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THE e-EXCHANGE BASIS GRAPH AND MATROID CONNECTEDNESS

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Abstract. Let M be a matroid and $e \in E(M)$. The e-exchange basis graph of M has vertices labeled by bases of M, and two vertices are adjacent when the bases labeling them have symmetric difference $\{e, x\}$ for some $x \in E(M)$. In this paper we show that a connected matroid is exactly a matroid with the property that for every element $e \in E(M)$, the e-exchange basis graph is connected.

1. Preliminaries

Terminology will follow Oxley [6]. Let M be a matroid. The set of bases of M is denoted by $\mathcal{B}(M)$. In the basis graph $G_{\mathcal{B}}(M)$, the vertices are labeled by the the elements of $\mathcal{B}(M)$. Let $B_1, B_2 \in \mathcal{B}(M)$. The vertices B_1 and B_2 are adjacent in $G_{\mathcal{B}}(M)$ exactly when $|B_1 \triangle B_2| = 2$.

It seems that interest in these structures for graphic matroids started in the mid 1960s [1] when they were called tree graphs. This name persisted even though Carlos Holzmann and Frank Harary studied them for general matroids [2]. Stephen Maurer completed the most in-depth treatment of the subject and called them basis graphs. His doctoral thesis and two resulting articles [3] and [4] were published in the early 1970s. More recently, they have been called basis exchange graphs or bases exchange graphs. This terminology has been attributed to Jack Edmonds [5]. While each name has particular advantages, we will refer to the structures by the shortest accepted name and call them *basis graphs*.

It is well known that the basis graph of a matroid is connected for any matroid. This is an easy result of the basis exchange axiom, and this result is listed as part of Theorem 2.1 in [3].

Lemma 1.1. *Let M be a matroid. Then $G_{\mathcal{B}}(M)$ is connected.*

It is easy to see that the basis graph of a matroid is isomorphic to the that of its dual.

Lemma 1.2. *Let M be a matroid. Then $G_{\mathcal{B}}(M^*) \cong G_{\mathcal{B}}(M)$. Further, $G_{\mathcal{B}}(M^*)$ is found by relabeling each vertex of $G_{\mathcal{B}}(M)$ by its complement in $E(M)$.*

Suppose a matroid M has element e. The *e-exchange basis graph* of M, which we will denote $G_{\mathcal{B},e}(M)$, is a subgraph of $G_{\mathcal{B}}(M)$. The vertex set of $G_{\mathcal{B},e}(M)$ is the same as that of $G_{\mathcal{B}}(M)$, and B_1B_2 is an edge for

$B_1, B_2 \in \mathcal{B}(M)$ exactly when $B_1 \triangle B_2 = \{e, f\}$ for some $f \in E(M)$. It is easy to see that this graph is bipartite with one part containing all the bases containing e. The next lemma is related to Lemma 1.2.

Lemma 1.3. *Let M be a matroid with $e \in E(M)$. Then $G_{\mathcal{B},e}(M^*) \cong G_{\mathcal{B},e}(M)$. Further, $G_{\mathcal{B},e}(M^*)$ is found by relabeling each vertex of $G_{\mathcal{B},e}(M)$ by its complement in $E(M)$.*

Now consider the case where e is a loop. Then e is avoided by every basis, and the following is an easy result.

Lemma 1.4. *The graph $G_{\mathcal{B},e}(M)$ has no edges if and only if e is a loop or coloop.*

The main part of our result has previously been established in the technical report [5], where machinery is developed showing that $G_{\mathcal{B},e}(M)$ is connected for every element e if and only if M is a connected matroid. The same result follows from elementary matroid operations, and this paper gives a shorter proof.

2. Main Result

While we have already seen that the basis graph is connected for every matroid, we see from Lemma 1.4 that there are matroids for which an e-exchange basis graph is not connected. In fact, we characterized the graphs having no edges, and this lemma points to the larger theorem that the connectedness of the e-exchange basis graphs of M is related to the connectedness of the underlying matroid. Our main result is that a matroid M is connected exactly when for every element $e \in E(M)$, the graph $G_{\mathcal{B},e}(M)$ is connected.

Theorem 2.1. *Let M be a matroid with at least two elements. The following are equivalent.*

(1) If $e \in E(M)$, then $G_{\mathcal{B},e}(M)$ is connected and has at least one edge.
(2) If $e, f \in E(M)$ are distinct, then there is a basis $B \in \mathcal{B}(M)$ so that $B \triangle \{e, f\} \in \mathcal{B}(M)$.
(3) M is connected.

Proof. First, we show that (2) and (3) are equivalent. A matroid M is connected exactly when for every pair of elements $\{e, f\} \in E(M)$ there is a circuit C containing $\{e, f\}$. This occurs exactly when there is a basis B containing f but not e, such that f is containted in the fundamental circuit of $B \cup e$. This occurs if and only if $B \triangle \{e, f\}$ is also a basis of M.

Now, we assume (1) and prove that (2) follows. Let $e, f \in E(M)$. Let V_e be the set of vertices of $G_{\mathcal{B}}(M)$ labeled by bases containing e. Let V_f be the set of vertices of $G_{\mathcal{B}}(M)$ labeled by bases containing f.

Suppose, for the sake of contradiction, that $V_e \subseteq V_f$. Then every basis containing e also contains f. As $G_{\mathcal{B},f}(M)$ is connected and bipartite with V_f as one part, there is a vertex, A, not in V_f. Observe that $e, f \notin A$. Since

$G_{\mathcal{B},e}(M)$ is connected and bipartite with V_e as one part, this graph contains an edge from vertex A to some vertex A' of V_e. Observe that A' contains e and f. As bases are equicardinal, $\{e, f\}$ is a proper subset of $A \triangle A'$ and $|A \triangle A'| > 2$, a contradiction. Therefore, the supposition is false and there is a basis, B_e of M so that $B_e \in V_e - V_f$. Symmetrically, there is a basis B_f of M so that $B_f \in V_f - V_e$.

Again, relying on (1), we know $G_{\mathcal{B},f}(M)$ is connected, so there is a path from B_e to B_f. Somewhere on this path, there is an edge connecting two bases with symmetric difference $\{e, f\}$. Thus (1) implies (2).

Finally, we demonstrate that (2) implies (1). Therefore, assume (2). For the sake of contradiction, we assume that there is an element $e \in E(M)$ so that $G_{\mathcal{B},e}(M)$ is not connected.

By assumption, we know for any $f \in E(M)-e$ that $\{e, f\}$ is the symmetric difference of the labels of their endvertices. Thus we conclude that the second part of (1) holds: $G_{\mathcal{B},e}(M)$ has at least one edge.

By our assumption, $G_{\mathcal{B},e}(M)$ is comprised of $n \geq 2$ components, $\chi_1, \chi_2, \dots, \chi_n$. By Lemma 1.1, $G_{\mathcal{B}}(M)$ is connected. So M has distinct bases B_i and B_j so that $B_i \in \chi_i$ and $B_j \in \chi_j$ and $B_i \triangle B_j = \{e_i, e_j\}$. We assume $e_i \in B_i$ and $e_j \in B_j$.

If $e \in \{e_i, e_j\}$, this would indicate an edge in $G_{\mathcal{B},e}(M)$ between different components, a contradiction. Thus either $e \in B_i \cap B_j$ or e avoids both bases. Relying on Lemma 1.3, up to duality, we may assume the former is true.

The set $H = \text{cl}(B_i - e)$ is a hyperplane. Notice $e_i \in H$. Label the complementary cocircuit, $C^* = E - H$. For $f \in C^* - e$, the set $B_i \triangle \{e, f\}$ is a basis of M. The element e_j is not in C^*. Otherwise $B_i \triangle \{e, e_j\} \in \mathcal{B}(M)$ making $B_i \triangle \{e, e_j\}$ adjacent to B_i and B_j in $G_{\mathcal{B},e}(M)$, a contradiction to their residing in different components. Because $e_j \in H$, clearly $H = \text{cl}(B_j - e)$. And so for $f \in C^* - e$, the set $B_j \triangle \{e, f\}$ is a basis of M.

In order to complete this proof, we will need to examine the structure of these two components in detail. It will be helpful to label the edges by the symmetric difference of the labels of the vertices.

Consider the subgraph of $G_{\mathcal{B},e}(M)$ induced by B_i and its neighbors. As $G_{\mathcal{B},e}(M)$ is bipartite, this graph is a star. The edges are labelled by $\{e, f\}$ for each element f in $C^* - e$. Exactly the same is true for the subgraph induced by B_j and its neighbors. Moreover the subgraph induced by B_j and its neighbors may be obtained from that induced by B_i and its neighbors by replacing B_i with B_j and replacing each occurrence of e_i with e_j in the vertex labels.

For each $f \in C^* - e$, the preceding argument may be repeated with $B_i \triangle \{e, f\}$ and $B_j \triangle \{e, f\}$ playing the roles of B_i and B_j, respectively. Extending this argument, we can make similar conclusions about all the vertices in the components χ_i and χ_j. Every basis labeling a vertex in χ_i contains e_i. Modifying χ_i by replacing the element e_i with e_j in the vertex labels produces χ_j. The set of edge labels in the first component is exactly the set of edge labels in the second component. Further, the same argument may be applied to every pair of components of $G_{\mathcal{B},e}(M)$, so the same edge labels appear in each component. Since e_i appears in every basis labeling a vertex of the component χ_i, the set $\{e, e_i\}$ does not appear as an edge label in the first component. Thus this set labels no edge in $G_{\mathcal{B},e}(M)$. This contradicts (2), proving our claim. □

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