Strongly Real Beauville Groups III

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Abstract Beauville surfaces are a class of complex surfaces defined by letting a finite group G act on a product of Riemann surfaces. These surfaces possess many attractive geometric properties several of which are dictated by properties of the group G. A particularly interesting subclass are the 'strongly real' Beauville surfaces that have an analogue of complex conjugation defined on them. In this survey we discuss these objects and in particular the groups that may be used to define them. *En route* we discuss several open problems, questions and conjectures and in places make some progress made on addressing these.

1 Introduction

The reader asking 'where are the first two instalments of this series?' should note the following. Morally the first instalment of this series (i.e. 'Strongly Real Beauville Groups I') is [29] whilst its sequel (i.e. 'Strongly Real Beauville Groups II') is [30]. Each instalment is fairly self-contained (to the point of having a fair amount of overlap in their introductory sections) and so hopefully the reader will lose little if they have neither read nor have to hand copies of these.

Roughly speaking (precise definitions will be given in the next section), a Beauville surface is a complex surface $\mathscr{S}$ defined by taking a pair of complex curves, i.e. Riemann surfaces, $\mathscr{C}_1$ and $\mathscr{C}_2$ and letting a finite group G act freely on their product to define $\mathscr{S}$ as a quotient $(\mathscr{C}_1 \times \mathscr{C}_2)/G$. These surfaces have a wide variety of attractive geometric properties: they are surfaces of general type; their automorphism groups [53] and fundamental groups [7, 23] are relatively easy to compute (being closely related to G); they are rigid surfaces in the sense of ad-

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mitting no nontrivial deformations [9] and thus correspond to isolated points in the moduli space of surfaces of general type [42].

Much of this good behaviour stems from the fact that the surface $(\mathscr{C}_1 \times \mathscr{C}_2)/G$ is uniquely determined by a particular pair of generating sets of G known as a 'Beauville structure'. This converts the study of Beauville surfaces to the study of groups with Beauville structures, i.e. Beauville groups.

Beauville surfaces were first defined by Catanese in [19] as a generalisation of an earlier example of Beauville [13, Exercise X.13(4)] (native English speakers may find the English translation [14] somewhat easier to read and get hold of) in which $\mathscr{C}_1 = \mathscr{C}_2$ and the curves are both the Fermat curve defined by the equation $X^5 + Y^5 + Z^5 = 0$ being acted on by the group $(\mathbb{Z}/5\mathbb{Z}) \times (\mathbb{Z}/5\mathbb{Z})$ (this choice of group may seem somewhat odd at first, but the reason will become clear later). Bauer, Catanese and Grunewald went on to use these surfaces to construct examples of smooth regular surfaces with vanishing geometric genus [10]. Early motivation came from the consideration of the 'Friedman-Morgan speculation' — a technical conjecture concerning when two algebraic surfaces are diffeomorphic which Beauville surfaces provide counterexamples to. More recently, they have been used to construct interesting orbits of the absolute Galois group Gal($\overline{\mathbb{Q}}/\mathbb{Q}$) (connections with Gothendeick's theory of *dessins d'enfants* make it possible for this group to act on the set of all Beauville surfaces). Indeed one of the more impressive applications of these surfaces is the proof by González-Diez and Jaikin-Zapirain in [44] that Gal($\overline{\mathbb{Q}}/\mathbb{Q}$) acts faithfully on the set of regular dessins by showing that it acts regularly on the set of Beauville surfaces.

Furthermore, Beauville's original example has also been used by Galkin and Shinder in [40] to construct examples of exceptional collections of line bundles.

Like any survey article, the topics discussed here reflect the research interests of the author. Slightly older surveys discussing related geometric and topological matters are given by Bauer, Catanese and Pignatelli in [11, 12]. Other notable works in the area include [6, 28, 54, 67, 73].

In Section 2 we provide preliminary information and in particular give specific definitions for the concepts we have only talked about very vaguely until now. In Section 3 we will discuss the finite simple groups before considering the more general case of characteristically simple groups in Section 4. In Section 5 we move to the opposite extreme by considering abelian and nilpotent groups. In Section 6 we will discuss recent work on Doubly Hurwitz Beauville groups and related constructions before finally in Section 7 discussing a number of smaller and more minor matters.

Acknowledgements The author wishes to express his deepest gratitude to the organisers of the 2018 London Mathematical Society Midlands Regional Meeting and the associated workshop on Galois covers, Grothendieck– Teichmüller theory and *Dessins d'enfants* without whose work this volume would not have appeared and to Emeritus Professor Robert Wilson for invaluable correspondence and computational assistance with the Baby Monster group. Finally the author also wishes to thank the anonymous referee whose numerous comment substantively improved the quality of the end result.

2 Preliminaries

We give the main definition.

Definition 1. A surface $\mathscr{S}$ is a *Beauville surface of unmixed type* if

- the surface $\mathscr{S}$ is isogenous to a higher product, that is, $\mathscr{S} \cong (\mathscr{C}_1 \times \mathscr{C}_2)/G$ where $\mathscr{C}_1$ and $\mathscr{C}_2$ are complex algebraic curves of genus at least 2 and G is a finite group acting faithfully on $\mathscr{C}_1$ and $\mathscr{C}_2$ by holomorphic transformations in such a way that it acts freely on the product $\mathscr{C}_1 \times \mathscr{C}_2$, and
- each $\mathscr{C}_i/G$ is isomorphic to the projective line $\mathbb{P}_1(\mathbb{C})$ and the corresponding covering map $\mathscr{C}_i \to \mathscr{C}_i/G$ is ramified over three points.

There also exists a concept of Beauville surfaces of mixed type in which the action of G interchanges the two curves $\mathscr{C}_1$ and $\mathscr{C}_2$ but these are much harder to construct and we shall not discuss these here. (For further details of the mixed case, the most up-to-date information at the time of writing may be found in the work of the author and Pierro in [36].)

In the first of the above conditions the genus of the curves in question needs to be at least 2. It was later proved by Fuertes, González-Diez and Jaikin-Zapirain in [38] that in fact we can take the genus as being at least 6. The second of the above conditions implies that each $\mathscr{C}_i$ carries a regular dessin in the sense of Grothendieck's theory of *dessins d'enfants* (children's drawings) [47]. Furthermore, by Belyĭ's Theorem [15] this ensures that $\mathscr{S}$ is defined over an algebraic number field in the sense that when we view each Riemann surface as being the zeros of some polynomial we find that the coeffcients of that polynomial belong to some number field. Equivalently they admit an orientably regular hypermap [58], with G acting as the orientation-preserving automorphism group. A modern account of *dessins d'enfants* and proofs of Belyĭ's theorem may be found in the recent book of Girondo and González-Diez [43].

These constructions can also be described instead in terms of uniformization and using the language of Fuchsian groups [46, 71].

What makes this class of surfaces so good to work with is the fact that all of the above definition can be 'internalised' into the group. It turns out that a group G can be used to define a Beauville surface if and only if it has a certain pair of generating sets known as a Beauville structure.

Definition 2. Let G be a finite group. For $x, y \in G$ let

$$\Sigma(x,y) := \bigcup_{i=1}^{|G|} \bigcup_{g \in G} \{(x^i)^g, (y^i)^g, ((xy)^i)^g\}.$$

An *unmixed Beauville structure* for the group G is a set of pairs of elements $\{\{x_1, y_1\}, \{x_2, y_2\}\} \subset G \times G$ with the property that $\langle x_1, y_1 \rangle = \langle x_2, y_2 \rangle = G$ such that

$$\Sigma(x_1, y_1) \cap \Sigma(x_2, y_2) = \{e\}.$$

If G has a Beauville structure we say that G is a *Beauville group*. Furthermore we say that the structure has *type*

$$((o(x_1),o(y_1),o(x_1y_1)),(o(x_2),o(y_2),o(x_2y_2))).$$

In some parts of the literature authors have defined the above structure in terms of so-called 'spherical systems of generators of length 3', meaning $\{x,y,z\} \subset G$ with $xyz = e$, but we omit $z = (xy)^{-1}$ from our notation in this survey. (The reader is warned that this terminology is a little misleading since the underlying geometry of Beauville surfaces is hyperbolic thanks to the below constraint on the orders of the elements.) Furthermore, many earlier papers on Beauville structures add the condition that for $i = 1,2$ we have that

$$\frac{1}{o(x_i)} + \frac{1}{o(y_i)} + \frac{1}{o(x_iy_i)} < 1,$$

but this condition was subsequently found to be unnecessary following Bauer, Catanese and Grunewald's investigation of the wall-paper groups in [8]. A triple of elements and their orders satisfying this condition are said to be hyperbolic. Geometrically, the type gives us considerable amounts of geometric information about the surface: the Riemann-Hurwitz formula

$$g(\mathscr{C}_i) = 1 + \frac{|G|}{2}\left(1 - \frac{1}{o(x_i)} - \frac{1}{o(y_i)} - \frac{1}{o(x_iy_i)}\right)$$

tells us the genus of each of the curves used to define the surface $\mathscr{S}$ and by a theorem of Zeuthen-Segre this in turn gives us the Euler number of the surface $\mathscr{S}$ since

$$e(\mathscr{S}) = 4\frac{(g(\mathscr{C}_1)-1)(g(\mathscr{C}_2)-1)}{|G|}$$

which in turn gives us the holomorphic Euler-Poincaré characteristic of $\mathscr{S}$ since $4\chi(\mathscr{S}) = e(\mathscr{S})$ (see [19, Theorem 3.4]). On a more practical and group theoretic note, the type is often useful for verifying that the critical condition that $\Sigma(x_1,y_1) \cap \Sigma(x_2,y_2) = \{e\}$ is satisfied since this will clearly hold whenever the number $o(x_1)o(y_1)o(x_1y_1)$ is coprime to the number $o(x_2)o(y_2)o(x_2y_2)$.

The abelian Beauville groups were essentially classified by Catanese in [19, page 24.] and the full argument is given explicitly in [8, Theorem 3.4] where the following is proved.

Theorem 1. *Let G be an abelian group. Then G is a Beauville group if, and only if, $G = (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})$ where $n > 1$ is coprime to* 6.

This explains why Beauville's original example used the group $(\mathbb{Z}/5\mathbb{Z}) \times (\mathbb{Z}/5\mathbb{Z})$ — it is the smallest abelian Beauville group.

Given any complex surface $\mathscr{S}$ it is natural to consider the complex conjugate surface $\overline{\mathscr{S}}$. In particular, it is natural to ask whether or not these two surfaces are biholomorphic.

Definition 3. Let $\mathscr{S}$ be a complex surface. We say that $\mathscr{S}$ is *real* if there exists a biholomorphism $\sigma : \mathscr{S} \to \overline{\mathscr{S}}$ such that σ^2 is the identity map.

(We remark that strictly speaking the above definition is not quite right, it being impossible to compose σ with itself. It is more accurate to talk of the composition $\sigma \circ \overline{\sigma}$ where $\overline{\sigma} : \overline{\mathscr{S}} \to \mathscr{S}$.)

As is often the case with Beauville surfaces, the above geometric condition can be translated into purely group theoretic terms.

Definition 4. Let G be a Beauville group and let $X = \{\{x_1, y_1\}, \{x_2, y_2\}\}$ be a Beauville structure for G. We say that G and X are *strongly real* if there exists an automorphism $\phi \in \mathrm{Aut}(G)$ and elements $g_i \in G$ for $i = 1, 2$ such that

$$g_i \phi(x_i) g_i^{-1} = x_i^{-1} \text{ and } g_i \phi(y_i) g_i^{-1} = y_i^{-1}$$

for $i = 1, 2$.

In practice we can always replace one generating pair by some generating pair that is conjugate to it and so we can take $g_1 = g_2 = e$ and this is often what is done in practice.

In [8] Bauer, Catanese and Grunewald show that a Beauville surface is real if, and only if, the corresponding Beauville group and structure are strongly real. This all comes from the study of the following related concept in the theory of Riemann surfaces. In Singerman's nomenclature of [66], a Riemann surface with a function behaving like the function σ in Definition 3 is said to be symmetric. The relationship with automorphisms of the corresponding group critically depends on the main result of [66]. The reader is warned, however, that some notable errors in [66] were subsequently found and are corrected by Jones, Singerman and Watson in [59]. More specifically, the condition that an automorphism like the above exists is sufficient but it is not necessary. This is corrected by Jones, Singerman and Watson by giving a complete list of conditions that are both necessary and sufficient in [59, Theorem 1.1]. We thus repeat the question first posed by the author as [30, Question 1].

Question 1. Are there interesting strongly real Beauville surfaces arising from the conditions given in [59, Theorem 1.1] but not [66, Theorem 2]?

We remark that symmetric Riemann surfaces are also connected to the theory of Klein surfaces. Real algebraic curves and compact Klein surfaces are equivalent in the same way that the categories of complex algebraic curves and compact Riemann surfaces are equivalent. Indeed, just as a compact, connected, orientable surface admits the structure of a complex analytic manifold of dimension 1 (this is, a Riemann surface structure) then a compact connected surface that is not necessarily orientable admits the structure of a complex *dianalytic* manifold of dimension 1, that is, a Klein surface structure. See [65] for an introductory discussion and [17] for a recent survey of these surfaces.

By way of immediate easy examples, note that the function $x \mapsto -x$ is an automorphism of any abelian group and so every Beauville group given by Theorem

1 is an example of a strongly real Beauville group. More generally the following question is the main subject of this article.

Question 2. Which groups are strongly real Beauville groups?

3 The Finite Simple Groups

Naturally, a necessary condition for being a strongly real Beauville group is being a Beauville group. Furthermore, a necessary condition for being a Beauville group is being 2-generated: we say that a group G is 2-generated if there exist two elements $x, y \in G$ such that $\langle x, y \rangle = G$. It is an easy exercise for the reader to show that the alternating groups A_n for $n \geq 3$ are 2-generated (see the work of Miller in [63]). In [68] Steinberg proved that all of the simple groups of Lie type are 2-generated and in [1] Aschbacher and Guralnick used cohomological methods to show that the larger of the sporadic simple groups are 2-generated, the smaller ones having been dealt with by numerous previous authors. These results rely heavily on the classification of finite simple groups. We thus have that all of the non-abelian finite simple groups are 2-generated making them natural candidates for Beauville groups. This led Bauer, Catanese and Grunewald to conjecture that aside from A_5, which is easily seen to not be a Beauville group, every non-abelian finite simple group is a Beauville group — see [8, Conjecture 1] and [9, Conjecture 7.17]. This suspicion was later proved correct [26, 27, 41, 48], indeed the full theorem proved by the author, Magaard and Parker in [27] is actually a more general statement about quasisimple groups (recall that a group G is quasisimple if it is generated by its commutators and the quotient by its center $G/Z(G)$ is a simple group.). A sketch of the proof of this Theorem is given by the author in [28, Section 3].

Having found that all but one of the non-abelian finite simple groups are Beauville groups, it is natural to ask which of the finite simple groups are strongly real Beauville groups. In [8, Section 5.4] Bauer, Catanese and Grunewald wrote

> There are 18 finite simple nonabelian groups of order ≤ 15000. By computer calculations we have found strongly [real] Beauville structures on all of them with the exceptions of A_5, $\mathrm{PSL}_2(7)$, A_6, A_7, $\mathrm{PSL}_3(3)$, $\mathrm{U}_3(3)$ and the Mathieu group M_{11}.

On the basis of this they made the following conjecture.

Conjecture 1 (The Weak Strongly Real Conjecture). All but finitely many of the finite simple groups are Strongly Real Beauville Groups.

In addition to the above, further ‘circumstantial evidence’ for this conjecture come from the following recent theorem of Malcolm [62] which suggests that if x and y can be simultaneously inverted by an inner automorphism, then we have plenty of control over $\Sigma(x, y)$.

Theorem 2. *Every element of every non-abelian finite simple group is a product of two strongly real elements.*

Subsequently, various infinite families of simple groups were shown to satisfy Conjecture 1 (including some alluded to above) and computations performed by the author lead to the following [29, Conjecture 1].

Conjecture 2 (The Strong Strongly Real Conjecture). All non-abelian finite simple groups apart from A_5, M_{11} and M_{23} are strongly real Beauville groups.

As far as the author is aware no advances in proving the conjecture has been made since [29] appeared so we refer the reader there for the specific information on the most recent progress on this conjecture.

4 Characteristically Simple Groups

Another class of finite groups that has recently been studied from the viewpoint of Beauville constructions, and appears to be fertile ground for providing further examples of strongly real Beauville groups, are the characteristically simple groups that we define as follows (the definition commonly given is somewhat different from the below but in the case finite groups it can easily be shown to be equivalent to the below).

Definition 5. A finite group G is said to be *characteristically simple* if G is isomorphic to some direct product S^k where S is a finite simple group.

For example, as we saw in Theorem 1, if $p > 3$ is prime then the abelian Beauville groups isomorphic to $(\mathbb{Z}/p\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z})$ are characteristically simple as is every finite simple group.

The study of characteristically simple Beauville groups was initiated by Jones in [55, 56] where the following conjecture is discussed.

Conjecture 3. Let G be a finite non-abelian characteristically simple group. Then G is a Beauville group if and only if it is a 2-generated group not isomorphic to A_5.

At the time that the previous installment of this series [30] was written the author was skeptical about 2-generated characteristically simple groups being strongly real Beauville groups. Consequently some calculations showing that S^k is strongly real for small values of k when $|S| < 100\,000$ was all that was given in [30, Section 4]. The author also gave a rather pithy conjecture regarding the groups of the form $S \times S$. Since then in [35] the author and Jones have generalised these results substantially with the following Theorem.

Theorem 3. *The 2-generated characteristically simple group S^k is a strongly real Beauville group if S is any of the following groups.*

(a) The alternating group A_n apart from $(n,k) = (5,1)$;
(b) The groups $L_2(q)$ apart from $(q,k) = (4,1)$ or $(5,1)$;
(c) The sporadic groups apart from $(S,k) = (M_{11},1)$ or $(M_{23},1)$ and

(d) Simple groups of order at most 10,000,000.

Simple groups of order at most 10,000,000 includes the groups $L_n(q)$ of order less than or equal to $|L_4(4)|$ and the ten smallest sporadic groups as well as the Tits group, ${}^2F_4(2)'$, among many others. A more comprehensive list may be found [22, pp. 239–240].

The above was proved as a step towards verifying the following, a substantial extension of Conjecture 2.

Conjecture 4 (The Strongly Strong Strongly Real Conjecture). Every 2-generated characteristically simple group S^k is a strongly real Beauville group apart from $(S,k) = (\mathrm{A}_5,1)$, $(\mathrm{M}_{11},1)$ or $(\mathrm{M}_{23},1)$.

5 Abelian and Nilpotent Groups

Recall that the abelian Beauville groups were classified in Theorem 1 and that an immediate corollary of this result is that every abelain Beauville group is strongly real.

Theorem 1 has been put to great use by González-Diez, Jones and Torres-Teigell in [45] where several structural results concerning the surfaces defined by abelian Beauville groups are proved. For each abelian Beauville group they describe all the surfaces arising from that group, enumerate them up to isomorphism and impose constraints on their automorphism groups. As a consequence they show that all such surfaces are defined over $\mathbb{Q}$.

After the abelian groups, the next most natural class of finite groups to consider are the nilpotent groups. In [2, Lemma 1.3] Barker, Boston and the author note the following easy Lemma.

Lemma 1. *Let G and G' be Beauville groups and let $\{\{x_1,y_1\},\{x_2,y_2\}\}$ and $\{\{x_1',y_1',\},\{x_2',y_2'\}\}$ be their respective Beauville structures. Suppose that*

$$gcd(o(x_i),o(x_i')) = gcd(o(y_i),o(y_i')) = 1$$

for $i = 1,2$. Then $\{\{(x_1,x_1',),(y_1,y_1')\},\{(x_2,x_2'),(y_2,y_2')\}\}$ is a Beauville structure for the group $G \times G'$.

Recall that a finite group is nilpotent if, and only if, it is isomorphic to the direct product of its Sylow subgroups. It thus follows that Lemma 1, and its easy to prove converse, reduces the study of nilpotent Beauville groups to that of Beauville p-groups. Note that Theorem 1 gives us infinitely many examples of Beauville p-groups for every prime $p > 3$: simply let n be any power of p. Early examples of Beauville 2-groups and 3-groups were constructed by Fuertes, González-Diez and Jaikin-Zapirain in [38] where a Beauville group of order 2^{12} and another of order 3^{12} were constructed. Even earlier than this, two (mixed) Beauville 2-groups of order 2^8 arose as part of a classification due to Bauer, Catanese and Grunewald in [10] of certain classes of surfaces of general type, which give rise to an example of an (unmixed) Beauville 2-group of order 2^7.

Subsequently, in [2] Barker, Boston and the author classified the Beauville p-groups of order at most p^4 and made substantial progress on the cases of groups of order p^5 and p^6. Later, in [69] Stix and Vdovina have constructed another infinite series of Beauville p-groups. In particular this gives the first examples of non-abelian Beauville p-groups of arbitrarily large order and any prime $p \geq 5$. To do this they make use of the theory of pro-p groups and in doing so provide generalisations of examples from [2]. The first explicit construction of an infinite family of Beauville 3-groups was recently given by Fernández-Alcober and Gül in [37] where they consider homomorphic images of the famous Nottingham group as well as providing other general constructions for Beauville p-groups. In doing so they settled several conjectures made in [2].

The earliest explicit infinite family of Beauville 2-groups were constructed by Barker, Boston, Peyerimhoff and Vdovina in [3, 4, 5] where, again, more general constructions are also considered. The most comprehensive surveys on Beauville p-groups in general are given by Boston in [16] and more recently by the author in [24].

Few of the known examples of Beauville p-groups are known to either be strongly real/non-strongly real. As far as the author is aware the earliest examples of non-abelian strongly real Beauville p-groups to be discovered were an isolated pair of examples of 2-groups constructed by the author in [30, Section 7] namely the groups
$$\langle u,v \,|\, (u^iv^j)^4,\, i,j=0,1,2,3\rangle$$
which has order 2^{14} and
$$\langle u,v \,|\, u^8, v^8, [u^2,v^2], (u^iv^j)^4,\, i,j=1,2,3\rangle$$
which has order 2^{13}.

Recently in [50] Gül constructed the first known infinite family of non-abelian strongly real Beauville p-groups and in particular discovered the first examples in which p is odd. More specifically, the main result of [50] is the following.

Theorem 4. *Let $F = \langle x,y \,|\, x^p, y^p\rangle$ be the free product of two cyclic groups of order p for an odd prime p and let $i = k(p-1)+1$ for $k \geq 1$. Then the quotient $F/\gamma_{i+1}(F)$ is a strongly real Beauville group.*

Subsequently in [51] Gül constructed further examples by considering quotients of certain triangle groups. More specifically Gül prove that there are non-abelian strongly real Beauville p-groups of order p^n for every $n \geq 3$, 5 or 7 for the primes $p \geq 5$, $p = 3$ and $p = 7$ respectively.

At around the same time the author constructed another infinite family of non-abelian strongly real Beauville p-groups for p odd in [32, 33] by proving the following.

Theorem 5. *Let p be an odd prime and let q and r be powers of p. If q and r are sufficiently large, then groups $C_q \wr C_r / Z(C_q \wr C_r)$ are strongly real Beauville groups.*

Unlike the groups given by Theorem 4 this theorem gives multiple non-isomorphic examples for infinitely many orders. For example when $(q,r)=(3^{28},3^3)$ or $(q,r)=(3^3,3^5)$ we obtain groups of order 3^{731} which cannot be isomorphic since they have centers of different orders.

By way of a new result we have the following.

Proposition 1. *For every prime p, the smallest non-abelian Beauville p-group is a strongly real Beauville group.*

Proof. The smallest non-abelian Beauville p-groups were determined by Barker, Boston and the author in [2]. For $p=2$ the smallest example is the group defined by the following presentation.

$$G := \langle u,v \,|\, (u^iv^j)^4 \text{ for } i,j=0,\ldots,3,\, (u^2v^2)^2,\, [u,v]^2,\, (uvuv^3)^2 \rangle$$

For an automorphism, we consider the group defined by the above presentation with an additional generator that we call t along with the new relations t^2, u^tu, v^tv. It is easy to see that if we take

$$x_1 := u, \quad y_1 := v, \quad x_2 := uvu \quad \text{and} \quad y_2 := uvuvu,$$

then $\langle x_1,y_1\rangle = \langle x_2,y_2\rangle = G$; conjugation by t inverts all of these elements and the conjugacy condition is easily checked computationally.

The case $p=3$ is similar.

The cases $p\geq 5$ are a special case of [34, Proposition 11]. □

One obvious place providing fertile ground for new examples of strongly real Beauville p-groups are subgroups of larger known Beauville p-groups because the exponent of a subgroup is at most that of the original group. The aforementioned group constructed in [30, Section 7] has an automorphism group of order 2^{25} suggesting its subgroup structure morally should work. Alas this good idea quickly falls down. None of the proper subgroups of order greater than 2^9 are even 2-generated, let alone are Beauville groups. As mentioned earlier in this section, no subgroup of order less than 2^7 is even a Beauville group, let alone a strongly real one suggesting that the subgroup structure of this group provides little in the way of new examples.

6 Doubly Hurwitz/Minimal Beauville Groups

We recall the following.

Definition 6. A finite group G is a *Hurwitz group* if it can be generated by an element of order 2 and an element of order 3 such that their product has order 7.

The study of these objects is motivated by Hurwitz's automorphisms theorem which states that the automorphism group of a compact Riemann surface of genus

$g \geq 2$ has order at most $84(g-1)$ with equality if and only if the automorphism group is a Hurwitz group. It is easy to show that a Hurwitz group is necessarily perfect making simple groups the natural starting point for investigating these objects.

Recently in [57] Jones and Pierro addressed a question of Zvonkin asking if there exist groups that act as Hurwitz groups in two essentially different ways, that is, which have two generating triples that together provide a Beauville structure.

Definition 7. A *doubly Hurwitz Beauville group* or *dHB group* is a Beauville group of type ((2,3,7),(2,3,7)).

The main results of [57] are summed up in the following.

Theorem 6.*(a) The following are doubly Hurwitz Beauville groups.*

(i) The alternating group A_n for all $n \geq 589$.
(ii) The groups $SL_n(q)$ and $L_n(q)$ for all $n \geq 631$ and prime powers q.

(b) None of the following are doubly Hurwitz Beauville groups.

(i) The sporadic simple groups.
(ii) The groups $L_n(q)$ for $n \leq 7$, ${}^2G_2(3^r)$, ${}^2F_4(2)'$, $G_2(q)$ and ${}^3D_4(q)$.

The basic question of which groups are doubly Hurwitz Beauville groups remains far from resolved, the content of [57] being just a first step, but despite this the following harder question still seems worth asking.

Question 3. Which groups have strongly real Beauville structures of type ((2,3,7),(2,3,7))?

For general discussions of the current knowledge of Hurwitz groups and their corresponding surfaces, see the two excellent surveys of Conder [20, 21] and the more historically-oriented survey of MacBeath [61].

Of course not all groups are Hurwitz groups however every group is the automorphism group of various Riemann surfaces and every group will attain the minimum genus on some surface. Given a group G its *strong symmetric genus* is the minimum genus of a compact Riemann surface on which G acts as a group of automorphisms preserving orientation. For groups that are not Hurwitz groups we can ask the more general analogous question replacing (2,3,7) with whatever type achieves the strong symmetric genus of the group.

Definition 8. A *doubly minimal Beauville group* or *dmB group* is a Beauville group G of type $((a,b,c),(a,b,c))$ where (a,b,c) attains the strong symmetric genus of G.

Question 4. Which Beauville groups are dmB groups?

For the sporadic groups we have the following.

Lemma 2. *None of the sporadic groups, except possibly the baby monster group $\mathbb{B}$, define Beauville surfaces corresponding to Riemann surfaces that attain their strong symmetric genus.*

Proof. The sporadic groups that are Hurwitz are dealt with in [57]. The generators that attain the strong symmetric genus of the remaining groups are given in Table 1. Each of M_{11}, M_{23}, J_3, McL and O'N have only one class of involutions. Similar arguments rule out the groups M_{12}, M_{22}, HS and Co_2. In the groups Co_1 and Fi_{23} all elements of order 8 power up to the same class of involutions. The group M_{24} is easily ruled out computationally (naively scrolling through the elements of the group, it is small enough to do this, shows that a (3,3,4) generating pair necessarily uses elements of class 3B). □

G	Type	G	Type	G	Type
M_{11}	(2,4,11)	M_{12}	(2,3,10)	M_{22}	(2,5,7)
M_{23}	(2,4,23)	HS	(2,3,11)	J_3	(2,4,5)
M_{24}	(3,3,4)	McL	(2,5,8)	Suz	(2,4,5)
O'N	(2,3,8)	Co_2	(2,3,11)	Fi_{23}	(2,3,8)
Co_1	(2,3,8)	$\mathbb{B}$	(2,3,8)		

Table 1 The sporadic simple groups that are not Hurwitz groups and the types of their generators that attain their symmetric genus.

The Baby Monster group $\mathbb{B}$ is famously computationally difficult to deal with: its lowest degree representation is in 4371 dimensions and its lowest degree permutation representation is on around 10^{10} points owing to have order around 10^{34}. Worse, the Baby Monster has far more conjugacy classes for us to worry about than the smaller cases. In [72] Wilson showed that the Baby Monster is not a Hurwitz group but is (2,3,8) generated. There are four classes of involutions; two classes of elements of order 3 and fourteen classes of elements of order 8! Structure constant calculations naively rule out very few cases without a detailed investigation of its 30 classes of maximal subgroups most of which contain elements of all of these orders. More worryingly we do not know the character tables of most of the maximal subgroups and calculating them in the larger cases a computationally taxing problem. Generating triples of classes $(2C,3B,8X)$ where X is any of N, M and K and $(2D,3B,8X)$ where X is one of N, M, K and I do exist but none using class $3A$ are known. It is unlikely that there are any suggesting that this case is the same as the other sporadic groups.

Problem 1. Settle the case of the Baby Monster.

One class of low-rank groups of Lie type not yet dealt with are ruled out by the following.

Lemma 3. *The Suzuki groups* ${}^2B_2(2^{2n+1})$ *are never dmB groups.*

Proof. These groups are well known to not be Hurwitz groups since they contain no elements of order 3. In [70] Suzuki showed that his now eponymous groups were (2,4,5) generated and it is this type that gives the strong symmetric genus of these groups. These groups however have only have one class of involutions making it impossible to have a Beauville structure of type ((2,4,5),(2,4,5)). □

We pose the analogue of Question 3 for dmB groups.

Question 5. Which groups have strongly real Beauville structures that make them dmB groups?

7 Miscelenia

In this final short section we briefly discuss a number of more minor matters.

7.1 Purity

In [34] the author initiated the study of the following.

Definition 9. A finite group G is a *Purely Strongly Real Beauville Group* if G is a Beauville group such that every Beauville structure of G is strongly real. A finite group G is a *Purely Non-Strongly Real Beauville Group* if G is a Beauville group such that none of its Beauville structures are strongly real.

The main results of [34] focus on constructing examples of these concepts and posing questions about them that are summarised as follows. We first highlight the fact that most Beauville groups appear to fit into neither category and various examples among the finite simple groups are constructed. For infinitely many examples we have the following.

Proposition 2. *The following are purely strongly real Beauville groups.*

(a) The groups $L_2(q)$ where $q > 4$ is even;
(b) abelian Beauville groups;
(c) the groups

$$\langle x,y,z \,|\, x^{p^n}, y^{p^n}, z^{p^r} [x,y] = z, [x,z], [y,z] \rangle$$

of order p^{2n+r} and where $p \geq 5$ is prime and $n \geq r \geq 1$ are integers.

Observe that the above gives no examples that are 2-groups or 3-groups.

Problem 2. Find other examples of purely strongly real Beauville groups.

In particular, we have the following question.

Question 6. Do there exist purely strongly real Beauville 2-groups and 3-groups?

The following provides us with infinitely many examples of purely non-strongly real Beauville groups.

Proposition 3. *If G and H are Beauville groups of coprime order, such that G is a purely non-strongly real Beauville group, then $G \times H$ is a purely non-strongly real Beauville group.*

As observed earlier (though not in the terminology defined in this section) the Matheiu groups M_{11} and M_{23} are purely non-strongly real Beauville groups which combined with the numerous examples of Beauville p-groups discussed earlier provides infinitely many examples of non-strongly real Beauville groups.

7.2 Higher Dimensional Analogues

In various parts of the literature many mathematicians have considered varieties isogenous to a higher product $(\mathscr{C}_1 \times \cdots \times \mathscr{C}_n)/G$, the case $n = 2$ (i.e. surfaces) simply being the most frequently studied. The property of Rigidity that makes Beauville surfaces stand out can easily be generalised to higher dimensions. The following definition was given by Jones in [52].

Definition 10. Let G be a finite group. The *Beauville dimension* of G is the least positive integer d such that there exist generating pairs $(x_1, y_1), \ldots, (x_d, y_d) \in G^2$ such that

$$\Sigma(x_1, y_1) \cap \cdots \cap \Sigma(x_d, y_d) = \{e\}.$$

We write $d_B(G)$ for the Beauville dimension of G. If no such integer exists then we say that the group has infinite Beauville dimension.

Beauville groups are simply groups of Beauville dimension 2. Groups with higher Beauville dimension correspond to higher dimensional complex varieties that also enjoy many of the nice properties of Beauville surfaces such as being rigid, being defined over algebraic number fields etc. By way of an easy example, we noted in section 3 the only finite simple group with Beauville dimension not equal to 2 is the alternating group A_5 which has infinite Beauville dimension since every sigma set must contain elements from the only class of cyclic subgroups of order 5. The following example is the earliest known given in [52].

Example 1. Consider the group

$$(\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z}) = \langle x, y \,|\, x^3, y^3, [x, y] \rangle.$$

The non-trivial elements of this group naturally partition into the four cyclic subgroups $\langle x \rangle$, $\langle y \rangle$, $\langle xy \rangle$ and $\langle xy^2 \rangle$. The sigma set of any generating set contains members from three of these subgroups. However we also have that

$$\Sigma(x, y) \cap \Sigma(x, y^2) \cap \Sigma(x, xy) \cap \Sigma(y, xy) = \{e\}$$

and so $d_B(G) = 4$.

For many years this and close relatives of it were the only known examples of groups with finite Beauville dimension greater than 2. Recent work of the author's PhD student, Ludovico Carta, to appear in [18], extends this example to infinite families of groups with Beauville dimensions 3 and 4.

Question 7. Are there any groups G such that $d_B(G)$ is finite and $d_B(G) > 4$?

Observe that the variety constructed in Example 1 combined with the earlier observations about abelian groups suggests the following.

Problem 3. Construct examples of groups G such that $d_B(G) > 2$ is finite and the corresponding variety is strongly real.

7.3 Reflection Groups

Another class of 2-generated finite groups that have only been partially investigated from the viewpoint of Beauville constructions are reflection groups. In [31] the author proves the following.

Theorem 7. *Every finite irreducible Coxeter group is a strongly real Beauville group aside from the groups of type:*

(a) A_n *for* $n \leq 3$*;*
(b) B_n *for* $n \leq 4$*;*
(c) D_n *for* $n \leq 4$*;*
(d) F_4*,* H_3 *and*
(e) $I_2(k)$ *for any* k*.*

Corollary 1. *No product of three or more irreducible Coxeter groups is a Beauville group. Furthermore,* $K_1 \times K_2$ *is a strongly real Beauville group if* K_1 *and* K_2 *are strongly real irreducible Coxeter Beauville groups not of type* B_n*.*

Corollary 2. *An irreducible Coxeter group is a Beauville group if and only if it is a strongly real Beauville group.*

Altogether the above goes most of the way to classifying which of the real reflection groups are strongly real Beauville groups, however completing the task is more difficult. Several examples are given in [31, Section 5] showing that $K_1 \times K_2$ can be a strongly real Beauville group, even when K_1 and/or K_2 are not.

Question 8. Which real reflection groups are Strongly Real Beauville Groups?

As far as the author is aware, nowhere in the literature considers the more general question of which finite reflection groups of any kind (the above completely ignores objects such as complex reflections groups and quarternionic reflection groups) are strongly real Beauville groups. For example, this wider class of groups includes all cyclic groups (which are clearly not even Beauville groups) but it also gives us examples like the following.

Example 2. In the Sheppard–Todd classification of complex reflection groups, the group denoted G_{24}, also denoted $W(\mathrm{J}_3(4))$, is isomorphic to the group $\mathrm{L}_2(7)\times C_2$. It is easily verified that if we take

$$x_1 := (1,4,8,3,6,5,7)(9,10),\, y_1 := (2,5,7,6,3,4,8)(9,10),$$

$$x_2 := (1,7,2,4)(3,6,8,5)(9,10),\, y_2 := (1,8,2,5)(3,6,4,7)$$

$$\text{and } t := (3,6)(4,7)(5,8),$$

then these permutations give a Beauville structure of type ((14,14,7),(4,4,4)) such that $x_i^t = x_i^{-1}$ and $y_i^t = y_i^{-1}$ for $i = 1,2$ showing that this is a Strongly Real Beauville Group.

Question 9. Which finite reflection groups are (strongly real) Beauville groups?

7.4 Beauville Spectra

The following definition was first made by Fuertes, González-Diez and Jaikin-Zapirain in [38, Definition 11].

Definition 11. Let G be a finite group. The *Beauville genus spectrum* of G, denoted $Spec(G)$, is the set of pairs of integers (g_1,g_2) such that $g_1 \leq g_2$ and there are curves $\mathscr{C}_1$ and $\mathscr{C}_2$ of genera g_1 and g_2 with the action of G on $\mathscr{C}_1\times\mathscr{C}_2$ such that $(\mathscr{C}_1\times\mathscr{C}_2)/G$ is a Beauville surface.

They went on in [38] to determine the Beauville genus spectra for the symmetric group S_5, the linear group $L_2(7)$ and abelian Beauville groups as well as showing that $Spec(S_6) \neq \varnothing$ (though clearly this last result has been generalised by any theorem proving that other groups are Beauville group). These calculations were later pushed further to other small almost-simple groups by Pierro in his PhD thesis [64], the largest group he considered being the Suzuki group Sz(8) (whose order is just 29120) there being 73 such pairs for this group. As the number of conjugacy classes of the groups grows, the size of the corresponding Beauville genus spectrum also grows making it difficult to push these calculations for almost-simple groups much further. Computer programmes written in GAP [39] can also be found in [64].

Imposing a restriction on the Beauville structures clearly makes this set smaller and thus the problem of determining such a spectrum is more tractable. The following natural definition was first made by the author in [24].

Definition 12. The *strongly real Beauville genus spectrum* of G, that we shall denote $SRSpec(G)$ is the set of pairs of integers (g_1,g_2) such that $g_1 \leq g_2$ and there are curves $\mathscr{C}_1$ and $\mathscr{C}_2$ of genera g_1 and g_2 with the action of G on $\mathscr{C}_1\times\mathscr{C}_2$ such that $(\mathscr{C}_1\times\mathscr{C}_2)/G$ is a real Beauville surface.

Since elements of larger order tend to have the property that no automorphism will map them to their inverses it seems likely to the author that $SRSpec(G)$ will in general be much smaller than $Spec(G)$ for most groups. In particular, if determining $Spec(G)$ for a given group G is difficult owing to its size, then the problem of performing the same task for $SRSPec(G)$ may be much more tractable.

Problem 4. Determine the strongly real Beauville genus spectrum of Beauville groups.

Many of the other problems raised here can be described in terms of this quantity. For example, determining if a group G is a strongly real Beauville group is the same as determining when $SRSPec(G) \neq \varnothing$; if G is a purely strongly real Beauville group, then it has the property that $Spec(G) = SRSpec(G)$ and if G is a purely non-strongly real Beauville group, then $SRSPec(G) = \varnothing$.

For most Beauville groups it is likely that $|SRSpec(G)| < |Spec(G)|$. This motivates the following interesting question first posed in the specific case of p-groups by the author as [24, Question 6.14].

Question 10. For a Beauville group G how does the size of $SRSpec(G)$ compare to $Spec(G)$? A little more specifically, how does $|SRSpec(G)|/|Spec(G)|$ behave as $|G| \to \infty$?

7.5 Beauville graphs

In recent years there has been a growing trend towards encoding generational problems for finite groups in graphs in the hope of using graph-theoretic techniques to address group-theoretic matters. The most common being the following.

Definition 13. Given a finite group G its *generating graph* is the graph $\Gamma(G)$ defined as follows. The vertices of $\Gamma(G)$ are the non-trivial elements of G with two elements x and y being adjoined by an edge if and only if $\langle x,y\rangle = G$.

It seems natural to translate the study of Beauville structures into graph-theoretic language. We thus make the following definition.

Definition 14. Given a finite group G its *Beauville generating graph* is the graph $\Gamma_B(G)$ defined as follows. The vertices of $\Gamma_B(G)$ are the sets $\Sigma(x,y)$ where $x,y \in G$ generate the group with two vertices $\Sigma(x,y)$ and $\Sigma(x',y')$ being adjoined by an edge if and only if $\Sigma(x,y) \cap \Sigma(x',y') = \{e\}$ or equivalently if $\{\{x,y\},\{x',y'\}\}$ is a Beauville structure for G.

Compared to the generating graph, the Beauville generating graph has far fewer vertices meaning that in principle it should be somewhat easier to study.

Example 3. Recall the classification of abelian Beauville groups in Theorem 1. In particular if $p \geq 5$ is prime, then $(\mathbb{Z}/p\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z})$ is a Beauville group. Similar to Example 1 the non-trivial elements of this group are partitioned into the $p+1$ sets of non-trivial elements of the cyclic subgroups. Any $\Sigma(x,y)$ consists of three of these and all possible combinations can be achieved. It follows that this graph is isomorphic to what graph theorists sometimes call the Kneser graph $K_{p+1}^{(3)}$ and general theorems regarding these objects reveal a flurry of properties.

- The graph is connected if and only $p > 5$ (the $p = 5$ case being a set of three disjoint edges.)
- These graphs are regular (that is, every vertex has the same degree) of degree $\binom{p-2}{3}$. Kneser graphs in general are in fact both vertex transitive and edge transitive (this is, any pair of vertices can be sent to each other by some automorphism of the graph and similarly for the edges).
- The chromatic number (the smallest number of colours that can be used to colour the vertices in such a way that adjacent vertices have different colours) is exactly $p-3$.
- If $p > 7$, then the girth (the length of the smallest cycle in the graph) is 3, in the case $p = 7$ it is 4.
- An application of the Erdős-Ko-Rado theorem tells us that the independence number (the largest size of a set of vertices such that no two are adjacent) is $\binom{p}{2}$.

For Beauville graphs more generally it is unlikely that properties anything like as nice as the above list will hold. Nonetheless the Beauville graphs of other Beauville groups may be of interest.

Lemma 4. *The Beauville graphs of the groups $L_2(p^r)$ are never connected.*

Proof. If q is odd then there is always a pair of type $((q+1)/2,(q-1)/2,p)$ by straightforward structure constant calculations. For such a pair we have that $\Sigma(x,y) = G$ and so this pair is guaranteed to generate the group and this vertex of the graph will be isolated. The case q even is similar. $\square$

Example 4. If G is the Mathieu group M_{11}, then the graph $\Gamma_B(G)$ is not connected since M_{11} is (5,6,11) generated and since the only primes dividing the order of the group are 2, 3, 5 and 11 and there is only one class of elements of order 2 so such a generating pair corresponds to an isolated vertex.

Question 11. Which Beauville groups have connected Beauville graphs? What other properties do these have?

Of course we can easily consider a more sparse graph that keep the strongly real condition in mind.

Definition 15. Let G be a finite group. The *strongly real Beauville graph* denoted $\Gamma_{SRB}(G)$ is defined the same way as the Beauville graph except adjacency is now defined by $\{\{x,y\},\{x',y'\}\}$ being a strongly real Beauville structure for G

Example 5. If G is a purely non-strongly real Beauville group, then $\Gamma_{SRB}(G) = \Gamma_B(G)$ so in particular some of these graphs are discussed in some detail in Example 3. If G is a purely non-strongly real Beauville group, then $\Gamma_{SRB}(G)$ is empty regardless of any properties of $\Gamma_B(G)$.

Problem 5. Investigate the properties of these graphs.

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