Edge growth in graph powers

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December 10, 2018

Abstract

For a graph G, its rth power G^r has the same vertex set as G, and has an edge between any two vertices within distance r of each other in G. We give a lower bound for the number of edges in the rth power of G in terms of the order of G and the minimal degree of G. As a corollary we determine how small the ratio $e(G^r)/e(G)$ can be for regular graphs of diameter at least r.

1 Introduction

We will consider both graphs that may have loops and graphs in which loops are explicitly forbidden. Loopless graphs will be denoted by Roman italic letters, such as "G", while graphs with loops allowed will be denoted by curly letters, such as "$\mathcal{G}$". For two vertices x and y (possibly $x = y$) we only allow one edge between x and y. The rth power of G, denoted G^r, is the graph with vertex set $V(G)$, and xy an edge whenever x and y are within distance r of each other. The *diameter* of a connected graph is the smallest r for which G^r is complete. For all standard notation we refer to [5].

For a connected graph of diameter at least r, one would expect G^r to have substantially more edges than G. In this note we examine how small the ratio $e(G^r)/e(G)$ can be, focusing primarily on the case when G is a regular graph.

The motivation for studying this comes from a corollary of the Cauchy-Davenport Theorem from additive number theory which we will now state. The Cayley graph of a subset $A \subseteq \mathbb{Z}_p$ is constructed on the vertex set $\mathbb{Z}_p$. For two distinct vertices $x, y \in \mathbb{Z}_p$, we define xy to be an edge whenever $x - y \in A$ or $y - x \in A$. The following is a consequence of the Cauchy-Davenport Theorem (usually stated in the language of additive number theory).

Theorem 1 (Cauchy, Davenport, [1, 2]). *Let p be a prime, G the Cayley graph of a set $A \subseteq \mathbb{Z}_p$, and r an integer such that $r < \mathrm{diam}(G)$. Then we have*

$$\frac{e(G^r)}{e(G)} \geq r. \tag{1}$$

One could ask whether inequalities similar to (1) hold for more general families of graphs. Motivated by the fact that Cayley graphs are regular, Hegarty asked this question for regular graphs and proved the following theorem.

Theorem 2 (Hegarty, [7])**.** *Let G be a regular, connected graph, with* $\mathrm{diam}(G) \geq 3$*. Then we have*

$$\frac{e(G^3)}{e(G)} \geq 1 + \epsilon. \tag{2}$$

Where $\epsilon \approx 0.087$.

The constant ϵ has since been improved to $\frac{1}{6}$ by the author [8] and to $\frac{3}{4}$ by DeVos and Thomassé [4]. The value $\epsilon = \frac{3}{4}$ is optimal in the sense that there exists a sequence of regular graphs of diameter greater than 3, G_m, satisfying $\frac{e(G_m^3)}{e(G_m)} \to \frac{7}{4}$ as $m \to \infty$ [4]. It is natural to ask what happens for other powers of G.

For G^2, Hegarty showed that no inequality similar to (2) with $\epsilon > 0$ can hold for regular graphs in general, by exhibiting a sequence of regular, connected graphs of diameter greater than 2, G_m, satisfying $\frac{e(G_m^2)}{e(G_m)} \to 1$ as $m \to \infty$ [7]. Goff [6] studied the 2nd power of regular graphs further and showed that for any d-regular connected graph G such that $\mathrm{diam}(G) > 2$, we have $\frac{e(G^2)}{e(G)} \geq 1 + \frac{3}{2d} - o\left(\frac{1}{d}\right)$. For general d-regular connected graphs G with $\mathrm{diam}(G) > 2$, the $\frac{3}{2d}$ term in this result cannot be replaced with $\frac{\lambda}{d}$ for any $\lambda > \frac{3}{2}$. However it is shown in [6] that with the exception of two families of exceptional graphs, we have $\frac{e(G^2)}{e(G)} \geq 1 + \frac{2}{d} - o\left(\frac{1}{d}\right)$ for all d-regular connected graphs with $\mathrm{diam}(G) > 2$.

In this note we consider all $r \geq 4$ and determine how small $\frac{e(G^r)}{e(G)}$ can be for G a regular, connected graph of diameter at least r. We prove the following theorem.

Theorem 3. *Let G be a connected, regular graph, and r a positive integer such that* $\mathrm{diam}(G) \geq r$*.*

- *If $r \equiv 0 \pmod 3$, then we have*

$$\frac{e(G^r)}{e(G)} \geq \frac{r+3}{3} - \frac{3}{2(r+3)}.$$

- *If $r \not\equiv 0 \pmod 3$, then we have*

$$\frac{e(G^r)}{e(G)} \geq \left\lceil \frac{r}{3} \right\rceil.$$

The case $r = 3$ of Theorem 3 is due to DeVos and Thomassé [4], and will not be proved here. Theorem 3 gives a lower bound on the ratio $\frac{e(G^r)}{e(G)}$ for regular graphs. The bounds on $\frac{e(G^r)}{e(G)}$ in Theorem 3 are optimal in the following sense. For each r, there exists a sequence of regular, connected graphs of diameter at least r, G_m, such that $\frac{e(G_m^r)}{e(G_m)}$ tends to the bound

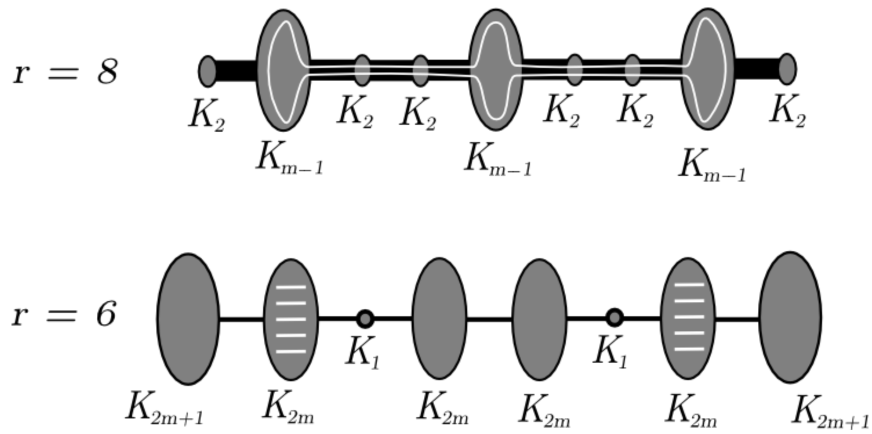

Figure 1: Graphs showing the cases "$r = 8$" and "$r = 6$" of Theorem 3 to be optimal. The grey ovals represent complete graphs of specified order. The black lines between the sets represent all the edges being present between them. The white cycle in the "$r = 8$" case represents the removal of a single cycle passing through all the vertices in the sets it intersects. The white matchings in the "$r = 6$" case represent a perfect matching being removed from the specified sets.

given by Theorem 3 as m tends to infinity. We refer to Figure 1 for a diagram of the sequences that we construct.

To see this for $r \not\equiv 0 \pmod 3$, we construct the following sequence of graphs G_m. Take disjoint sets of vertices $N_0, ..., N_r$, with $|N_i| = m-1$ if $i \equiv 1 \pmod 3$ and $|N_i| = 2$ otherwise. Add all the edges between N_i and N_{i+1} for $i = 0, 1, \ldots, r-1$. Add all the edges within N_i for all i. Remove a cycle passing through all the vertices in $N_1 \cup ... \cup N_{r-1}$. It is easy to see that G_m is m-regular and has diameter r. If $r \equiv 1 \pmod 3$ then $|G_m| = \frac{1}{3}(rm + 2m + 3r)$ will hold. Since G_m is m-regular, we have $e(G_m) = \frac{1}{6}(rm + 2m + 3r)m$. Since G_m^r is complete, we have $e(G_m^r) = \frac{1}{18}(rm + 2m + 3r)(rm + 2m + 3r - 1)$. This implies that $\frac{e(G_m^r)}{e(G_m)} \to \lceil \frac{r}{3} \rceil$ as $m \to \infty$. A similar calculation can be used to show that the same limit holds when $r \equiv 2 \pmod 3$.

For $r \equiv 0 \pmod 3$, we construct the following sequence of graphs H_m to show that Theorem 3 is optimal. Take disjoint sets of vertices $N_0, ..., N_{r+1}$. Let $|N_0| = |N_{r+1}| = 2m + 1$, $|N_i| = 1$ if $i \equiv 2 \pmod 3$, and $|N_i| = 2m$ otherwise. Add all the edges between N_i and N_{i+1} for $i = 0, 1, \ldots, r$. Add all the edges within N_i for all i. Delete a perfect matching from each of the sets N_2 and N_r. This will ensure that H_m is $4m$-regular and has diameter $r + 1$. Note that $|H_m| = \frac{1}{3}(4rm + r + 12m + 6)$, and so we have $e(H_m) = \frac{1}{6}(4rm + r + 12m + 6)4m$. The only edges missing from H_m^r will be between N_0 and N_{r+1}, so we have $e(H_m^r) = \frac{1}{18}(4rm + r + 12m + 6)(4rm + r + 12m + 5) - (2m + 1)^2$. This implies that $\frac{e(H_m^r)}{e(H_m)} \to \frac{r+3}{3} - \frac{3}{2(r+3)}$ as $m \to \infty$. This construction is a generalization of one from [4].

All the examples constructed above have their diameter close to r. If a graph G has diameter larger than r, it seems that the bounds of Theorem 3 can be improved. Some results in this direction have been obtained DeVos, McDonald and Scheide [3].

The requirement of G being regular in the above theorems is quite restrictive. Following [4], we will instead assume that G has minimum degree $\delta(G)$, and give the following bound on $e(G^r)$ in terms of $|G|$ and $\delta(G)$.

Theorem 4. *Let G be a connected graph, and r a positive integer such that* $\mathrm{diam}(G) \geq r$.

- *If $r \equiv 0 \pmod 3$, then we have*

$$e(G^r) \geq \left(\frac{r+3}{6} - \frac{3}{4(r+3)}\right) \delta(G)|G|.$$

- *If $r \not\equiv 0 \pmod 3$, then we have*

$$e(G^r) \geq \frac{1}{2} \left\lceil \frac{r}{3} \right\rceil \delta(G)|G|.$$

The case $r = 3$ of Theorem 4 is due to DeVos and Thomassé [4], and will not be proved here. Theorem 4 easily implies Theorem 3.

2 Proof of Theorem 4

We will prove a version of Theorem 4 for graphs which may contain loops since in that setting the proof seems more natural.

The neighbourhood of a vertex x, $N(x)$, is defined as the set of vertices adjacent to x. (If there is a loop at x, then $N(x)$ will contain x itself.) The degree of x is $|N(x)|$. For graphs with loops allowed, $\mathcal{G}^r$ is defined identically to how it was defined for loopless graphs. Note that if $\mathcal{G}$ is a graph with loops allowed, then $\mathcal{G}^r$ always has a loop at each vertex. For two sets of vertices X and Y, let $d(X, Y)$ denote the length of a shortest path between a vertex in X and a vertex in Y. If X is a set of vertices, let $N^r(X)$ be the set of vertices at distance at most r from X. We abbreviate $N^r(\{x\})$ as $N^r(x)$ and $d(\{x\}, \{y\})$ as $d(x, y)$.

We prove the following theorem, and then deduce Theorem 4 as a corollary. Several ideas in the proof of Theorem 5 are taken from [4]. In particular, Claims 11 and 12 are analogues of claims proved in [4].

Theorem 5. *Let $\mathcal{G}$ be a connected graph, and r a positive integer such that $r \geq 6$ and* $\mathrm{diam}(G) \geq r$.

- *If $r \equiv 0 \pmod 3$, then we have*

$$e(\mathcal{G}^r) \geq \left(\frac{r+3}{6} - \frac{3}{4(r+3)}\right) \delta(\mathcal{G})|\mathcal{G}| + \frac{1}{2}|\mathcal{G}|.$$

- *If $r \not\equiv 0 \pmod 3$, then we have*

$$e(\mathcal{G}^r) \geq \frac{1}{2} \left\lceil \frac{r}{3} \right\rceil \delta(\mathcal{G})|\mathcal{G}| + \frac{1}{2}|\mathcal{G}|.$$

Proof. For convenience, we will set $\delta = \delta(\mathcal{G})$. If P is a path between two vertices x and y, we say that P is a *geodesic* if the length of P is $d(x, y)$. The notion of a geodesic was used in [4], and is useful because the neighbourhood of a geodesic must be quite large. This is quantified in the following claim.

Claim 6. *Let P be a length k geodesic. Then $|N(P)| \geq \left(\left\lfloor \frac{k}{3} \right\rfloor + 1\right) \delta$ holds.*

Proof. If $x_0,\ x_1, \ldots, x_k$ are the vertices of P (in the order in which they occur along the path), then $N(x_0),\ N(x_3), \ldots, N(x_{3\lfloor \frac{k}{3} \rfloor})$ are all disjoint, contained in $N(P)$, and of order at least δ. This implies the result. □

We now prove the case "$r \not\equiv 0 \pmod 3$" of the theorem.

The diameter of $\mathcal{G}$ is at least r, so $\mathcal{G}$ contains a length r geodesic, P. Claim 6 implies that the following holds:

$$|\mathcal{G}| \geq |N(P)| \geq \left(\left\lfloor \frac{r}{3} \right\rfloor + 1\right) \delta \geq \left\lceil \frac{r}{3} \right\rceil \delta. \tag{3}$$

Note that $\mathcal{G}^r$ contains a loop at every vertex, so we have $e(\mathcal{G}^r) = \sum_{v \in V(\mathcal{G})} \left(\frac{1}{2}|N^r(v)| + \frac{1}{2}\right)$. Thus to prove Theorem 5 it is sufficent to exhibit $\left\lceil \frac{r}{3} \right\rceil \delta$ elements of $N^r(v)$ for each vertex $v \in V(G)$.

Let v be a vertex in G. Suppose that there exists a length $r-1$ geodesic P_v starting from v. Then $N(P_v)$ is contained in $N^r(v)$, giving

$$|N^r(v)| \geq |N(P_v)| \geq \left(\left\lfloor \frac{r-1}{3} \right\rfloor + 1\right) \delta = \left\lceil \frac{r}{3} \right\rceil \delta.$$

The second inequality is an application of Claim 6.

Suppose that all the vertices in $\mathcal{G}$ are within distance $r-1$ of v. In this case we have $N^r(v) = V(\mathcal{G})$, which is of order at least $\left\lceil \frac{r}{3} \right\rceil \delta$ by (3). This completes the proof of the case "$r \not\equiv 0 \pmod 3$" of the theorem.

For the rest of the proof fix r such that $r \equiv 0 \pmod 3$ and $r \geq 6$.

If v is a vertex of $\mathcal{G}$, we say that v is *sufficient* if $|N^r(v)| \geq \left(\frac{r}{3} + 1\right) \delta$. Otherwise we say that v is *insufficient*.

The following is a useful property of insufficient vertices.

Claim 7. *Let v be an insufficient vertex. Then there is some vertex at distance $r+1$ from v.*

Proof. Since $\text{diam}(\mathcal{G}) \geq r$, Claim 6 implies that $|\mathcal{G}| \geq \left(\frac{r}{3} + 1\right) \delta$. By assumption $N^r(v) < \left(\frac{r}{3} + 1\right) \delta$, so v cannot be within distance r from all the vertices in the graph. □

The following three claims will allow us to bound the number of insufficient vertices in $\mathcal{G}$.

Claim 8. *If $2 < d(x, y) < r$ holds for $x, y \in V(\mathcal{G})$, then either x or y is sufficient.*

Proof. Suppose that x is insufficient. By Claim 7, we can find a length r geodesic starting from x with vertex sequence x, x_1, $x_2, \dots, x_r$.

Suppose that $N(y) \cap N(x_i) \neq \emptyset$ for some i with $3 \leq i \leq r-3$. In this case $N(x)$, $N(x_3)$, $N(x_6), \dots, N(x_r)$ are all contained in $N^r(y)$. There are $\frac{r}{3}+1$ of these, they are all disjoint (since x, x_1, $x_2, \dots, x_r$ form a geodesic), and are of order at least δ. Hence y is sufficient.

Otherwise $N(y) \cap N(x_i) = \emptyset$ for all $3 \leq i \leq r-3$. In this case $N(x)$, $N(y)$, $N(x_3)$, $N(x_6), \dots, N(x_{r-3})$ are all disjoint and contained in $N^r(x)$. This contradicts our initial assumption that x is insufficient. □

Claim 9. *Let x and y be two vertices in $\mathcal{G}$ such that $d(x,y) = r$ or $d(x,y) = r+1$. If there exists a vertex $z \in \mathcal{G}$ such that $d(z,x),\ d(z,y) \geq r-1$, then either x or y is sufficient.*

Proof. Choose any z in $N^{r-1}(\{x,y\}) \setminus N^{r-2}(\{x,y\}$. This set is nonempty by the second assumption of the claim. We will have $d(z,x), d(z,y) \geq r-1$ and either $d(z,x) = r-1$ or $d(z,y) = r-1$. Without loss of generality assume that $d(z,x) = r-1$ and $d(z,y) \geq r-1$.

We will show that x is sufficient. Let $x,\ x_1, \dots, x_{d(x,y)-1},\ y$ be a geodesic between x and y. For $i = 1, \dots, d(x,y)-1$, the triangle inequality implies that

$$d(x_i,z) \geq d(x,z) - d(x,x_i) = d(x,z) - i, \tag{4}$$

$$d(x_i,z) \geq d(y,z) - d(y,x_i) = d(y,z) - d(x,y) + i. \tag{5}$$

Averaging (4) and (5), and use the inequalities $d(z,x), d(z,y) \geq r-1$ and $d(x,y) \leq r+1$ gives

$$d(x_i,z) \geq \frac{r-3}{2}. \tag{6}$$

If $r \geq 9$, then (6) implies that $d(x_i,z) \geq 3$ for all i. Hence $N(x)$, $N(z)$, $N(x_3)$, $N(x_6), \dots, N(x_{r-3})$ are all disjoint and contained in $N^r(x)$. Hence x is sufficient.

If $r = 6$, then (4) and (5) imply that $d(x_i,z) \geq 3$ for all x_i except possibly x_3 or x_4. In this case $N(z)$, $N(x_2)$ and $N(x_5)$ are all disjoint and contained in $N^6(x)$. Hence x is sufficient. □

Claim 10. *If $d(x,y) = r$ holds for $x, y \in V(\mathcal{G})$, then either x or y is sufficient.*

Proof. Suppose that x and y are insufficient. By Claim 7 there exists $z \in V(\mathcal{G})$ such that $d(x,z) = r+1$. Let $x,\ x_1, \dots, x_{r-1}, y$ be a geodesic between x and y. Since x and y are insufficient, Claim 9 implies that we have $d(z,y) < r-1$. Note that $d(x,z) = r+1$ implies that $N(z) \cap N(x_i) = \emptyset$ for all $i \leq r-2$. Thus $N(z)$, $N(x_1)$, $N(x_4), \dots, N(x_{r-2})$ are all disjoint and contained in $N^r(y)$. This contradicts our assumption that y is insufficient. □

Let X be the set of insufficient vertices in $\mathcal{G}$. We define an equivalence relation "$\sim$" on X by letting $x \sim y$ if $d(x,y) \leq 2$. For $r \geq 6$, Claim 8 implies that this is an equivalence relation. Let $X_1, \dots, X_l$ be the equivalence classes of $\sim$.

The following claim gives a lower bound on the order of $\mathcal{G}$.

Claim 11. $|\mathcal{G}| \geq \left(\frac{r+3}{6}\right) \delta l$

Proof. Claims 8 and 10 imply that $d(X_i, X_j) \geq r+1$ for all $i \neq j$. If $d(X_i, X_j) = r+1$ for some i and j, then Claim 9 implies that we have $d(X_i, z) < r-1$ or $d(X_j, z) < r-1$ for all $z \in V(\mathcal{G})$. Then, Claim 8 implies that all the vertices outside of X_i and X_j are sufficient. This gives us two cases to consider:

(i) $d(X_i, X_j) \geq r+2$ for all $i \neq j$.

(ii) $d(X_1, X_2) = r+1$.

Suppose that (i) holds (this includes the case when $l = 1$). For each i, choose x_i to be a vertex in X_i. Note that $N^{\lfloor \frac{r}{2} \rfloor}(x_i)$ contains a length $\lfloor \frac{r}{2} \rfloor$ geodesic, P_i. Using Claim 6 gives

$$\left| N^{\lfloor \frac{r}{2} \rfloor + 1}(X_i) \right| \geq |N(P_i)| \geq \left(\left\lfloor \frac{1}{3} \left\lfloor \frac{r}{2} \right\rfloor \right\rfloor + 1 \right) \delta \geq \left(\frac{r+3}{6} \right) \delta.$$

For the last inequality we are using the fact that $r \equiv 0 \pmod 3$. Note that (i) implies that $N^{\lfloor \frac{r}{2} \rfloor + 1}(X_i) \cap N^{\lfloor \frac{r}{2} \rfloor + 1}(X_j) = \emptyset$ for all i, j. This implies that the following holds:

$$|V(\mathcal{G})| \geq \sum_{i=1}^{l} \left| N^{\lfloor \frac{r}{2} \rfloor + 1}(X_i) \right| \geq \left(\frac{r+3}{6} \right) \delta l.$$

Suppose that (ii) holds. Using Claim 6 we obtain

$$|V(\mathcal{G})| \geq \left(\frac{r}{3} + 1 \right) \delta = \left(\frac{r+3}{6} \right) \delta l.$$

□

When x is insufficient, the following claim gives a lower bound on the order of $N^r(x)$.

Claim 12. *Suppose that x is an insufficient vertex in the equivalence class X_i. Then, $|N^r(x)| \geq |X_i| + \frac{r}{3}\delta$ holds.*

Proof. By Claim 7, we can choose a length r geodesic from x. Let $x, x_1, \ldots, x_r$ be the vertices of this geodesic. Suppose that $X_i \cap N(x_j)$ is nonempty for some x_j. Choose $y \in X_i \cap N(x_j)$. Clearly $j \leq 1$ must hold, since otherwise $N(x)$, $N(x_3), N(x_6), \ldots, N(x_r)$ would all be contained in $N^r(y)$, contradicting that y is insufficient (since $y \in X_i$).

Hence X_i, $N(x_2)$, $N(x_5), \ldots, N(x_{r-1})$ are all disjoint and contained in $N^r(x)$ proving the claim. □

Claims 11 and 12 are all that is needed to prove Theorem 5, as follows

$$
\begin{aligned}
2e(\mathcal{G}^r) - \left(\frac{r+3}{3} - \frac{3}{2(r+3)}\right)\delta|\mathcal{G}| - |\mathcal{G}| &= \sum_{x\in V(\mathcal{G})} |N^r(x)| - \left(\frac{r+3}{3} - \frac{3}{2(r+3)}\right)\delta|\mathcal{G}| \\
&\geq \frac{3}{2(r+3)}\delta|\mathcal{G}| + \sum_{i=1}^{l}\left(|X_i|^2 - |X_i|\delta\right) \\
&\geq \frac{1}{4}\delta^2 l + \sum_{i=1}^{l}\left(|X_i|^2 - |X_i|\delta\right) \\
&= \sum_{i=1}^{l}\left(|X_i|^2 - |X_i|\delta + \frac{1}{4}\delta^2\right) \\
&= \sum_{i=1}^{l}\left(|X_i| - \frac{1}{2}\delta\right)^2 \\
&\geq 0.
\end{aligned}
$$

The first equality uses the fact that $\mathcal{G}^r$ contains a loop at every vertex, hence $2e(\mathcal{G}^r) = \sum_{x\in V(\mathcal{G})} |N^r(x)| + |\mathcal{G}|$. The first inequality follows from the definition of "sufficent vertex", Claim 12 and rearranging, while the second follows from Claim 11. This completes the proof. □

Proof of Theorem 4. Let $\mathcal{G}$ be a copy of G with a loop added at every vertex. Then $\mathcal{G}^r$ will be isomorphic to G^r with a loop added at every vertex. Note that we have $e(\mathcal{G}^r) = e(G^r) + |G|$, and $\delta(\mathcal{G}) = \delta(G) + 1$. Substitute these into Theorem 5 obtain the following.

- If $r \equiv 0 \pmod 3$, then we have

$$e(G^r) \geq \left(\frac{r+3}{6} - \frac{3}{4(r+3)}\right)\delta(G)|G| + \left(\frac{r+3}{6} - \frac{3}{4(r+3)} - \frac{1}{2}\right)|G|.$$

- If $r \not\equiv 0 \pmod 3$, then we have

$$e(G^r) \geq \frac{1}{2}\left\lceil\frac{r}{3}\right\rceil \delta(G)|G| + \left(\frac{1}{2}\left\lceil\frac{r}{3}\right\rceil - \frac{1}{2}\right)|G|.$$

Note that for $r \geq 3$, both $\frac{r+3}{6} - \frac{3}{4(r+3)} - \frac{1}{2}$ and $\frac{1}{2}\left\lceil\frac{r}{3}\right\rceil - \frac{1}{2}$ are non-negative, so Theorem 4 follows. □

Acknowledgment

The author would like to thank his supervisors Jan van den Heuvel and Jozef Skokan for advice and discussions.

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