Maximal Length Elements of Excess Zero in Finite Coxeter Groups

S.B. Hart and P.J. Rowley*

Abstract

Here we prove that for W a finite Coxeter group and C a conjugacy class of W, there is always an element of C of maximal length in C which has excess zero. An element $w \in W$ has excess zero if there exists elements $\sigma, \tau \in W$ such that $\sigma^2 = \tau^2 = 1, w = \sigma\tau$ and $\ell(w) = \ell(\sigma) + \ell(\tau)$, ℓ being the length function on W.
(MSC2000: 20F55; keywords: Coxeter group, length, conjugacy class, excess)

1 Introduction

Conjugacy classes of finite Coxeter groups have long been of interest, the correspondence between partitions and conjugacy classes of the symmetric groups having been observed by Cauchy [4] in the early days of group theory. For Coxeter groups of type B_n and D_n, descriptions of their conjugacy classes, by Specht [13] and Young [14], have also been known for a long time. In 1972, Carter [2] gave a uniform and systematic treatment of the conjugacy classes of Weyl groups. More recently, Geck and Pfeiffer [6] reworked Carter's descriptions from more of an algorithmic standpoint. Motivation for investigating the conjugacy classes of finite Coxeter groups, and principally those of the irreducible finite Coxeter groups, has come from many directions, for example in the representation theory of these groups and the classification of maximal tori in groups of Lie type (see [3]). The behaviour of length in a conjugacy class is frequently important. Of particular interest are those elements of minimal and maximal lengths in their class. Instrumental to Carter's work was establishing the fact that in a finite Coxeter group every element is either an involution or a product of two involutions. Given the importance of the length function, it is natural to ask whether for an element w it is possible to choose two involutions σ and τ with $w = \sigma\tau$ in such a way that combining a reduced expression for σ with one for τ produces a reduced expression for w. That is, can we ensure that the length $\ell(w)$ is given by $\ell(w) = \ell(\sigma) + \ell(\tau)$? Not surprisingly, the answer to this is, in general, no. This naturally leads to introducing the concept of *excess* of w, denoted by $e(w)$, and defined by

$$e(w) = \min\{\ell(\sigma) + \ell(\tau) - \ell(w) : \sigma\tau = w, \sigma^2 = \tau^2 = 1\}.$$

In [7], [8] and [9], various properties of excess were investigated. It was shown, among other things, that in every conjugacy class of a Coxeter group W there is an element of w of minimal length in the conjugacy class, such that the excess of

*The authors wish to acknowledge support for this work from a London Mathematical Society Research in Pairs grant.

w is zero [8, Theorem 1.1]. This raises the question as to whether there is also an element of maximal length and excess zero.

In this paper we address this question and show that elements of maximal length and excess zero do indeed exist.

Theorem 1.1. *Let W be a finite Coxeter group and C a conjugacy class of W. Then there exists an element w of maximal length in C such that $e(w) = 0$.*

In the course of proving this result we need a workable description of representatives of maximal length in conjugacy classes of Coxeter groups of types A_n, B_n and D_n. Minimal length elements in conjugacy classes of Coxeter groups have received considerable attention – see [6]. Now every finite Coxeter group W possesses a (unique) element w_0 of maximal length in W. For C a conjugacy class of W, set $C_0 = Cw_0 = \{ww_0 : w \in C\}$. If, as happens in many cases, $w_0 \in Z(W)$, then C_0 is also a conjugacy class of W. Moreover, $w \in C$ has minimal length in C if and only if ww_0 has maximal length in C_0. Thus information about maximal length elements in a conjugacy class may be obtained from that known about minimal length elements. Among the finite irreducible Coxeter groups, only those of type I_m (m odd), A_n, D_n (n odd) and E_6 have $w_0 \notin Z(W)$. The first of these, being just dihedral groups, are quickly dealt with. Descriptions of maximal length elements in conjugacy classes of type A_n were given by Kim [11] and for E_6 see Table III of [5]. In Section 3 of this paper we deal with type D_n (and in doing so give a result for type B_n at the same time). Representatives of maximal length for type D_n could be extracted from Section 4 of [5], but here we give a more direct treatment that deals with both type B_n and type D_n and gives more information about the number of long and short roots taken negative by elements of maximal length. Theorem 3.1 gives an expression for the maximal length of elements in a given conjugacy class for type D_n while Theorem 1.2 below gives a list of maximal length class representatives in types B_n and D_n, and this is what we require for our work on elements of excess zero.

Theorems 1.2 and 3.1 are consequences of a more general result, Theorem 3.6, concerning D-lengths and B-lengths of elements in a Coxeter group W of type B_n (D-length and B-length will be defined in Section 3). Suppose $\hat{W}$ is of type D_n. Then we may regard $\hat{W}$ as a canonical index 2 subgroup of W where W is a Coxeter group of type B_n. Let C be a conjugacy class of W that is contained in $\hat{W}$. In the case when n is odd, $w_0 \neq \hat{w}_0$ (the longest element of $\hat{W}$) and consequently $C_0 = Cw_0$ is not even a subset of $\hat{W}$, much less a conjugacy class of $\hat{W}$. However, working in the wider context of W, we are able to obtain elements of maximal D-length in C from suitable elements of minimal B-length in C_0. Therefore, in the course of establishing Theorem 3.1, we also produce representative elements of maximal length in their conjugacy class. To describe these elements, we will take for our group of type B_n the group of signed permutations; that is, permutations w of $\{1, \ldots, n, -1, \ldots, -n\}$ such that $w(-i) = -w(i)$ for $1 \leq i \leq n$. Signed permutations can be written as permutations where each number has either a plus or a minus sign above it. So, for example, if $w = (\overset{+}{1}\ \overset{-}{2}\ \overset{-}{3})$, then $w(1) = 2$, $w(-1) = -2$, $w(2) = -3$, $w(-2) = 3$, $w(3) = -1$ and $w(-3) = 1$. The set of signed permutations where an even number of minus signs appear is a subgroup which is of type D_n. Conjugacy classes in types B_n and D_n are parameterized by signed cycle type (this will be described fully in Section 3), with some classes splitting in type D_n.

For n a natural number, an ordered sequence $\lambda = (\lambda_1, \ldots, \lambda_m)$ with $\lambda_1 + \cdots + \lambda_m = n$ is called a *composition* of n. A *partition* of n is a composition of n, $\lambda = (\lambda_1, \ldots, \lambda_m)$, with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$. So as there is no confusion between compositions and permutations, for cycles we do not use commas but space out the elements of the cycle. Thus, for example, $(1, 3, 2)$ is a composition of 6 while $(1\ 3\ 2)$ is a permutation in $\mathrm{Sym}(3)$. Now, let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a composition of n, and let $\rho \geq 0$. For ease of notation set $\mu_i = \sum_{j=1}^{i-1} \lambda_j$ (and by convention $\mu_1 = 0$). We then define the *corresponding signed element* $w_{\lambda,\rho}$ to be $w_{\lambda,\rho} = w_1 \cdots w_m$ where

$$w_i = \begin{cases} (\overset{-}{\mu_i{+}1}\ \ \overset{-}{\mu_i{+}2}\ \cdots\ \overset{-}{\mu_{i+1}{-}1}\ \ \overset{-}{\mu_{i+1}}) & \text{if } 1 \leq i \leq \rho; \\ (\overset{-}{\mu_i{+}1}\ \ \overset{-}{\mu_i{+}2}\ \cdots\ \overset{-}{\mu_{i+1}{-}1}\ \ \overset{+}{\mu_{i+1}}) & \text{if } \rho < i \leq m. \end{cases}$$

We call λ a *maximal split partition* (with respect to ρ) if $\lambda_1 \geq \cdots \geq \lambda_\rho$ and $\lambda_{\rho+1} \geq \cdots \geq \lambda_m$.

For example, if $\lambda = (5, 2, 4, 3)$ and $\rho = 2$, then

$$w_{\lambda,\rho} = (\overset{-}{1}\ \overset{-}{2}\ \overset{-}{3}\ \overset{-}{4}\ \overset{-}{5})(\overset{-}{6}\ \overset{-}{7})(\overset{-}{8}\ \ \overset{-}{9}\ \ \overset{-}{10}\ \ \overset{+}{11})(\overset{-}{12}\ \ \overset{-}{13}\ \ \overset{+}{14}).$$

Our second main result in this paper is the following.

Theorem 1.2. *Let W be of type B_n and $\hat{W}$ its canonical subgroup of type D_n. Every conjugacy class of W contains an element $w_{\lambda,\rho}$, where λ is a maximal split partition with respect to ρ. Each element $w_{\lambda,\rho}$ has maximal B-length and maximal D-length in its conjugacy class of W. Moreover, the excess of $w_{\lambda,\rho}$ is zero, both with respect to the length function of W and, if $w_{\lambda,\rho} \in \hat{W}$, with respect to the length function of $\hat{W}$.*

Representatives of minimal length in conjugacy classes of types B_n and D_n appear in Theorems 3.4.7 and 3.4.12 of [6]. However we need additional information about elements of minimal length in W-conjugacy classes, which gives as a byproduct (in Corollaries 3.4 and 3.5) an alternative proof that the representatives given in [6] are indeed of minimal length.

In the rest of this section we briefly discuss the proof of Theorem 1.1. Given a root system Φ for a Coxeter group W, we have that Φ is the disjoint union of the set of positive roots Φ^+ and the set of negative roots $\Phi^- = -\Phi^+$. For detail on root systems, including these observations, see for example Chapter 5 of [10]. It is well known (for example Proposition 5.6 of [10]) that for any w in W, the length $\ell(w)$ is given by

$$\ell(w) = |N(w)| = |\{\alpha \in \Phi^+ : w(\alpha) \in \Phi^-\}|.$$

That is, $\ell(w)$ is the number of positive roots taken negative by w. We emphasise here that, in line with other work on Coxeter groups, elements of the group will act on the left. It is easy to show that if $w = gh$ for some $g, h \in W$, then

$$\ell(w) = \ell(g) + \ell(h) - 2|N(g) \cap N(h^{-1})|. \tag{1}$$

(Equation (1) is well known but is stated and proved as part of Lemma 2.1 in [8].) Our method of proving Theorem 1.1 for the classical Weyl groups will be as follows. First we will establish a collection of elements w constituting a representative of maximal length for each conjugacy class of the group under consideration. For each such w, we will obtain involutions σ and τ such that $N(\sigma)\cap N(\tau)=\emptyset$ and $\sigma\tau=w$. It follows from Equation (1) that the excess of w is zero. We conclude this section with two lemmas which will be useful later.

Lemma 1.3. *Let W be a Coxeter group. Let $g,h\in W$ and suppose $N(g)\cap N(h^{-1})=\emptyset$. Then $N(gh)=N(h)\dot{\cup}h^{-1}(N(g))$.*

Proof. Note that $|N(h)\cap h^{-1}N(g)|=|hN(h)\cap N(g)|\leq|\Phi^-\cap N(g)|=0$. So $N(h)$ and $h^{-1}(N(g))$ are indeed disjoint. Suppose $\alpha\in N(h)$. Then $gh(\alpha)\in\Phi^+$ would imply that $h(\alpha)\in -N(g)$, which implies $-h(\alpha)\in N(h^{-1})\cap N(g)$, a contradiction. Hence $gh(\alpha)\in\Phi^-$, meaning $N(h)\subseteq N(gh)$. Now suppose $\alpha\in N(gh)\setminus N(h)$. Then $h(\alpha)\in\Phi^+$ but $gh(\alpha)\in\Phi^-$. Therefore $\alpha\in h^{-1}(N(g))$. Conversely, since $N(h^{-1})\cap N(g)=\emptyset$, we have $h^{-1}(N(g))\subseteq\Phi^+$ and so $h^{-1}(N(g))\subset N(gh)$. Therefore $N(gh)=N(h)\dot{\cup}h^{-1}(N(g))$. □

Lemma 1.4. *Let W be a Coxeter group. Suppose $t_1,t_2,\ldots,t_m$ are involutions with the property that whenever $i\neq j$ we have $t_i(N(t_j))=N(t_j)$. Then $N(t_1\cdots t_m)=\dot{\cup}_{i=1}^m N(t_i)$.*

Proof. The result clearly holds when $m=1$. Assume the result holds for $m=k$. Set $u_k=t_1t_2\cdots t_k$. Then inductively $N(u_k)=\dot{\cup}_{i=1}^k N(t_i)$. If $\alpha\in N(u_k)$ then $\alpha\in N(t_i)$ for some $i\leq k$ and so $t_{k+1}(\alpha)\in N(t_i)\subseteq\Phi^+$. Thus $N(u_k)\cap N(t_{k+1})=\emptyset$. Lemma 1.3 now gives $N(u_{k+1})=N(t_{k+1})\dot{\cup}t_{k+1}(N(u_k))=\dot{\cup}_{i=1}^{k+1}N(t_i)$. The result follows by induction. □

Finally for $\sigma\in\mathrm{Sym}(n)$, the *support* of σ, denoted $\mathrm{supp}(\sigma)$ is simply the set of points not fixed by σ. That is,

$$\mathrm{supp}(\sigma)=\{i\in\{1,\ldots,n\}:\sigma(i)\neq i\}.$$

We thank the referee for his/her helpful comments.

2 Type A_{n-1}

The permutation group $\mathrm{Sym}(n)$ is a Coxeter group of type A_{n-1}. So throughout this section we will set $W=\mathrm{Sym}(n)$. In this context then, the length of an element w is the number of inversions, that is the number of pairs (i,j) with $1\leq i<j\leq n$ such that $w(i)>w(j)$. We can also think of this in terms of the root system (which we can consider as a warm up for the type B_n and D_n cases). For the root system Φ we can take

$$\Phi^+=\{e_i-e_j:1\leq i<j\leq n\}$$

and $\Phi^-=-\Phi^+$. Hence

$$N(w)=\{e_i-e_j:i<j,w(i)>w(j)\}.$$

For what follows it will sometimes be helpful to consider intervals $[a,b]$ for $1\leq a<b\leq n$. The group $\mathrm{Sym}([a,b])$ is a standard parabolic subgroup of W, and by $\Phi^+_{[a,b]}$ we mean $\{e_i-e_j:a\leq i<j\leq b\}$. We note that if $w\in\mathrm{Sym}([a,b])$ then $N(w)\subseteq\Phi^+_{[a,b]}$. The conjugacy classes of W are parameterized by partitions of n. Kim [11] has described a set of representative elements of maximal length in conjugacy classes of $\mathrm{Sym}(n)$, using the 'stair form'. Following [11] we give the following definition.

Definition 2.1. *Let n be a positive integer.*

(i) Define the sequence $a_1, a_2, \ldots, a_n$ by $a_{2i-1} = i$ and $a_{2i} = n - (i-1)$. (So $a_1 = 1$. $a_2 = n$, $a_3 = 2$, $a_4 = n-1$ and so on.)

(ii) Given a composition $\lambda = (\lambda_1, \ldots, \lambda_m)$ of n, its corresponding element *is the element of* $\mathrm{Sym}(n)$ *defined by*

$$w_\lambda = w_1 w_2 \cdots w_m$$

where $w_i = (a_{\lambda_1+\cdots+\lambda_{i-1}+1}\ \ a_{\lambda_1+\cdots+\lambda_{i-1}+2}\ \ \cdots\ \ a_{\lambda_1+\cdots+\lambda_{i-1}+\lambda_i})$.

(iii) Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a composition of n. Then λ is a maximal composition *of n if there exists ℓ, with $0 \le \ell \le m$ such that $\lambda_1, \ldots, \lambda_\ell$ are even numbers in any order, and $\lambda_{\ell+1}, \ldots, \lambda_m$ are odd numbers in decreasing order.*

For example, given the maximal composition (4,5) of 9, the corresponding element is $(1\,9\,2\,8)(3\,7\,4\,6\,5)$. Any partition of n can be reordered so as to produce a maximal composition. Therefore each conjugacy class can be represented by a maximal composition. We can now state the main result of [11].

Theorem 2.2 (Kim, [11])**.** *Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a maximal composition of n. The corresponding element w_λ of λ has maximal length in its conjugacy class.*

Given a sequence $b_1, b_2, \ldots b_k$, of distinct elements in $\{1, \ldots, n\}$, we define $g_{b_1,\cdots,b_k}$ to be the permutation that reverses the sequence and fixes all other $c \in \{1, \ldots, n\}$, so that $g(b_i) = b_{k+1-i}$. That is,

$$g = (b_1\ b_k)(b_2\ b_{k-1}) \cdots (b_{\lfloor k/2 \rfloor}\ b_{\lceil k/2 \rceil + 1}).$$

In particular, $g_{b_1,\cdots,b_k}$ is an involution.

Let w be the k-cycle $(b_1\, b_2\ \cdots\ b_k)$. Define

$$\sigma(w) = \begin{cases} g_{b_1,\ldots,b_k} & \text{if } k \text{ even} \\ g_{b_2,\ldots,b_k} & \text{if } k \text{ odd} \end{cases} \tag{2}$$

$$\tau(w) = \begin{cases} g_{b_1,\ldots,b_{k-1}} & \text{if } k \text{ even} \\ g_{b_1,\ldots,b_k} & \text{if } k \text{ odd} \end{cases} \tag{3}$$

Lemma 2.3. *Let w be a cycle of* $\mathrm{Sym}(n)$*. Then writing $\sigma = \sigma(w)$ and $\tau = \tau(w)$ we have that $w = \sigma\tau$, where σ and τ are both involutions.*

Proof. It is clear from the definitions that σ and τ are involutions. Let $w = (b_1 \cdots b_k)$. If k is even, then by (2) and (3) we see that for $i \le k-1$ we have $\sigma\tau(b_i) = \sigma(b_{k-i}) = b_{(k+1)-(k-i)} = b_{i+1}$, and $\sigma\tau(b_k) = \sigma(b_k) = b_1$. Therefore $w = \sigma\tau$. If k is odd then $\sigma(b_j) = b_{k+2-j}$ when $2 \le j \le k$, and $\sigma(b_1) = b_1$. Therefore, when $i \le k-1$ we have $\sigma\tau(b_i) = \sigma(b_{k+1-i}) = b_{k+2-(k+1-i)} = b_{i+1}$ and $\sigma\tau(b_k) = \sigma(b_1) = b_1$. Again we get $w = \sigma\tau$. $\square$

Before we go further we introduce some additional notation. Any composition λ (via its corresponding element w_λ) induces a partition $X = (X_1, \ldots, X_m)$ of $\{1, \ldots, \lceil \frac{n}{2} \rceil\}$ and a partition $Y = (Y_1, \ldots, Y_m)$ of $\{\lceil \frac{n}{2} \rceil + 1, \ldots, n\}$ by setting

$$X_k = \{1, \ldots, \lceil \tfrac{n}{2} \rceil\} \cap \mathrm{supp}(w_k);$$
$$Y_k = \{\lceil \tfrac{n}{2} \rceil + 1, \ldots, n\} \cap \mathrm{supp}(w_k).$$

By definition of w_k we see that X_k is an interval $[\underline{x}_k, \overline{x}_k]$ where $\underline{x}_k$ and $\overline{x}_k$ are, respectively, the minimal and maximal elements of X_k appearing in $\mathrm{supp}(w_k)$. Similarly we may write $Y_k = [\underline{y}_k, \overline{y}_k]$ for appropriate $\underline{y}_k$ and $\overline{y}_k$. For example, if $\lambda = (8, 5)$, then $w_\lambda = (1\ 13\ 2\ 12\ 3\ 11\ 4\ 10)(5\ 9\ 6\ 8\ 7)$ and we have $X_1 = \{1, 2, 3, 4\} = [1, 4]$, $X_2 = \{5, 6, 7\}$, $Y_1 = \{10, 11, 12, 13\}$, $Y_2 = \{8, 9\}$. Note also that $\sigma(w_1) = (1\ 10)(2\ 11)(3\ 12)(4\ 13)$, $\sigma(w_2) = (6\ 8)(7\ 9)$, $\tau(w_1) = (1\ 4)(2\ 3)(11\ 13)$ and $\tau(w_2) = (5\ 7)(8\ 9)$. We will see that $\tau(w_k)$ leaves the sets X_k and Y_k invariant, and $\sigma(w_k)$ interchanges, in an order-preserving way, nearly all, if not all, elements of X_k and Y_k.

Proposition 2.4. *Let λ be a maximal composition of n and let C be the corresponding conjugacy class of* $\mathrm{Sym}(n)$. *Then the corresponding element w_λ has excess zero.*

Proof. Write $\lambda = (\lambda_1, \ldots, \lambda_m)$. Set $w = w_\lambda = w_1 \cdots w_m$ where w_i is as given in Definition 2.1. For each i set $\sigma_i = \sigma(w_i)$ and $\tau_i = \tau(w_i)$. Since the supports (in other words the sets of non-fixed points) of σ and τ are subsets of the support of w_i, it is clear that both σ_i and τ_i commute with both σ_j and τ_j whenever $i \neq j$. Hence $\sigma = \sigma_1 \cdots \sigma_m$ and $\tau = \tau_1 \cdots \tau_m$ are involutions with the property that $\sigma\tau = w$. We must show that $N(\sigma) \cap N(\tau) = \emptyset$. This will imply by Equation (1) that $e(w) = 0$.

Consider the cycle w_k of w. Then $w_k = (a_{L+1}\ \ a_{L+2}\ \ \cdots\ \ a_{L+\lambda_k})$ (setting $L = \sum_{j=1}^{k-1} \lambda_j$). This means, depending on the parity of L, that there is some $i \geq 1$ for which w_k is either $(i\ \ n{-}i{+}1\ \ \ i{+}1\ \ \ n{-}i\ \ \cdots)$ or $(n{-}i{+}2\ \ \ i\ \ \ n{-}i{+}1\ \ \cdots)$. The support of w_k is $X_k \cup Y_k$.

Let us consider $\tau_k = \tau(w_k)$. Now if λ_k is even, we have $\tau_k = \prod_{i=1}^{(\lambda_k/2-1)} (a_{L+i}\ \ a_{L+\lambda_k-i})$. If λ_k is odd then $\tau_k = \prod_{i=1}^{\lfloor \lambda_k/2 \rfloor} (a_{L+i}\ \ a_{L+\lambda_k+1-i})$. In both cases τ_k is mapping odd terms of the sequence (a_i) to odd terms and even terms to even terms. In particular, $\tau_k \in \mathrm{Sym}(X_k) \times \mathrm{Sym}(Y_k)$. Therefore

$$N(\tau_k) \subseteq \{e_i - e_j : \underline{x}_k \leq i < j \leq \overline{x}_k\} \cup \{e_i - e_j : \underline{y}_k \leq i < j \leq \overline{y}_k\}. \tag{4}$$

(If λ_k is odd then we have equality here and $\tau_k = g_{\underline{x}_k, \ldots, \overline{x}_k} g_{\underline{y}_k, \ldots, \overline{y}_k}$.)

Next we look at σ_k. If λ_k is even, then setting $\mu = \lfloor \frac{\lambda_k}{2} \rfloor$ we have

$$\sigma_k = \prod_{i=1}^{\mu} (a_{L+i}\ \ a_{L+\lambda_k+1-i}).$$

If λ_k is odd then $\sigma_k = \prod_{i=2}^{\mu+1} (a_{L+i}\ \ a_{L+\lambda_k+2-i})$. What happens this time is that σ_k is the order preserving bijection between the highest μ elements of X_k and the lowest μ elements of Y_k. Therefore

$$\begin{aligned} N(\sigma_k) = \{&e_i - e_j : \underline{x}_k \leq \overline{x}_k + 1 - \mu \leq i \leq \overline{x}_k < j < \underline{y}_k\} \\ &\cup \{e_i - e_j : \overline{x}_k < i < \underline{y}_k \leq j \leq \underline{y}_k + \mu - 1 \leq \overline{y}_k\}. \end{aligned} \tag{5}$$

Now for $\ell \neq k$, we have that σ_ℓ fixes all i for $i \notin X_\ell \cup Y_\ell$ and interchanges various elements of X_ℓ and Y_ℓ. Therefore $\sigma_\ell(N(\sigma_k)) = N(\sigma_k)$. So we may apply Lemma 1.4 to conclude that $N(\sigma) = \dot\bigcup_{k=1}^{m} N(\sigma_k)$. Similarly since τ_ℓ fixes all i for $i \notin X_\ell \cup Y_\ell$, we can deduce that $N(\tau) = \dot{\bigcup}_{k=1}^m N(\tau_k)$. Looking at Equations (4) and (5) it is clear that $N(\tau) \cap N(\sigma) = \emptyset$. Therefore by Equation (1) we see that $\ell(w) = \ell(\sigma) + \ell(\tau)$ and hence $e(w) = 0$, as required. $\square$

We remark that Theorem 1.1 for type A_n follows immediately from Theorem 2.2 and Proposition 2.4.

3 Maximal lengths in types B_n and D_n

Throughout this section, W is assumed to be a Coxeter group of type B_n containing $\hat{W}$, the canonical index 2 subgroup of type D_n. We will view elements of W as signed cycles. A cycle is called negative if it has an odd number of minus signs above its entries, and positive otherwise. The conjugacy classes of W are parameterized by signed cycle type. So for X a subset of a conjugacy class of W, this data may be encoded by

$$\lambda(X) = (\lambda_1, \dots, \lambda_{\nu_X}; \lambda_{\nu_X+1}, \dots, \lambda_{z_X})$$

where ν_X is the number of negative cycles, z_X is the total number of cycles, and $\lambda_1 \leq \ldots \leq \lambda_{\nu_X}$, respectively $\lambda_{\nu_X+1} \leq \cdots \leq \lambda_{z_X}$, are the lengths of the negative, respectively positive, cycles of X. So any element of X has $\lambda(X)$ as its signed cycle type.

Our main aim in this section is to prove the first part of Theorem 1.2 along with the following result.

Theorem 3.1. *Suppose $\hat{W}$ is a Coxeter group of type D_n, and let $\hat{C}$ be a conjugacy class of $\hat{W}$. Set $C = \hat{C}_0 = \hat{C}w_0$, where w_0 is the longest element of W, and assume that $\lambda(C) = (\lambda_1, \dots, \lambda_{\nu_C}; \lambda_{\nu+1}, \dots, \lambda_{z_C})$. Then the maximal length in $\hat{C}$ is*

$$n^2 + z_C - 2\sum_{i=1}^{\nu_C - 1} (\nu_C - i)\lambda_i.$$

Let Φ be the root system of W. We employ the usual description of Φ (as given, for example in [10]). So the positive long roots are $\Phi^+_{\text{long}} = \{e_i \pm e_j : 1 \leq i < j \leq n\}$, the negative long roots are $\Phi^-_{\text{long}} = -\Phi^+_{\text{long}}$ and $\Phi_{\text{long}} = \Phi^+_{\text{long}} \cup \Phi^-_{\text{long}}$. The short roots are $\Phi^+_{\text{short}} = \{e_i : 1 \leq i \leq n\}$, $\Phi^-_{\text{short}} = -\Phi^+_{\text{short}}$ and $\Phi_{\text{short}} = \Phi^+_{\text{short}} \cup \Phi^-_{\text{short}}$. Finally the positive roots are $\Phi^+ = \Phi^+_{\text{long}} \cup \Phi^+_{\text{short}}$, the negative roots are $\Phi^- = \Phi^-_{\text{long}} \cup \Phi^-_{\text{short}}$ and $\Phi = \Phi^+ \cup \Phi^-$. We note that the set of positive roots for $\hat{W}$ is Φ^+_{long}. We recall our convention will be that the action of a group element is on the left of the root, so that for example $(\overset{-}{1}\overset{+}{3}\overset{+}{8})(e_1) = (\overset{+}{1}\overset{+}{3}\overset{+}{8})(\overset{-}{1})(e_1) = -e_3$.

For $w \in W$, we define the following two sets.

$$\Lambda(w) = \{\alpha \in \Phi^+_{\text{long}} : w(\alpha) \in \Phi^-\};$$

$$\Sigma(w) = \{\alpha \in \Phi^+_{\text{short}} : w(\alpha) \in \Phi^-\}.$$

Set $l_B(w) = |\Lambda(w)| + |\Sigma(w)|$ and $l_D(w) = |\Lambda(w)|$. By [10] $l_B(w)$ is the length of w and, should $w \in \hat{W}$, then $l_D(w)$ is the length of w viewed as an element of $\hat{W}$. We call $l_B(w)$ the B-length of w and $l_D(w)$ the D-length of w. Given $w \in W$, let $\overline{w}$ be the corresponding element of $\mathrm{Sym}(n)$. So, for example, if $w = (\overset{-}{1}\overset{+}{3}\overset{+}{8})$, then

$\overline{w} = (138)$.
Observe that for $w \in W$, by a slight abuse of notation, we can write

$$w = \overline{w}\left(\prod_{e_i \in \Sigma(w)} (\overset{-}{i})\right).$$

Hence, in our above example, $(\overset{-}{1}\overset{+}{3}\overset{+}{8}) = (138)(\overset{-}{1})$.

Later when we talk about excess in these groups, to avoid ambiguity we will use the notation $e_B(w)$ to mean the excess $e(w)$ when w is viewed as an element of W, and $e_D(w)$ to mean the excess $e(w)$ when w is viewed (where appropriate) as an element of $\hat{W}$. That is, for all w in W we define

$$\begin{aligned} e_B(w) &= \min\{\ell_B(\sigma) + \ell_B(\tau) - \ell_B(w) : \sigma, \tau \in W, w = \sigma\tau, \sigma^2 = \tau^2 = 1\}; \\ e_D(w) &= \min\{\ell_D(\sigma) + \ell_D(\tau) - \ell_D(w) : \sigma, \tau \in \hat{W}, w = \sigma\tau, \sigma^2 = \tau^2 = 1\}. \end{aligned}$$

As noted earlier, conjugacy classes of W are parameterized by signed cycle type. So, for example, if W is of type B_9 and C is the W-conjugacy class of $w = (\overset{+}{1}\,\overset{+}{2})(\overset{+}{3}\,\overset{-}{4}\,\overset{+}{5})(\overset{-}{6}\,\overset{+}{7}\,\overset{+}{8})(\overset{+}{9})$, then the signed cycle type $\lambda(C)$ of C is $\lambda(C) = (3, 3; 1, 2)$. In $\hat{W}$, conjugacy classes are also parameterized by signed cycle type, with the exception that there are two classes for each signed cycle type consisting only of even length, positive cycles. (The length profiles in each pair of split classes are identical, because the classes are interchanged by the length-preserving graph automorphism.)

Lemma 3.2. *Let C be a conjugacy class of W, and $w \in C$. Then*

$$|\Lambda(w)| \geq n - z_C + 2\sum_{i=1}^{\nu_C - 1} (\nu_C - i)\lambda_i.$$

Moreover $|\Sigma(w)| \geq \nu_C$.

Proof. Set $\nu = \nu_C$ and $z = z_C$. Write w as a product of disjoint cycles, $w = \sigma_1\sigma_2\cdots\sigma_z$, where $\sigma_1, \ldots, \sigma_\nu$ are negative cycles and the remaining cycles are positive. Also, order the negative cycles such that $i < j$ if and only if the minimal element in $\mathrm{supp}(\overline{\sigma_i})$ is smaller than the minimal element in $\mathrm{supp}(\overline{\sigma_j})$. Our approach is to consider certain $\langle w \rangle$-orbits of roots.
Firstly, let σ be a positive k-cycle of w and consider the orbits consisting of roots of the form $e_a - e_b$, for $a, b \in \mathrm{supp}(\overline{\sigma})$ and $a \neq b$. Each such orbit has length k. There are $2\binom{k}{2}$ roots of this form, and hence $k-1$ such orbits. Let c be the maximal element in $\mathrm{supp}(\overline{\sigma})$. Then each orbit contains both $e_a - e_c$ and $e_c - e_b$ for some $a, b \in \mathrm{supp}(\overline{\sigma})$. Now $e_a - a_c \in \Phi^+$ and $e_c - e_b \in \Phi^-$. Therefore each orbit includes a transition from positive to negative (that is, a positive root α for which $w(\alpha)$ is negative). Hence each orbit contributes at least one root to $\Lambda(w)$. Therefore each positive k-cycle contributes at least $k-1$ roots to $\Lambda(w)$.

Next suppose σ is a negative k-cycle of w. This time we consider orbits consisting of roots of the form $\pm e_a \pm e_b$, for $a, b \in \mathrm{supp}(\overline{\sigma})$ and $a \neq b$. Each such orbit has length $2k$. There are $4\binom{k}{2}$ roots of this form, and hence $k-1$ such orbits. Moreover if α lies in one of these orbits, then $-\alpha$ lies in the same orbit. Thus again each orbit includes a transition from positive to negative and hence contributes at least one root to $\Lambda(w)$. Therefore each negative k-cycle contributes at least $k-1$ roots to $\Lambda(w)$.

Now suppose σ_i and σ_j are negative cycles, with $i < j$, and consider the union of all orbits consisting of roots of the form $\pm e_a \pm e_b$, where $a \in \operatorname{supp}(\overline{\sigma_i})$ and $b \in \operatorname{supp}(\overline{\sigma_j})$. Suppose $|\operatorname{supp}(\overline{\sigma_i})| = k$ and $|\operatorname{supp}(\overline{\sigma_j})| = \ell$. Let c be minimal in $\operatorname{supp}(\overline{\sigma_i})$. Then every orbit contains some $\pm e_c \pm e_b$ for some $b \in \operatorname{supp}(\overline{\sigma_j})$. For every root of the form $e_c \pm e_b$, we have $w^k(e_c \pm e_b) = -e_c \pm e_{b'}$ and $w^{2k}(e_c \pm e_{b'}) = e_c \pm e_{b''}$ for some $b', b'' \in \operatorname{supp}(\overline{\sigma_j})$. Now $e_c \pm e_b$ and $e_c \pm e_{b''}$ are positive roots, but $-e_c \pm e_{b'}$ is negative. Therefore in this orbit or part of orbit there is at least one transition from positive to negative. There are 2ℓ roots of the form $e_c \pm e_b$, and hence each pair σ_i, σ_j of negative cycles with $i < j$ contributes at least $2|\operatorname{supp}(\overline{\sigma_j})|$ roots to $\Lambda(w)$. For example, letting i range from 1 to $\nu - 1$, we get a total of $(\nu - 1) \times 2|\operatorname{supp}(\overline{\sigma_\nu})|$ roots from pairs σ_i and σ_ν.

Combining these three observations and writing k_i for $|\operatorname{supp}(\overline{\sigma_i})|$, we see that

$$\Lambda(w) \geq \sum_{i=1}^{z}(k_i - 1) + 2\sum_{i=2}^{\nu}(i-1)k_i.$$

Since $\{k_1, \dots, k_\nu\} = \{\lambda_1, \dots, \lambda_\nu\}$, and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_\nu$, it is clear that

$$\begin{aligned}\sum_{i=2}^{\nu}(i-1)k_i &= k_2 + 2k_3 + \cdots + (\nu-1)k_\nu \\ &\geq \lambda_{\nu-1} + 2\lambda_{\nu-2} + \cdots + (\nu-1)\lambda_1 \\ &= \sum_{i=1}^{\nu-1}(\nu - i)\lambda_i.\end{aligned}$$

Therefore

$$|\Lambda(w)| \geq n - z + 2\sum_{i=1}^{\nu-1}(\nu - i)\lambda_i.$$

It only remains to show that $|\Sigma(w)| \geq \nu$. This trivially follows from the fact that there are ν negative cycles and each negative cycle must contain at least one minus sign. Therefore there are at least ν roots e_a for which $w(e_a) \in \Phi^-$. Thus $|\Sigma(w)| \geq \nu$ and the proof of the lemma is complete. $\square$

Next, given a conjugacy class C of W we define a particular element u_C of C (which will turn out to have minimal B-length). Recall that the signed cycle type of C is

$$\lambda(C) = (\lambda_1, \lambda_2, \dots, \lambda_{\nu_C}; \lambda_{\nu_C+1}, \dots, \lambda_{z_C}),$$

and write $\mu_i = n - \sum_{j=1}^{i} \lambda_j$ for $1 \leq i < z_C$. Set $\nu = \nu_C$ and $z = z_C$. Then define u_C to be the following element of C.

$$\begin{aligned}u_C = &(\overset{+}{1}\,\overset{+}{2} \cdots \overset{+}{\lambda_z})(\overset{+}{\mu_{z-1}+1} \cdots \overset{+}{\mu_{z-2}}) \cdots (\overset{+}{\mu_{\nu+1}+1} \quad \overset{+}{\mu_{\nu+1}+2} \cdots \overset{+}{\mu_\nu}) \cdot \\ &\cdot (\overset{+}{\mu_\nu+1} \quad \overset{+}{\mu_\nu+2} \cdots \overset{+}{\mu_{\nu-1}-1} \quad \overset{-}{\mu_{\nu-1}}) \cdots (\overset{+}{\mu_1+1} \cdots \overset{+}{n-1} \quad \overset{-}{n})\end{aligned}$$

As an example, let $w = (\overset{-}{1}\,\overset{+}{7}\,\overset{-}{2}\,\overset{-}{9})(\overset{-}{3}\,\overset{+}{4}\,\overset{-}{6})(\overset{+}{5}\,\overset{-}{8})$ and let C be the conjugacy class of w in type B_9. Then $\lambda_C = (2,4;3)$, $\nu_C = 2$, $z_C = 3$, $\mu_1 = 7$ and $\mu_2 = 3$. This gives $u_C = (\overset{+}{1}\,\overset{+}{2}\,\overset{+}{3})(\overset{+}{4}\,\overset{+}{5}\,\overset{+}{6}\,\overset{-}{7})(\overset{+}{8}\,\overset{-}{9})$.

Lemma 3.3. *Suppose $w = u_C$ for some conjugacy class C of W. Then $|\Sigma(w)| = \nu_C$ and $|\Lambda(w)| = n - z_C + 2\sum_{i=1}^{\nu_C-1}(\nu_C - i)\lambda_i$*

Proof. Again set $z = z_C$ and $\nu = \nu_C$. The size of $\Sigma(w)$ is simply the number of minus signs appearing in the expression for w. Here, $\Sigma(w) = \{e_n, e_{\mu_1}, \ldots, e_{\mu_{\nu-1}}\}$ and $|\Sigma(w)| = \nu$.
To find $\Lambda(w)$, consider pairs (i,j) with $1 \leq i < j \leq n$. Suppose first that i and j are in the same cycle of $\overline{w}$. Then $e_i \notin \Sigma(w)$ because only the maximal element of each negative cycle has a minus sign above it. If $j = \mu_k$ for some k, or if $j = n$, then exactly one of $e_i + e_j \in \Lambda(w)$ or $e_i - e_j \in \Lambda(w)$ occurs (depending whether $k < \nu$). Otherwise, $e_i - e_j \notin \Lambda(w)$ and $e_i + e_j \notin \Lambda(w)$. Hence a cycle $(\overset{+}{\mu_{k+1}{+}1}\ \cdots\ \overset{+}{\mu_k{-}1}\ \ \overset{\pm}{\mu_k})$ contributes exactly $\lambda_{k+1} - 1$ roots to $\Lambda(w)$.

Now suppose that i and j are in different cycles. Hence $\overline{w}(i) < \overline{w}(j)$. It is a simple matter to check that if $e_i \in \Sigma(w)$, then $\{e_i + e_j, e_i - e_j\} \subseteq \Lambda(w)$, whereas if $e_i \notin \Sigma(w)$, then $e_i - e_j$ and $e_i + e_j$ are not in $\Lambda(w)$. Therefore each i with $e_i \in \Sigma(w)$ contributes exactly $2(n-i)$ additional roots to $\Lambda(w)$, and no roots are contributed when $e_i \notin \Sigma(w)$.

Therefore

$$\begin{aligned}
|\Lambda(w)| &= \sum_{k=1}^{z}(\lambda_{k+1} - 1) + \sum_{k:e_k \in \Sigma(w)} 2(n-k) \\
&= (n-z) + 2\left((n-n) + (n-\mu_1) + (n-\mu_2) + \cdots + (n-\mu_{\nu-1})\right) \\
&= (n-z) + 2\sum_{i=1}^{\nu-1}\sum_{j=1}^{i}\lambda_j \\
&= n - z + 2\sum_{i=1}^{\nu-1}(\nu - i)\lambda_i.
\end{aligned}$$

Therefore $|\Lambda(w)| = n - z_C + 2\sum_{i=1}^{\nu_C-1}(\nu_C - i)\lambda_i$ and $|\Sigma(w)| = \nu_C$. □

Corollary 3.4. *Let C be a conjugacy class of W. Then the minimal B-length in C is*

$$n + \nu_C - z_C + 2\sum_{i=1}^{\nu_C-1}(\nu_C - i)\lambda_i.$$

If $w \in C$ has minimal B-length, then $|\Lambda(w)| = n - z_C + 2\sum_{i=1}^{\nu_C-1}(\nu_C - i)\lambda_i$ and $|\Sigma(w)| = \nu_C$. Moreover, u_C is a representative of minimal B-length in C.

In the next corollary the element u_C^t is the element obtained from u_C by taking its shortest positive cycle (which in this context will be the cycle $(\overset{+}{n}\ \overset{+}{n{-}1}\ \cdots\ \overset{+}{m})$ for some odd m), and putting minus signs over n and $n-1$. In other words it is the conjugate of u_C by $t = (\bar{n})$. Conjugation by $(\bar{n})$ is the length preserving automorphism of $\hat{W}$ induced by the graph automorphism of the Coxeter graph D_n.

Corollary 3.5. *Let C be a conjugacy class of W. If C is also a conjugacy class, or a union of conjugacy classes, of $\hat{W}$, then the minimal D-length of elements in the class(es) is $n - z_C + 2\sum_{i=1}^{\nu_C - 1}(\nu_C - i)\lambda_i$. Moreover u_C and u_C^t are representatives of minimal D-length in the class(es), with one in each $\hat{W}$-class if the class C splits.*

Theorem 3.6. *Let C be a conjugacy class of W and $w \in C$. Let C_0 be the conjugacy class of ww_0 where w_0 is the longest element of W. Then the maximal B-length of elements of C is $n^2 - |\Lambda(u_{C_0})| - |\Sigma(u_{C_0})|$, with $u_{C_0}w_0$ being an element of maximal B-length. If C is a conjugacy class or union of conjugacy classes of $\hat{W}$, the maximal D-length of elements of C is $n^2 - n - |\Lambda(u_{C_0})|$. Moreover $u_{C_0}w_0$ and $u_{C_0}^t w_0$ are representatives of maximal D-length in the class(es), with one in each $\hat{W}$-class if C is a split class.*

Proof. Let C be a conjugacy class of W. Since w_0 is central, the W-conjugacy class C_0 of ww_0 is just Cw_0. Moreover, for any root α we have $w_0(\alpha) = -\alpha$. Therefore for all $x \in W$, $|\Lambda(xw_0)| = (n^2 - n) - |\Lambda(x)|$ and $|\Sigma(xw_0)| = n - |\Sigma(x)|$. (Note that there are $n^2 - n$ long positive roots and n short positive roots.) Let $u = u_{C_0}$. Then by Lemma 3.2 and Lemma 3.3, we have that for all $v \in C_0$, $|\Lambda(v)| \geq |\Lambda(u)|$ and $|\Sigma(v)| \geq |\Sigma(u)|$. Now every $x \in C$ is of the form vw_0 for some $v \in C_0$. Hence for every $x \in C$, we have

$$\begin{aligned} |\Lambda(x)| &\leq n^2 - n - |\Lambda(u)| \quad \text{and} \\ |\Sigma(v)| &\leq n - |\Sigma(u)|. \end{aligned}$$

Also $|\Lambda(uw_0)| = n^2 - n - |\Lambda(u)|$ and $|\Sigma(uw_0)| = n - |\Sigma(u)|$. Therefore the maximal B-length in C is $n^2 - n - |\Lambda(u)| + n - |\Sigma(u)| = n^2 - |\Lambda(u)| - |\Sigma(u)|$ and this is attained by the element uw_0. Moreover, if C is a conjugacy class (or union of conjugacy classes) of $\hat{W}$, then the maximal D-length is $n^2 - n - |\Lambda(u)|$ and this is attained by uw_0 (or $(uw_0)^t$ if the class splits). $\square$

Theorem 3.1 now follows immediately from Theorem 3.6 and Lemma 3.3. All that remains in this section is to prove the following corollary, which is the first part of Theorem 1.2.

Corollary 3.7. *Every conjugacy class of W contains an element $w_{\lambda,\rho}$, where λ is a maximal split partition with respect to ρ. Each element $w_{\lambda,\rho}$ has maximal B-length and maximal D-length in its conjugacy class of W.*

Proof. Note that each element $w_{\lambda,\rho}$, where $\lambda = (\lambda_1, \ldots, \lambda_m)$ is a maximal split partition with respect to ρ, is of the form $w_0 u_C$ for some u_C. In particular we have $z_C = m$ and $\nu_C = m - \rho$. Thus each element $w_{\lambda,\rho}$ has maximal B-length and maximal D-length in the class Cw_0. Therefore, given any class C' of W, setting $C = C'w_0$ we see that $w_0 u_C$ is $w_{\lambda,\rho}$ for some suitable λ, ρ, and so $w_{\lambda,\rho}$ is of maximal B-length and D-length in C'. $\square$

It is the task of the next section to show that these elements $w_{\lambda,\rho}$ have excess zero.

4 Excess zero in types B_n and D_n

The aim of this section is to prove Theorems 1.1 and 1.2 for W and $\hat{W}$. In order to do this we will show that the elements $w_{\lambda,\rho}$ described in Theorem 1.2 have excess zero both in W and (if applicable) in $\hat{W}$. To obtain the required involutions σ and τ such that $N(\sigma) \cap N(\tau) = \emptyset$ and $\sigma\tau = w$, we modify the definition of $g_{b_1,\dots,b_k}$ given in Section 2. We will only need to consider sequences of consecutive integers here though. Let $\{a+1, a+2, \dots, a+k\}$ be a sequence of consecutive positive integers in $\{1, \dots, n\}$. Define $g_{a;k}$ to be the permutation of W that reverses the sequence and fixes all other $c \in \{1, \dots, n\}$. (Essentially this is just $g_{b_1,\cdots,b_k}$ where $b_1 = a+1$, $b_2 = a+2$, $\dots$, $b_k = a+k$, but viewed as an element of W rather than $\mathrm{Sym}(n)$.) Thus $g_{a;k}(a+i) = a+k+1-i$ for $1 \leq i \leq k$. That is,

$$g_{a;k} = (\overset{+}{a+1}\ \ \overset{+}{a+k})(\overset{+}{a+2}\ \ \overset{+}{a+k-1})\cdots(\overset{+}{a+\lfloor k/2\rfloor}\ \ \overset{+}{a+\lceil k/2\rceil+1}).$$

In particular, $g_{a;k}$ is an involution.

We also define $h_{a;k}$ to be $g_{a;k}$ with the plus signs replaced by minus signs. Thus $h_{a;k}(a+i) = -(a+k+1)+i$ for $1 \leq i \leq k$. So

$$h_{a;k} = \begin{cases} (\overset{-}{a+1}\ \ \overset{-}{a+k})(\overset{-}{a+2}\ \ \overset{-}{a+k-1})\cdots(\overset{-}{a+\frac{k}{2}}\ \ \overset{-}{a+\frac{k}{2}+1}) & \text{if } k \text{ even;} \\ (\overset{-}{a+1}\ \ \overset{-}{a+k})(\overset{-}{a+2}\ \ \overset{-}{a+k-1})\cdots(\overset{-}{a+\lfloor\frac{k}{2}\rfloor}\ \ \overset{-}{a+\lceil\frac{k}{2}\rceil+1})(\overset{-}{a+\lceil\frac{k}{2}\rceil}) & \text{if } k \text{ odd.} \end{cases}$$

In particular, $h_{a;k}$ is an involution. Moreover $h_{a;k}$ is order preserving on the intervals $[1,a]$, $[a+1,a+k]$ and $[a+k+1,n]$.

As an example $g_{1;6} = (\overset{+}{2}\,\overset{+}{7})(\overset{+}{3}\,\overset{+}{6})(\overset{+}{4}\,\overset{+}{5})$ and $h_{3;5} = (\overset{-}{4}\,\overset{-}{8})(\overset{-}{5}\,\overset{-}{7})(\overset{-}{6})$.

Next we define two kinds of cycle and some involutions which are relevant to our analysis of the elements $w_{\lambda,\rho}$. Define

$$\begin{aligned}
w^-_{a;k} &= (\overset{-}{a+1}\ \ \overset{-}{a+2}\ \ \cdots\ \ \overset{-}{a+k-1}\ \ \overset{-}{a+k}) \\
\sigma(w^-_{a;k}) &= h_{a;k} \\
\tau(w^-_{a;k}) &= g_{a;k-1} \\
w^+_{a;k} &= (\overset{-}{a+1}\ \ \overset{-}{a+2}\ \ \cdots\ \ \overset{-}{a+k-1}\ \ \overset{+}{a+k}) \\
\sigma(w^+_{a;k}) &= h_{a+1;k-1} \\
\tau(w^+_{a;k}) &= g_{a;k}
\end{aligned}$$

Lemma 4.1. *Let w be either $w^-_{a;k}$ or $w^+_{a;k}$. Then writing $\sigma = \sigma(w)$ and $\tau = \tau(w)$ we have that $w = \sigma\tau$, where σ and τ are both involutions.*

Proof. It is clear from the definitions that σ and τ are involutions. First we consider $w = w^-_{a;k}$. Then if $1 \leq i \leq k-1$ we have $\sigma\tau(a+i) = \sigma(a+k-i) = -(a+k+1) + (k-i) = -(a+i+1)$, whereas $\sigma\tau(a+k) = \sigma(a+k) = -(a+k+1)+k = -(a+1)$. Therefore $w = \sigma\tau$ in this case. Now consider $w = w^+_{a;k}$. Then if $1 \leq i \leq k-1$ we have $\sigma\tau(a+i) = \sigma(a+k+1-i) = \sigma((a+1)+(k-i)) = -((a+1)+(k-1)+1)+(k-i) = -(a+i+1)$, whereas $\sigma\tau(a+k) = \sigma(a+1) = a+1$. Thus again $w = \sigma\tau$ and the proof is complete. $\square$

Proposition 4.2. *Let $w = w_{\lambda,\rho}$ be the corresponding signed element of the maximal split partition $\lambda = (\lambda_1, \dots, \lambda_m)$ (with respect to ρ). Then $e_B(w) = e_D(w) = 0$.*

Proof. By definition, and recalling that $\mu_i = \sum_{j=1}^{i-1} \lambda_j$ we have $w = w_1 \cdots w_m$ where

$$w_i = \begin{cases} (\overset{-}{\mu_i{+}1}\ \ \overset{-}{\mu_i{+}2}\ \cdots\ \overset{-}{\mu_{i+1}{-}1}\ \ \overset{-}{\mu_{i+1}}) & \text{if } 1 \le i \le \rho; \\ (\overset{-}{\mu_i{+}1}\ \ \overset{-}{\mu_i{+}2}\ \cdots\ \overset{-}{\mu_{i+1}{-}1}\ \ \overset{+}{\mu_{i+1}}) & \text{if } \rho < i \le m. \end{cases}$$

Therefore

$$w_i = \begin{cases} w^-_{\mu_i;\lambda_i} & \text{if } 1 \le i \le \rho; \\ w^+_{\mu_i;\lambda_i} & \text{if } \rho < i \le m. \end{cases}$$

For each i set $\sigma_i = \sigma(w_i)$ and $\tau_i = \tau(w_i)$. Since the supports of $\overline{\sigma_i}$ and $\overline{\tau_i}$ are subsets of the support of w_i, it is clear that both σ_i and τ_i commute with both σ_j and τ_j whenever $i \neq j$. Hence $\sigma = \sigma_1 \cdots \sigma_m$ and $\tau = \tau_1 \cdots \tau_m$ are involutions with the property that $\sigma\tau = w$. We must show that $N(\sigma) \cap N(\tau) = \emptyset$.

Consider a cycle w_k of w. Then $\tau(w_k)$ is either $g_{\mu_k;\lambda_k - 1}$ or $g_{\mu_k;\lambda_k}$. The action of g is to reverse the order of the sequence $\mu_k + 1, \ldots, \mu_k + \lambda_k$, reverse the order of the sequence $-\mu_k, \ldots, -\mu_k - \lambda_k$ and fix all other integers. Hence

$$N(\tau(w_k)) \subseteq \{e_i - e_j : \mu_k < i < j \le \mu_{k+1}\}. \tag{6}$$

On the other hand $\sigma(w_k)$ is either $h_{\mu_k;\lambda_k}$ or $h_{\mu_k+1;\lambda_k-1}$, so is of the form $h_{a;b}$ where $a \ge \mu_k$ and $a + b = \mu_{k+1}$. We observe that

$$\Sigma(h_{a;b}) = \{e_{a+1}, \ldots, e_{a+b}\} \subseteq \{e_{\mu_k+1}, \ldots, e_{\mu_{k+1}}\}. \tag{7}$$

Recall that $h_{a;b}$ fixes e_i for all $i \notin \{a+1, \ldots, a+b\}$ and $h_{a;b}(e_i) = -e_{2a+b+1-i}$ if $i \in \{a+1, \ldots, a+b\}$. From this we see that

$$\Lambda(h_{a;b}) = \{e_i + e_j : a < i < j \le a+b\} \cup \{e_i \pm e_j : a < i < a+b < j \le n\}. \tag{8}$$

Therefore

$$\Lambda(\sigma(w_k)) \subseteq \{e_i + e_j : \mu_k < i < j \le \mu_{k+1}\} \cup \{e_i \pm e_j : \mu_k < i < \mu_{k+1} < j \le n\}. \tag{9}$$

For $l \neq k$, we note that σ_l and τ_l fix all i for $i \notin \{\mu_l + 1, \ldots, \mu_{l+1}\}$. In particular they stabilise (setwise) the sets $\{1, \ldots, \mu_k\}$, $\{\mu_k + 1, \ldots, \mu_{k+1}\}$ and $\{\mu_{k+1} + 1, \ldots, n\}$. Therefore $\sigma_l(N(\sigma_k)) = N(\sigma_k)$. So we may apply Lemma 1.4 to conclude that $N(\sigma) = \dot\bigcup_{i=1}^m N(\sigma_k)$ and that $N(\tau) = \dot\bigcup_{i=1}^m N(\tau_k)$.

Equations (6), (7) and (9) now imply that $N(\tau) \cap N(\sigma) = \emptyset$. Therefore by Equation (1) we see that $\ell_B(w) = \ell_B(\sigma) + \ell_B(\tau)$ and therefore $e_B(w) = 0$. But also $N(\tau) \cap N(\sigma) = \emptyset$ implies that $\Lambda(\tau) \cap \Lambda(\sigma) = \emptyset$, and so we also have $\ell_D(w) = \ell_D(\sigma) + \ell_D(\tau)$, giving $e_D(w) = 0$ as required. □

We observe that Theorem 1.2 now follows immediately from Corollary 3.7 and Proposition 4.2.

Corollary 4.3. *Theorem 1.1 holds for Coxeter groups of type B_n and D_n.*

Proof. If W is of type B_n, then by Theorem 1.2 every conjugacy class C of W contains an element of the form $w_{\lambda,\rho}$ for suitable λ and ρ, and this element has excess zero and maximal B-length in C. Now consider $\hat{W}$ of type D_n, and let C

be a conjugacy class of $\hat{W}$. If C is also a conjugacy class of W, then again C contains some $w_{\lambda,\rho}$, which has maximal D-length and excess zero. If C is not a conjugacy class of W then $C \cup C^{(\bar{n})}$ is a conjugacy class of W (as conjugation by $(\bar{n})$ is a length preserving map corresponding to the non-trivial graph automorphism of D_n), so for some $w = w_{\lambda,\rho}$ we have either w or $w^{(\bar{n})} \in C$. Now $e(w) = 0$, which means there are σ, τ involutions such that $w = \sigma\tau$ and $\ell(w) = \ell(\sigma) + \ell(\tau)$. Hence $w^{(\bar{n})} = \sigma^{(\bar{n})}\tau^{(\bar{n})}$ and, since conjugation by $(\bar{n})$ is a length-preserving map, we have $\ell(w^{(\bar{n})}) = \ell(\sigma^{(\bar{n})}) + \ell(\tau^{(\bar{n})})$. Hence either w or $w^{(\bar{n})}$ is an element of maximal D-length and excess zero in C. $\square$

5 Conclusion

Proof of Theorem 1.1 Observe that every finite Coxeter group W is a direct product of irreducible Coxeter groups. If $W = W_1 \times \cdots \times W_n$ for some W_i, then it is easy to see that for $w = (w_1, \ldots, w_n) \in W$, we have $\ell(w) = \ell(w_1) + \cdots + \ell(w_n)$ and $e(w) = e(w_1) + \cdots + e(w_n)$. Moreover w is of maximal length in some conjugacy class C of W if and only if each w_i is of maximal length in a conjugacy class of W_i. Therefore Theorem 1.1 holds if and only if it holds for all finite irreducible Coxeter groups. Theorem 1.1 has already been proved for types A_n, B_n and D_n (Proposition 2.4 and Corollary 4.3). The exceptional groups E_6, E_7, E_8, F_4, H_3 and H_4 were checked using the computer algebra package MAGMA[1]. In each case there is at least one (usually many) elements of maximal length and excess zero in every conjugacy class. Finally it is easy to check that every element of the dihedral group has excess zero, so the result is trivially true. Thus Theorem 1.1 holds for every finite irreducible Coxeter group, and hence for all finite Coxeter groups. $\square$

Theorem 1.1 shows the existence of at least one element of maximal length and excess zero in every conjugacy class of a finite Coxeter group. However, if one looks at some small examples in the classical Weyl groups, it appears that every element of maximal length in a conjugacy class has excess zero. It is natural to ask whether this holds in general. It turns out that it does not – although the number of elements for which it fails seems to be small. For example, if W is of type E_6, then in 23 of the 25 conjugacy classes every element of maximal length has excess zero. If W is of type E_7, then every element of maximal length in 59 of the 60 conjugacy classes has excess zero. In the remaining class, which consists of elements of order 3, there are 708 elements of maximal length, all but 50 of which have excess zero.
For the classical Weyl groups, we have checked all conjugacy classes of these groups for n up to 10, and in each case every element of maximal length in a conjugacy class has excess zero. However, as Lemma 5.1 shows, there are examples of elements w of maximal length with an arbitrarily large number of pairs of involutions xy with $w = xy$, such that only one such pair has the property that $\ell(w) = \ell(x) + \ell(y)$. Elements like these 'only just' have zero excess. Because of near misses such as this, we are not sufficiently confident that the pattern of maximal length elements having zero excess will continue, even in classical groups. It would be interesting to know whether it does.

Lemma 5.1. *Let W be of type B_n, for $n \geq 2$. There are at least 2^n pairs of* *involutions (x, y) such that $xy = (\overset{-}{1}\,\overset{+}{2})$, but only one of these pairs has the property that $\ell(x) + \ell(y) = \ell((\overset{-}{1}\overset{+}{2}))$.*

Proof. The element $w = (\overset{-}{1}\,\overset{+}{2})$ is certainly of maximal length in its conjugacy class, by Theorem 1.2. If x is an involution such that $xy = w$ for some involution y, then $w^x = w^{-1}$. Thus $x = x_1 x_2$ where x_1 and x_2 are commuting involutions, x_2 fixes 1 and 2, and x_1 is either $(\overset{-}{1})$, $(\overset{-}{2})$, $(\overset{-}{1}\overset{-}{2})$ or $(\overset{+}{1}\overset{+}{2})$. We can then determine y, and the upshot is that we get the following possibilities, where here z is any involution fixing 1 and 2.

$$\begin{array}{cc}
x & y \\
(\overset{-}{1})z & (\overset{-}{1}\overset{-}{2})z \\
(\overset{-}{2})z & (\overset{+}{1}\overset{+}{2})z \\
(\overset{-}{1}\overset{-}{2})z & (\overset{-}{2})z \\
(\overset{+}{1}\overset{+}{2})z & (\overset{-}{1})z
\end{array}$$

If $N(x) \cap N(y) = \emptyset$ then clearly we must have $z = 1$. It is now a quick check to show that the only possibility is $x = (\overset{-}{2})$, $y = (\overset{+}{1}\overset{+}{2})$. The number of possible pairs (x, y) is four times the number of involutions in a Coxeter group of type B_{n-2}, which is at least 2^{n-2}, because for all subsets $\{a_1, \ldots, a_k\}$ of size k of $\{3, 4, \ldots, n\}$ the element $(\overset{-}{a_1}) \cdots (\overset{-}{a_k})$ is an involution. $\square$

References

[1] Cannon, J.J. and Playoust, C. *An Introduction to Algebraic Programming with* MAGMA [draft], Springer-Verlag (1997).

[2] Carter, R.W. *Conjugacy Classes in the Weyl Group*, Compositio. Math. **25**, Facs. 1 (1972), 1–59.

[3] Carter, R. W. *Finite groups of Lie type. Conjugacy classes and complex characters* Wiley Classics Library. John Wiley and Sons, Ltd., Chichester, 1993. 544 pp.

[4] Cauchy, A. *Exercises d'analyse et de physique mathematique*, 1844.

[5] Geck, M., Kim, S. and Pfeiffer, G. *Minimal length elements in twisted conjugacy classes of finite Coxeter groups*, J. Algebra 229 (2000), no. 2, 570–600.

[6] Geck, M. and Pfeiffer, G. *Characters of Finite Coxeter Groups and Iwahori-Hecke Algebras*, LMS Monographs, New Series 21, (2000).

[7] Hart, S.B. and Rowley, P.J. *Involution products in Coxeter groups*, J. Group Theory 14 (2011), no.2, 251–259.

[8] Hart, S.B. and Rowley, P.J. *Zero excess and minimal length in finite coxeter groups*, J. Group Theory 15 (2012), no.4, 497–512.

[9] Hart, S.B. and Rowley, P.J. *On excess in finite Coxeter groups*, Journal of Pure and Applied Algebra 219 (2015), no.5, 1657-1669.

[10] Humphreys, J.E. *Reflection Groups and Coxeter Groups*, Cambridge Studies in Advanced Mathematics, 29 (1990).

[11] Kim, S. *Mots de longueur maximale dans une classe de conjugaison d'un groupe symtrique, vu comme groupe de Coxeter (French) [Words of maximal length in a conjugacy class of a symmetric group, viewed as a Coxeter group]*, C. R. Acad. Sci. Paris Sr. I Math. 327 (1998), no. 7, 617–622.

[12] Shi, Jian-Yi. *The enumeration of Coxeter elements*, J. Algebraic Combin. 6 (1997), no. 2, 161–171.

[13] Specht, W. *Darstellungstheorie der Hyperoktaedergruppe*, Math. Z. 42 (1937), 629 – 640.

[14] Young, A. *The collected papers of Alfred Young, 1873 – 1940. With a foreword by G. de B. Robinson and a biography by H. W. Turnbull*, Mathematical Expositions, 21, University of Toronto Press, Toronto, 1977.