Highly linked tournaments

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Abstract

A (possibly directed) graph is k-linked if for any two disjoint sets of vertices $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_k\}$ there are vertex disjoint paths $P_1, \ldots, P_k$ such that P_i goes from x_i to y_i. A theorem of Bollobás and Thomason says that every $22k$-connected (undirected) graph is k-linked. It is desirable to obtain analogues for directed graphs as well. Although Thomassen showed that the Bollobás-Thomason Theorem does not hold for general directed graphs, he proved an analogue of the theorem for tournaments—there is a function $f(k)$ such that every strongly $f(k)$-connected tournament is k-linked. The bound on $f(k)$ was reduced to $O(k \log k)$ by Kühn, Lapinskas, Osthus, and Patel, who also conjectured that a linear bound should hold. We prove this conjecture, by showing that every strongly $452k$-connected tournament is k-linked.

1 Introduction

A graph is connected if there is a path between any two vertices. A graph is k-connected if it remains connected after the removal of any set of $(k-1)$-vertices. This could be seen as a notion of how robust the graph is. For example, if the graph represents a communication network, then the connectedness measures how many nodes need to fail before communication becomes impossible.

Similar notions make sense for directed graphs, except in that context we usually want a *directed path* between every pair of vertices. If this holds, we say that the directed graph is *strongly connected*. A directed graph is strongly k-connected it remains strongly connected after the removal of any set of $(k-1)$-vertices. In this paper, when dealing with connectedness of directed graphs we will always mean strong connectedness.

Connectedness is a fundamental notion in graph theory, and there are countless theorems which involve it. Perhaps the most important of these is Menger's Theorem, which provides an alternative characterization of k-connectedness. Menger's Theorem says that a graph is k-connected if, and only if, there are k internally vertex-disjoint paths between any pair of vertices. Menger's Theorem has the following simple corollary:

Corollary 1.1. *If G is k-connected then for any two disjoint sets of vertices $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_k\}$ there are vertex-disjoint paths $P_1, \ldots, P_k$ such that P_i goes from x_i to $y_{\sigma(i)}$ for some permutation σ of $[k]$.*

This corollary is proved by constructing a new graph H from G by adding two vertices x and y such that x is joined to $\{x_1, \ldots, x_k\}$ and y is joined to $\{y_1, \ldots, y_k\}$. It is easy to see that H is k-connected, and so, by Menger's Theorem, has k vertex-disjoint $x-y$ paths. Removing the vertices x and y produces the required paths $P_1, \ldots, P_k$. It is not hard to see that the converse of Corollary 1.1 holds for graphs on at least $2k$ vertices as well.

Notice that in Corollary 1.1, we had no control over where the path P_i starting at x_i ends up—it could end at any of the vertices $y_1, \ldots, y_k$. In practice we might want to have control over this. This leads to the notion of k-linkedness. A graph is k-linked if for any two disjoint sets of vertices $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_k\}$ there are vertex disjoint paths $P_1, \ldots, P_k$ such that P_i goes from x_i to y_i.

Linkedness is a stronger notion than connectedness. A natural question is whether a k-connected graph must also be ℓ-linked for some ℓ (which may be smaller than k). Larman and Mani [7], and Jung [4] were the first to show that this is indeed the case—they showed that there is a function $f(k)$ such that every $f(k)$-connected graph is k-linked. This result uses a theorem of Mader [8] about the existence of large topological complete minors in graphs with many edges. The first bounds on $f(k)$ were exponential in k, but Bollobás and Thomason showed that a linear bound on the connectedness suffices [3].

Theorem 1.2 (Bollobás and Thomason)**.** *Every $22k$-connected graph is k-linked.*

The constant 22 has since been reduced to 10 by Thomas and Wollan [10].

Much of the above discussion holds true for directed graphs as well (when talking about *strong* k-connectedness and *directed* paths). Menger's Theorem remains true, as does Corollary 1.1. A directed graph is k-linked if for two disjoint sets of vertices $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_k\}$ there are vertex disjoint directed paths $P_1, \ldots, P_k$ such that P_i goes from x_i to y_i. Somewhat surprisingly there is no function $f(k)$ such that every strongly $f(k)$-connected directed graph is k-linked. Indeed Thomassen constructed directed graphs of arbitrarily high connectedness which are not even 2-linked [12]. Thus, there is a real difference between the directed and undirected cases. For tournaments however the situation is better (A tournament is a directed graph which has exactly one directed edge between any two vertices). There Thomassen showed that there is a constant C, such that every $Ck!$-connected tournament is k-linked [11]. Kühn, Lapinskas, Osthus, and Patel improved the bound on the connectivity to $10^4 k \log k$.

Theorem 1.3 (Kühn, Lapinskas, Osthus, and Patel, [5]). *All strongly $10^4 k \log k$-connected tournaments are k-linked.*

This theorem is proved using a beautiful construction utilizing the asymptotically optimal sorting networks of Ajtai, Komlós, and Szemerédi [1]. The proof is based on building a small sorting network inside the tournament, which is combined with the directed version of Corollary 1.1 in order to reorder the endpoints of the paths so that P_i goes from x_i to y_i. We refer to [5] for details.

Since sorting networks on k inputs require size at least $k \log k$, it is unlikely that this approach can give a $o(k \log k)$ bound in Theorem 1.3. Nevertheless, Kühn, Lapinskas, Osthus, and Patel conjectured that a linear bound should be possible.

Conjecture 1.4 (Kühn, Lapinskas, Osthus, and Patel, [5]). *There is a constant C such that every strongly Ck-connected tournament is k-linked.*

There has also been some work for small k. Bang-Jensen showed that every 5-connected tournament is 2-linked [2]. Here the value "5" is optimal.

The main result of this paper is a proof of Conjecture 1.4.

Theorem 1.5. *Every strongly $452k$-connected tournament is k-linked.*

The above theorem is proved using the method of "linkage structures in tournaments" recently introduced in [5] and [6]. Informally, a linkage structure L in a tournament T, is a small subset of $V(T)$ with the property that for many pairs of vertices x, y outside L, there is a path from x to y most of whose vertices are contained in L. Such structures can be found in highly connected tournaments, and they have various applications such as finding Hamiltonian cycles [5, 9] or partitioning tournaments into highly connected subgraphs [6]. Linkage structures were introduced in the same paper where Conjecture 1.4 was made. However in the past they were constructed *using Theorem 1.3* to first show that a tournament is highly linked. In our paper the perspective is different—the linkage structures are built using only connectedness, and then linkedness follows as a corollary of the presence of the linkage structures.

It would be interesting to reduce the constant 452 in Theorem 1.5. It is not hard to find minor improvements to our proof in Section 2 which improve this constant by a little bit. It is not clear what the correct value of the constant should be, and we are not aware of any non-trivial constructions for large k. In view of the Bollobás-Thomason Theorem, we pose the following problem.

Problem 1.6. *Show that every strongly $22k$-connected tournament is k-linked.*

2 Proof of Theorem 1.5

A directed path P is a sequence of vertices $v_1, v_2, \dots, v_k$ in a directed graph such that $v_i v_{i+1}$ is an edge for all $i = 1, \dots, k-1$. The vertex v_1 is called the *start* of P, and v_k the *end* of P. The *length* of P is the number of edges it has which is $|P|-1$. The vertices $v_2,\dots,v_{k-1}$ are the *internal vertices* of P. Two paths are said to be internally disjoint if their internal vertices are distinct.

A tournament T is transitive if for any three vertices $x,y,z\in V(T)$, if xy and yz are both edges, then xz is also an edge. It's easy to see that a tournament is transitive exactly when it has an ordering $(v_1,v_2,\dots,v_k)$ of $V(T)$ such that the edges of T are $\{v_iv_j : i<j\}$. We say that v_1 is the *tail* of T, and v_k is the *head* of T.

The *out-neighbourhood* of a vertex v in a directed graph, denoted $N^+(v)$, is the set of vertices u for which vu is an edge. Similarly, the *in-neighbourhood*, denoted $N^-(v)$, is the set of vertices u for which uv is an edge. The *out-degree* of v is $d^+(v)=|N^+(v)|$, and the *in-degree* of v is $d^-(v)=|N^-(v)|$. A useful fact is that every tournament, T, has a vertex of out-degree at least $(|T|-1)/2$, and a vertex of in-degree at least $(|T|-1)/2$. To see this, observe that since T has $\binom{|T|}{2}$ edges, its average in and out-degrees are both $(|T|-1)/2$.

We'll need the following lemma which says that in any tournament, we can find two large sets such that there is a linkage between them.

Lemma 2.1. *Let n and m be two integers with $m\le n/11$. Every tournament T on n vertices, contains two disjoint sets of vertices $\{x_1,\dots,x_m\}$ and $\{y_1,\dots,y_m\}$ such that for any permutation σ of $[m]$, there are vertex-disjoint paths $P_1,\dots,P_m$ such that P_i goes from x_i to $y_{\sigma(i)}$.*

Proof. Let $x_1,\dots,x_m$ be a set of m vertices in T of largest out-degrees i.e. any set such that any vertex u outside it satisfies $d^+(u)\le d^+(x_i)$ for all i. Let $y_1,\dots,y_m$ be a set of m vertices in T of largest in-degrees. Since $m\le n/11$, we can choose $\{x_1,\dots,x_m\}$ and $\{y_1,\dots,y_m\}$ to be disjoint.

Recall that every tournament T has a vertex of out-degree at least $(|T|-1)/2$. This means that $d^+(x_i)\ge(n-m)/2$ for each $i=1,\dots,m$ (since otherwise, there would be a vertex in $T\setminus\{x_1,\dots,x_m\}+x_i$ of out-degree larger than x_i, contradicting the choice of x_i). Similarly, we obtain $d^-(y_i)\ge(n-m)/2$ for each $i=1,\dots,m$.

For each i and $j\le m$, let $X_{i,j}=\big(N^+(x_i)+x_i\big)\setminus\big(N^-(y_j)+y_j\big)$, $Y_{i,j}=\big(N^-(y_j)+y_j\big)\setminus\big(N^+(x_i)+x_i\big)$, $I_{i,j}=N^+(x_i)\cap N^-(y_j)$, and $M_{i,j}$ a maximum matching of edges directed from $X_{i,j}$ to $Y_{i,j}$.

Notice that we have

$$|X_{i,j}\setminus V(M_{i,j})|=|N^+(x_i)+x_i-y_i|-|I_{i,j}|-e(M_{i,j})\ge\frac{1}{2}(n-m)-|I_{i,j}|-e(M_{i,j}).$$

Similarly, we obtain $|Y_{i,j}\setminus V(M_{i,j})|\ge\frac{1}{2}(n-m)-|I_{i,j}|-e(M_{i,j})$.

Since M is maximal, all the edges between $X_{i,j}\setminus V(M_{i,j})$ and $Y_{i,j}\setminus V(M_{i,j})$ go from $Y_{i,j}$ to $X_{i,j}$. Therefore, if $\frac{1}{2}(n-m)-|I_{i,j}|-e(M_{i,j})\ge m$ holds, then the lemma follows by choosing $x'_1,\dots,x'_m$ to be any m vertices in $Y_{i,j}\setminus V(M_{i,j})$, and $y'_1,\dots,y'_m$ to be any m vertices in $X_{i,j}\setminus V(M_{i,j})$. This ensures that we can always choose length 1 paths $P_1,\dots,P_m$ as in the lemma.

Therefore, we can suppose that $\frac{1}{2}(n-m)-|I_{i,j}|-e(M_{i,j})<m$. Combining this with $m\le n/11$ we obtain that $|I_{i,j}|+e(M_{i,j})>4m$ for every i and j.

Notice that for all i and j, there are $|I_{i,j}| + e(M_{i,j}) \geq 4m+1$ internally vertex disjoint paths of length ≤ 3 between x_i and y_j. This allows us to construct vertex disjoint paths $P_1, \dots, P_m$ each of length ≤ 3, such that P_i goes from x_i to $y_{\sigma(i)}$ (where σ is an arbitrary permutation of $[m]$). Indeed assuming we have constructed the paths $P_1, \dots, P_k$, then we have $|V(P_1) \cup \dots \cup V(P_k)| \leq 4m$, and so one of the $4m+1$ internally vertex disjoint paths between x_{k+1} and y_{k+1} must be disjoint from $V(P_1) \cup \dots \cup V(P_k)$. We let P_{k+1} be this path, and then repeat this process until we have the required m paths. □

A set of vertices S in-dominates another set B, if for every $b \in B \setminus S$, there is some $s \in S$ such that bs is an edge. Notice that by this definition, a set in-dominates itself. A *in-dominating set* in a tournament T is any set S which in-dominates $V(T)$. Notice that by repeatedly pulling out vertices of largest in-degree and their in-neighbourhoods from T, we can find an in-dominating set of order at most $\lceil \log_2 |T| \rceil$. For our purposes we'll study sets which are constructed by pulling out some fixed number of vertices by this process.

Definition 2.2. *We say that a sequence $(v_1, v_2, \dots, v_k)$ of vertices of a tournament T is a partial greedy in-dominating set if v_1 is a maximum in-degree vertex in T, and for each i, v_i is a maximum in-degree vertex in the subtournament of T on $N^+(v_1) \cap N^+(v_2) \cap \dots \cup N^+(v_{i-1})$.*

Partial greedy out-dominating sets are defined similarly, by letting v_i be a maximum out-degree vertex in $N^-(v_1) \cap N^-(v_2) \cap \dots \cap N^-(v_{i-1})$ at each step.

Notice that every partial greedy in-dominating set is a transitive tournament with head v_k and tail v_1.

For small k, partial greedy in-dominating sets do not necessarily dominate all the vertices in a tournament. A crucial property of partial greedy in-dominating sets is that the vertices they don't dominate have large out-degree. The following is a version of a lemma appearing in [5].

Lemma 2.3. *Let $(v_1, v_2, \dots, v_k)$ be a partial greedy in-dominating set in a tournament T. Let E be the set of vertices which are not in-dominated by A. Then every $u \in E$ satisfies $d^+(u) \geq 2^{k-1}|E|$.*

Proof. The proof is by induction on k. The initial case is when $k = 1$. In this case we have $E = N^+(v_1)$ where v_1 is a maximum in-degree vertex in T. For any $u \in E$, we must have $d^-(u) \leq d^-(v_1) = |T \setminus E - v_1| = |T| - |E| - 1$. Therefore we have $d^+(u) = |T \setminus N^-(u) - u| = |T| - d^-(u) - 1 \geq |E|$ as required.

Now suppose that the lemma holds for $k = k_0$. Let $(v_1, \dots, v_{k_0+1})$ be a partial greedy in-dominating set in T, and let $E_0 = N^+(v_1) \cap \dots \cap N^+(v_{k_0})$. By induction we have $d^+(u) \geq 2^{k_0-1}|E_0|$ for every $u \in E_0$. By definition v_{k_0+1} is a maximum in-degree vertex in E_0. Let $E = E_0 \cap N^+(v_{k_0+1})$ be the set of vertices not in-dominated by $(v_1, \dots, v_{k_0+1})$. Since v_{k_0+1} is a maximum in-degree vertex in E_0, we have $|N^-(v_{k_0+1}) \cap E_0| \geq (|E_0| - 1)/2$ which implies $|E| = |E_0| - |(N^-(v_{k_0+1}) + v_{k_0+1}) \cap E_0| \leq |E_0|/2$. Combining this with the inductive hypothesis, we obtain $d^+(u) \geq 2^{k_0-1}|E_0| \geq 2^{k_0}|E|$, completing the proof. □

We are now ready to prove the main result of this paper.

Proof of Theorem 1.5. Let T be a strongly $452k$-connected tournament. Notice that this means that all vertices in T have in-degree and out-degree at least $452k$.

Let $x_1, \dots, x_k$ and $y_1, \dots, y_k$ be vertices in T as in the definition of k-linkedness. We will construct vertex disjoint paths from x_i to y_i. Let $T' = T \setminus \{x_1, \dots, x_k, y_1, \dots, y_k\}$.

Let D_1^- be a partial greedy in-dominating set in T' on 2 vertices. Then, for all $i = 2, \dots, 55k$, let D_i^- be a partial greedy in-dominating set on 2 vertices in $T' \setminus (D_1^- \cup \dots \cup D_{i-1}^-)$.

Similarly, let D_1^+ be a partial greedy out-dominating set on 2 vertices in $T' \setminus (D_1^- \cup \dots \cup D_{55k}^-)$. Then, for all $i = 2, \dots, 55k$, let D_i^+ be a partial greedy out-dominating set on 2 vertices in $T' \setminus (D_1^+ \cup \dots \cup D_{i-1}^+ \cup D_1^- \cup \dots \cup D_{55k}^-)$.

Let $X = D_1^+ \cup \dots \cup D_{55k}^+ \cup D_1^- \cup \dots \cup D_{55k}^- \cup \{x_1, \dots, x_k, y_1, \dots, y_k\}$. For each i, let E_i^- be the set of vertices in $T \setminus X$ which aren't in-dominated by D_i^-, and E_i^+ the set of vertices in $T \setminus X$ which aren't out-dominated by D_i^+. By Lemma 2.3, we have $d^+(v) \geq 2|E_i^-|$ for every $v \in E_i^-$, and also $d^-(v) \geq 2|E_i^+|$ for every $v \in E_i^+$.

Let T^- be the set of heads of $D_1^-, \dots, D_{55k}^-$, and T^+ the set of tails of $D_1^+, \dots, D_{55k}^+$. Apply Lemma 2.1 to T^- in order to find two subsets X^- and Y^- of order $5k$ of $V(T^-)$, such that for any bijection $f : X^- \to Y^-$, there is a set of $5k$ vertex-disjoint paths in T^- with each path joining x to $f(x)$ for some $x \in X^-$. Apply Lemma 2.1 to T^+ in order to find two subsets X^+ and Y^+ of order $5k$ of $V(T^+)$, such that for any bijection $f : X^+ \to Y^+$, there is a set of $5k$ vertex-disjoint paths in T^+ with each path joining x to $f(x)$ for some $x \in X^+$. Reorder $(D_1^-, \dots, D_{55k}^-)$ so that X^- is the set of heads of $D_1^-, \dots, D_{5k}^-$. Reorder $(D_1^+, \dots, D_{55k}^+)$ so that Y^+ is the set of tails of $D_1^+, \dots, D_{5k}^+$. Notice that since each partial greedy dominating set is on 2 vertices, we have $|X| \leq 222k$. By Menger's Theorem, since T is 452-connected, there is a set of vertex-disjoint paths $Q_1, \dots, Q_{5k}$ in $(T \setminus X) \cup Y^- \cup X^+$ such that each path Q_i starts in Y^- and ends in X^+.

Recall that all out-degrees in T are at least $452k$ and $|X| \leq 222k$. Therefore, for each $i = 1, \dots, k$ we can choose an out-neighbour x_i' of x_i which is not in X. Similarly for each i we can choose an in-neighbour y_i' of y_i which is not in X. In addition we can ensure that $x_1', \dots, x_k', y_1', \dots, y_k'$ are all distinct. Let $X' = X \cup \{x_1', \dots, x_k', y_1', \dots, y_k'\}$.

Notice that each vertex $v \in E_i^-$ satisfies $d^+(v) \geq 2|E_i^-|$ and $2|X'| + 4k$. Averaging these, we get $d^+(v) \geq |E_i^-| + |X'| + 2k$ and so v has at least $2k$ out-neighbours outside of $E_i^- \cup X'$. Similarly each $v \in E_i^+$ has at least $2k$ in-neighbours outside of $E_i^+ \cup X'$. Therefore, for each i, we choose x_i'' to be either equal to x_i' if $x_i' \notin E_i^-$ or we choose x_i'' to be an out-neighbour of x_i' in $T \setminus (E_i^- \cup X')$. Similarly, for each i, we choose y_i'' to be either equal to y_i' if $y_i' \notin E_i^+$ or we choose y_i'' to be an in-neighbour of y_i' in $T \setminus (E_i^+ \cup X')$. We can also choose the vertices $x_1'', \dots, x_k'', y_1'', \dots, y_k''$ so that they are all distinct (since when $x'' \neq x'$ and $y'' \neq y'$ are always at least $2k$ choices for x_i'' and y_i'' respectively).

For each $i = 1, \dots, k$, let Q_i^- be a path from x_i'' to the head of D_i^- whose internal vertices are all in D_i^-. The facts that D_i^- is transitive and $x_i'' \notin E_i^-$ ensure that we can do this. Similarly, for each i let Q_i^+ be a path from the tail of D_i^+ to y_i'' whose internal vertices are all in D_i^+.

Notice that at least k of the paths $Q_1, \ldots, Q_{5k}$ are disjoint from $\{x'_1, \ldots, x'_k, y'_1, \ldots, y'_k, x''_1, \ldots, x''_k, y''_1, \ldots, y''_k\}$. Let $Q'_1, \ldots, Q'_k$ be some choice of such paths.

Since Q_i^- ends in X^- and Q'_i starts in Y^-, Lemma 2.1 implies that we can choose disjoint paths $P_1^-, \ldots, P_k^-$ in T^- such that P_i^- is from the end of Q_i^- to the start of Q'_i. Similarly we can choose disjoint paths $P_1^+, \ldots, P_k^+$ in T^+ such that P_i^+ is from the end of Q'_i to the start of Q_i^+.

Now for each i we join x_i to x'_i to Q_i^- to P_i^- to Q'_i to P_i^+ to Q_i^+ to y'_i to y_i in order to obtain the required vertex-disjoint paths from the x_is to the y_is. □

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