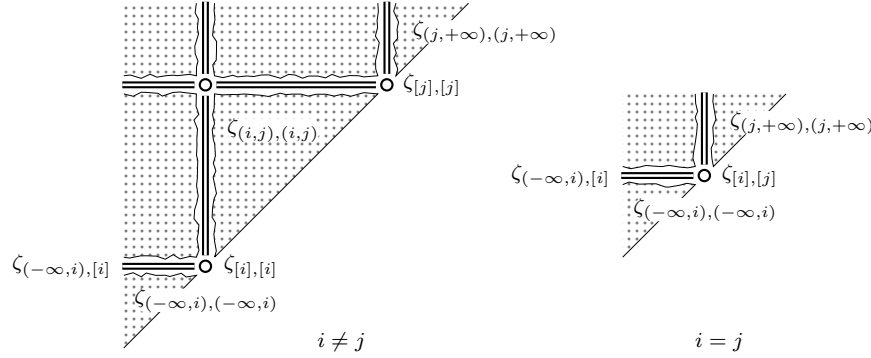
is shown in Fig. 2. Note that the construction of  $\mathfrak{R}_\varphi^{(0,0)}$  requires  $\omega^2$  applications of  $\text{cl}$ .

**THEOREM 3.2.** *An  $\mathcal{HS}_{horn}^\square$ -formula  $\varphi$  is  $\langle a, b \rangle$ -satisfiable in  $\mathfrak{T}(\triangleleft)$  if and only if  $\perp @ \langle x, y \rangle \notin \text{cl}^*(\mathfrak{V}_\varphi)$ , for any  $\langle x, y \rangle$ . Furthermore, if some model  $\mathfrak{M}$  over  $\mathfrak{T}(\triangleleft)$  satisfies  $\varphi$  at  $\langle a, b \rangle$ , then  $\mathfrak{R}_\varphi^{(a,b)}, \langle a, b \rangle \models \varphi$  and, for any  $\langle x, y \rangle \in \text{int}(\mathfrak{T})$  and any variable  $p$ ,  $\mathfrak{R}_\varphi^{(a,b)}, \langle x, y \rangle \models p$  implies  $\mathfrak{M}, \langle x, y \rangle \models p$ .*

**PROOF.** Suppose  $\perp @ \langle x, y \rangle \notin \text{cl}^*(\mathfrak{V}_\varphi)$ . It is easily shown by induction that we have  $\lambda @ \langle x, y \rangle \in \text{cl}^*(\mathfrak{V}_\varphi)$  iff  $\mathfrak{R}_\varphi^{(a,b)}, \langle x, y \rangle \models \lambda$ . It follows that  $\mathfrak{R}_\varphi^{(a,b)}, \langle a, b \rangle \models \varphi$ . Suppose also that  $\mathfrak{M}, \langle a, b \rangle \models \varphi$ , for some model  $\mathfrak{M}$  over  $\mathfrak{T}(\triangleleft)$ . Denote by  $\mathfrak{V}$  the set of  $\lambda @ \langle x, y \rangle$  such that  $\lambda$  occurs in  $\varphi$ ,  $\langle x, y \rangle \in \text{int}(\mathfrak{T})$  and  $\mathfrak{M}, \langle x, y \rangle \models \lambda$ . Clearly,  $\mathfrak{V}$  is closed under the rules for  $\text{cl}$ , and so  $\text{cl}^*(\mathfrak{V}_\varphi) \subseteq \mathfrak{V}$ . This observation also shows that if  $\varphi$  is  $\langle a, b \rangle$ -satisfiable in  $\mathfrak{T}(\triangleleft)$  then  $\perp @ \langle x, y \rangle \notin \text{cl}^*(\mathfrak{V}_\varphi)$ .  $\square$

If  $\perp @ \langle x, y \rangle \notin \text{cl}^*(\mathfrak{V}_\varphi)$ , we call  $\mathfrak{R}_\varphi^{(a,b)}$  the *canonical model of  $\varphi$  based on  $\mathfrak{T}(\triangleleft)$* . Our next aim is to show that if (i)  $\mathfrak{T} \in \text{Dis}$  and  $\triangleleft$  is  $\leq$ , or (ii)  $\mathfrak{T} \in \text{Den}$  and  $\triangleleft \in \{\leq, <\}$ , then there is a bounded-size multi-modal Kripke frame  $\mathfrak{Z}^{(a,b)}$  with a set of worlds  $Z$  and an accessibility relation  $R$ , for every interval relation  $R$ , and a surjective map  $f: \text{int}(\mathfrak{T}) \rightarrow Z$  such that the following conditions hold:

- (p1) if  $\langle x, y \rangle R \langle x', y' \rangle$  then  $f(\langle x, y \rangle) R f(\langle x', y' \rangle)$ ;
- (p2) if  $z R z'$  then, for every  $\langle x, y \rangle \in f^{-1}(z)$ , there is  $\langle x', y' \rangle \in f^{-1}(z')$  with  $\langle x, y \rangle R \langle x', y' \rangle$ ;
- (p3) for any variable  $p$  and any  $z \in Z$ , either  $f^{-1}(z) \cap \nu(p) = \emptyset$  or  $f^{-1}(z) \subseteq \nu(p)$ .

Fig. 3. Zones in the canonical models over  $\text{Dis}(\leq)$  and  $\text{Den}(\leq)$ .

In modal logic, a surjection respecting the first two properties is called a *p-morphism* (or *bounded morphism*) from  $\mathfrak{F}_{\mathfrak{T}}$  to  $\mathfrak{Z}^{(a,b)}$  (see, e.g., [Chagroff and Zakharyashev 1997; Goranko and Otto 2006]). It is well-known that if  $f$  is a p-morphism from  $\mathfrak{F}_{\mathfrak{T}}$  to  $\mathfrak{Z}^{(a,b)}$  and  $\varphi$  is  $f(\langle a, b \rangle)$ -satisfiable in  $\mathfrak{Z}^{(a,b)}$  then  $\varphi$  is  $\langle a, b \rangle$ -satisfiable in  $\mathfrak{T}(\triangleleft)$ . Moreover, if the third condition also holds and  $\mathfrak{R}_{\varphi}^{(a,b)}, \langle a, b \rangle \models \varphi$ , then  $\varphi$  is  $f(\langle a, b \rangle)$ -satisfiable in  $\mathfrak{Z}^{(a,b)}$ . Indeed, in this case  $f$  is a p-morphism from the canonical model  $\mathfrak{R}_{\varphi}^{(a,b)}$  onto the model  $(\mathfrak{Z}^{(a,b)}, \nu')$ , where  $\nu'(p) = \{z \mid f^{-1}(z) \subseteq \nu(p)\}$ .

To construct  $\mathfrak{Z}^{(a,b)}$  and  $f$ , we require a few definitions. If  $a < b$ , we denote by  $\text{sec}_{\mathfrak{T}}(a, b)$  the set of *non-empty* subsets of  $T$  of the form  $(-\infty, a)$ ,  $[a, a]$ ,  $(a, b)$ ,  $[b, b]$  and  $(b, \infty)$ , where  $(-\infty, a) = \{x \in T \mid x < a\}$  and  $(b, \infty) = \{x \in T \mid x > b\}$ . If  $a = b$ , then  $\text{sec}_{\mathfrak{T}}(a, b)$  consists of non-empty sets of the form  $(-\infty, a)$ ,  $[a, a]$  and  $(a, \infty)$ . We call each  $\sigma \in \text{sec}_{\mathfrak{T}}(a, b)$  an  $(a, b)$ -*section* of  $\mathfrak{T}$ . Clearly,  $\text{sec}_{\mathfrak{T}}(a, b)$  is a partition of  $T$ . Given  $\sigma, \sigma' \in \text{sec}_{\mathfrak{T}}(a, b)$ , we write  $\sigma \preceq \sigma'$  if there exist  $x \in \sigma$  and  $x' \in \sigma'$  such that  $\langle x, x' \rangle \in \text{int}(\mathfrak{T})$ . The definition of  $\mathfrak{Z}^{(a,b)}$  depends on the type of the linear order  $\mathfrak{T}$  and the semantics for the interval relations.

*Case  $\mathfrak{T}(\leq)$ , for  $\mathfrak{T} \in \text{Dis} \cup \text{Den}$ .* If  $\mathfrak{T} = (T, \leq)$  is a linear order from  $\text{Dis}$  or  $\text{Den}$  and the semantics is reflexive, then we divide  $\text{int}(\mathfrak{T})$  into *zones* of the form

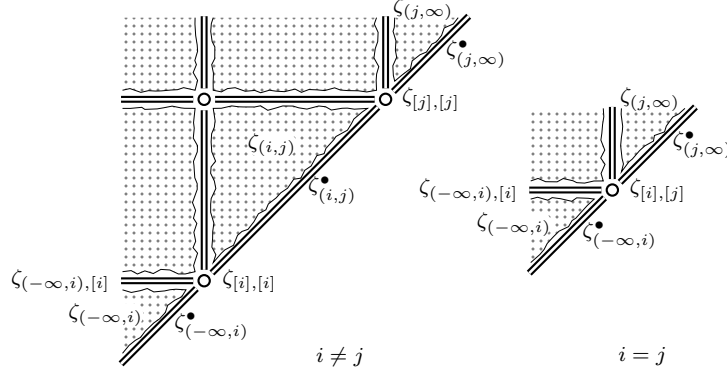
$$\text{--- } \zeta_{\sigma, \sigma'} = \{\langle x, x' \rangle \in \text{int}(\mathfrak{T}) \mid x \in \sigma, x' \in \sigma'\}, \text{ where } \sigma, \sigma' \in \text{sec}_{\mathfrak{T}}(a, b) \text{ and } \sigma \preceq \sigma'.$$

For  $a < b$  (or  $a = b$ ), there are at most 15 (respectively, at most 6) disjoint non-empty zones covering  $\text{int}(\mathfrak{T})$ ; see Fig. 3. These zones form the set  $Z$  of worlds in the frame  $\mathfrak{Z}^{(a,b)}$ , and for any  $\zeta, \zeta' \in Z$  and any interval relation  $R$ , we set  $\zeta R \zeta'$  iff there exist  $\langle x, y \rangle \in \zeta$  and  $\langle x', y' \rangle \in \zeta'$  such that  $\langle x, y \rangle R \langle x', y' \rangle$ . Finally, we define a map  $f: \text{int}(\mathfrak{T}) \rightarrow Z$  by taking  $f(\langle x, y \rangle) = \zeta$  iff  $\langle x, y \rangle \in \zeta$ . By definition,  $f$  is 'onto' and satisfies (p1). Condition (p2) is checked by direct inspection of Fig. 3, while condition (p3) is an immediate consequence of the following lemma:

**LEMMA 3.3.** *For any zone  $\zeta$  and any literal  $\lambda$  in  $\varphi$ , if  $\mathfrak{R}_{\varphi}^{(i,j)}, \langle x, y \rangle \models \lambda$  for some  $\langle x, y \rangle \in \zeta$ , then  $\mathfrak{R}_{\varphi}^{(i,j)}, \langle x, y \rangle \models \lambda$  for all  $\langle x, y \rangle \in \zeta$ .*

**PROOF.** It suffices to show that if  $\lambda @ \langle x, y \rangle \in \text{cl}^{\alpha+1}(\mathfrak{W}_{\varphi})$  for some  $\langle x, y \rangle \in \zeta$ , then  $\lambda @ \langle x', y' \rangle \in \text{cl}^{\alpha+1}(\mathfrak{W}_{\varphi})$  for all  $\langle x', y' \rangle \in \zeta$ , assuming that  $\text{cl}^{\alpha}(\mathfrak{W}_{\varphi})$  satisfies this property, which is the case for  $\alpha = 0$ .

Suppose  $\langle x, y \rangle \in \zeta$  and  $\lambda @ \langle x, y \rangle \in \text{cl}^{\alpha+1}(\mathfrak{W}_{\varphi})$  is obtained by an application of (cl1) to  $[R]\lambda @ \langle u, v \rangle \in \text{cl}^{\alpha}(\mathfrak{W}_{\varphi})$  with  $\langle u, v \rangle R \langle x, y \rangle$  and  $\langle u, v \rangle \in \zeta'$ . Take any  $\langle x', y' \rangle \in \zeta$ . By (p2), there

Fig. 4. Zones in the canonical models over  $\text{Den}(\prec)$ .

is  $\langle u', v' \rangle \in \zeta'$  such that  $\langle u', v' \rangle R \langle x', y' \rangle$ . By our assumption,  $[R]\lambda \langle u', v' \rangle \in \text{cl}^\alpha(\mathfrak{A}_\varphi)$ , and so an application of (c1) to it gives  $\lambda @ \langle x', y' \rangle \in \text{cl}^{\alpha+1}(\mathfrak{A}_\varphi)$ .

Suppose next that  $\langle x, y \rangle \in \zeta$  and  $[R]\lambda @ \langle x, y \rangle \in \text{cl}^{\alpha+1}(\mathfrak{A}_\varphi)$  is obtained by an application of (c2). Then  $\lambda \langle u, v \rangle \in \text{cl}^\alpha(\mathfrak{A}_\varphi)$  for all  $\langle u, v \rangle$  with  $\langle x, y \rangle R \langle u, v \rangle$ . Take any  $\langle x', y' \rangle \in \zeta$ . We show that  $\lambda \langle u', v' \rangle \in \text{cl}^\alpha(\mathfrak{A}_\varphi)$  for every  $\lambda \langle u', v' \rangle$  with  $\langle x', y' \rangle R \langle u', v' \rangle$ , from which  $[R]\lambda @ \langle x', y' \rangle \in \text{cl}^{\alpha+1}(\mathfrak{A}_\varphi)$  will follow. Let  $\langle u', v' \rangle \in \zeta'$ . By (p1),  $\zeta R \zeta'$  and, by (p2),  $\langle x, y \rangle R \langle u, v \rangle$  for some  $\langle u, v \rangle \in \zeta'$  such that  $\langle x, y \rangle R \langle u, v \rangle$ . Then  $\lambda \langle u, v \rangle \in \text{cl}^\alpha(\mathfrak{A}_\varphi)$  and, by our assumption,  $\lambda \langle u', v' \rangle \in \text{cl}^\alpha(\mathfrak{A}_\varphi)$ .

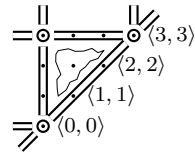
The case of rule (c3) is obvious.  $\square$

Note that Lemma 3.3 does not hold for  $\mathfrak{T}(\prec)$ . Indeed, we may have punctual intervals  $\langle y, y \rangle$  (for  $y \notin \{a, b\}$ ) such that  $\mathfrak{R}_\varphi^{\langle a, b \rangle}, \langle y, y \rangle \models [E]\perp$  but  $\mathfrak{R}_\varphi^{\langle a, b \rangle}, \langle x, y \rangle \not\models [E]\perp$  for  $x < y$ , with  $\langle x, y \rangle$  from the same zone as  $\langle y, y \rangle$ .

*Case  $\mathfrak{T}(\prec)$ , for  $\mathfrak{T} \in \text{Den}$ .* If  $\mathfrak{T}$  is a dense linear order and the semantics is irreflexive, we divide  $\text{int}(\mathfrak{T})$  into zones of three types:

- $\zeta_{\sigma, \sigma'} = \{\langle x, x' \rangle \in \text{int}(\mathfrak{T}) \mid x \in \sigma, x' \in \sigma'\}$ , where  $\sigma, \sigma' \in \text{sec}_{\mathfrak{T}}(a, b)$ ,  $\sigma \preceq \sigma'$  and  $\sigma \neq \sigma'$ ;
- $\zeta_\sigma = \{\langle x, x' \rangle \in \text{int}(\mathfrak{T}) \mid x, x' \in \sigma, x \neq x'\}$ , where  $\sigma \in \text{sec}_{\mathfrak{T}}(a, b)$ ;
- $\zeta_\sigma^\bullet = \{\langle x, x \rangle \in \text{int}(\mathfrak{T}) \mid x \in \sigma\}$ , where  $\sigma \in \text{sec}_{\mathfrak{T}}(a, b)$ .

Now, for  $a < b$  (or  $a = b$ ), we have at most 18 (respectively, at most 8) disjoint non-empty zones covering  $\text{int}(\mathfrak{T})$ ; see Fig. 4. It is again easy to see that the map  $f: \text{int}(\mathfrak{T}) \rightarrow Z$  defined by taking  $f(\langle x, y \rangle) = \zeta$  iff  $\langle x, y \rangle \in \zeta$  satisfies (p1)–(p3). The fact that  $\mathfrak{T}$  is dense is required for (p2). For discrete  $\mathfrak{T}$ , condition (p2) does not hold. For example, for  $\mathfrak{T} = (\mathbb{Z}, \triangleleft)$ ,  $a = 0$  and  $b = 3$ , we have  $\zeta_{(a,b)}^\bullet \bar{E} \zeta_{(a,b), (a,b)}$  but for  $\langle 2, 2 \rangle \in \zeta_{(a,b)}^\bullet$  there is no  $\langle x', y' \rangle \in \zeta_{(a,b), (a,b)}$  such that  $\langle 2, 2 \rangle \bar{E} \langle x', y' \rangle$  as shown in the picture below:



Thus, in both cases the constructed function  $f: \text{int}(\mathfrak{T}) \rightarrow Z$  satisfies conditions (p1)–(p3), and so, using Theorem 3.2, we obtain:

**THEOREM 3.4.** *Suppose  $\mathfrak{T} \in \text{Dis}$  and  $\triangleleft$  is  $\leq$ , or  $\mathfrak{T} \in \text{Den}$  and  $\triangleleft \in \{\leq, <\}$ . Then an  $\mathcal{HS}_{\text{horn}}^{\square}$ -formula  $\varphi$  is  $\langle a, b \rangle$ -satisfiable in  $\mathfrak{T}(\triangleleft)$  iff  $\varphi$  is  $f(\langle a, b \rangle)$ -satisfiable in  $\mathfrak{Z}^{\langle a, b \rangle}$ .*

To check whether  $\varphi$  is  $f(\langle a, b \rangle)$ -satisfiable in  $\mathfrak{Z}^{\langle a, b \rangle}$ , we take the set

$$\mathfrak{U}_{\varphi} = \{\lambda @ f(\langle a, b \rangle) \mid \lambda \text{ an initial condition of } \varphi\} \cup \{\top @ \zeta \mid \zeta \in Z\}$$

and apply to it the following obvious modifications of rules (cl1)–(cl3):

- if  $[R]\lambda @ \zeta \in \mathfrak{U}_{\varphi}$ , then we add to  $\mathfrak{U}_{\varphi}$  all  $\lambda @ \zeta'$  such that  $\zeta R \zeta'$ ;
- if  $\lambda @ \zeta' \in \mathfrak{U}_{\varphi}$  for all  $\zeta' \in Z$  with  $\zeta R \zeta'$  and  $[R]\lambda$  occurs in  $\varphi$ , then we add  $[R]\lambda @ \zeta$  to  $\mathfrak{U}_{\varphi}$ ;
- if  $[U](\lambda_1 \wedge \dots \wedge \lambda_k \rightarrow \lambda)$  occurs in  $\varphi$  and  $\lambda_i @ \zeta \in \mathfrak{U}_{\varphi}$ ,  $1 \leq i \leq k$ , then add  $\lambda @ \zeta$  to  $\mathfrak{U}_{\varphi}$ .

It is readily seen that at most  $|Z| \cdot |\varphi|$  applications are enough to construct a fixed point  $\text{cl}^*(\mathfrak{U}_{\varphi})$ . Similarly to Theorem 3.2, we then show that  $\varphi$  is  $f(\langle a, b \rangle)$ -satisfiable in  $\mathfrak{Z}^{\langle a, b \rangle}$  iff  $\text{cl}^*(\mathfrak{U}_{\varphi})$  does not contain  $\perp @ f(\langle a, b \rangle)$ .

**THEOREM 3.5.** *Suppose  $\text{Dis}' \subseteq \text{Dis}$  and  $\text{Den}' \subseteq \text{Den}$  are non-empty. Then  $\text{Dis}'(\leq)$ -,  $\text{Den}'(\leq)$ - and  $\text{Den}'(<)$ -satisfiability of  $\mathcal{HS}_{\text{horn}}^{\square}$ -formulas are all P-complete.*

**PROOF.** Observe first that, for each of  $\text{Dis}'(\leq)$ ,  $\text{Den}'(\leq)$ ,  $\text{Den}'(<)$ , there are at most 8 pairwise non-isomorphic frames of the form  $\mathfrak{Z}^{\langle a, b \rangle}$ . As we saw above, checking whether  $\varphi$  is satisfiable in one of them can be done in polynomial time. It remains to apply Theorem 3.4. The matching lower bound holds already for propositional Horn formulas; see, e.g., [Dantsin et al. 2001, Theorem 4.2] and references therein.  $\square$

It is readily seen that, in fact, Theorem 3.5 also holds for  $\text{Lin}'(\leq)$ , where  $\text{Lin}'$  is any non-empty subclass of  $\text{Lin}$ .

### 3.1. Ontology-based access to temporal data with extensions of $\mathcal{HS}_{\text{horn}}^{\square}$

We now briefly discuss how extensions of  $\mathcal{HS}_{\text{horn}}^{\square}$  can be used to facilitate access to temporal data; for more details and experiments consult [Kontchakov et al. 2016].

*Querying historical data.* Suppose that a non-IT expert user would like to query the historical data provided by the STOLE<sup>6</sup> ontology that extracts facts about the Italian Public Administration from journal articles [Adorni et al. 2015]. The STOLE dataset,  $\mathcal{D}$ , contains facts about institutions, legal systems, events, and people such as:

$$\begin{aligned} & \text{LegalSystem}(\text{regno\_di\_sardegna})@[1720, 1861], \\ & \text{Institution}(\text{consiglio\_di\_intendenza})@[1806, 1865]. \end{aligned}$$

The former one, for example, states that Regno di Sardegna was a legal system in the period between 1720 and 1861. Suppose now that the user is searching for institutions founded during the Regno di Sardegna period. To simplify the user's task, we can create an ontology,  $\mathcal{O}$ , with the single clause

$$[U]\forall x (\text{Institution}(x) \wedge \langle B \rangle \langle \bar{D} \rangle \text{LegalSystem}(\text{regno\_di\_sardegna}) \rightarrow \text{RdSIInstitution}(x)).$$

The user's query can now be very simple:  $q(x, t, s) = \text{RdSIInstitution}(x)@[t, s]$ . However, the query-answering system has to find *certain answers* to the *ontology-mediated query*  $(\mathcal{O}, q(x, t, s))$  over  $\mathcal{D}$ , which are triples  $(a, m, n)$  such that  $\text{RdSIInstitution}(a)@[m, n]$  holds in all models of  $\mathcal{O}$  and  $\mathcal{D}$ . As shown by Kontchakov et al. [2016], this ontology-mediated query can be 'rewritten' into a standard datalog query  $(\Pi, G(x, t, s))$ , where  $\Pi$  is a datalog program  $\Pi$  and  $G(x, t, s)$  a goal, such that the certain answers to  $(\mathcal{O}, q(x, t, s))$  over  $\mathcal{D}$  coincide with the answers to  $(\Pi, G(x, t, s))$  over  $\mathcal{D}$ .

<sup>6</sup>For STORIA LEgislativa della pubblica amministrazione italiana.

The ontology language in this case is a straightforward datalog extension of  $\mathcal{HS}_{horn}^\square$ . However, to represent temporal data, we require more complex initial conditions compared to  $\mathcal{HS}_{horn}^\square$ , namely, facts of the form  $P(a_1, \dots, a_l)@[n, m]$ , where  $\langle n, m \rangle$  is an interval. The zonal representation of canonical models above can be extended to this case, but the number of zones will be quadratic in the number of the initial conditions.

We next show an application that requires a *multi-dimensional* version of  $\mathcal{HS}_{horn}^\square$ .

*Querying sensor data.* Consider a turbine monitoring system that is receiving from sensors a stream of data of the form  $Blade(id)@[l_1, l_2]$ , where  $id$  is a turbine blade ID and  $l_2$  is the temperature range over  $(\mathbb{R}, <)$  observed during the time interval  $l_1$  over  $(\mathbb{Z}, \leq)$ . Suppose also that the user wants to find the blades and time intervals where the temperature was rising. Thinking of a pair  $\iota = (l_1, l_2)$  as a rectangle in the two-dimensional space  $(\mathbb{Z}, \leq) \times (\mathbb{R}, <)$  and using the operators  $\langle R \rangle_\ell$  in dimension  $\ell \in \{1, 2\}$  coordinate-wise (that is,  $\iota R_\ell \iota'$  iff  $\iota_\ell R'_\ell$  and  $\iota_i = \iota'_i$ , for  $i \neq \ell$ ), we can define rectangles with rising temperature by the clause

$$[U]\forall x (\langle \bar{A} \rangle_1 \langle \bar{O} \rangle_2 BladeTemp(x) \wedge \langle A \rangle_1 \langle O \rangle_2 BladeTemp(x) \rightarrow TempRise(x))$$

saying that the temperature of a blade  $x$  is rising over a rectangle  $(l_1, l_2)$  if  $BladeTemp(x)$  holds at some rectangles  $(l_1^-, l_2^-)$  and  $(l_1^+, l_2^+)$  located as shown in Fig. 5.

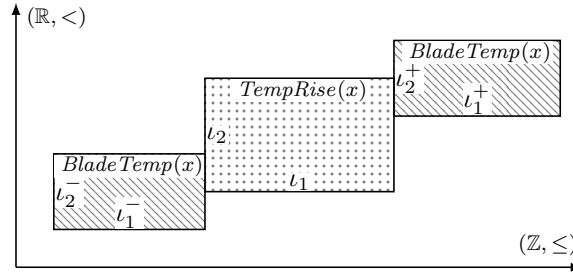


Fig. 5. Rectangles with rising temperature.

Note that relation algebras over (hyper)rectangles are well-known in temporal and spatial knowledge representation: the rectangle/block algebra RA [Balbiani et al. 2002] that extends Allen's interval algebra; see also [Navarrete et al. 2013; Cohn et al. 2014; Zhang and Renz 2014] and references therein. This multi-dimensional  $\mathcal{HS}_{horn}^\square$  is capable of expressing rules such as 'if  $A$  holds at  $\iota$  and  $A'$  at  $\iota'$ , then  $B$  holds at the intersection  $\kappa$  of  $\iota$  and  $\iota'$  (or at the smallest rectangle  $\kappa$  covering  $\iota$  and  $\iota'$ )' as shown in Fig. 6.

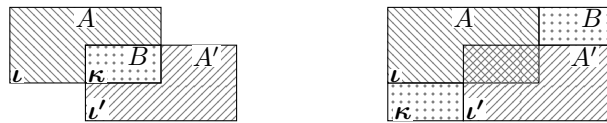


Fig. 6. Expressing simple rules in multi-dimensional  $\mathcal{HS}_{horn}^\square$ .

Answering ontology-mediated queries with ontologies in the datalog extension of multi-dimensional  $\mathcal{HS}_{horn}^\square$  is P-complete for data complexity and can also be done via rewriting into standard datalog queries over the data. The reasonable scalability of

this approach was shown experimentally by Kontchakov et al. [2016] for both one- and two-dimensional cases using standard off-the-shelf datalog tools.

#### 4. LOWER BOUNDS

In this section, we show that tractability results such as Theorem 3.5 are not possible when some kind of ‘controlled infinity’ becomes expressible in the formalism.

##### 4.1. Methodology

When simulating complex problems in  $\mathcal{HS}$ -models, we always begin by singling out those intervals—call them *units*—that are used in the simulation. It should be clear that if an  $\mathcal{HS}$ -fragment is capable of

- (i) forcing an  $\omega$ -type infinite (or unbounded finite) sequence of units, and
- (ii) passing polynomial-size information from one unit to the next,

then it is PSPACE-hard (because polynomial space bounded Turing machine computations can be encoded). It is readily seen that  $\mathcal{HS}_{horn}$  can easily do both (i) and (ii). We show that, in certain situations, Horn clauses can be encoded by means of core clauses, which gives (i) and (ii) already in the core fragments. In particular, this is the case:

- for  $\mathcal{HS}_{core}$  over any class of unbounded timelines under arbitrary semantics (Theorem 4.1), and even
- for  $\mathcal{HS}_{core}^{\square}$  over any class of unbounded discrete timelines under the irreflexive semantics (Theorem 4.2).

Further, if a fragment is expressive enough to

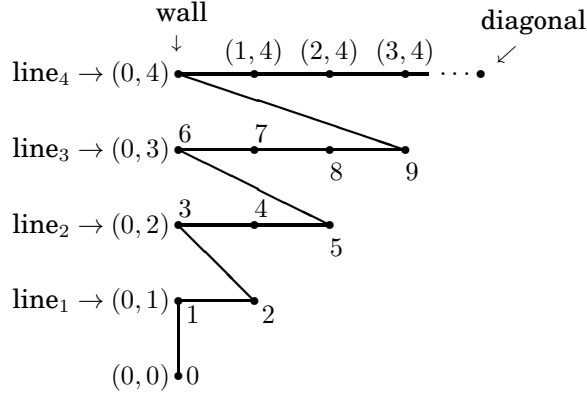
- (iii) force an  $\omega \times \omega$ -like grid-structure of units, and
- (iv) pass (polynomial-size) information from each unit representing some grid-point to the unit representing its right- and up-neighbours in the grid,

then it becomes possible to encode undecidable problems such as  $\omega \times \omega$ -tilings, Turing or counter machine computations. We show this to be the case for the following fragments:

- $\mathcal{HS}_{horn}^{\diamond}$  over any class of unbounded timelines under arbitrary semantics (Theorem 4.3),
- $\mathcal{HS}_{core}$  over any class of unbounded timelines under the irreflexive semantics (Theorem 4.4), and
- $\mathcal{HS}_{horn}^{\square}$  over any class of unbounded discrete timelines under the irreflexive semantics (Theorem 4.5).

Although  $\mathcal{HS}$ -models are always grid-like by definition, it is not straightforward to achieve (iii)–(iv) in them. Even if we consider the irreflexive semantics and discrete underlying linear orders,  $\mathcal{HS}$  does not provide us with horizontal and vertical next-time operators. The undecidability proofs for (Boolean)  $\mathcal{HS}$ -satisfiability given by Halpern and Shoham [1991] and Marx and Reynolds [1999] (for irreflexive semantics), by Reynolds and Zakharyashev [2001] and Gabbay et al. [2003] (for arbitrary semantics), and by Bresolin et al. [2008] (for the BE,  $\bar{B}E$  and  $\bar{B}\bar{E}$  fragments with irreflexive semantics) all employ the following solution to this problem:

- (v) Instead of using a grid-like subset of an  $\mathcal{HS}$ -model as units representing grid-locations, we use some Cantor-style enumeration of either the whole  $\omega \times \omega$ -grid or its north-western octant  $nw_{\omega \times \omega}$  (see Fig. 7), and then force a *unique* infinite (or unbounded finite) sequence of units representing this enumeration (or an unbounded finite prefix of it).

Fig. 7. An enumeration of the  $nw_{\omega \times \omega}$ -grid.

(vi) Then we use some ‘up- and right-pointers’ in the model to access the unit representing the grid-location immediately above and to the right of the current one.

Here, we follow a similar approach. The proofs of Theorems 4.3–4.5 differ in how (v) and (vi) are achieved by the capabilities of the different formalisms.

- In the proof of Theorem 4.3, the encoding of the  $\omega \times \omega$ -grid resembles that of [Marx and Reynolds 1999; Reynolds and Zakharyashev 2001; Gabbay et al. 2003] for modal products of linear orders, and [Gabelaia et al. 2005b] for modal products of various transitive (not necessarily linear) relations, regardless whether the relations are irreflexive or reflexive. In particular, in the reflexive semantics the uniqueness constraints in (v) are usually not satisfiable, so instead it is forced that all points encoding the same unit behave in the same way. It turns out that, with some additional ‘tricks’, this technique is applicable to  $\mathcal{HS}_{horn}^{\diamond}$ -formulas.
- It is not clear whether the above method can be applied to the case of  $\mathcal{HS}_{core}$ . In the proof of Theorem 4.4, we achieve (for the irreflexive semantics) (v) and (vi) in a different way, similar to that of [Halpern and Shoham 1991].
- Both techniques above make an essential use of  $\langle R \rangle$ -operators. In order to achieve (v) and (vi) using  $\mathcal{HS}_{horn}^{\square}$ -formulas with the irreflexive semantics and discrete linear orders, in the proof of Theorem 4.5 we provide a completely different encoding the  $nw_{\omega \times \omega}$ -grid.

## 4.2. Turing machines

We begin by fixing the notation and terminology regarding Turing machines. A *single-tape right-infinite deterministic Turing Machine (TM)*, for short) is a tuple  $\mathcal{A} = (Q, \Sigma, q_0, q_f, \delta_{\mathcal{A}})$ , where  $Q$  is a finite set of *states* containing, in particular, the *initial state*  $q_0$  and the *halt state*  $q_f$ ,  $\Sigma$  is the *tape alphabet* (with a distinguished *blank* symbol  $\sqcup \in \Sigma$ ), and  $\delta_{\mathcal{A}}$  is the *transition function*, where we use the symbol  $\ell \notin \Sigma$  to mark the leftmost cell of the tape:

$$\delta_{\mathcal{A}} : (Q - \{q_f\}) \times (\Sigma \cup \{\ell\}) \rightarrow Q \times (\Sigma \cup \{\ell, r\}).$$

The transition function transforms each pair of the form  $(q, s)$  into one of the following pairs:

- $(q', s')$  (write  $s'$  and change the state to  $q'$ );
- $(q', \ell)$  (move one cell left and change the state to  $q'$ );

—  $(q', r)$  (move one cell right and change the state to  $q'$ ),

where  $l$  and  $r$  are fresh symbols. We assume that if  $s = \mathcal{L}$  (i.e., the leftmost cell of the tape is active) then  $\delta_{\mathcal{A}}(q, s) = (q', r)$  (that is, having reached the leftmost cell, the machine always moves to the right). We set  $size(\mathcal{A}) = |Q \cup \Sigma \cup \delta_{\mathcal{A}}|$ . *Configurations* of  $\mathcal{A}$  are infinite sequences of the form

$$C = (s_0, s_1, \dots, s_i, \dots, s_n, \sqcup, \dots),$$

where either  $s_0 = \mathcal{L}$  and all  $s_1, \dots, s_n$  save one, say  $s_i$ , are in  $\Sigma$ , while  $s_i$  belongs to  $Q \times \Sigma$  and represents the *active cell* and the *current state*, or  $s_0 = (q, \mathcal{L})$  for some  $q \in Q$  ( $s_0$  is the active cell), and all  $s_1, \dots, s_n$  are in  $\Sigma$ . In both cases, all cells of the tape located to the right of  $s_n$  contain  $\sqcup$ . We assume that the machine always starts with the empty tape (all cells of which are blank), and so the *initial configuration* is represented by the sequence

$$C_0 = ((q_0, \mathcal{L}), \sqcup, \sqcup, \dots).$$

We denote by  $(C_n \mid n < H)$  the unique sequence of subsequent configurations of  $\mathcal{A}$  starting with the empty tape—the unique *computation of  $\mathcal{A}$  with empty input*—where

$$H = \begin{cases} n + 1, & n \text{ is the smallest number with } (q_f, s) \text{ occurring in } C_n \text{ for some } s, \\ \omega, & \text{otherwise.} \end{cases}$$

If  $H < \omega$ , we say that  $\mathcal{A}$  *halts with empty input*, and call  $C_{H-1}$  the *halting configuration* of  $\mathcal{A}$ . If  $H = \omega$ , we say that  $\mathcal{A}$  *diverges with empty input*. We denote by  $C_n(m)$  the  $m$ th symbol in  $C_n$ .

In our lower bound proofs, we use the following Turing machine problems [Moret 1998]:

HALTING: ( $\Sigma_1^0$ -hard)

Given a Turing machine  $\mathcal{A}$ , does it halt with empty input?

NON-HALTING: ( $\Pi_1^0$ -hard)

Given a Turing machine  $\mathcal{A}$ , does it diverge with empty input?

PSPACE-BOUND HALTING: (PSPACE-hard)

Given a Turing machine  $\mathcal{A}$  whose computation with empty input uses at most  $poly(size(\mathcal{A}))$  tape cells for some polynomial function  $poly()$ , does  $\mathcal{A}$  halt on empty input?

PSPACE-BOUND NON-HALTING: (PSPACE-hard)

Given a Turing machine  $\mathcal{A}$  whose computation with empty input uses at most  $poly(size(\mathcal{A}))$  tape cells for some polynomial function  $poly()$ , does  $\mathcal{A}$  diverge on empty input?

### 4.3. PSPACE-hardness of core fragments

As we have already observed, proving PSPACE-hardness in the case of  $\mathcal{H}S_{horn}$  is relatively easy. In order to do this in the case  $\mathcal{H}S_{core}$ , we use the following *binary implication trick* to capture at least some of the Horn features in  $\mathcal{H}S_{core}$ . For any literals  $\lambda_1$ ,  $\lambda_2$ , and  $\lambda$ , we define the formula  $[\lambda_1 \wedge \lambda_2 \Rightarrow_H \lambda]$  as the conjunction of

$$[U](\lambda_1 \rightarrow \langle A \rangle \mu_1) \wedge [U](\lambda_2 \rightarrow \langle A \rangle \mu_2), \quad (4)$$

$$[U](\mu_2 \rightarrow \neg \langle \bar{B} \rangle \mu_1), \quad (5)$$

$$[U](\mu_1 \rightarrow \mu \wedge [\bar{B}] \mu) \wedge [U](\mu_2 \rightarrow [B] \mu), \quad (6)$$

$$[U](\langle A \rangle \mu \rightarrow \lambda), \quad (7)$$



where  $\mu_1$ ,  $\mu_2$ , and  $\mu$  are fresh variables (the  $H$  in  $\Rightarrow_H$  stands for ‘horizontal’). The following claim holds for arbitrary semantics:

**CLAIM 4.1.** *Suppose  $\mathfrak{M}$  is an  $\mathcal{HS}$ -model based on some linear order  $\mathfrak{T}$  and satisfying  $[\lambda_1 \wedge \lambda_2 \Rightarrow_H \lambda]$ . For all  $y$  in  $\mathfrak{T}$ , if there exist  $x_1, x_2 \leq y$  such that  $\mathfrak{M}, \langle x_1, y \rangle \models \lambda_1$  and  $\mathfrak{M}, \langle x_2, y \rangle \models \lambda_2$ , then  $\mathfrak{M}, \langle x, y \rangle \models \lambda$  for all  $x \leq y$ .*

**PROOF.** Suppose  $\mathfrak{M}, \langle x_1, y \rangle \models \lambda_1$  and  $\mathfrak{M}, \langle x_2, y \rangle \models \lambda_2$ . Take some  $x \leq y$ . By (4), there exist  $z_1, z_2 \geq y$  such that  $\mathfrak{M}, \langle y, z_1 \rangle \models \mu_1$  and  $\mathfrak{M}, \langle y, z_2 \rangle \models \mu_2$ . Then  $z_1 \leq z_2$  by (5). So  $\mathfrak{M}, \langle y, z \rangle \models \mu$  for all  $z \geq y$  by (6), and therefore  $\mathfrak{M}, \langle x, y \rangle \models \lambda$  by (7).  $\square$

**Soundness:** Observe that in order to satisfy  $[\lambda_1 \wedge \lambda_2 \Rightarrow_H \lambda]$  the following are necessary:

- $\lambda$  is *horizontally stable*: for every  $y$ , we have  $\mathfrak{M}, \langle x, y \rangle \models \lambda$  iff  $\mathfrak{M}, \langle x', y \rangle \models \lambda$  for all  $x'$ ;
- if  $\mathfrak{M}, \langle x', y \rangle \not\models \lambda$  (and so  $\mathfrak{M}, \langle x, y \rangle \not\models \lambda$  for all  $x$ ) and  $\mathfrak{M}, \langle x'', y \rangle \models \lambda_1$  for some  $x', x''$ , then  $\mathfrak{M}, \langle x, y \rangle \not\models \lambda_2$  should hold for all  $x$ .

We use the binary implication trick to prove the following:

**THEOREM 4.1. ( $\mathcal{HS}_{core}$ , arbitrary semantics)**

(i) For any class  $\mathcal{C}$  of linear orders containing an infinite order,  $\mathcal{C}$ -satisfiability of  $\mathcal{HS}_{core}$ -formulas is PSPACE-hard. (ii) Fin-satisfiability of  $\mathcal{HS}_{core}$ -formulas is PSPACE-hard.

**PROOF.** (i) We reduce PSPACE-BOUND NON-HALTING to  $\mathcal{C}$ -satisfiability. Let  $\mathcal{A}$  be a Turing machine whose computation on empty input uses  $< poly(size(\mathcal{A}))$  tape cells for some polynomial function  $poly()$ , and let  $N = poly(size(\mathcal{A}))$ . Then we may assume that each configuration  $\mathcal{C}$  of  $\mathcal{A}$  is not infinite but of length  $N$ , and  $\mathcal{A}$  never visits the last cell of any configuration. Let  $\Gamma_{\mathcal{A}} = \Sigma \cup \{\mathcal{L}\} \cup (Q \times (\Sigma \cup \{\mathcal{L}\}))$ . For each  $i < N$  and  $z \in \Gamma_{\mathcal{A}}$ , we introduce two propositional variables:  $cell_i^z$  (to encode that ‘the content of the  $i$ th cell is  $z$ ’) and its ‘copy’  $\overline{cell}_i^z$ .

Then we can express the uniqueness of cell-contents by

$$\bigwedge_{i < N} \bigwedge_{z \neq z' \in \Gamma_{\mathcal{A}}} [U](cell_i^z \rightarrow \neg cell_i^{z'}), \quad (8)$$

and initialise the computation by

$$cell_0^{(q_0, \mathcal{L})} \wedge \bigwedge_{0 < i < N} cell_i^{\perp}. \quad (9)$$

Now we pass information from one configuration to the next, using the ‘copy’ variables and the ‘binary implication trick’:

$$[U](cell_i^{(q,s)} \rightarrow \langle A \rangle \overline{cell}_i^{(q,s)}), \quad \text{for } i < N, (q, s) \in (Q - \{q_f\}) \times (\Sigma \cup \{\mathcal{L}\}), \quad (10)$$

$$[cell_i^{(q,s)} \wedge cell_j^z \Rightarrow_H \langle A \rangle \overline{cell}_j^z], \quad \text{for } i, j < N, (q, s) \in (Q - \{q_f\}) \times (\Sigma \cup \{\mathcal{L}\}), z \in \Sigma \cup \{\mathcal{L}\}, \quad (11)$$

$$[U](\overline{cell}_i^{(q,s)} \rightarrow \neg \langle B \rangle \overline{cell}_j^z). \quad (12)$$

We can force that all  $\overline{cell}_i^{(q,s)}$ -intervals are different (meaning none of them is punctual) by the conjunction of, say,

$$[U](\overline{cell}_i^{(q,s)} \rightarrow \text{unit}), \quad \text{for } i < N, (q, s) \in Q \times (\Sigma \cup \{\mathcal{L}\}), \quad (13)$$

$$[U](\text{unit} \rightarrow \neg [D]\text{unit}). \quad (14)$$

Finally, we can ensure that the information passed in fact encodes the computation steps of  $\mathcal{A}$  by the following formulas. For all  $(q, s) \in (Q - \{q_f\}) \times (\Sigma \cup \{\mathcal{L}\})$  and  $z \in \Sigma \cup \{\mathcal{L}\}$ ,

— if  $\delta_{\mathcal{A}}(q, s) = (q', s')$ , then take the conjunction of

$$[\mathbf{U}](\overline{\text{cell}}_i^{(q,s)} \rightarrow \text{cell}_i^{(q',s')}), \quad \text{for } i < N, \quad (15)$$

$$[\overline{\text{cell}}_i^{(q,s)} \wedge \langle \bar{\mathbf{B}} \rangle \overline{\text{cell}}_j^z \Rightarrow_H \text{cell}_j^z], \quad \text{for } i, j < N, j \neq i; \quad (16)$$

— if  $\delta_{\mathcal{A}}(q, s) = (q', r)$ , then take the conjunction of

$$[\mathbf{U}](\overline{\text{cell}}_i^{(q,s)} \rightarrow \text{cell}_i^r), \quad \text{for } i < N - 1, \quad (17)$$

$$[\overline{\text{cell}}_i^{(q,s)} \wedge \langle \bar{\mathbf{B}} \rangle \overline{\text{cell}}_{i+1}^z \Rightarrow_H \text{cell}_{i+1}^{(q',z)}], \quad \text{for } i < N - 1, \quad (18)$$

$$[\overline{\text{cell}}_i^{(q,s)} \wedge \langle \bar{\mathbf{B}} \rangle \overline{\text{cell}}_j^z \Rightarrow_H \text{cell}_j^z], \quad \text{for } i < N - 1, j < N, j \neq i, i + 1; \quad (19)$$

— if  $\delta_{\mathcal{A}}(q, s) = (q', l)$ , then take the conjunction of (17) for  $0 < i < N$  and

$$[\overline{\text{cell}}_i^{(q,s)} \wedge \langle \bar{\mathbf{B}} \rangle \overline{\text{cell}}_{i-1}^z \Rightarrow_H \text{cell}_{i-1}^{(q',z)}], \quad \text{for } 0 < i < N, \quad (20)$$

$$[\overline{\text{cell}}_i^{(q,s)} \wedge \langle \bar{\mathbf{B}} \rangle \overline{\text{cell}}_j^z \Rightarrow_H \text{cell}_j^z], \quad \text{for } 0 < i < N, j < N, j \neq i, i - 1. \quad (21)$$

Finally, we force non-halting with

$$[\mathbf{U}](\text{cell}_i^{(q_f,s)} \rightarrow \perp), \quad \text{for } i < N, s \in \Sigma \cup \{\mathcal{L}\}. \quad (22)$$

**CLAIM 4.2.** *Let  $\Phi_{\mathcal{A}}$  be the conjunction of (8)–(22). If  $\Phi_{\mathcal{A}}$  is satisfiable in an  $\mathcal{HS}$ -model, then  $\mathcal{A}$  diverges with empty input.*

**PROOF.** Take any  $\mathcal{HS}$ -model  $\mathfrak{M}$  based on a linear order  $\mathfrak{T}$ . Suppose  $\mathfrak{M}, \langle r, r' \rangle \models \Phi_{\mathcal{A}}$ . Then it is not hard to show by induction on  $n$  that there exists an infinite sequence  $u_0 \leq u_1 < u_2 < \dots < u_n < \dots$  of points in  $\mathfrak{T}$  such that  $u_0 = r$ ,  $u_1 = r'$ , and for all  $n < \omega$ , the interval  $\langle u_n, u_{n+1} \rangle$  ‘represents’ the  $n$ th configuration  $\mathcal{C}_n$  in the infinite computation of  $\mathcal{A}$  with empty input in the following sense:

$$\mathfrak{M}, \langle u_n, u_{n+1} \rangle \models \text{cell}_i^z \quad \text{iff} \quad \mathcal{C}_n(i) = z,$$

for all  $i < N$  and  $z \in \Gamma_{\mathcal{A}}$ .  $\square$

On the other hand, if  $\mathcal{A}$  diverges on empty input, then take some linear order  $\mathfrak{T}$  containing an infinite ascending chain  $t_0 < t_1 < \dots$  and define an  $\mathcal{HS}$ -model  $\mathfrak{M} = (\mathfrak{F}_{\mathfrak{T}}, \nu)$  by taking, for all  $i < N$  and  $z \in \Gamma_{\mathcal{A}}$ ,

$$\begin{aligned} \nu(\text{unit}) &= \{\langle t_{2n}, t_{2n+2} \rangle \mid n < \omega\}, \\ \nu(\text{cell}_i^z) &= \{\langle x, t_{2n+2} \rangle \mid \mathcal{C}_n(i) = z, n < \omega, x \leq t_{2n+2}\}, \\ \nu(\overline{\text{cell}}_i^z) &= \begin{cases} \{\langle t_{2n+2}, t_{2n+5} \rangle \mid \mathcal{C}_n(i) = z, n < \omega\}, & \text{if } z \in \Sigma \cup \{\mathcal{L}\}, \\ \{\langle t_{2n+2}, t_{2n+4} \rangle \mid \mathcal{C}_n(i) = z, n < \omega\}, & \text{if } z \in Q \times (\Sigma \cup \{\mathcal{L}\}). \end{cases} \end{aligned}$$

We claim that it is possible to evaluate the fresh auxiliary variables in the binary trick formulas (11), (16), (18)–(21) so that  $\mathfrak{M}, \langle t_0, t_2 \rangle \models \Phi_{\mathcal{A}}$  with arbitrary semantics. Indeed, for example, fix some  $(q, s) \in (Q - \{q_f\}) \times (\Sigma \cup \{\mathcal{L}\})$  with  $\delta_{\mathcal{A}}(q, s) = (q', r)$ ,  $z \in \Sigma \cup \{\mathcal{L}\}$ , and  $i < N - 1$ , and consider the corresponding instance of conjunct (18):

$$[\overline{\text{cell}}_i^{(q,s)} \wedge \langle \bar{\mathbf{B}} \rangle \overline{\text{cell}}_{i+1}^z \Rightarrow_H \text{cell}_{i+1}^{(q',z)}].$$

If we take

$$\begin{aligned} \nu(\mu_1) &= \{\langle t_{2n+4}, t_{2n+6} \rangle \mid C_n(i) = (q, s), n < \omega\}, \\ \nu(\mu_2) &= \{\langle x, t_{2n+7} \rangle \mid t_{2n+2} \leq x \leq t_{2n+5}, C_n(i+1) = z, n < \omega\}, \end{aligned}$$

then it is easy to check that

$$[U](\overline{\text{cell}}_i^{(q,s)} \rightarrow \langle A \rangle \mu_1) \wedge [U](\langle \bar{B} \rangle \overline{\text{cell}}_{i+1}^z \rightarrow \langle A \rangle \mu_2) \quad \text{and} \quad [U](\mu_2 \rightarrow \neg \langle \bar{B} \rangle \mu_1)$$

hold in  $\mathfrak{M}$  (at all points); see Fig. 8. Further, if we take

$$\begin{aligned} \nu(\mu) &= \{\langle t_{2n+4}, y \rangle \mid y \geq t_{2n+6}, C_n(i) = (q, s), n < \omega\} \cup \\ &\quad \{\langle x, y \rangle \mid t_{2n+2} \leq x \leq t_{2n+5}, x \leq y \leq t_{2n+7}, C_n(i+1) = z, n < \omega\}, \end{aligned}$$

then it is straightforward to see that  $[U](\mu_1 \rightarrow \mu \wedge \langle \bar{B} \rangle \mu) \wedge [U](\mu_2 \rightarrow \langle B \rangle \mu)$  also holds in  $\mathfrak{M}$ . Finally, we claim that  $[U](\langle A \rangle \mu \rightarrow \text{cell}_{i+1}^{(q',z)})$  holds in  $\mathfrak{M}$  as well.

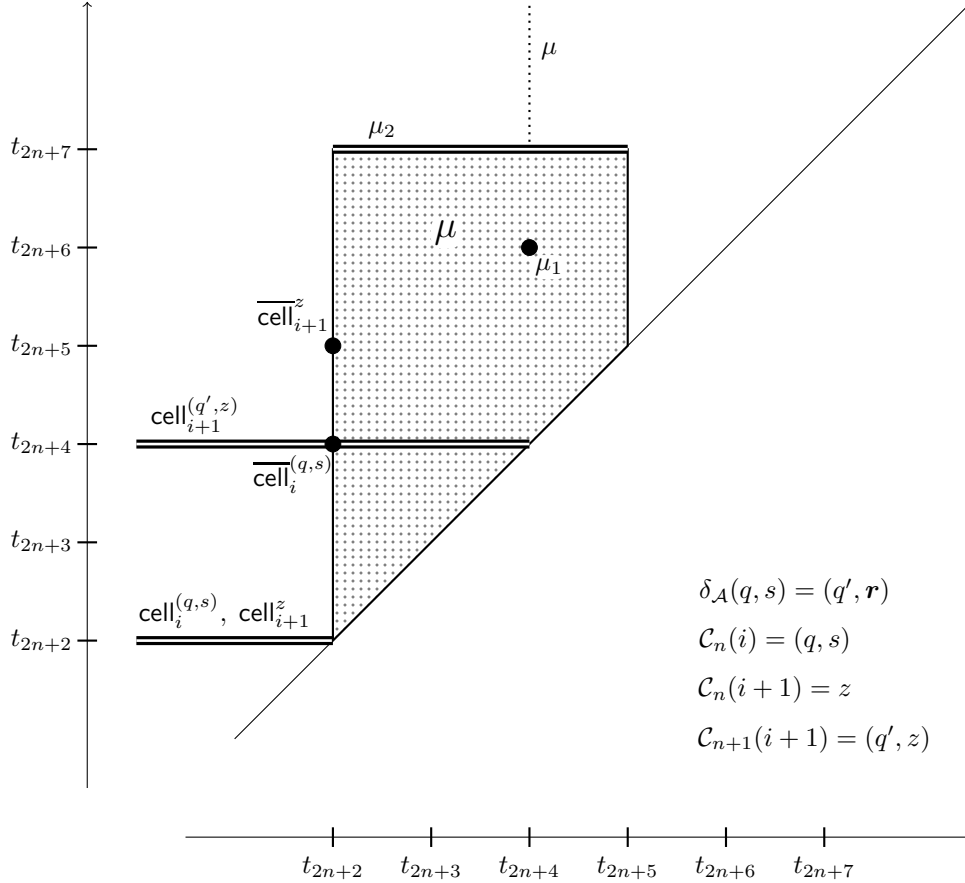


Fig. 8. Satisfying  $[\overline{\text{cell}}_i^{(q,s)} \wedge \langle \bar{B} \rangle \overline{\text{cell}}_{i+1}^z \Rightarrow_H \text{cell}_{i+1}^{(q',z)}]$  in  $\mathfrak{M}$ .

Indeed, suppose  $\mathfrak{M}, \langle x, y \rangle \models [A] \mu$  for some  $x \leq y$ . Then there exist  $y_1, y_2$  such that  $\mathfrak{M}, \langle y, y_1 \rangle \models \mu_1$  and  $\mathfrak{M}, \langle y, y_2 \rangle \models \mu_2$ . Thus, there is  $n < \omega$  such that  $y = t_{2n+4}$  and

$\mathcal{C}_n(i) = (q, s)$ , and there is  $m < \omega$  such that  $t_{2m+2} \leq y \leq t_{2m+5}$  and  $\mathcal{C}_m(i+1) = z$ . It follows that either  $m = n+1$  or  $m = n$ . If  $m = n+1$  were the case, then both  $\mathcal{C}_n(i) = (q, s)$  and  $\mathcal{C}_{n+1}(i+1) = z$  would hold, which is not possible when the head moves to the right. So  $m = n$ , and we have  $\mathcal{C}_n(i) = (q, s)$  and  $\mathcal{C}_n(i+1) = z$ . Therefore,  $\mathcal{C}_{n+1}(i+1) = (q', z)$ , and so  $\mathfrak{M}, \langle x, t_{2n+4} \rangle \models \text{cell}_{i+1}^{(q', z)}$ , as required. Checking the other conjuncts in  $\Phi_{\mathcal{A}}$  is similar and left to the reader.

The case when  $\mathfrak{T}$  contains an infinite descending chain requires ‘symmetrical versions’ of the used formulas and is also left to the reader.

(ii) In the finite case, we reduce PSPACE-BOUND HALTING to Fin-satisfiability. To achieve this, we just omit the conjunct (22) from  $\Phi_{\mathcal{A}}$ . Now, (10) together with the finiteness of the models force the computation to reach the halting state.  $\square$

**THEOREM 4.2.** ( $\mathcal{HS}_{core}^{\square}$ , **discrete orders, irreflexive semantics**)

(i) For any class  $\text{Dis}^{\infty}$  of discrete linear orders containing an infinite order,  $\text{Dis}^{\infty}(<)$ -satisfiability of  $\mathcal{HS}_{core}^{\square}$ -formulas is PSPACE-hard. (ii)  $\text{Fin}(<)$ -satisfiability of  $\mathcal{HS}_{core}^{\square}$ -formulas is PSPACE-hard.

**PROOF.** (i) We again reduce PSPACE-BOUND NON-HALTING to the satisfiability problem. Take any  $\mathcal{HS}$ -model  $\mathfrak{M}$  based on a discrete linear order  $\mathfrak{T}$ , and consider the irreflexive semantics of the interval relations. In this case, we can single out unit-intervals with the formula

$$[\text{U}](\text{unit} \rightarrow [\text{E}] \perp) \wedge [\text{U}](\text{[E]} \perp \rightarrow \text{unit}). \quad (23)$$

It should be clear that if  $\mathfrak{M} \models (23)$  then, for all  $\langle x, x' \rangle$  in  $\text{int}(\mathfrak{T})$ , we have  $\mathfrak{M}, \langle x, x' \rangle \models \text{unit}$  iff  $x = x'$ . Further, it is easy to pass information from one unit-interval to the next, as we have a ‘next-time operator w.r.t.’ the above unit-sequence. Namely,

$$[\text{U}](\text{[B]} \lambda \rightarrow \text{[E]} \lambda')$$

forces  $\lambda'$  to be true at a unit-interval, whenever  $\lambda$  is true at the previous one.

To replace the binary implication trick with one using only  $\mathcal{HS}_{core}^{\square}$ -formulas, we employ the following *binary implication trick for the diagonal*. For any literals  $\lambda_1, \lambda_2$  and  $\lambda$ , we define the formula  $[\lambda_1 \wedge \lambda_2 \Rightarrow_H^d \lambda]$  as the conjunction of

$$\begin{aligned} & [\text{U}](\lambda_1 \rightarrow [\text{A}] \mu), \\ & [\text{U}](\lambda_2 \rightarrow [\text{A}] [\bar{\text{E}}] \mu), \\ & [\text{U}](\text{[A]} [\bar{\text{A}}] \mu \rightarrow \lambda), \end{aligned}$$

where  $\mu$  is a fresh variable. Then we clearly have the following:

**CLAIM 4.3.** *Suppose  $\mathfrak{M}$  satisfies  $[\lambda_1 \wedge \lambda_2 \Rightarrow_H^d \lambda]$ . If  $\mathfrak{M}, \langle u_n, u_n \rangle \models \lambda_1 \wedge \lambda_2$  then  $\mathfrak{M}, \langle x, u_{n+1} \rangle \models \lambda$  for all  $x \leq u_{n+1}$ .*

**Soundness:** Observe again that to satisfy  $[\lambda_1 \wedge \lambda_2 \Rightarrow_H^d \lambda]$  it is necessary that  $\lambda$  is horizontally stable in the model.

Now suppose that  $\mathcal{A}$  is a Turing machine whose computation with empty input uses  $< \text{poly}(\text{size}(\mathcal{A}))$  tape cells for some polynomial function  $\text{poly}()$ , and let  $\Phi_{\mathcal{A}}^d$  be the conjunction of unit, (8), (9), (22), (23), and the following formulas, for all  $(q, s) \in (Q - \{q_f\}) \times (\Sigma \cup \{\mathcal{L}\})$  and  $z \in \Sigma \cup \{\mathcal{L}\}$ :

$$[\text{U}](\text{cell}_i^{(q, s)} \rightarrow \neg[\bar{\text{B}}] \perp), \quad \text{for } i < N,$$

and

— if  $\delta_{\mathcal{A}}(q, s) = (q', s')$ , then

$$\begin{aligned} & [\text{U}]([\text{B}]\text{cell}_i^{(q,s)} \rightarrow [\text{E}]\text{cell}_i^{(q',s')}), \quad \text{for } i < N, \\ & [\text{cell}_i^{(q,s)} \wedge \text{cell}_j^z \Rightarrow_H^d \text{cell}_j^z], \quad \text{for } i, j < N, j \neq i; \end{aligned} \quad (24)$$

— if  $\delta_{\mathcal{A}}(q, s) = (q', r)$ , then

$$\begin{aligned} & [\text{U}]([\text{B}]\text{cell}_i^{(q,s)} \rightarrow [\text{E}]\text{cell}_i^r), \quad \text{for } i < N - 1, \\ & [\text{cell}_i^{(q,s)} \wedge \text{cell}_{i+1}^z \Rightarrow_H^d \text{cell}_{i+1}^{(q',z)}], \quad \text{for } i < N - 1, \\ & [\text{cell}_i^{(q,s)} \wedge \text{cell}_j^z \Rightarrow_H^d \text{cell}_j^z], \quad \text{for } i < N - 1, j < N, j \neq i, i + 1; \end{aligned} \quad (25)$$

— if  $\delta_{\mathcal{A}}(q, s) = (q', l)$ , then (25) for  $0 < i < N$  and

$$\begin{aligned} & [\text{cell}_i^{(q,s)} \wedge \text{cell}_{i-1}^z \Rightarrow_H^d \text{cell}_{i-1}^{(q',z)}], \quad \text{for } 0 < i < N, \\ & [\text{cell}_i^{(q,s)} \wedge \text{cell}_j^z \Rightarrow_H^d \text{cell}_j^z], \quad \text{for } 0 < i < N, j < N, j \neq i, i - 1. \end{aligned}$$

**CLAIM 4.4.** *If  $\Phi_{\mathcal{A}}^d$  is satisfiable in an  $\mathcal{HS}$ -model based on a discrete linear order, then  $\mathcal{A}$  diverges with empty input.*

**PROOF.** Take any  $\mathcal{HS}$ -model  $\mathfrak{M}$  based on a discrete linear order  $\mathfrak{T}$ , and suppose  $\mathfrak{M}, \langle r, r' \rangle \models \Phi_{\mathcal{A}}^d$  with the irreflexive semantics. Then it is not hard to show by induction on  $n$  that there exists an infinite sequence  $r = r' = u_0 < u_1 < u_2 < \dots < u_n < \dots$  of subsequent points in  $\mathfrak{T}$  such that for all  $n < \omega$ , the interval  $\langle u_n, u_n \rangle$  ‘represents’ the  $n$ th configuration  $\mathcal{C}_n$  in the infinite computation of  $\mathcal{A}$  with empty input in the following sense:

$$\mathfrak{M}, \langle u_n, u_n \rangle \models \text{cell}_i^z \quad \text{iff} \quad \mathcal{C}_n(i) = z,$$

for all  $i < N$  and  $z \in \Gamma_{\mathcal{A}}$ .  $\square$

On the other hand, if  $\mathcal{A}$  diverges on empty input, then take some discrete linear order  $\mathfrak{T}$  containing an infinite ascending chain  $t_0 < t_1 < \dots$  of subsequent points. Define an  $\mathcal{HS}$ -model  $\mathfrak{M} = (\mathfrak{F}_{\mathfrak{T}}, \nu)$  by taking, for all  $i < N$  and  $z \in \Gamma_{\mathcal{A}}$ ,

$$\begin{aligned} \nu(\text{unit}) &= \{\langle x, x \rangle \mid x \text{ in } \mathfrak{T}\}, \\ \nu(\text{cell}_i^z) &= \{\langle x, t_n \rangle \mid x \leq t_n, \mathcal{C}_n(i) = z, n < \omega\}. \end{aligned}$$

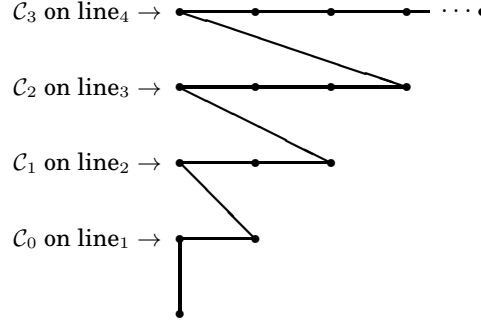
We claim that it is possible to evaluate the fresh auxiliary variable in each binary trick formula so that  $\mathfrak{M}, \langle t_0, t_0 \rangle \models \Phi_{\mathcal{A}}^d$  with the irreflexive semantics. Indeed, for example, fix some  $(q, s) \in (Q - \{q_f\}) \times (\Sigma \cup \{\mathcal{L}\})$  with  $\delta_{\mathcal{A}}(q, s) = (q', s')$ ,  $z \in \Sigma \cup \{\mathcal{L}\}$ , and  $i, j < N$ ,  $j \neq i$ , and consider the corresponding instance of conjunct (24):

$$[\text{cell}_i^{(q,s)} \wedge \text{cell}_j^z \Rightarrow_H^d \text{cell}_j^z].$$

Take

$$\nu(\mu) = \{\langle x, y \rangle \mid x \leq t_n, y \geq t_{n+1}, \mathcal{C}_n(i) = (q, s), \mathcal{C}_n(j) = z, n < \omega\}.$$

Then it is straightforward to see that  $[\text{U}](\text{cell}_i^{(q,s)} \rightarrow [\text{A}]\mu)$  and  $[\text{U}](\text{cell}_j^z \rightarrow [\text{A}][\bar{\text{E}}]\mu)$  both hold in  $\mathfrak{M}$ . We claim that  $[\text{U}](\text{cell}_i^{(q,s)} \rightarrow [\text{A}][\bar{\text{A}}]\mu \rightarrow \text{cell}_j^z)$  holds in  $\mathfrak{M}$  as well. Indeed, suppose  $\mathfrak{M}, \langle x, y \rangle \models [\text{A}][\bar{\text{A}}]\mu$  for some  $x \leq y$ . Then  $y = t_{n+1}$  for some  $n < \omega$  such that  $\mathcal{C}_n(i) = (q, s)$  and  $\mathcal{C}_n(j) = z$ . Thus,  $\mathcal{C}_{n+1}(j) = z$ , and so  $\mathfrak{M}, \langle x, t_{n+1} \rangle \models \text{cell}_j^z$ , as required. Checking the other conjuncts in  $\Phi_{\mathcal{A}}^d$  is similar and left to the reader.

Fig. 9. Placing the computation of  $\mathcal{A}$  on the  $nw_{\omega \times \omega}$ -grid.

The case when  $\mathfrak{T}$  contains an infinite descending chain of immediate predecessor points requires ‘symmetrical versions’ of the used formulas and is also left to the reader.

(ii) We reduce PSPACE-BOUND HALTING to  $\text{Fin}(<)$ -satisfiability. To achieve this, we omit the conjunct (22) from  $\Phi_{\mathcal{A}}^d$  above in order to force the computation to reach the halting state.  $\square$

#### 4.4. Undecidability

In our undecidability proofs, we ‘represent’ Turing machine computations on the  $nw_{\omega \times \omega}$ -grid as follows. Given any Turing machine  $\mathcal{A}$ , observe that for any computation of  $\mathcal{A}$  in the  $n$ th step the head can never move further than the  $n$ th cell. If  $\mathcal{A}$  starts with empty input, this means that  $C_n(m) = \sqcup$  for all  $n < H$  and  $n < m < \omega$ . Because of this we may actually assume that  $C_n$  is not of infinite length but of *finite* length  $n + 2$ . (Thus,  $C_0 = ((q_0, \mathcal{L}), \sqcup)$  and  $\mathcal{A}$  never visits the last cell of any  $C_n$ , so it is always  $\sqcup$ .) So we can place the subsequent finite configurations of the computation on the subsequent horizontal *lines* of the  $nw_{\omega \times \omega}$ -grid, continuously one after another (until we reach  $C_{H-2}$ , if  $H < \omega$ ), as depicted in Fig. 9.

Observe also that only the active cell and its neighbours can be changed by the transition to the next configuration, while all other cells remain the same. So instead of using the transition function  $\delta_{\mathcal{A}}$ , we can have the same information in the form of a ‘triples to cells’ function  $\tau_{\mathcal{A}}$  defined as follows. Let  $\Gamma_{\mathcal{A}} = \Sigma \cup \{\mathcal{L}\} \cup (Q \times (\Sigma \cup \{\mathcal{L}\}))$  and let  $W_{\mathcal{A}} \subseteq \Gamma_{\mathcal{A}} \times \Gamma_{\mathcal{A}} \times \Gamma_{\mathcal{A}}$  consist of those triples that can occur as three subsequent cells in the continuous enumeration of the configurations of the computation, that is, let

$$W_{\mathcal{A}} = ((Q^- \times \Sigma) \times \Sigma \times \Sigma) \cup (\Sigma \times (Q^- \times \Sigma) \times \Sigma) \cup (\Sigma \times \Sigma \times (Q^- \times \Sigma)) \cup \\ (\mathbf{LEnd} \times \Sigma \times \Sigma) \cup (\{\sqcup\} \times \mathbf{LEnd} \times \Sigma) \cup (\Sigma \times \{\sqcup\} \times \mathbf{LEnd}) \cup \{((q_0, \mathcal{L}), \sqcup, \mathcal{L})\},$$

where  $Q^- = Q - \{q_f\}$  and  $\mathbf{LEnd} = \{\mathcal{L}\} \cup (Q^- \times \{\mathcal{L}\})$ . We define a function  $\tau_{\mathcal{A}} : W_{\mathcal{A}} \rightarrow \Gamma_{\mathcal{A}}$  by taking, for all  $(x, y, z) \in W_{\mathcal{A}}$ ,

$$\tau_{\mathcal{A}}(x, y, z) = \begin{cases} (q', y), & \text{if either } x \in (Q - \{q_f\}) \times (\Sigma \cup \{\mathcal{L}\}) \text{ and } \delta_{\mathcal{A}}(x) = (q', r), \\ & \text{or } z \in (Q - \{q_f\}) \times \Sigma \text{ and } \delta_{\mathcal{A}}(z) = (q', l), \\ (q', y'), & \text{if } y \in (Q - \{q_f\}) \times \Sigma \text{ and } \delta_{\mathcal{A}}(y) = (q', y'), \\ y' & \text{if } y = (q, y') \text{ and } \delta_{\mathcal{A}}(y) = (q', M) \text{ for } M = l, r, \\ y, & \text{otherwise.} \end{cases} \quad (26)$$

Then it is easy to see that  $\tau_{\mathcal{A}}$  indeed determines the computation of  $\mathcal{A}$ , that is, for all  $0 < n < H$ ,  $\mathcal{C}_n(n+1) = \sqcup$  and for all  $m \leq n$ ,

$$\mathcal{C}_n(m) = \begin{cases} \tau_{\mathcal{A}}(\sqcup, \mathcal{C}_{n-1}(0), \mathcal{C}_{n-1}(1)), & \text{if } m = 0, \\ \tau_{\mathcal{A}}(\mathcal{C}_{n-1}(m-1), \mathcal{C}_{n-1}(m), \mathcal{C}_{n-1}(m+1)), & \text{if } 0 < m < n, \\ \tau_{\mathcal{A}}(\mathcal{C}_{n-1}(n-1), \sqcup, \mathcal{C}_n(0)), & \text{if } m = n. \end{cases}$$

**THEOREM 4.3. ( $\mathcal{HS}_{\text{horn}}^{\diamond}$ , arbitrary semantics)**

(i) For any class  $\mathcal{C}$  of linear orders containing an infinite order,  $\mathcal{C}$ -satisfiability of  $\mathcal{HS}_{\text{horn}}^{\diamond}$ -formulas is undecidable. (ii) Fin-satisfiability of  $\mathcal{HS}_{\text{horn}}^{\diamond}$ -formulas is undecidable.

**PROOF.** (i) We reduce NON-HALTING to  $\mathcal{C}$ -satisfiability. We discuss only the case when  $\mathcal{C}$  contains some linear order  $\mathfrak{T}$  having an infinite ascending chain. (The case when  $\mathfrak{T}$  contains an infinite descending chain requires ‘symmetrical versions’ of the used formulas and it is left to the reader.)

To make the main ideas more transparent, first we assume the *irreflexive semantics* for the interval relations, and then we show how to modify the proof for *arbitrary semantics*. Take any  $\mathcal{HS}$ -model  $\mathfrak{M}$  based on some linear order  $\mathfrak{T}$ . We begin with forcing a *unique* infinite unit-sequence in  $\mathfrak{M}$ , using the conjunction of (14) and

$$\text{unit} \wedge [\text{U}](\text{unit} \rightarrow \langle \text{A} \rangle \text{unit}), \quad (27)$$

$$[\text{U}](\text{unit} \rightarrow \neg \langle \text{E} \rangle \text{unit} \wedge \neg \langle \bar{\text{B}} \rangle \text{unit} \wedge \neg \langle \text{D} \rangle \text{unit} \wedge \neg \langle \text{O} \rangle \text{unit}). \quad (28)$$

Then it is straightforward to show the following:

**CLAIM 4.5.** *Let  $\phi_{\text{enum}}$  be the conjunction of (14), (27) and (28), and suppose that  $\mathfrak{M}, \langle r, r' \rangle \models \phi_{\text{enum}}$ . Then there is an infinite sequence  $u_0 < u_1 < \dots < u_n < \dots$  of points in  $\mathfrak{T}$  such that for all  $r \leq x$  and all  $r' \leq x'$ , we have  $\mathfrak{M}, \langle x, x' \rangle \models \text{unit}$  iff  $x = u_n$  and  $x' = u_{n+1}$  for some  $n < \omega$ .*

Next, we use this unit-sequence to encode the enumeration of the  $nw_{\omega \times \omega}$ -grid depicted in Fig. 7. Observe that for this particular enumeration the right-neighbour of a grid-location is the next one in the enumeration. As we generated our unit-sequence with (27), we have access from one unit-interval to the next by the A interval relation. So, to encode the  $nw_{\omega \times \omega}$ -grid, it is enough to use ‘up-pointers’. We force the proper placement of ‘up-pointers’ in a particular way, by using the following properties of this enumeration:

- (a.1) 0 is on the diagonal, and  $\text{up\_neighbour\_of}(0) = 1$ .
- (a.2) If  $n$  is on the diagonal, then  $\text{up\_neighbour\_of}(n) + 1$  is on the diagonal, for every  $n < B$ .
- (a.3) If  $n$  is the up-neighbour of some location, then  $n$  is not on the diagonal, for every  $n < B$ .
- (a.4) If  $n$  is not on the diagonal, then  $\text{up\_neighbour\_of}(n+1) = \text{up\_neighbour\_of}(n) + 1$ , for every  $n+1 < B$ .
- (a.5) If  $n$  is on the diagonal, then  $\text{up\_neighbour\_of}(n+1) = \text{up\_neighbour\_of}(n) + 2$ , for every  $n+1 < B$ .

**CLAIM 4.6.** *Properties (a.1)–(a.5) uniquely determine<sup>7</sup> the enumeration in Fig. 7.*

**PROOF.** We prove by induction on  $n < B$  that for every  $k \leq n$ ,

- (i)  $k = \langle x, y \rangle$  is like it should be in Fig. 7.
- (ii)  $k$  is on the diagonal iff  $k = \langle x, x \rangle$  for some  $x$ .

<sup>7</sup>Among those that contain the enumeration of the diagonal locations as  $(0, 0), \dots, (1, 1), \dots, (2, 2), \dots$

Indeed, for  $n = 1$  (i) follows from (a.1), and (b) follows from (a.3). Now suppose inductively that (i)–(ii) hold for all  $k \leq n$  for some  $0 < n < B$ , and let  $n + 1 < B$ . There are three cases.

If  $n$  is on the diagonal, then by (ii),  $n = (x, x)$  for some  $x > 0$ . Let  $m = (x - 1, x - 1)$ . Then  $m < n$  by (i) and so by (ii),  $m$  is on the diagonal. So by (a.5),  $n + 1 = \text{up\_neighbour\_of}(m + 1)$ , proving (i). Now (ii) follows from (a.3).

If  $n$  is not on the diagonal and  $n = (x, y)$  for some  $y$  and  $x < y - 1$ , then let  $m = (x, y - 1)$ . Then  $m < n$  by (i) and so by (ii),  $m$  is not on the diagonal. So by (a.4),  $n + 1 = \text{up\_neighbour\_of}(m + 1)$ , proving (i). Now (ii) follows from (a.3).

If  $n$  is not on the diagonal and  $n = (y - 1, y)$  for some  $y$ , then let  $m = (y - 1, y - 1)$ . Then  $m < n$  by (i) and so by (ii),  $m$  is on the diagonal. By (a.2),  $n + 1$  is on the diagonal, so it should be the next ‘unused’ diagonal location, which is  $(y, y)$ , proving both (i) and (ii).  $\square$

Next, given a unique infinite unit-sequence  $\mathcal{U} = (\langle u_n, u_{n+1} \rangle \mid n < \omega)$  as in Claim 4.5 above, we express ‘horizontal’ and ‘vertical next-time’ in  $\mathfrak{M}$  ‘with respect to  $\mathcal{U}$ ’. Given literals  $\lambda_1$  and  $\lambda_2$ , let  $\text{grid\_succ}_{\rightarrow}[\lambda_1, \lambda_2]$  denote the conjunction of

$$\begin{aligned} & [\text{U}](\lambda_1 \rightarrow \neg \langle \text{E} \rangle \lambda_1) \wedge [\text{U}](\lambda_2 \rightarrow \neg \langle \text{E} \rangle \lambda_2), \\ & [\text{U}](\lambda_1 \rightarrow \langle \text{E} \rangle \lambda_2), \\ & [\text{U}](\lambda_1 \rightarrow [\text{E}](\langle \text{E} \rangle \lambda_2 \rightarrow \neg \langle \text{B} \rangle \text{unit})), \end{aligned} \quad (29)$$

and similarly, let  $\text{grid\_succ}_{\uparrow}[\lambda_1, \lambda_2]$  denote the conjunction of

$$\begin{aligned} & [\text{U}](\lambda_1 \rightarrow \neg \langle \bar{\text{B}} \rangle \lambda_1) \wedge [\text{U}](\lambda_2 \rightarrow \neg \langle \bar{\text{B}} \rangle \lambda_2), \\ & [\text{U}](\lambda_1 \rightarrow \langle \bar{\text{B}} \rangle \lambda_2), \\ & [\text{U}](\lambda_1 \rightarrow [\bar{\text{B}}](\langle \bar{\text{B}} \rangle \lambda_2 \rightarrow \neg \langle \text{E} \rangle \text{unit})). \end{aligned} \quad (30)$$

It is straightforward to show the following:

**CLAIM 4.7.** *Suppose  $\mathfrak{M}, \langle u_m, u_n \rangle \models \lambda_1$  for some  $m, n < \omega$ .*

- *Suppose  $\mathfrak{M}$  satisfies  $\text{grid\_succ}_{\rightarrow}[\lambda_1, \lambda_2]$ . Then, for all  $x$ ,  $\mathfrak{M}, \langle x, u_n \rangle \models \lambda_2$  iff  $x = u_{m+1}$ , and  $\mathfrak{M}, \langle x, u_n \rangle \models \lambda_1$  iff  $x = u_m$ .*
- *Suppose  $\mathfrak{M}$  satisfies  $\text{grid\_succ}_{\uparrow}[\lambda_1, \lambda_2]$ . Then, for all  $y$ ,  $\mathfrak{M}, \langle u_m, y \rangle \models \lambda_2$  iff  $y = u_{n+1}$ , and  $\mathfrak{M}, \langle u_m, y \rangle \models \lambda_1$  iff  $y = u_n$ .*

Now we can encode (a.1)–(a.5) as follows. We use a propositional variable  $\text{up}$  to mark up-pointers, variables  $\text{diag}$  and  $\bar{\text{diag}}$  to mark those respective unit-points that are on the diagonal and not on the diagonal, and further fresh variables  $\text{now}$ ,  $\text{up}_{\uparrow}$ ,  $\text{up}_{\rightarrow}$ ,  $\text{up}^+$  (see Fig. 10 for the intended placement of the variables). Then we express (a.1) by the conjunction of

$$\text{unit} \wedge \text{diag} \wedge \text{now}, \quad (31)$$

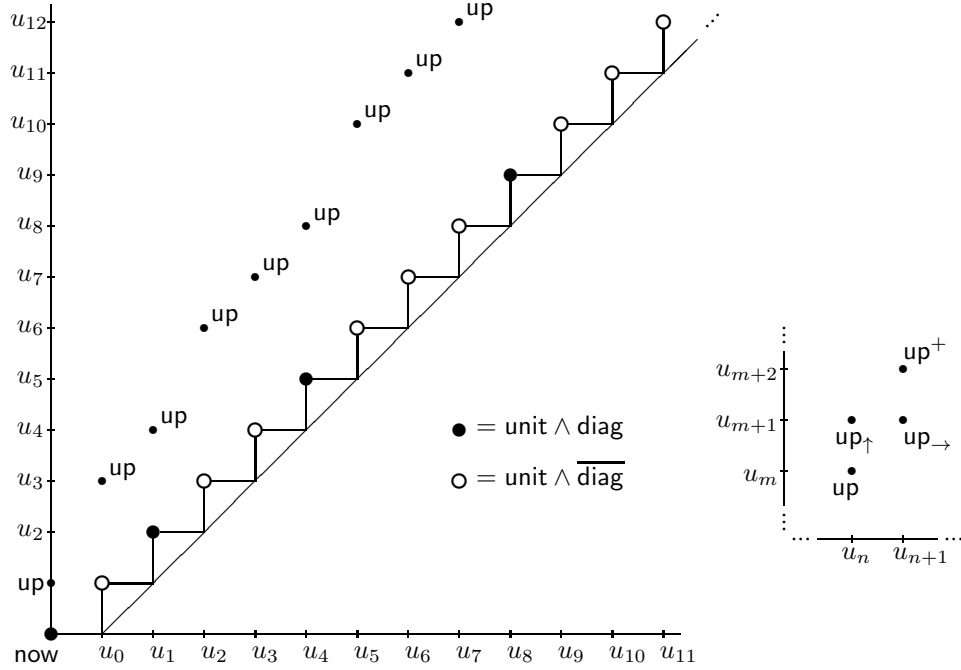
$$\text{grid\_succ}_{\uparrow}[\text{now}, \text{up}], \quad (32)$$

(a.2) by the conjunction of

$$\text{grid\_succ}_{\uparrow}[\text{up}, \text{up}_{\uparrow}], \quad (33)$$

$$[\text{U}](\text{unit} \wedge \text{diag} \rightarrow [\bar{\text{B}}](\text{up}_{\uparrow} \rightarrow [\text{E}](\text{unit} \rightarrow \text{diag}))), \quad (34)$$



Fig. 10. Encoding the  $nw_{\omega \times \omega}$ -grid in an  $\mathcal{HS}$ -model: version 1.

(a.3) by the conjunction of

$$[U](\text{up} \rightarrow [E](\text{unit} \rightarrow \overline{\text{diag}})), \quad (35)$$

$$[U](\text{diag} \wedge \overline{\text{diag}} \rightarrow \perp), \quad (36)$$

(a.4) by

$$[U]\left(\text{unit} \wedge \overline{\text{diag}} \rightarrow [\bar{B}](\text{up}_{\uparrow} \rightarrow [E](\text{up}_{\rightarrow} \rightarrow \text{up}))\right), \quad (37)$$

and (a.5) by the conjunction of

$$\text{grid\_succ}_{\rightarrow}[\text{up}_{\uparrow}, \text{up}_{\rightarrow}], \quad (38)$$

$$\text{grid\_succ}_{\uparrow}[\text{up}_{\rightarrow}, \text{up}^+], \quad (39)$$

$$[U]\left[\text{unit} \wedge \text{diag} \rightarrow [\bar{B}](\text{up}_{\uparrow} \rightarrow [E](\text{up}_{\rightarrow} \rightarrow [\bar{B}](\text{up}^+ \rightarrow \text{up}))\right)]. \quad (40)$$

It is not hard to show the following:

**CLAIM 4.8.** *Suppose  $\mathfrak{M}, \langle r, r' \rangle \models \phi_{\text{enum}} \wedge \phi_{\text{grid}}$ , where  $\phi_{\text{grid}}$  is the conjunction of (31)–(40). Then now,  $\overline{\text{diag}}$ ,  $\overline{\text{diag}}$  and  $\text{up}$  are properly placed (see Fig. 10).*

Given a Turing machine  $\mathcal{A}$ , we will use the function  $\tau_{\mathcal{A}}$  (defined in (26)) to force a diverging computation of  $\mathcal{A}$  with empty input as follows. We introduce (with a slight abuse of notation) a propositional variable  $x$  for each  $x \in \Gamma_{\mathcal{A}}$ . Then we formulate gen-

eral constraints as

$$[U](x \rightarrow \text{unit}), \quad \text{for } x \in \Gamma_{\mathcal{A}}, \quad (41)$$

$$[U](x \rightarrow \neg y), \quad \text{for } x \neq y, x, y \in \Gamma_{\mathcal{A}}, \quad (42)$$

and then force the computation steps by the conjunction of

$$\langle A \rangle(q_0, \mathcal{L}), \quad (43)$$

$$[U](\text{diag} \rightarrow \sqcup), \quad (44)$$

$$[U]\left(y \wedge \langle A \rangle z \wedge \langle \bar{A} \rangle x \rightarrow [\bar{B}](\text{up} \rightarrow [E](\text{unit} \rightarrow \tau_{\mathcal{A}}(x, y, z)))\right), \quad \text{for } (x, y, z) \in W_{\mathcal{A}}. \quad (45)$$

Finally, we force non-halting with

$$[U]((q_f, s) \rightarrow \perp), \quad \text{for } s \in \Sigma \cup \{\mathcal{L}\}. \quad (46)$$

Using Claims 4.5–4.8, now it is straightforward to prove the following:

**CLAIM 4.9.** *Let  $\Psi_{\mathcal{A}}$  be the conjunction of  $\phi_{enum}$ ,  $\phi_{grid}$  and (41)–(46). If  $\Psi_{\mathcal{A}}$  is satisfiable in an  $\mathcal{HS}$ -model, then  $\mathcal{A}$  diverges with empty input.*

On the other hand, Fig. 10 shows how to satisfy  $\phi_{enum} \wedge \phi_{grid}$  (using the irreflexive semantics) in an  $\mathcal{HS}$ -model that is based on some linear order  $\mathfrak{T}$  having an infinite ascending chain  $u_0 < u_1 < \dots$ . If  $\mathcal{A}$  diverges with empty input, then we can add, for all  $x \in \Gamma_{\mathcal{A}}$ ,

$$\nu(x) = \{\langle u_{n-1}, u_n \rangle \mid n > 0, C_j(i) = x \text{ and the } n\text{th point in the grid-enumeration is } (i, j + 1)\} \quad (47)$$

to obtain an  $\mathcal{HS}$ -model  $\mathfrak{M} = (\mathfrak{T}_{\mathfrak{T}}, \nu)$  satisfying (41)–(46) as well.

Next, we show how to modify the formula  $\Psi_{\mathcal{A}}$  above in order to be satisfiable with arbitrary semantics of the interval relations. ‘Uniqueness forcing’ constraints like (28), (29), and (30) above are clearly not satisfiable with the reflexive semantics. Expanding on an idea of [Spaan 1993], [Reynolds and Zakharyashev 2001; Gabbay et al. 2003; Gabelaia et al. 2005b], we use the following *chessboard trick* to solve this problem and kind of ‘discretise’ the  $\mathcal{HS}$ -model. Take two fresh propositional variables  $\text{Htick}$  and  $\text{Vtick}$ , and make the  $\mathcal{HS}$ -model  $\mathfrak{M}$  ‘chessboard-like’ by the formula

$$[U](\text{Htick} \rightarrow [\bar{B}]\text{Htick}) \wedge [U](\text{Vtick} \rightarrow [E]\text{Vtick}). \quad (48)$$

However, to make it a real chessboard, we also need to have ‘cover’ by these variables and their negations, that is, for every interval in  $\mathfrak{M}$ ,  $\text{Htick} \vee \neg\text{Htick}$  and  $\text{Vtick} \vee \neg\text{Vtick}$  should hold. In order to express these by  $\mathcal{HS}_{horn}^{\diamond}$ -formulas, we use the following *cover trick* of [Artale et al. 2007, p. 11]. For any literals  $\lambda$  and  $\bar{\lambda}$ , let  $\text{Cover}_{\leftrightarrow}[\lambda, \bar{\lambda}]$  denote the conjunction of

$$[U](\top \rightarrow \langle \bar{B} \rangle (M_{\lambda} \wedge \langle E \rangle X_{\lambda} \wedge \langle E \rangle Y_{\lambda})), \quad (49)$$

$$[U](X_{\lambda} \wedge Y_{\lambda} \rightarrow \perp),$$

$$[U]\left(\langle \bar{B} \rangle (M_{\lambda} \wedge \langle E \rangle (Y_{\lambda} \wedge \langle E \rangle X_{\lambda})) \rightarrow \lambda\right),$$

$$[U]\left(\langle \bar{B} \rangle (M_{\lambda} \wedge \langle E \rangle (X_{\lambda} \wedge \langle E \rangle Y_{\lambda})) \rightarrow \bar{\lambda}\right),$$

$$[U](\lambda \wedge \bar{\lambda} \rightarrow \perp),$$

where  $M_{\lambda}$ ,  $X_{\lambda}$ , and  $Y_{\lambda}$  are fresh variables.

*Soundness:* Observe that  $\text{Cover}_{\leftrightarrow}[\lambda, \bar{\lambda}]$  forces the model to be infinite. Also, it always implies that both  $\lambda$  and  $\bar{\lambda}$  are vertically stable, that is,

$$[\text{U}](\lambda \rightarrow [\bar{\text{B}}]\lambda) \wedge [\text{U}](\bar{\lambda} \rightarrow [\bar{\text{B}}]\bar{\lambda}).$$

holds. We can define  $\text{Cover}_{\uparrow}[\lambda, \bar{\lambda}]$  similarly, for horizontally stable  $\lambda$  and  $\bar{\lambda}$ . Now we take fresh variables  $\overline{\text{Htick}}$  and  $\overline{\text{Vtick}}$ , and define Chessboard by taking

$$\text{Chessboard} := \text{Cover}_{\leftrightarrow}[\overline{\text{Htick}}, \overline{\text{Htick}}] \wedge \text{Cover}_{\uparrow}[\overline{\text{Vtick}}, \overline{\text{Vtick}}]. \quad (50)$$

Then (48) and the similar formula for  $\overline{\text{Htick}}$  and  $\overline{\text{Vtick}}$  follow. Suppose that  $\mathfrak{M}$  is an  $\mathcal{HS}$ -model based on some linear order  $\mathfrak{T} = (T, \leq)$  satisfying Chessboard. We define two new binary relations  $\prec_{\rightarrow}^{\mathfrak{M}}$  and  $\prec_{\uparrow}^{\mathfrak{M}}$  on  $T$  by taking, for all  $u, v \in T$ ,

$$\begin{aligned} u \prec_{\rightarrow}^{\mathfrak{M}} v \quad \text{iff} \quad & \exists z \left( u \leq z \leq v \text{ and} \right. \\ & \left. \forall y \left( \text{if } \langle z, y \rangle \text{ is in } \mathfrak{M}, \text{ then } (\mathfrak{M}, \langle u, y \rangle \models \text{Htick} \leftrightarrow \mathfrak{M}, \langle z, y \rangle \models \neg \text{Htick}) \right) \right); \\ u \prec_{\uparrow}^{\mathfrak{M}} v \quad \text{iff} \quad & \exists z \left( u \leq z \leq v \text{ and} \right. \\ & \left. \forall x \left( \text{if } \langle x, u \rangle \text{ is in } \mathfrak{M}, \text{ then } (\mathfrak{M}, \langle x, u \rangle \models \text{Vtick} \leftrightarrow \mathfrak{M}, \langle x, z \rangle \models \neg \text{Vtick}) \right) \right). \end{aligned}$$

Then it is straightforward to check that both  $\prec_{\rightarrow}^{\mathfrak{M}}$  and  $\prec_{\uparrow}^{\mathfrak{M}}$  imply  $\leq$ , and both are transitive and irreflexive. (They are not necessarily linear orders.) We call a non-empty subset  $I \subseteq T$  a *horizontal  $\mathfrak{M}$ -interval* (shortly, an *h-interval*), if  $I$  is maximal with the following two properties:

- for all  $x, y, z \in T$ , if  $x \leq y \leq z$  and  $x, z \in I$  then  $y \in I$ ;
- either  $\mathfrak{M}, \langle x, y \rangle \models \text{Htick}$ , for all  $x \in I$  and  $y \in T$  such that  $\langle x, y \rangle$  is in  $\mathfrak{M}$ , or  $\mathfrak{M}, \langle x, y \rangle \models \neg \text{Htick}$ , for all  $x \in I$  and  $y \in T$  such that  $\langle x, y \rangle$  is in  $\mathfrak{M}$ .

For any  $x \in T$ , let  $h\_int(x)$  denote the unique h-interval  $I$  with  $x \in I$ . We define *v-intervals* and  $v\_int(x)$  similarly, using  $\prec_{\uparrow}^{\mathfrak{M}}$ . A set  $S$  of the form  $S = I \times J$  for some h-interval  $I$  and v-interval  $J$  is called a *square*. For any  $\langle x, y \rangle$  in  $\mathfrak{M}$ , let  $square(x, y)$  denote the unique square  $S$  with  $\langle x, y \rangle \in S$ .

Now we define horizontal and vertical successor squares. Given propositional variables  $P$  and  $Q$ , let  $\text{succ\_sq}_{\rightarrow}[P, Q]$  be the conjunction of

$$[\text{U}](P \wedge \text{Htick} \rightarrow \langle \text{E} \rangle (Q \wedge \overline{\text{Htick}})), \quad (51)$$

$$[\text{U}](P \wedge \bar{P} \rightarrow \perp),$$

$$[\text{U}](P \wedge \text{Htick} \rightarrow [\text{E}] P'), \quad (52)$$

$$[\text{U}](P' \wedge \overline{\text{Htick}} \rightarrow (\bar{P} \wedge [\text{E}] \bar{P})),$$

$$[\text{U}](Q \wedge \bar{Q} \rightarrow \perp),$$

$$[\text{U}](Q \wedge \overline{\text{Htick}} \rightarrow [\text{E}] Q'),$$

$$[\text{U}](Q' \wedge \text{Htick} \rightarrow (\bar{Q} \wedge [\text{E}] \bar{Q})),$$

$$[\text{U}](P' \wedge \text{Htick} \wedge \langle \text{E} \rangle (Q \wedge \overline{\text{Htick}}) \rightarrow P), \quad (53)$$

$$[\text{U}](P' \wedge \overline{\text{Htick}} \wedge \langle \text{E} \rangle Q \rightarrow Q), \quad (54)$$

$$[\text{U}](Q \wedge P \rightarrow \perp), \quad (55)$$

$$[\text{U}](Q \wedge \langle \text{E} \rangle P \rightarrow \perp) \quad (56)$$

plus similar formulas for the ‘ $P \wedge \overline{\text{Htick}}$ ’ case (here  $\overline{P}$ ,  $\overline{Q}$ ,  $P'$  and  $Q'$  are fresh variables). One can define  $\text{succ\_sq}_{\uparrow}[P, Q]$  similarly. Finally, we let

$$\text{fill}[P] = \text{succ\_sq}_{\rightarrow}[P_l, P] \wedge \text{succ\_sq}_{\rightarrow}[P, P_r] \wedge \text{succ\_sq}_{\uparrow}[P_d, P] \wedge \text{succ\_sq}_{\uparrow}[P, P_u],$$

where  $P_l, P_r, P_d,$  and  $P_u$  are fresh variables.

**CLAIM 4.10.** *Suppose  $\mathfrak{M}$  satisfies Chessboard and  $\text{succ\_sq}_{\rightarrow}[P, Q]$ . Then the following hold, for all  $x, y, z, w$ :*

- (i) *If  $\mathfrak{M}, \langle x, y \rangle \models P$ , then there is  $v$  such that  $x \prec_{\rightarrow}^{\mathfrak{M}} v$  and  $\mathfrak{M}, \langle v, y \rangle \models Q$ .*
- (ii) *If  $\mathfrak{M}, \langle x, y \rangle \models P$  and  $x \prec_{\rightarrow}^{\mathfrak{M}} z$ , then  $\mathfrak{M}, \langle z, y \rangle \not\models P$ .*
- (iii) *If  $\mathfrak{M}, \langle x, y \rangle \models Q$  and  $x \prec_{\rightarrow}^{\mathfrak{M}} z$ , then  $\mathfrak{M}, \langle z, y \rangle \not\models Q$ .*
- (iv) *If  $\mathfrak{M}, \langle x, y \rangle \models P$ ,  $z \in h\_int(x)$ ,  $x \leq z$ , then  $\mathfrak{M}, \langle z, y \rangle \models P$ .*
- (v) *If  $\mathfrak{M}, \langle x, y \rangle \models P$ ,  $\mathfrak{M}, \langle z, y \rangle \models Q$ ,  $w \in h\_int(z)$  and  $w \leq z$ , then  $\mathfrak{M}, \langle w, y \rangle \models Q$ .*
- (vi) *If  $\mathfrak{M}, \langle x, y \rangle \models P$  and  $\mathfrak{M}, \langle z, y \rangle \models Q$ , then  $x \prec_{\rightarrow}^{\mathfrak{M}} z$  and there is no  $t$  with  $x \prec_{\rightarrow}^{\mathfrak{M}} t \prec_{\rightarrow}^{\mathfrak{M}} z$ .*

*Similar statements hold if  $\mathfrak{M}$  satisfies  $\text{succ\_sq}_{\uparrow}[P, Q]$ . Therefore,*

- (vii) *if  $\mathfrak{M}$  satisfies  $\text{fill}[P]$  and  $\mathfrak{M}, \langle x, y \rangle \models P$  then  $\mathfrak{M}, \langle x', y' \rangle \models P$  for all  $\langle x', y' \rangle \in \text{square}(x, y)$ .*

**PROOF.** It is mostly straightforward. We show the trickiest case, (vi) We have  $x \leq z$  by (55). Suppose, say, that  $\mathfrak{M}, \langle x, y \rangle \models \text{Htick}$ . By (i), there is  $v$  such that  $x \prec_{\rightarrow}^{\mathfrak{M}} v$  and  $\mathfrak{M}, \langle v, y \rangle \models Q$ , and so  $\mathfrak{M}, \langle v, y \rangle \models \overline{\text{Htick}}$ . Then  $z \in h\_int(v)$  follows by (iii), and so  $x \prec_{\rightarrow}^{\mathfrak{M}} z$ . Now let  $t$  be such that  $x \leq t \leq z$ . If  $\mathfrak{M}, \langle t, y \rangle \models \text{Htick}$ , then  $\mathfrak{M}, \langle t, y \rangle \models P$  by (52) and (53), and so  $t \in h\_int(x)$  by (ii). If  $\mathfrak{M}, \langle t, y \rangle \models \overline{\text{Htick}}$ , then  $\mathfrak{M}, \langle t, y \rangle \models Q$  by (52) and (54), and so  $t \in h\_int(z)$  by (iii).  $\square$

**Soundness:** If  $\mathfrak{M}$  satisfies  $\text{fill}[P]$  then  $P$  must be both ‘horizontally and vertically square-unique’ in the following sense: if  $\mathfrak{M}, \langle x, y \rangle \models P$  and  $\mathfrak{M}, \langle x', y' \rangle \models P$  for some  $x \prec_{\rightarrow}^{\mathfrak{M}} x'$  and  $y \prec_{\uparrow}^{\mathfrak{M}} y'$ , then  $\text{square}(x, y) = \text{square}(x', y')$  must follow.

Now, using this ‘chessboard trick’, we can modify the formula  $\Psi_{\mathcal{A}}$  above for any semantical choice of the interval relations. To begin with, instead of using  $\phi_{\text{enum}}$ , we force a unique infinite sequence of unit-squares by introducing a fresh variable  $\text{next}$ , and taking the conjunction  $\phi_{\text{enum}}^r$  of the following formulas:

$$\begin{aligned} & \text{Chessboard} \wedge \text{fill}[\text{unit}] \wedge \text{fill}[\text{next}], \\ & \text{unit} \wedge \text{succ\_sq}_{\rightarrow}[\text{unit}, \text{next}], \\ & \text{succ\_sq}_{\uparrow}[\text{next}, \text{unit}]. \end{aligned} \tag{57}$$

Then we have the following generalisation of Claim 4.5:

**CLAIM 4.11.** *Suppose  $\mathfrak{M}, \langle r, r' \rangle \models \phi_{\text{enum}}^r$ . Then there exist infinite sequences  $(x_n \mid n < \omega)$  and  $(y_n \mid n < \omega)$  of points in  $\mathfrak{T}$  such that the following hold:*

- (i)  *$r = x_0 \prec_{\rightarrow}^{\mathfrak{M}} x_1 \prec_{\rightarrow}^{\mathfrak{M}} \dots \prec_{\rightarrow}^{\mathfrak{M}} a_n \prec_{\rightarrow}^{\mathfrak{M}} \dots$  and  $r' = y_0 \prec_{\uparrow}^{\mathfrak{M}} y_1 \prec_{\uparrow}^{\mathfrak{M}} \dots \prec_{\uparrow}^{\mathfrak{M}} y_n \prec_{\uparrow}^{\mathfrak{M}} \dots$*
- (ii) *There is no  $x$  with  $x_n \prec_{\rightarrow}^{\mathfrak{M}} x \prec_{\rightarrow}^{\mathfrak{M}} x_{n+1}$  and there is no  $y$  with  $y_n \prec_{\uparrow}^{\mathfrak{M}} y \prec_{\uparrow}^{\mathfrak{M}} y_{n+1}$ , for any  $n < \omega$ .*
- (iii) *For all  $x, y$ ,  $\mathfrak{M}, \langle x, y \rangle \models \text{unit}$  iff  $\langle x, y \rangle \in \text{square}(x_n, y_n)$  for some  $n < \omega$ .*

In order to show the soundness of  $\phi_{\text{enum}}^r$ , let  $\mathfrak{T} = (T, \leq)$  be a linear order containing an infinite ascending chain  $u_0 < u_1 < \dots$

**CLAIM 4.12.**  *$\phi_{\text{enum}}^r$  is satisfiable in an HS-model based on  $\mathfrak{T}$  under arbitrary semantics.*

PROOF. For each  $n < \omega$ , we let

$$U_n = \{x \in T \mid u_n \leq x < u_{n+1}\}.$$

It is straightforward to check that the following  $\mathcal{HS}$ -model  $\mathfrak{M} = (\mathfrak{F}_{\mathfrak{X}}, \nu)$  satisfies  $\text{Cover}_{\leftrightarrow}[\text{Htick}, \overline{\text{Htick}}]$ :

$$\begin{aligned} \nu(\text{Htick}) &= \{\langle x, y \rangle \in \text{int}(\mathfrak{F}) \mid x \in U_n, n \text{ is even}\}, \\ \nu(\overline{\text{Htick}}) &= \{\langle x, y \rangle \in \text{int}(\mathfrak{F}) \mid x \in U_n, n \text{ is odd}\}, \\ \nu(\text{M}_{\text{Htick}}) &= \{\langle x, y \rangle \in \text{int}(\mathfrak{F}) \mid x \in U_m, y \in U_n, \text{ both } m, n \text{ are even, or both } m, n \text{ are odd}\}, \\ \nu(\text{X}_{\text{Htick}}) &= \{\langle x, y \rangle \in \text{int}(\mathfrak{F}) \mid x \in U_n, y \in U_{n+1} \cup U_{n+2}, n \text{ is even}\}, \\ \nu(\text{Y}_{\text{Htick}}) &= \{\langle x, y \rangle \in \text{int}(\mathfrak{F}) \mid x \in U_n, y \in U_{n+1} \cup U_{n+2}, n \text{ is odd}\}. \end{aligned}$$

$\text{Cover}_{\uparrow}[\text{Vtick}, \overline{\text{Vtick}}]$  can be satisfied similarly. The rest is obvious.  $\square$

Next, consider the formula  $\phi_{\text{grid}}$  defined in Claim 4.8. Let  $\phi_{\text{grid}}^r$  be obtained from  $\phi_{\text{grid}}$  by replacing each occurrence of  $\text{grid\_succ}_{\rightarrow}$  by  $\text{succ\_sq}_{\rightarrow}$  and each occurrence of  $\text{grid\_succ}_{\uparrow}$  by  $\text{succ\_sq}_{\uparrow}$ , and adding the conjuncts  $\text{fill}[P]$  for  $P \in \{\text{now}, \text{unit}, \text{diag}, \overline{\text{diag}}, \text{up}, \text{up}_{\uparrow}, \text{up}_{\rightarrow}, \text{up}^+\}$ . Using Claim 4.11, it is straightforward to show that we have the analogue of Claim 4.8 for squares.

Finally, given a Turing machine  $\mathcal{A}$ , let  $\Psi_{\mathcal{A}}^r$  be the conjunction of  $\phi_{\text{enum}}^r, \phi_{\text{grid}}^r$ , (41)–(46), and  $\text{fill}[x]$  for each  $x \in \Gamma_{\mathcal{A}}$ . Then we have:

CLAIM 4.13. *If  $\Psi_{\mathcal{A}}^r$  is satisfiable in an  $\mathcal{HS}$ -model, then  $\mathcal{A}$  diverges with empty input.*

On the other hand, using Fig. 10, Claim 4.12 and (47) it is easy to show how to satisfy  $\Psi_{\mathcal{A}}^r$  in an  $\mathcal{HS}$ -model that is based on some linear order  $\mathfrak{X}$  having an infinite ascending chain  $u_0 < u_1 < \dots$ , regardless which semantics of the interval relations is considered.

(ii) We reduce ‘halting’ to Fin-satisfiability. We show how to modify the formula  $\Psi_{\mathcal{A}}^r$  above to achieve this. To begin with, ‘generating’ conjuncts like (49) and its ‘vertical’ version in Chessboard, and (51) and its  $\overline{\text{Htick}}$  version in  $\text{succ\_sq}_{\rightarrow}[\text{unit}, \text{next}]$  of (57) are not satisfiable in  $\mathcal{HS}$ -models based on finite orders. In order to obtain a finitely satisfiable version, we introduce a fresh variable  $\text{end}$ , replace (46) with the conjunction of

$$[\text{U}](\text{end} \rightarrow \text{unit}), \tag{58}$$

$$[\text{U}](\text{end} \wedge x \rightarrow \perp), \quad \text{for } x \in \Sigma \cup \{\mathcal{L}\} \cup (Q^- \times (\Sigma \cup \{\mathcal{L}\})), \tag{59}$$

then replace conjunct (49) in  $\text{Cover}_{\leftrightarrow}[\lambda, \bar{\lambda}]$  with the conjunction of

$$[\text{U}](\langle \text{R} \rangle \text{end} \rightarrow \langle \bar{\text{B}} \rangle (\text{M}_{\lambda} \wedge \langle \text{E} \rangle \text{X}_{\lambda} \wedge \langle \text{E} \rangle \text{Y}_{\lambda})), \quad \text{for } \text{R} \in \{\text{A}, \bar{\text{B}}, \bar{\text{D}}, \text{L}, \text{O}\},$$

(and similarly in  $\text{Cover}_{\uparrow}[\lambda, \bar{\lambda}]$ ), and then replace conjunct (51) in  $\text{succ\_sq}_{\rightarrow}[\text{unit}, \text{next}]$  with the conjunction of

$$[\text{U}](\langle \text{R} \rangle \text{end} \wedge \text{unit} \wedge \text{Htick} \rightarrow \langle \text{E} \rangle (\text{next} \wedge \overline{\text{Htick}})), \quad \text{for } \text{R} \in \{\text{A}, \bar{\text{B}}, \bar{\text{D}}, \text{L}, \text{O}\}$$

(and do similarly for the ‘ $\overline{\text{Htick}}$ -version’, and for the ‘generating’ conjuncts in  $\text{succ\_sq}_{\uparrow}[\text{next}, \text{unit}]$ ).  $\square$

THEOREM 4.4. ( $\mathcal{HS}_{\text{core}}$ , *irreflexive semantics*)

(i) For any class  $\mathcal{C}$  of linear orders containing an infinite order,  $\mathcal{C}(\langle \cdot \rangle)$ -satisfiability of  $\mathcal{HS}_{\text{core}}$ -formulas is undecidable. (ii)  $\text{Fin}(\langle \cdot \rangle)$ -satisfiability of  $\mathcal{HS}_{\text{core}}$ -formulas is undecidable.

PROOF. (i) We reduce NON-HALTING to  $\mathcal{C}(<)$ -satisfiability. Given an  $\mathcal{HS}$ -model  $\mathfrak{M}$  based on some linear order  $\mathfrak{T}$ , observe that the formula  $\phi_{enum}$  (defined in Claim 4.5) that forces a unique infinite unit-sequence  $(\langle u_n, u_{n+1} \rangle \mid n < \omega)$  in  $\mathfrak{M}$  is within  $\mathcal{HS}_{core}$ . However, the formula  $\phi_{grid}$  (defined in Claim 4.8) we used in the proof of Theorem 4.3 to encode the  $nw_{\omega \times \omega}$ -grid in  $\mathfrak{M}$  with the help of properly placed up-pointers contains several seemingly ‘non- $\mathcal{HS}_{core}$ -able’ conjuncts. In order to fix this, below we will force the proper placement of up-pointers in a different way.

Consider again the enumeration of  $nw_{\omega \times \omega}$  in Fig. 7. Observe that the enumerated points can be organized in (horizontal) *lines*:  $line_1 = (1, 2)$ ,  $line_2 = (3, 4, 5)$ ,  $line_3 = (6, 7, 8, 9)$ , and so on. Consider the following properties of this enumeration (different from the ones listed as (a.1)–(a.5) in the proof of Theorem 4.3 above):

- (b.1)  $start\_of(line_1) = 1$ , and  $up\_neighbour\_of(0) = 1$ .
- (b.2)  $start\_of(line_{i+1}) = end\_of(line_i) + 1$ , for all  $i > 0$ .
- (b.3) Every line starts with some  $n$  on the wall and ends with some  $m$  on the diagonal.
- (b.4) If  $n$  is in  $line_i$ , then  $up\_neighbour\_of(n)$  is in  $line_{i+1}$ , for all  $i$ .
- (b.5) For every  $m, n$ , if  $m < n$  then  $up\_neighbour\_of(m) < up\_neighbour\_of(n)$ .
- (b.6) For every  $n > 0$  on the wall, there is  $m$  with  $up\_neighbour\_of(m) = n$ .
- (b.7) For every  $n$ , if  $n$  is neither on the wall nor on the diagonal, then there is  $m$  with  $up\_neighbour\_of(m) = n$ .

Observe that (b.1) and (b.2) imply that every  $n$  in the enumeration belongs to  $line_i$  for some  $i$ . Also, by (b.2) and (b.3), for every  $i$  there is a unique  $m$  in  $line_i$  that is on the diagonal (its last according to the enumeration). As  $up\_neighbour\_of$  is an injective function, by (b.4) we have that

$$\text{number of points in } line_i \leq \text{number of points in } line_{i+1}.$$

Further, by (b.4), (b.6) and (b.7),

$$\text{number of non-diagonal points in } line_{i+1} \leq \text{number of points in } line_i.$$

Therefore,

$$\text{length\_of}(line_{i+1}) = \text{length\_of}(line_i) + 1 \text{ for all } i.$$

Finally, by (b.4) and (b.5) we obtain that  $line_i$  is what it should be in Fig. 7, and so we have:

CLAIM 4.14. *Properties (b.1)–(b.7) uniquely determine<sup>8</sup> the enumeration in Fig. 7.*

Given a unique infinite unit-sequence  $\mathcal{U} = (\langle u_n, u_{n+1} \rangle \mid n < \omega)$  in  $\mathfrak{M}$  as in Claim 4.5 above, we now encode (b.1)–(b.7) as follows. In addition to  $up$ ,  $diag$ , and  $now$ , we will also use a variable  $wall$  to mark those unit-points that are on the wall, and a variable  $line$  to mark lines in the following sense:  $\mathfrak{M}, \langle x, y \rangle \models line$  iff  $x = u_m, y = u_n$  and  $(m + 1, \dots, n)$  is a line (see Fig. 11 for the intended placement of the variables).

To begin with, we express that  $up\_neighbour\_of$  is an injective function by

$$[U](up \rightarrow \neg(E)up \wedge \neg(\bar{B})up), \quad (60)$$

then we express (b.1) by the conjunction of

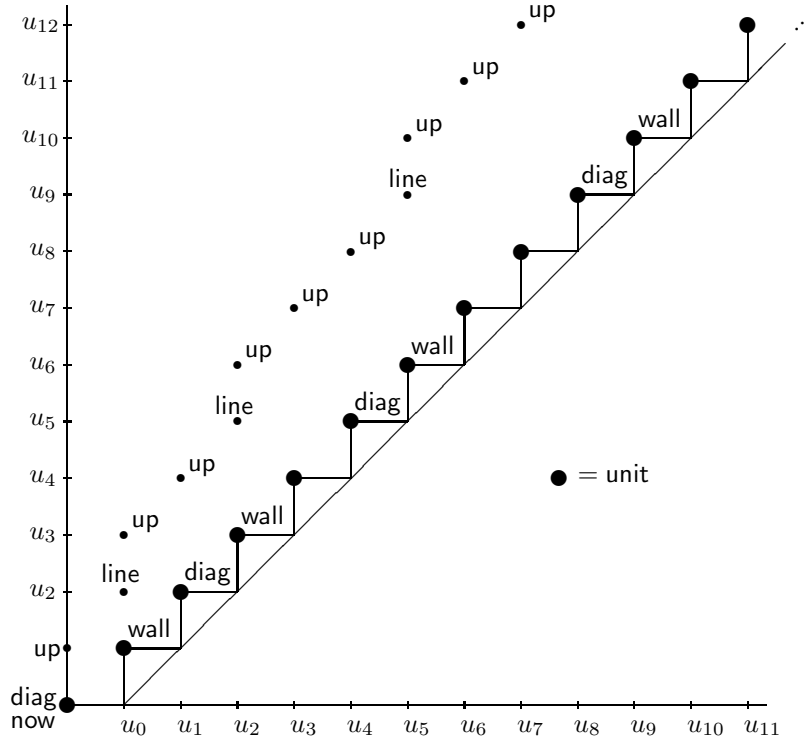
$$now \wedge \langle A \rangle line, \quad (61)$$

$$[U](up \rightarrow \neg(D)now), \quad (62)$$

(b.2) by

$$[U](line \rightarrow \langle A \rangle line), \quad (63)$$

<sup>8</sup>among those that contain the enumeration of the diagonal locations as  $(0, 0), \dots, (1, 1), \dots, (2, 2), \dots$

Fig. 11. Encoding the  $nw_{\omega} \times \omega$ -grid in an  $\mathcal{HS}$ -model: version 2.

(b.3) by the conjunction of

$$[U](\text{wall} \rightarrow \text{unit}), \quad (64)$$

$$[U](\text{diag} \rightarrow \text{unit}), \quad (65)$$

$$[U](\text{line} \rightarrow \langle E \rangle \text{diag} \wedge \langle B \rangle \text{wall}), \quad (66)$$

(b.4) by the conjunction of

$$[U](\text{unit} \rightarrow \langle \bar{B} \rangle \text{up}), \quad (67)$$

$$[U](\text{up} \rightarrow \langle E \rangle \text{unit} \wedge \langle B \rangle \text{unit}), \quad (68)$$

$$[U](\text{up} \rightarrow \neg \langle \bar{B} \rangle \text{line} \wedge \neg \langle D \rangle \text{line}), \quad (69)$$

(b.5) by

$$[U](\text{up} \rightarrow \neg \langle D \rangle \text{up}), \quad (70)$$

(b.6) by

$$[U](\text{wall} \rightarrow \langle \bar{E} \rangle \text{up}). \quad (71)$$

Finally, we can express (b.7) by

$$[\langle \bar{D} \rangle \text{line} \wedge \text{unit} \Rightarrow_H \langle A \rangle \langle \bar{A} \rangle \text{up}], \quad (72)$$

using the 'binary implication trick' introduced in Section 4.3.

Now it is not hard to show the following:

**CLAIM 4.15.** *Suppose  $\mathfrak{M}, \langle r, r' \rangle \models \phi_{enum} \wedge \phi_{grid}^{core}$ , where  $\phi_{grid}^{core}$  is a conjunction of (60)–(72). Then now, wall, diag, line, and up are properly placed (see Fig. 11).*

On the other hand, using Fig. 11 it is not hard to see that  $\phi_{grid}^{core}$  is satisfiable (using the irreflexive semantics) in an  $\mathcal{HS}$ -model that is based on some linear order  $\mathfrak{T}$  having an infinite ascending chain  $u_0 < u_1 < \dots$ . In particular, conjunct (72) is satisfiable because of the following:  $\langle A \rangle \langle \bar{A} \rangle \text{up}$  is clearly horizontally stable, and it is easy to check that for every  $x, n$  with  $\mathfrak{M}, \langle x, u_n \rangle \models \neg \langle A \rangle \langle \bar{A} \rangle \text{up}$ , we have  $\mathfrak{M}, \langle x, u_n \rangle \models \neg \langle \bar{D} \rangle \text{line}$ .

Given a Turing machine  $\mathcal{A}$ , consider the conjuncts (41)–(46) above, and observe that the only non- $\mathcal{HS}_{core}$  conjuncts among them are (45) for  $(x, y, z) \in W_{\mathcal{A}}$ . In order to replace these with  $\mathcal{HS}_{core}$ -formulas we introduce the following fresh propositional variables:

- $(y, z)$  and  $\overline{(y, z)}$ , for all  $y, z \in \Gamma_{\mathcal{A}}$ , and
- $(x, y, z)$  and  $\overline{(x, y, z)}$ , for all  $(x, y, z) \in W_{\mathcal{A}}$ .

Then we again use the ‘binary implication trick’ of Section 4.3 (and its ‘vertical’ version), and take the conjunction of the following formulas, for all  $y, z \in \Gamma_{\mathcal{A}}$  and all  $(x, y, z) \in W_{\mathcal{A}}$ :

$$\begin{aligned} & [\langle \bar{A} \rangle y \wedge z \Rightarrow_V \overline{(y, z)}], \\ & [\text{U}](\overline{(y, z)} \rightarrow \langle \bar{A} \rangle (y, z)), \\ & [\text{U}](\overline{(y, z)} \rightarrow \text{unit}), \\ & [\langle A \rangle (y, z) \wedge x \Rightarrow_H \overline{(x, y, z)}], \\ & [\text{U}](\overline{(x, y, z)} \rightarrow \langle A \rangle (x, y, z)), \\ & [\text{U}](\overline{(x, y, z)} \rightarrow \text{up} \wedge \langle E \rangle \tau_{\mathcal{A}}(x, y, z)). \end{aligned}$$

Fig. 12 shows the intended meaning of these formulas, and also how to satisfy them in the  $\mathcal{HS}$ -model  $\mathfrak{M}$  defined in (47).

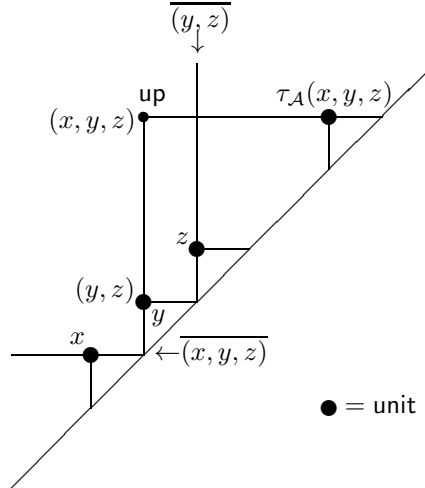


Fig. 12. Encoding formula (45) in  $\mathcal{HS}_{core}$ .

(ii) We reduce HALTING to  $\text{Fin}(<)$ -satisfiability. In order to achieve this, we introduce a fresh variable  $\text{end}$ , replace (46) with the conjunction of (58) and (59), and replace the ‘generating’ conjunct (27) of  $\phi_{enum}$  with

$$\text{unit} \wedge [\langle L \rangle \text{end} \wedge \text{unit} \Rightarrow_H \langle A \rangle \text{unit}], \quad (73)$$



using the binary implication trick.  $\square$

**THEOREM 4.5. ( $\mathcal{HS}_{horn}^\square$ , *discrete orders, irreflexive semantics*)**

(i) For any class  $\text{Dis}^\infty$  of discrete linear orders containing an infinite order,  $\text{Dis}^\infty(\langle \cdot \rangle)$ -satisfiability of  $\mathcal{HS}_{horn}^\square$ -formulas is undecidable. (ii)  $\text{Fin}(\langle \cdot \rangle)$ -satisfiability of  $\mathcal{HS}_{horn}^\square$ -formulas is undecidable.

**PROOF.** (i) We again reduce ‘non-halting’ to satisfiability, modifying the techniques employed in the proofs of Theorems 4.3 and 4.4. In both of these proofs, ‘positive’  $\langle R \rangle$ -operators are used for two purposes. First, they help to ‘generate’ an infinite unit-sequence; see formula (27). Second, they help to ‘generate’ appropriate pointers for the encoding of the  $nw_{\omega \times \omega}$ -grid via the enumeration in Fig. 7; see formulas `grid_succ $\rightarrow$` , `grid_succ $\uparrow$` , (63), (66)–(68), (71) and (72). Below, we show how to ‘mimic’ these features within  $\mathcal{HS}_{horn}^\square$ . Recall that formulas of the form  $[U](\varphi \rightarrow \neg[R]\perp)$  are within  $\mathcal{HS}_{horn}^\square$ .

Take any  $\mathcal{HS}$ -model  $\mathfrak{M}$  based on a discrete linear order  $\mathfrak{T}$ , and consider the irreflexive semantics of the interval relations. In case of these semantical choices, we can single out unit-intervals as follows. Let  $\phi_{enum}^\square$  denote the formula

$$[U](\text{unit} \rightarrow \neg[E]\perp \wedge [E][E]\perp) \wedge [U](\langle E \rangle[E]\perp \wedge [E][E]\perp \rightarrow \text{unit}).$$

It is not hard to see that if  $\mathfrak{M}$  satisfies  $\phi_{enum}^\square$  then, for all  $\langle x, x' \rangle$  in  $\text{int}(\mathfrak{T})$ , we have  $\mathfrak{M}, \langle x, x' \rangle \models \text{unit}$  iff  $x'$  is an immediate successor of  $x$  in  $\mathfrak{T}$ . (Note that this is not the same unit-sequence as in the proof of Theorem 4.2.) This unit-sequence has the useful property of having access to the ‘next’ and ‘previous’ unit-intervals with the  $A$  and  $\bar{A}$  interval relations, respectively.

The following *nw-next trick* will also be essential. For any finite conjunction  $\varphi$  of literals and any literal  $\lambda$ , we define the formula  $[\varphi \Rightarrow \odot \lambda]$  as the conjunction of

$$\begin{aligned} & [U](\varphi \rightarrow \lambda_\downarrow \wedge [B]\lambda_\downarrow \wedge [\bar{B}]\lambda_\uparrow), \\ & [U](\lambda_\uparrow \wedge [B]\lambda_\downarrow \rightarrow \lambda_*), \\ & [U](\lambda_* \rightarrow \lambda_\rightarrow \wedge [E]\lambda_\rightarrow \wedge [\bar{E}]\lambda_\leftarrow), \\ & [U](\lambda_\leftarrow \wedge [E]\lambda_\rightarrow \rightarrow \lambda), \end{aligned}$$

where  $\lambda_\downarrow, \lambda_\uparrow, \lambda_\rightarrow, \lambda_\leftarrow$  and  $\lambda_*$  are fresh variables. Now suppose  $u_0 < u_1 < \dots < u_n < \dots$  is an infinite sequence of subsequent points in  $\mathfrak{T}$ . (We will ‘force’ its existence with the formula (74) below.) It is easy to see the following:

**CLAIM 4.16.** *If  $\mathfrak{M} \models [\varphi \Rightarrow \odot \lambda]$  and  $\mathfrak{M}, \langle u_i, u_j \rangle \models \varphi$ , then  $\mathfrak{M}, \langle u_{i-1}, u_{j+1} \rangle \models \lambda$ .*

**Soundness:** Observe that in order to satisfy  $[\varphi \Rightarrow \odot \lambda]$  there are certain restrictions on  $\varphi$  and  $\lambda$ . For example, there is no problem whenever they are both ‘horizontally and vertically unique in  $\mathfrak{M}$ ’ in the following sense: If  $\mathfrak{M}, \langle x, y \rangle \models \varphi$  then  $\mathfrak{M}, \langle x', y \rangle \not\models \varphi$  and  $\mathfrak{M}, \langle x, y' \rangle \not\models \varphi$  for any  $x' \neq x, y' \neq y$  (and similarly for  $\lambda$ ).

Next, we force the proper placement of line- and up-pointers of the  $nw_{\omega \times \omega}$ -grid in Fig. 7 in a novel way, different from the ones in the proofs of Theorems 4.3 and 4.4. In representing this enumeration by our unit-sequence, each line will be followed by a ‘mirror’-unit, then by a ‘mirrored copy’ of the next line with its locations listed in reverse order, and then by a proper listing of the next line’s locations. In order to achieve this, we introduce the following fresh propositional variables:

- `grid_proper`, `wall` and `diag` (to mark those unit-intervals that represent line-locations and the respective wall- and diagonal-ends of each line);
- `grid_copy` (to mark unit-intervals representing the mirror-copies of proper line locations);

- up and mirror (to mark pointers helping to access the up-neighbour of each location);
- first\_mirror, last\_mirror and last\_up (to mark the beginning and end of each ‘north-west going’ mirror- and up-sequence, respectively).

See Fig. 13 for the intended placement of these variables, and for an example of how to access, say, grid-location (1, 4) from (1, 3), and (1, 3) from (1, 2) with the help of up- and mirror-pointers.

We force the proper placement of these variables by the conjunction  $\phi_{grid}^{\square}$  of the following formulas:

$$\begin{aligned}
& \text{init} \wedge [\text{init} \Rightarrow \textcircled{\times} \text{last\_up}], \\
& [\text{U}](\text{init} \rightarrow \text{unit} \wedge \text{wall}), \\
& [\text{U}](\text{unit} \wedge \langle \bar{\text{E}} \rangle \text{last\_up} \rightarrow \text{diag}), \\
& [\text{U}](\text{diag} \rightarrow [\text{A}](\text{unit} \rightarrow \text{first\_mirror})), \\
& [\text{first\_mirror} \Rightarrow \textcircled{\times} \text{mirror}], \\
& [\text{wall} \Rightarrow \textcircled{\times} \text{up}], \\
& [\text{U}](\text{unit} \wedge \langle \bar{\text{E}} \rangle \text{up} \rightarrow \text{grid\_proper}), \\
& [\text{mirror} \wedge \langle \text{B} \rangle \text{grid\_proper} \Rightarrow \textcircled{\times} \text{mirror}], \\
& [\text{U}](\text{mirror} \wedge \langle \text{B} \rangle \text{wall} \rightarrow \text{last\_mirror}), \\
& [\text{U}](\text{unit} \wedge \langle \bar{\text{E}} \rangle \text{mirror} \rightarrow \text{grid\_copy}), \\
& [\text{U}](\text{unit} \wedge \langle \bar{\text{E}} \rangle \text{last\_mirror} \rightarrow \text{wall}), \\
& [\text{up} \wedge \langle \text{B} \rangle \text{grid\_copy} \Rightarrow \textcircled{\times} \text{up}], \\
& [\text{U}](\text{up} \wedge \langle \text{B} \rangle \text{first\_mirror} \rightarrow \text{last\_up}).
\end{aligned}$$

Then it is not hard to show the following:

**CLAIM 4.17.** *If  $\mathfrak{M}, \langle u_0, u_1 \rangle \models \phi_{enum}^{\square} \wedge \phi_{grid}^{\square}$ , then all variables are placed as in Fig. 13.*

Finally, given a Turing machine  $\mathcal{A}$ , we again place the subsequent configurations of its computation with empty input on the subsequent lines of the  $nw_{\omega \times \omega}$ -grid (see Fig. 9), using the function  $\tau_{\mathcal{A}}$  defined in (26). We define the formula  $\Psi_{\mathcal{A}}^{\square}$  as follows. First, we ensure that there are infinitely many unit-intervals with

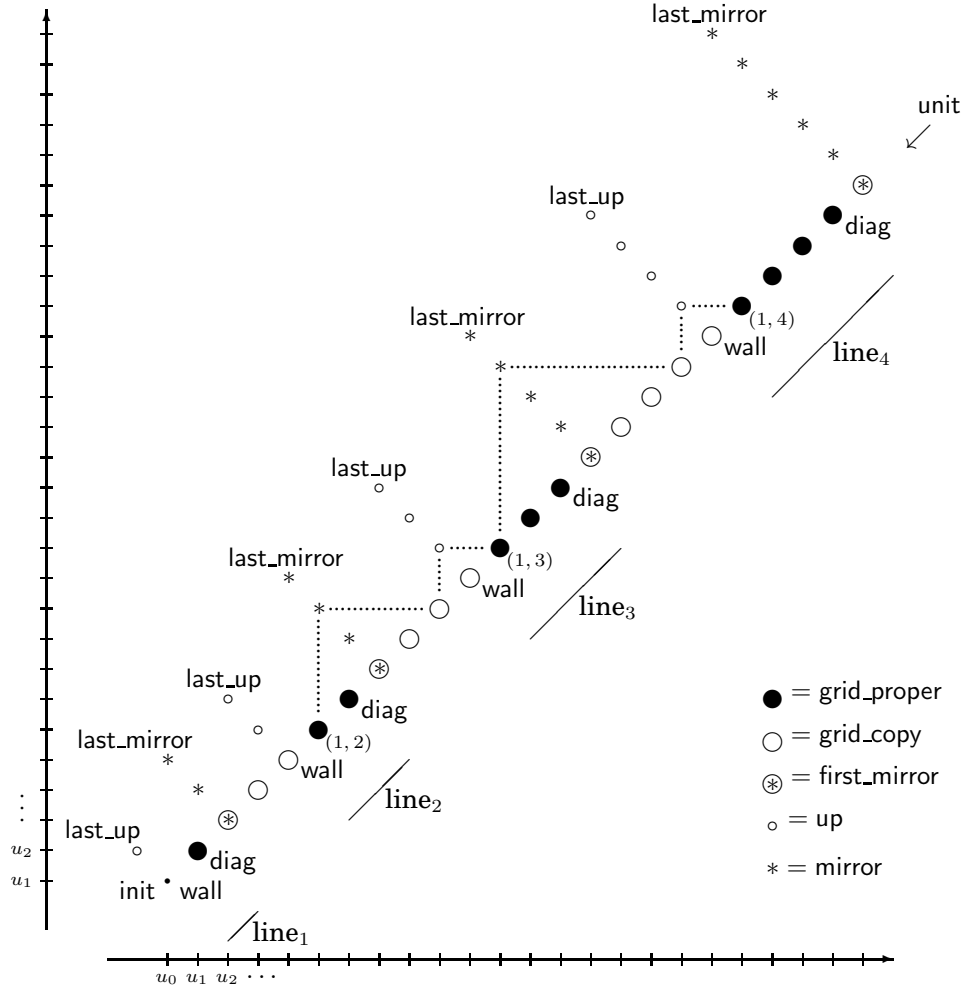
$$[\text{U}](\text{unit} \wedge x \rightarrow \neg[\bar{\text{B}}]\perp), \quad \text{for } x \in \Sigma \cup \{\mathcal{L}\} \cup ((Q - \{q_f\}) \times (\Sigma \cup \{\mathcal{L}\})). \quad (74)$$

Next, we take the general constraints (41) and (42), then initialize the computation with

$$[\text{U}](\text{init} \rightarrow (q_0, \mathcal{L})),$$

and then force the computation steps with the conjunction of (44) and

$$\begin{aligned}
& [\text{U}](\text{first\_mirror} \rightarrow \mathcal{L}), \\
& [\text{U}]\left(\text{grid\_proper} \wedge y \wedge \langle \text{A} \rangle z \wedge \langle \bar{\text{A}} \rangle x \rightarrow [\bar{\text{B}}](\text{mirror} \rightarrow [\text{E}](\text{unit} \rightarrow \tau_{\mathcal{A}}(x, y, z)))\right), \\
& \hspace{20em} \text{for } (x, y, z) \in W_{\mathcal{A}}, \\
& [\text{U}]\left(\text{wall} \wedge y \wedge \langle \text{A} \rangle z \rightarrow [\bar{\text{B}}](\text{mirror} \rightarrow [\text{E}](\text{unit} \rightarrow \tau_{\mathcal{A}}(\perp, y, z)))\right), \quad \text{for } (\perp, y, z) \in W_{\mathcal{A}}, \\
& [\text{U}]\left(\text{grid\_copy} \wedge \langle \bar{\text{B}} \rangle \text{up} \wedge x \rightarrow [\bar{\text{B}}](\text{up} \rightarrow [\text{E}](\text{unit} \rightarrow x))\right), \quad \text{for } x \in \Gamma_{\mathcal{A}}.
\end{aligned}$$

Fig. 13. Encoding the  $nw_\omega \times \omega$ -grid in an  $\mathcal{HS}$ -model: version 3.

Then we force non-halting with (46). Using Claim 4.17, now it is straightforward to prove the following:

**CLAIM 4.18.** *If  $\Psi_{\mathcal{A}}^\square$  is satisfiable in an  $\mathcal{HS}$ -model based on a discrete linear order, then  $\mathcal{A}$  diverges with empty input.*

On the other hand, using Fig. 13 it is not hard to see that  $\phi_{enum}^\square \wedge \phi_{grid}^\square$  is satisfiable (using the irreflexive semantics) in an  $\mathcal{HS}$ -model that is based on some discrete linear order  $\mathfrak{T}$  having an infinite ascending chain  $u_0 < u_1 < \dots$  of subsequent points. If  $\mathcal{A}$  diverges with empty input, then it is not hard to modify the  $\mathcal{HS}$ -model  $\mathfrak{M}$  given in (47) to obtain a model satisfying  $\Psi_{\mathcal{A}}^\square$ . The case when  $\mathfrak{T}$  contains an infinite descending chain of immediate predecessor points requires ‘symmetrical versions’ of the used formulas and is left to the reader.

(ii) We reduce ‘halting’ to  $\text{Fin}(<)$ -satisfiability. In order to achieve this, we omit (46). This completes the proof of the theorem.  $\square$

## 5. CONCLUSIONS AND OPEN PROBLEMS

Our motivation for introducing the Horn fragments of  $\mathcal{HS}$  and investigating their computational behaviour comes from two sources. The first one is applications for ontology-based access to temporal data, where an ontology provides definitions of complex temporal predicates that can be employed in user queries. Atemporal ontology-based data access (OBDA) [Poggi et al. 2008] with Horn description logics and profiles of *OWL2* is now paving its way to industry [Kharlamov et al. 2015], supported by OBDA systems such as Stardog [Pérez-Urbina et al. 2012], Ultrawrap [Sequeda et al. 2014], and the Optique platform [Giese et al. 2015; Rodriguez-Muro et al. 2013; Kontchakov et al. 2014]. However, OBDA ontology languages were not designed for applications with *temporal* data (sensor measurements, historical records, video or audio annotations, etc.). That the datalog extension of (multi-dimensional)  $\mathcal{HS}_{horn}^{\square}$  is sufficiently expressive for defining useful temporal predicates over historical and sensor data was shown by Kontchakov et al. [2016], who also demonstrated experimentally the efficiency of  $\mathcal{HS}_{horn}^{\square}$  for query answering. We briefly discussed these applications in Section 3.1. (Other temporal ontology languages have been developed based on Horn fragments of the linear temporal logic LTL [Artale et al. 2015a; Gutiérrez-Basulto et al. 2015; Gutiérrez-Basulto et al. 2016a], computational tree logic CTL [Gutiérrez-Basulto et al. 2014], and metric temporal logic MTL [Gutiérrez-Basulto et al. 2016b; Brandt et al. 2017].)

Our second motivation originates in multi-dimensional modal logic [Gabbay et al. 2003; Kurucz 2007]. Its formalisms try to capture the interactions between modal operators representing time, space, knowledge, actions, etc., and are closely connected not only to  $\mathcal{HS}$  but also to finite variable fragments of various kinds of predicate logics (as first-order quantifiers can be regarded as propositional modal operators over interacting universal relations). While the satisfiability problem of the two-variable fragment of classical predicate logic is NEXPTIME-complete [Grädel et al. 1997], taming even two-dimensional propositional modal logics over interacting transitive but not equivalence relations by designing their interesting fragments turned out to be a difficult task. Introducing syntactical restrictions (guards, monodicity) [Hodkinson 2006; Degtyarev et al. 2006; Hodkinson et al. 2000; Hodkinson et al. 2002; Hodkinson et al. 2003] and/or modifying the semantics by allowing various subsets of product-like domains [Gabelaia et al. 2005a; Gabelaia et al. 2006; Hampson and Kurucz 2015] or restricting the available valuations [Göller et al. 2015] might result in decidable logics that are still very complex, ranging from EXPSpace to non-primitive recursive. In this context, it would be interesting to see whether Horn fragments of multi-dimensional modal formalisms exhibit more acceptable computational properties. Here, we make a step in this direction.

This paper has launched an investigation of Horn fragments of the Halpern-Shoham interval temporal logic  $\mathcal{HS}$ , which provides a powerful framework for temporal representation and reasoning on the one hand, but is notorious for its nasty computational behaviour on the other. We classified the Horn fragments of  $\mathcal{HS}$  along the four axes:

- the type of interval modal operators available in the fragment: boxes  $[R]$  or diamonds  $\langle R \rangle$ , or both;
- the type of the underlying timelines: discrete or dense linear orders;
- the type of semantics for the interval relations: reflexive or irreflexive; and
- the number of literals in Horn clauses: two in the *core* fragment or more.

Both positive and negative results were obtained. The most unexpected negative results are the undecidability of (i)  $\mathcal{HS}_{core}$  with both box and diamond operators under the irreflexive semantics, and of (ii)  $\mathcal{HS}_{horn}^{\square}$  over discrete orders under the irreflexive semantics. Compared with (i) and (ii), the ubiquitous undecidability of  $\mathcal{HS}_{horn}^{\diamond}$  might look like a natural feature. Fortunately, we have also managed to identify a ‘chink in  $\mathcal{HS}$ ’s armour’ by proving that  $\mathcal{HS}_{horn}^{\square}$  turns out to be *tractable* (P-complete) over both discrete and dense orders under the reflexive semantics and over dense orders under the irreflexive semantics. First applications of the  $\mathcal{HS}_{horn}^{\square}$  fragment to ontology-based data access over temporal databases or streamed data have been found by Kontchakov et al. [2016].

Recently, Wałęga [2017] has considered a hybrid version of  $\mathcal{HS}_{horn}^{\square}$  (with nominals and the @-operator) and proved that it is NP-complete over discrete and dense orders under the reflexive semantics and over dense orders under the irreflexive semantics.

In order to prove the undecidability results mentioned above as well as PSPACE-hardness of  $\mathcal{HS}_{core}$  under the reflexive semantics and of  $\mathcal{HS}_{core}^{\square}$  over discrete orders under the irreflexive semantics, we developed a powerful toolkit that utilises the 2D character of  $\mathcal{HS}$  and builds on various techniques and tricks from many-dimensional modal logic. However, we still do not completely understand the computational properties of the core fragment of  $\mathcal{HS}$ , leaving the following questions open:

**QUESTION 1.** *Are  $\mathcal{HS}_{core}$  and  $\mathcal{HS}_{core}^{\diamond}$  decidable over any unbounded class of timelines under the reflexive semantics? What is the computational complexity?*

**QUESTION 2.** *Is  $\mathcal{HS}_{core}^{\square}$  decidable over any unbounded class of discrete timelines under the irreflexive semantics? What is the computational complexity?*

In our Horn- $\mathcal{HS}$  logics, we did not restrict the set of available interval relations, which used to be one of the ways of obtaining decidable fragments. Classifying Horn fragments of  $\mathcal{HS}$  along this axis can be an interesting direction for further research in the area. Syntactically, all of our Horn- $\mathcal{HS}$  logics are different. However, we do not know whether they are distinct in terms of their expressive power. Establishing an expressivity hierarchy of Horn fragments of  $\mathcal{HS}$  (taking into account different semantical choices) can also be an interesting research question.

## ACKNOWLEDGMENTS

We are grateful to Przemysław Wałęga for spotting and correcting a mistake in the preliminary version of the proof of Theorem 4.2. Thanks are also due to the anonymous reviewers for their useful suggestions.

## REFERENCES

- ADORNI, G., MARATEA, M., PANDOLFO, L., AND PULINA, L. 2015. An ontology for historical research documents. In *Proc. of the 9th Int. Conf. on Web Reasoning and Rule Systems (RR 2015)*. LNCS Series, vol. 9209. Springer, 11–18.
- ALLEN, J. F. 1983. Maintaining knowledge about temporal intervals. *Commun. ACM* 26, 11, 832–843.
- ALLEN, J. F. 1984. Towards a general theory of action and time. *Artif. Intell.* 23, 2, 123–154.
- ARTALE, A., KONTCHAKOV, R., KOVTUNOVA, A., RYZHIKOV, V., WOLTER, F., AND ZAKHARYASCHEV, M. 2015a. First-order rewritability of temporal ontology-mediated queries. In *Proc. of the 24th Int. Joint Conf. on Artificial Intelligence (IJCAI 2015)*. AAAI Press, 2706–2712.
- ARTALE, A., KONTCHAKOV, R., LUTZ, C., WOLTER, F., AND ZAKHARYASCHEV, M. 2007. Temporalising tractable description logics. In *Proc. of the 20th Int. Symp. on Temporal Representation and Reasoning (TIME 2013)*. IEEE, 11–22.
- ARTALE, A., KONTCHAKOV, R., RYZHIKOV, V., AND ZAKHARYASCHEV, M. 2013. The complexity of clausal fragments of LTL. In *Proc. of the 19th Int. Conf. on Logic for Programming, Artificial Intelligence, and Reasoning (LPAR)*. LNCS Series, vol. 8312. Springer, 35–52.

- ARTALE, A., KONTCHAKOV, R., RYZHIKOV, V., AND ZAKHARYASCHEV, M. 2014. A cookbook for temporal conceptual data modelling with description logics. *ACM Trans. Comput. Log.* 15, 3, 25:1–25:50.
- ARTALE, A., KONTCHAKOV, R., RYZHIKOV, V., AND ZAKHARYASCHEV, M. 2015b. Tractable interval temporal propositional and description logics. In *Proc. of the 29th Conf. on Artificial Intelligence (AAAI 2015)*. AAAI Press, 1417–1423.
- BALBIANI, P., CONDOTTA, J., AND DEL CERRO, L. F. 2002. Tractability results in the block algebra. *J. Log. Comput.* 12, 5, 885–909.
- BÖHLEN, M. H., SNODGRASS, R. T., AND SOO, M. D. 1996. Coalescing in temporal databases. In *Proc. of the 22nd Int. Conf. on Very Large Data Bases (VLDB'96)*. Morgan Kaufmann, 180–191.
- BRANDT, S., KALAYCI, E. G., KONTCHAKOV, R., RYZHIKOV, V., XIAO, G., AND ZAKHARYASCHEV, M. 2017. Ontology-based data access with a Horn fragment of metric temporal logic. In *Proc. of the 31st Conf. on Artificial Intelligence (AAAI 2017)*. AAAI Press, 1070–1076.
- BRESOLIN, D., DELLA MONICA, D., GORANKO, V., MONTANARI, A., AND SCIAVICCO, G. 2008. Decidable and undecidable fragments of Halpern and Shoham's interval temporal logic: towards a complete classification. In *Proc. of the 15th Int. Conf. on Logic for Programming, Artificial Intelligence, and Reasoning (LPAR)*. LNCS Series, vol. 5330. Springer, 590–604.
- BRESOLIN, D., DELLA MONICA, D., MONTANARI, A., SALA, P., AND SCIAVICCO, G. 2012a. Interval temporal logics over finite linear orders: the complete picture. In *Proc. of the 20th European Conf. on Artificial Intelligence (ECAI)*. Frontiers in Artificial Intelligence and Applications Series, vol. 242. IOS Press, 199–204.
- BRESOLIN, D., DELLA MONICA, D., MONTANARI, A., SALA, P., AND SCIAVICCO, G. 2012b. Interval temporal logics over strongly discrete linear orders: the complete picture. In *Proc. of the 3rd Int. Symp. on Games, Automata, Logics, and Formal Verification (GANDALF 2012)*. EPTCS Series, vol. 96. 155–168.
- BRESOLIN, D., DELLA MONICA, D., MONTANARI, A., SALA, P., AND SCIAVICCO, G. 2015. On the complexity of fragments of the modal logic of Allen's relations over dense structures. In *Proc. of the 9th Int. Conf. on Language and Automata Theory and Applications (LATA)*. LNCS Series, vol. 8977. Springer, 511 – 523.
- BRESOLIN, D., DELLA MONICA, D., MONTANARI, A., AND SCIAVICCO, G. 2014a. The light side of interval temporal logic: the Bernays-Schönfinkel fragment of CDT. *Ann. Math. Artif. Intell.* 71, 1-3, 11–39.
- BRESOLIN, D., MUÑOZ-VELASCO, E., AND SCIAVICCO, G. 2014b. Sub-propositional fragments of the interval temporal logic of Allen's relations. In *Proc. of the 14th European Conference on Logics in Artificial Intelligence (JELIA 2014)*. LNCS Series, vol. 8761. Springer, 122–136.
- CAU, A., HALE, R., DIMITROV, J., ZEDAN, H., MOSZKOWSKI, B. C., MANJUNATHAIAH, M., AND SPIVEY, M. 2002. A compositional framework for hardware/software co-design. *Design Autom. for Emb. Sys.* 6, 4, 367–399.
- CHAGROV, A. AND ZAKHARYASCHEV, M. 1997. *Modal Logic*. Clarendon Press, Oxford.
- CHEN, C.-C. AND LIN, I.-P. 1993. The computational complexity of satisfiability of temporal Horn formulas in propositional linear-time temporal logic. *Information Processing Letters* 45, 3, 131–136.
- CHEN, C.-C. AND LIN, I.-P. 1994. The computational complexity of the satisfiability of modal Horn clauses for modal propositional logics. *Theor. Comput. Sci.* 129, 1, 95–121.
- CIMATTI, A., ROVERI, M., AND TONETTA, S. 2015. HRETL: A temporal logic for hybrid systems. *Inf. Comput.* 245, 54–71.
- COHN, A. G., LI, S., LIU, W., AND RENZ, J. 2014. Reasoning about topological and cardinal direction relations between 2-dimensional spatial objects. *J. Artif. Intell. Res. (JAIR)* 51, 493–532.
- DANTSIN, E., EITER, T., GOTTLÖB, G., AND VORONKOV, A. 2001. Complexity and expressive power of logic programming. *ACM Comput. Surv.* 33, 3, 374–425.
- DEGTYAREV, A., FISHER, M., AND KONEV, B. 2006. Monodic temporal resolution. *ACM Trans. Comput. Log.* 7, 108–150.
- DELLA MONICA, D., GORANKO, V., MONTANARI, A., AND SCIAVICCO, G. 2011. Interval temporal logics: a journey. *Bulletin of EATCS* 105, 73 – 99.
- FARIÑAS DEL CERRO, L. AND PENTTONEN, M. 1987. A note on the complexity of the satisfiability of modal Horn clauses. *J. Logic Programming* 4, 1, 1–10.
- GABBAY, D., KURUCZ, A., WOLTER, F., AND ZAKHARYASCHEV, M. 2003. *Many-Dimensional Modal Logics: Theory and Applications*. Studies in Logic and the Foundations of Mathematics Series, vol. 148. Elsevier.
- GABELAIA, D., KONTCHAKOV, R., KURUCZ, A., WOLTER, F., AND ZAKHARYASCHEV, M. 2005a. Combining spatial and temporal logics: expressiveness vs. complexity. *J. Artif. Intell. Res. (JAIR)* 23, 167–243.
- GABELAIA, D., KURUCZ, A., WOLTER, F., AND ZAKHARYASCHEV, M. 2005b. Products of 'transitive' modal logics. *J. Symbolic Logic* 70, 993–1021.

- GABELAIA, D., KURUCZ, A., WOLTER, F., AND ZAKHARYASCHEV, M. 2006. Non-primitive recursive decidability of products of modal logics with expanding domains. *Ann. Pure Appl. Logic* 142, 245–268.
- GIESE, M., SOYLU, A., VEGA-GORGOJO, G., WAALER, A., HAASE, P., JIMÉNEZ-RUIZ, E., LANTI, D., REZK, M., XIAO, G., ÖZÇEP, Ö., AND ROSATI, R. 2015. Optique: Zooming in on big data. *IEEE Computer* 48, 3, 60–67.
- GÖLLER, S., JUNG, J., AND LOHREY, M. 2015. The complexity of decomposing modal and first-order theories. *ACM Trans. Comput. Log.* 16, 9:1–9:43.
- GOLUMBIC, M. C. AND SHAMIR, R. 1993. Complexity and algorithms for reasoning about time: A graph-theoretic approach. *J. ACM* 40, 5, 1108–1133.
- GORANKO, V. AND OTTO, M. 2006. Model theory of modal logic. In *Handbook of Modal Logic*, P. Blackburn, J. van Benthem, and F. Wolter, Eds. Studies in Logic and Practical Reasoning Series, vol. 3. Elsevier, 255–325.
- GRÄDEL, E., KOLAITIS, P. G., AND VARDI, M. Y. 1997. On the decision problem for two-variable first-order logic. *Bull. Symbolic Logic* 3, 1, 53–69.
- GUTIÉRREZ-BASULTO, V., JUNG, J. C., AND KONTCHAKOV, R. 2016a. Temporalized EL ontologies for accessing temporal data: Complexity of atomic queries. In *Proc. of the 25th Int. Joint Conf. on Artificial Intelligence, (IJCAI 2016)*. IJCAI/AAAI Press, 1102–1108.
- GUTIÉRREZ-BASULTO, V., JUNG, J. C., AND OZAKI, A. 2016b. On metric temporal description logics. In *Proc. of the 22nd European Conference on Artificial Intelligence (ECAI 2016) - Including Prestigious Applications of Artificial Intelligence (PAIS 2016)*. Frontiers in Artificial Intelligence and Applications Series, vol. 285. IOS Press, 837–845.
- GUTIÉRREZ-BASULTO, V., JUNG, J. C., AND SCHNEIDER, T. 2014. Lightweight description logics and branching time: A troublesome marriage. In *Proc. of the 14th Int. Conf. on the Principles of Knowledge Representation and Reasoning (KR 2014)*. AAAI Press.
- GUTIÉRREZ-BASULTO, V., JUNG, J. C., AND SCHNEIDER, T. 2015. Lightweight temporal description logics with rigid roles and restricted Tboxes. In *Proc. of the 24th Int. Joint Conf. on Artificial Intelligence, (IJCAI 2015)*. AAAI Press, 3015–3021.
- HALPERN, J. AND SHOHAM, Y. 1991. A propositional modal logic of time intervals. *J. ACM* 38, 4, 935–962.
- HAMPSON, C. AND KURUCZ, A. 2015. Undecidable propositional bimodal logics and one-variable first-order linear temporal logics with counting. *ACM Trans. Comput. Log.* 16, 3, 27:1–27:36.
- HODKINSON, I. 2006. Complexity of monodic guarded fragments over linear and real time. *Ann. Pure Appl. Logic* 138, 94–125.
- HODKINSON, I., KONTCHAKOV, R., KURUCZ, A., WOLTER, F., AND ZAKHARYASCHEV, M. 2003. On the computational complexity of decidable fragments of first-order linear temporal logics. In *Proc. of the 10th Int. Symp. on Temporal Representation and Reasoning and 4th Int. Conf. on Temporal Logic (TIME ICTL 2003)*. IEEE, 91–98.
- HODKINSON, I., WOLTER, F., AND ZAKHARYASCHEV, M. 2000. Decidable fragments of first-order temporal logics. *Ann. Pure Appl. Logic* 106, 85–134.
- HODKINSON, I., WOLTER, F., AND ZAKHARYASCHEV, M. 2002. Decidable and undecidable fragments of first-order branching temporal logics. In *Procs. of 17th Annual IEEE Symp. on Logic in Comp. Sci. (LICS 2002)*. IEEE, 393–402.
- HUSTADT, U., MOTIK, B., AND SATTLER, U. 2007. Reasoning in description logics by a reduction to disjunctive datalog. *J. Autom. Reasoning* 39, 3, 351–384.
- KHARLAMOV, E., HOVLAND, D., JIMÉNEZ-RUIZ, E., LANTI, D., LIE, H., PINKEL, C., REZK, M., SKJÆVELAND, M. G., THORSTENSEN, E., XIAO, G., ZHELEZNYAKOV, D., AND HORROCKS, I. 2015. Ontology based access to exploration data at Statoil. In *Proc. of the 14th Int. Semantic Web Conf. (ISWC 2015), Part II*. LNCS Series, vol. 9367. Springer, 93–112.
- KONTCHAKOV, R., PANDOLFO, L., PULINA, L., RYZHIKOV, V., AND ZAKHARYASCHEV, M. 2016. Temporal and spatial OBDA with many-dimensional Halpern-Shoham logic. In *Proc. of the 25th Int. Joint Conf. on Artificial Intelligence (IJCAI 2016)*. IJCAI/AAAI Press.
- KONTCHAKOV, R., REZK, M., RODRIGUEZ-MURO, M., XIAO, G., AND ZAKHARYASCHEV, M. 2014. Answering SPARQL queries over databases under OWL 2 QL entailment regime. In *Proc. of the 13th Int. Semantic Web Conf. (ISWC 2014), Part I*. LNCS Series, vol. 8796. Springer, 552–567.
- KRÖTZSCH, M., RUDOLPH, S., AND HITZLER, P. 2013. Complexities of Horn description logics. *ACM Trans. Comput. Log.* 14, 1, 2:1–2:36.
- KULKARNI, K. G. AND MICHELS, J. 2012. Temporal features in SQL: 2011. *SIGMOD Record* 41, 3, 34–43.
- KURUCZ, A. 2007. Combining modal logics. In *Handbook of Modal Logic*, P. Blackburn, J. van Benthem, and F. Wolter, Eds. Studies in Logic and Practical Reasoning Series, vol. 3. Elsevier, 869–924.

- LODAYA, K. 2000. Sharpening the undecidability of interval temporal logic. In *Proc. of the 6th Asian Comp. Sci. Conf. on Advances in Comp. Sci.* LNCS Series, vol. 1961. Springer, 290–298.
- LUTZ, C., WOLTER, F., AND ZAKHARYASCHEV, M. 2008. Temporal description logics: A survey. In *Proc. of the 15th Int. Symp. on Temporal Representation and Reasoning (TIME 2008)*. IEEE, 3–14.
- MARCINKOWSKI, J. AND MICHALISZYN, J. 2014. The undecidability of the logic of subintervals. *Fundam. Inform.* 131, 2, 217–240.
- MARX, M. AND REYNOLDS, M. 1999. Undecidability of compass logic. *J. Logic Comput.* 9, 6, 897–914.
- MONTANARI, A., PRATT-HARTMANN, I., AND SALA, P. 2010a. Decidability of the logics of the reflexive sub-interval and super-interval relations over finite linear orders. In *Proc. of the 17th Int. Symp. on Temporal Representation and Reasoning (TIME 2010)*. IEEE, 27–34.
- MONTANARI, A., PUPPIS, G., AND SALA, P. 2010b. Maximal decidable fragments of Halpern and Shoham’s modal logic of intervals. In *Proc. of the 37th Int. Coll. on Automata, Languages and Programming (ICALP), Part II*. LNCS Series, vol. 6199. Springer, 345–356.
- MONTANARI, A., SCIAVICCO, G., AND VITACOLONNA, N. 2002. Decidability of interval temporal logics over split-frames via granularity. In *Proc. of the 8th European Conf. on Logics in Artificial Intelligence (JELIA)*. LNAI Series, vol. 2424. Springer, 259–270.
- MORET, B. M. 1998. *The Theory of Computation*. Addison-Wesley.
- MUÑOZ-VELASCO, E., PELEGRÍN-GARCÍA, M., SALA, P., AND SCIAVICCO, G. 2015. On coarser interval temporal logics and their satisfiability problem. In *Proc. of the 16th Conf. of the Spanish Association for Artificial Intelligence (CAEPIA)*. LNCS Series, vol. 9422. Springer, 105–115.
- NAVARRETE, I., MORALES, A., SCIAVICCO, G., AND VIEDMA, M. A. C. 2013. Spatial reasoning with rectangular cardinal relations - the convex tractable subalgebra. *Ann. Math. Artif. Intell.* 67, 1, 31–70.
- NGUYEN, L. 2005. On the complexity of fragments of modal logics. In *Advances in Modal Logic, Vol.5*, R. Schmidt, I. Pratt-Hartmann, M. Reynolds, and H. Wansing, Eds. King’s College Publications, 249–268.
- PÉREZ-URBINA, H., RODRÍGUEZ-DÍAZ, E., GROVE, M., KONSTANTINIDIS, G., AND SIRIN, E. 2012. Evaluation of query rewriting approaches for OWL 2. In *Proc. of Joint Workshop on Scalable and High-Performance Semantic Web Systems (SSWS+HPCSW 2012)*. CEUR-WS Series, vol. 943.
- POGGI, A., LEMBO, D., CALVANESE, D., DE GIACOMO, G., LENZERINI, M., AND ROSATI, R. 2008. Linking data to ontologies. *J. Data Semantics X*, 133–173.
- PRATT-HARTMANN, I. 2005. Temporal prepositions and their logic. *Artif. Intell.* 166, 1-2, 1–36.
- REYNOLDS, M. AND ZAKHARYASCHEV, M. 2001. On the products of linear modal logics. *J. Logic Comput.* 11, 6, 909–931.
- RODRIGUEZ-MURO, M., KONTCHAKOV, R., AND ZAKHARYASCHEV, M. 2013. Ontology-based data access: Ontop of databases. In *Proc. of the 12th Int. Semantic Web Conf. (ISWC 2013)*. LNCS Series, vol. 8218. Springer, 558–573.
- SCHWENTICK, T. AND ZEUME, T. 2010. Two-variable logic with two order relations. In *Computer Science Logic*, A. Dawar and H. Veith, Eds. LNCS Series, vol. 6247. Springer, 499–513.
- SEQUEDA, J. F., ARENAS, M., AND MIRANKER, D. P. 2014. OBDA: query rewriting or materialization? in practice, both! In *Proc. of the 13th Int. Semantic Web Conf. (ISWC 2014), Part I*. LNCS Series, vol. 8796. Springer, 535–551.
- SPAAN, E. 1993. Complexity of modal logics. Ph.D. thesis, University of Amsterdam.
- TERENZIANI, P. AND SNODGRASS, R. T. 2004. Reconciling point-based and interval-based semantics in temporal relational databases: A treatment of the Telic/Atelic distinction. *IEEE Trans. Knowl. Data Eng.* 16, 5, 540–551.
- VENEMA, Y. 1990. Expressiveness and completeness of an interval tense logic. *Notre Dame J. Formal Logic* 31, 4, 529–547.
- VENEMA, Y. 1991. A modal logic for chopping intervals. *J. Logic Comput.* 1, 4, 453–476.
- W3C OWL WORKING GROUP. 2012. OWL 2 web ontology language document overview. <http://www.w3.org/TR/owl2-overview/>.
- WAŁĘGA, P. A. 2017. Computational complexity of a hybridized Horn fragment of Halpern-Shoham logic. In *Proc. of the 7th Indian Conf. on Logic and Its Applications (ICLA 2017)*. Springer, 224–238.
- ZHANG, P. AND RENZ, J. 2014. Qualitative spatial representation and reasoning in angry birds: The extended rectangle algebra. In *Proc. of the 14th Int. Conf. on the Principles of Knowledge Representation and Reasoning (KR 2014)*. AAAI Press.
- ZHOU, C. AND HANSEN, M. R. 2004. *Duration Calculus: A Formal Approach to Real-Time Systems*. EATCS: Monographs in Theoretical Computer Science. Springer.