More on Strongly Real Beauville Groups

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Abstract Beauville surfaces are a class of complex surfaces defined by letting a finite group G act on a product of Riemann surfaces. These surfaces possess many attractive geometric properties several of which are dictated by properties of the group G. A particularly interesting subclass are the 'strongly real' Beauville surfaces that have an analogue of complex conjugation defined on them. In this survey we discuss these objects and in particular the groups that may be used to define them. *En route* we discuss several open problems, questions and conjectures and discuss some progress made on addressing these.

1 Introduction

Roughly speaking (precise definitions will be given in the next section), a Beauville surface is a complex surface $\mathscr{S}$ defined by taking a pair of complex curves, i.e. Riemann surfaces, $\mathscr{C}_1$ and $\mathscr{C}_2$ and letting a finite group G act freely on their product to define $\mathscr{S}$ as a quotient $(\mathscr{C}_1 \times \mathscr{C}_2)/G$. These surfaces have a wide variety of attractive geometric properties: they are surfaces of general type; their automorphism groups [40] and fundamental groups [16] are relatively easy to compute (being closely related to G); they are rigid surfaces in the sense of admitting no nontrivial deformations [8] and thus correspond to isolated points in the moduli space of surfaces of general type [31].

Much of this good behaviour stems from the fact that the surface $(\mathscr{C}_1 \times \mathscr{C}_2)/G$ is uniquely determined by a particular pair of generating sets of G known as a 'Beauville structure'. This converts the study of Beauville surfaces to the study of groups with Beauville structures, i.e. Beauville groups.

Beauville surfaces were first defined by Catanese in [16] as a generalisation of an earlier example of Beauville [12, Exercise X.13(4)] (native English speakers may find the English translation [13] somewhat easier to read and get hold of) in which $\mathscr{C}_1 = \mathscr{C}_2$ and the curves are both the Fermat curve defined by the equation $X^5 + Y^5 + Z^5 = 0$ being acted on by the group $(\mathbb{Z}/5\mathbb{Z}) \times (\mathbb{Z}/5\mathbb{Z})$ (this choice of group may seem somewhat odd at first, but the reason will become clear later). Bauer, Catanese and Grunewald went on to use these surfaces to construct examples of smooth regular surfaces with vanishing geometric genus [9]. Early motivation came from the consideration of the 'Friedman-Morgan speculation' — a technical conjecture concerning when two algebraic surfaces are diffeomorphic which Beauville surfaces provide counterexamples to. More recently, they have been used to construct interesting orbits of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (connections with Gothendeick's theory of *dessins d'enfant* make it possible for this group to act on the set of all Beauville surfaces). Indeed one of the more impressive applications of these surfaces is the recent proof by González-Diez and Jaikin-Zapirain in [33] that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on the set of regular dessins by showing that it acts regularly on the set of Beauville surfaces.

Furthermore, Beauville's original example has also recently been used by Galkin and Shinder in [29] to construct examples of exceptional collections of line bundles.

Like any survey article, the topics discussed here reflect the research interests of the author. Slightly older surveys discussing related geometric and topological matters are given by Bauer, Catanese and Pignatelli in [10, 11]. Other notable

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works in the area include [6, 21, 41, 48, 53]. Whilst this article is largely expository in nature we also report incremental progress on various different problems that will appear here. Indeed, this work can be naturally viewed as a sequel to the author's earlier article [22], though the reader will lose little if they have neither read nor have a copy of [22] to hand.

We remark that throughout we shall use the standard 'Atlas' notation for finite groups and related concepts as described in [19], excepting that we will occasionally deviate to minimise confusion with similar notation for geometric concepts. In particular, given two groups A and B we use the following notation.

- We write $A \times B$ for the direct product of A and B, that is, the group whose members are ordered pairs (a,b) with $a \in A$ and $b \in B$ such that for two pairs $(a,b),(a',b') \in A \times B$ we have the multiplication $(a,b)(a',b') = (aa',bb')$. Given a positive integer k we write A^k for the direct product of k copies of A.
- We write $A.B$ for the extension of A by B, that is, a group with a normal subgroup isomorphic to A whose quotient is B (such groups are not necessarily direct products — for instance $\mathrm{SL}_2(5)$=2.$\mathrm{PSL}_2(5)$).
- We write $A : B$ for a semi-direct product of A and B, also known as a split extension A and B, that is, there is a homomorphism $\phi\colon B \to Aut(A)$ with elements of this group being ordered pairs (b,a) with $a \in A$ and $b \in B$ such that for $(b,a),(b',a') \in A : B$ we have the multiplication $(b,a)(b',a') = (bb', a^{\phi(b')}a')$.
- We write $A \wr B$ for the wreath product of A and B, that is, if B is a permutation group on n points then we have the split extension $A^n : B$ with B acting in a way that permutes the n copies of A.

In Section 2 we provide preliminary information and in particular give specific definitions for the concepts we have only talked about very vaguely until now. In Section 3 we will discuss the case of the finite simple groups. In Sections 4,5 and 6 we will discuss families of groups closely related to these such as characteristically simple groups and almost simple groups. Finally, in Section 7 we will conclude with a brief discussion of the question of which of the abelian and nilpotent groups are strongly real Beauville groups.

2 Preliminaries

We give the main definition.

Definition 1. A surface $\mathscr{S}$ is a **Beauville surface of unmixed type** if

- the surface $\mathscr{S}$ is isogenous to a higher product, that is, $\mathscr{S} \cong (\mathscr{C}_1 \times \mathscr{C}_2)/G$ where $\mathscr{C}_1$ and $\mathscr{C}_2$ are algebraic curves of genus at least 2 and G is a finite group acting faithfully on $\mathscr{C}_1$ and $\mathscr{C}_2$ by holomorphic transformations in such a way that it acts freely on the product $\mathscr{C}_1 \times \mathscr{C}_2$, and
- each $\mathscr{C}_i/G$ is isomorphic to the projective line $\mathbb{P}_1(\mathbb{C})$ and the corresponding covering map $\mathscr{C}_i \to \mathscr{C}_i/G$ is ramified over three points.

There also exists a concept of Beauville surfaces of mixed type in which the action of G interchanges the two curves $\mathscr{C}_1$ and $\mathscr{C}_2$ but these are much harder to construct and we shall not discuss these here. (For further details of the mixed case, the most up-to-date information at the time of writing may be found in the work of the author and Pierro in [25].)

In the first of the above conditions the genus of the curves in question needs to be at least 2. It was later proved by Fuertes, González-Diez and Jaikin-Zapirain in [27] that in fact we can take the genus as being at least 6. The second of the above conditions implies that each $\mathscr{C}_i$ carries a regular dessin in the sense of Grothendieck's theory of *dessins d'enfants* (children's drawings) [36]. Furthermore, by Belyĭ's Theorem [14] this ensures that $\mathscr{S}$ is defined over an algebraic number field in the sense that when we view each Riemann surface as being the zeros of some polynomial we find that the coeffcients of that polynomial belong to some number field. Equivalently they admit an orientably regular hypermap [42], with G acting as the orientation-preserving automorphism group. A modern account of dessins d'enfants and proofs of Belyĭ's theorem may be found in the recent book of Girondo and González-Diez [32].

These constructions can also be described instead in terms of uniformisation and using the language of Fuchsian groups [35, 51].

What makes this class of surfaces so good to work with is the fact that all of the above definition can be 'internalised' into the group. It turns out that a group G can be used to define a Beauville surface if and only if it has a certain pair of generating sets known as a Beauville structure.

Definition 2. Let G be a finite group. Let $x, y \in G$ and let

$$\Sigma(x,y) := \bigcup_{i=1}^{|G|} \bigcup_{g\in G} \{(x^i)^g, (y^i)^g, ((xy)^i)^g\}.$$

An **unmixed Beauville structure** for the group G is a set of pairs of elements $\{\{x_1,y_1\},\{x_2,y_2\}\} \subset G\times G$ with the property that $\langle x_1,y_1\rangle = \langle x_2,y_2\rangle = G$ such that

$$\Sigma(x_1,y_1)\cap\Sigma(x_2,y_2) = \{e\}.$$

If G has a Beauville structure we say that G is a **Beauville group**. Furthermore we say that the structure has **type**

$$((o(x_1),o(y_1),o(x_1y_1)),(o(x_2),o(y_2),o(x_2y_2))).$$

Traditionally, authors have defined the above structure in terms of so-called 'spherical systems of generators of length 3', meaning $\{x,y,z\} \subset G$ with $xyz = e$, but we omit $z = (xy)^{-1}$ from our notation in this survey. (The reader is warned that this terminology is a little misleading since the underlying geometry of Beauville surfaces is hyperbolic thanks to the below constraint on the orders of the elements.) Furthermore, many earlier papers on Beauville structures add the condition that for $i = 1,2$ we have that

$$\frac{1}{o(x_i)} + \frac{1}{o(y_i)} + \frac{1}{o(x_iy_i)} < 1,$$

but this condition was subsequently found to be unnecessary following Bauer, Catanese and Grunewald's investigation of the wall-paper groups in [7]. A triple of elements and their orders satisfying this condition are said to be hyperbolic. Geometrically, the type gives us considerable amounts of geometric information about the surface: the Riemann-Hurwitz formula

$$g(\mathscr{C}_i) = 1 + \frac{|G|}{2}\left(1 - \frac{1}{o(x_i)} - \frac{1}{o(y_i)} - \frac{1}{o(x_iy_i)}\right)$$

tells us the genus of each of the curves used to define the surface $\mathscr{S}$ and by a theorem of Zeuthen-Segre this in turn gives us the Euler number of the surface $\mathscr{S}$ since

$$e(\mathscr{S}) = 4\frac{(g(\mathscr{C}_1)-1)(g(\mathscr{C}_2)-1)}{|G|}$$

which in turn gives us the holomorphic Euler-Poincaré characteristic of $\mathscr{S}$, namely $4\chi(\mathscr{S}) = e(\mathscr{S})$ (see [16, Theorem 3.4]). On a more practical and group theoretic note, the type is often useful for verifying that the condition that $\sigma(x_1,y_1)\cap\sigma(x_2,y_2) = \{e\}$ is satisfied since this will clearly hold whenever the number $o(x_1)o(y_1)o(x_1y_1)$ is coprime to the number $o(x_2)o(y_2)o(x_2y_2)$.

Furthermore, if a group can be generated by a pair of elements of orders a and b whose product has order c then G is a homomorphic image of the triangle group

$$\Delta(a,b,c) = \langle x,y,z \,|\, x^a = y^b = z^c = xyz = e\rangle.$$

Homomorphic images of the triangle group $\Delta(2,3,7)$ are known as Hurwitz groups. In several places in the literature, knowing that a particular group is a Hurwitz group has proved useful for deciding its status as a Beauville group. For a discussion of known results on Hurwitz groups see the excellent surveys of Conder [17, 18] and for a more historically oriented discussion see the brief account given by Murray Macbeath in [45].

The abelian Beauville groups were essentially classified by Catanese in [16, page 24.] and the full argument is given explicitly in [7, Theorem 3.4] where the following is proved.

Theorem 1. *Let G be an abelian group. Then G is a Beauville group if, and only if, $G = (\mathbb{Z}/n\mathbb{Z})\times(\mathbb{Z}/n\mathbb{Z})$ where $n > 1$ is coprime to* 6.

This explains why Beauville's original example used the group $(\mathbb{Z}/5\mathbb{Z})\times(\mathbb{Z}/5\mathbb{Z})$ — it is the smallest abelian Beauville group.

Given any complex surface $\mathscr{S}$ it is natural to consider the complex conjugate surface $\overline{\mathscr{S}}$. In particular, it is natural to ask whether or not these two surfaces are biholomorphic.

Definition 3. Let $\mathscr{S}$ be a complex surface. We say that $\mathscr{S}$ is **real** if there exists a biholomorphism $\sigma : \mathscr{S} \to \overline{\mathscr{S}}$ such that σ^2 is the identity map.

(We remark that strictly speaking the above definition is not quite right, it being impossible to compose σ with itself. It is more accurate to talk of the composition $\sigma \circ \overline{\sigma}$ where $\overline{\sigma} \colon \overline{\mathscr{S}} \to \mathscr{S}$.)

As is often the case with Beauville surfaces, the above geometric condition can be translated into purely group theoretic terms.

Definition 4. Let G be a Beauville group and let $X = \{\{x_1, y_1\}, \{x_2, y_2\}\}$ be a Beauville structure for G. We say that G and X are **strongly real** if there exists an automorphism $\phi \in \mathrm{Aut}(G)$ and elements $g_i \in G$ for $i = 1, 2$ such that

$$g_1 \phi(x_i) g_1^{-1} = x_i^{-1} \text{ and } g_2 \phi(y_i) g_2^{-1} = y_i^{-1}$$

for $i = 1, 2$.

In practice we can always replace one generating pair by some generating pair that is conjugate to it and so we can take $g_1 = g_2 = e$ and this is often what is done in practice.

In [7] Bauer, Catanese and Grunewald show that a Beauville surface is real if, and only if, the corresponding Beauville group and structure are strongly real. This all comes from the study of the following related concept in the theory of Riemann surfaces. In Singerman's nomenclature of [47], a Riemann surface with a function behaving like the function σ in Definition 3 are said to be **symmetric**. The relationship with automorphisms of the corresponding group critically depends on the main result of [47]. The reader is warned, however, that some notable errors in [47] have recently been found and are corrected by Jones, Singerman and Watson in [43]. More specifically, the condition that an automorphism like the above exists is sufficient but it is not necessary. This is corrected by Jones, Singerman and Watson by giving a complete list of conditions that are both necessary and sufficient in [43, Theorem 1.1].

Question 1. Are there interesting strongly real Beauville surfaces arising from the conditions given in [43, Theorem 1.1] but not [47, Theorem 2]?

We remark that symmetric Riemann surfaces are also connected to the theory of Klein surfaces. Real algebraic curves and compact Klein surfaces are equivalent in the same way that the categories of complex algebraic curves and compact Riemann surfaces are equivalent. Indeed, just as a compact, connected, orientable surface admits the structure of a complex analytic manifold of dimension 1 (this is, a Riemann surface structure) then a compact connected surface that is not necessarily orientable admits the structure of a complex *dianalytic* manifold of dimension 1, that is, a Klein surface structure. See [46] for an introductory discussion and [15] for a recent survey of these surfaces.

By way of immediate easy examples, note that the function $x \mapsto -x$ is an automorphism of any abelian group and so every Beauville group given by Theorem 1 is an example of a strongly real Beauville group. More generally the following question is the main subject of this article.

Question 2. Which groups are strongly real Beauville groups?

3 The Finite Simple Groups

Naturally, a necessary condition for being a strongly real Beauville group is being a Beauville group. Furthermore, a necessary condition for being a Beauville group is being 2-generated: we say that a group G is 2-generated if there exist two elements $x, y \in G$ such that $\langle x, y \rangle = G$. It is an easy exercise for the reader to show that the alternating groups A_n for $n \geq 3$ are 2-generated (see Miller [44]). In [49] Steinberg proved that all of the simple groups of Lie type are 2-generated and in [1] Aschbacher and Guralnick used cohomological methods to show that all of the sporadic simple groups are 2-generated. We thus have that all of the non-abelian finite simple groups are 2-generated making them natural candidates for Beauville groups. This lead Bauer, Catanese and Grunewald to conjecture that aside from A_5, which is easily seen to not be a Beauville group, every non-abelian finite simple group is a Beauville group — see [7, Conjecture 1] and [8, Conjecture 7.17]. This suspicion was later proved correct [23, 24, 30, 37], indeed the full theorem proved by the author, Magaard and Parker in [24] is actually a more general statement about quasisimple groups (recall that a group G is quasisimple if it is generated by its commutators and the quotient by its center $G/Z(G)$ is a simple group.). A sketch of the proof of this Theorem is given by the author in [21, Section 3].

Having found that almost all of the non-abelian finite simple groups are Beauville groups, it is natural to ask which of the non-abelian finite simple groups are strongly real Beauville groups. In [7, Section 5.4] Bauer, Catanese and Grunewald wrote

> There are 18 finite simple nonabelian groups of order ≤ 15000. By computer calculations we have found strongly [real] Beauville structures on all of them with the exceptions of A_5, $\mathrm{PSL}_2(7)$, A_6, A_7, $\mathrm{PSL}_3(3)$, $\mathrm{U}_3(3)$ and the Mathieu group M_{11}.

On the basis of these computations they conjectured that all but finitely many of the non-abelian finite simple groups are strongly real Beauville groups. Several authors have worked on this conjecture and consequently many special cases are now known to be true.

- In [26] Fuertes and González-Diez showed that the alternating groups A_n ($n \geq 7$) and the symmetric groups S_n ($n \geq 5$) are strongly real Beauville groups by explicitly writing down permutations for their generators and the automorphisms and applying some of the classical theory of permutation groups to show that their elements had the properties they claimed. Subsequently the alternating group A_6 was also shown to be a strongly real Beauville group.
- In [28] Fuertes and Jones prove that the simple groups $PSL_2(q)$ for prime powers $q > 5$ and the quasisimple groups $SL_2(q)$ for prime powers $q > 5$ are strongly real Beauville groups. As with the alternating and symmetric groups, these results are proved by writing down explicit generators, this time combined with a celebrated theorem usually (but historically inaccurately) attributed to Dickson for the maximal subgroups of $PSL_2(q)$. General lemmas for lifting Beauville structures from a group to its covering groups are also used.
- Settling the case of the sporadic simple groups makes no impact on the Bauer, Catanese and Grunewald's original conjecture, there being only 26 of them. Nonetheless, for reasons we shall return to below, in [20] the author determined which of the sporadic simple groups are strongly real Beauville groups, including the '27^{th} sporadic simple group', the Tits group ${}^2\mathrm{F}_4(2)'$. Of all the sporadic simple groups only the Mathieu groups M_{11} and M_{23} are not strongly real. For all of the other sporadic groups smaller than the Baby Monster group $\mathbb{B}$ explicit words in the 'standard generators' [52] for a strongly real Beauville structure are given. For the Baby Monster group $\mathbb{B}$ and Monster group $\mathbb{M}$ character theoretic methods are used.
- In [22, Theorem 2] the author verifies the conjecture for the Suzuki groups, again making use of knowledge of the subgroup structure of these groups and writing down explicit matrices in the natural 4 dimensional representations of these groups.
- In unpublished calculations, the author has pushed the original computations of Bauer, Catanese and Grunewald to every non-abelian finite simple group of order at most 100 000 000.

Many of the smaller groups seem to require the use of outer automorphisms to make their Beauville structures strongly real, which explains much of the above difficulty in finding strongly real Beauville structures in certain groups. Slightly larger groups have enough conjugacy classes for inner automorphisms to be used instead. Consequently, it seems that 'small' non-abelian finite simple groups fail to be strongly real if they have too few conjugacy classes (as is the case with A_5 and as we would intuitively expect) or if they have no outer automorphisms — a phenomenon that is extremely rare but is true of both of the groups M_{11} and M_{23}. We are thus lead to assert the following somewhat bolder strengthening of the above.

Conjecture 1. All non-abelian finite simple groups apart from A_5, M_{11} and M_{23} are strongly real Beauville groups.

4 Characteristically Simple Groups

Another class of finite groups that has recently been studied from the viewpoint of Beauville constructions, and appears to be fertile ground for providing further examples of strongly real Beauville groups, are the characteristically simple groups that we define as follows (the definition commonly given definition is somewhat different from the below but in the case finite groups it can easily be shown that that below is equivalent to it).

Definition 5. A finite group G is said to be **characteristically simple** if G is isomorphic to some direct product H^k where H is a finite simple group.

For example, as we saw in Theorem 1, if $p > 3$ is prime then the abelian Beauville groups isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ are characteristically simple.

Characteristically simple Beauville groups have recently been investigated by Jones in [38, 20] where the following conjecture is discussed.

Conjecture 2. Let G be a finite non-abelian characteristically simple group. Then G is a Beauville group if and only if it is a 2-generator group not isomorphic to A_5.

In particular, the main results of [38, 39] verify this conjecture in the cases where H is any of the alternating groups; the linear groups $PSL_2(q)$ and $PSL_3(q)$; the unitary groups $PSU_3(q)$; the Suzuki groups ${}^2B_2(2^{2n+1})$; the small Ree groups ${}^2G_2(3^{2n+1})$ and the sporadic simple groups. Furthermore, as discussed in the previous section this conjecture is true for all simple groups in the case $k = 1$.

For large values of k, the group H^k will not be 2-generated despite the fact that H will be as discussed in Section 3. The values of k for which H^k is 2-generated can be surprisingly large. For example, a special case of the results alluded to in the previous paragraph is the somewhat amusing fact that

$$A_5 \times A_5 \times A_5 \times A_5 \times A_5 \times A_5 \times A_5 \times A_5 \times A_5 \times A_5 \times$$

$$A_5 \times A_5 \times A_5 \times A_5 \times A_5 \times A_5 \times A_5 \times A_5 \times A_5$$

is a Beauville group, despite the fact that A_5 itself is not a Beauville group.

The question of which characteristically simple Beauville groups are strongly real was first investigated by the author in [22, Section 3]. More specifically the following conjecture was investigated.

Conjecture 3. If H is a finite simple group of order greater than 3, then the group $H \times H$ is a strongly real Beauville group.

In many cases previously known strongly real Beauville structures of simple groups H can be used to provide strongly real Beauville structures for the groups $H \times H$. In particular we have the following.

Theorem 2. *If H is any of the following groups, then $H \times H$ is a strongly real Beauville group.*

(a) The cyclic groups of prime order $p > 3$;
(b) The alternating groups A_n for $n \geq 5$;
(c) The linear groups $PSL_2(q)$ for prime powers $q > 5$;
(d) The Suzuki groups ${}^2B_2(2^{2n+1})$;
(e) All simple groups of order at most 100 000 000;
(f) The sporadic simple groups.

The proofs of several of these cases relied on the following general construction.

Theorem 3. *Let G be a strongly real Beauville group with strongly real Beauville structure $\{\{x_1,y_1\},\{x_2,y_2\}\}$ such that the numbers $o(x_1)o(y_1)o(x_1y_1)$ is coprime to $o(x_2)o(y_2)o(x_2y_2)$. Furthermore, suppose that there is an automorphism $\phi \in Aut(G)$ such that*

$$\phi(x_1) = x_1^{-1},\ \phi(y_1) = y_1^{-1},\ \phi(x_2) = x_2^{-1} \text{ and } \phi(y_2) = y_2^{-1}.$$

Then the group $G \times G$ is a strongly real Beauville group.

Proof. See [22, Theorem 3].

The reader may be somewhat suspicious of the cases of the alternating group A_5 as well as the Mathieu groups M_{11} and M_{23} given these groups non-status as strongly real Beauville groups, in addition to asking why we are not being ambitious in considering larger direct products. Each of the groups $\mathrm{A}_5 \times \mathrm{A}_5$, $\mathrm{M}_{11} \times \mathrm{M}_{11}$ and $\mathrm{M}_{23} \times \mathrm{M}_{23}$ are indeed strongly real Beauville groups, the automorphisms used to invert the elements of the structures being ones that interchanges the two factors of the direct products. The fact that the none of the corresponding simple groups in these cases are strongly real means that this automorphism cannot be extended to higher products so in particular none of the groups $\mathrm{A}_5 \times \mathrm{A}_5 \times \mathrm{A}_5$, $\mathrm{M}_{11} \times \mathrm{M}_{11} \times \mathrm{M}_{11}$ and $\mathrm{M}_{23} \times \mathrm{M}_{23} \times \mathrm{M}_{23}$ are strongly real despite the fact that the automorphisms making the double products strongly real can be adapted to make each of the groups $\mathrm{A}_5 \times \mathrm{A}_5 \times \mathrm{A}_5 \times \mathrm{A}_5$, $\mathrm{M}_{11} \times \mathrm{M}_{11} \times \mathrm{M}_{11} \times \mathrm{M}_{11}$ and $\mathrm{M}_{23} \times \mathrm{M}_{23} \times \mathrm{M}_{23} \times \mathrm{M}_{23}$ are strongly real. In short, any precise statement concerning which higher products of simple groups are strongly real must be much more complicated.

Despite the remarks made in the previous paragraph, the following are proved by the author in [22, Lemma 5 and 6].

Lemma 1.

(a) Let $n \geq 11$ be odd and let $k \leq (n-6)/2$ be positive integers. Then A_n^k is a strongly real Beauville group.
(b) Let $n \geq 12$ be an even integer and let $k \leq (n-8)/4$. Then A_n^k is a strongly real Beauville group.

More generally the following question seems natural.

Question 3. Given a finite simple group H for which values of k is the characteristically simple group H^k a strongly real Beauville group.

By way of a partial answer to this question the author has computed values of k such that H^r is a strongly real Beauville group for every $r \leq k$ for every simple group of order at most 30 000 (with the exception of the alternating group A_5 and the Mathieu group M_{11} for which we have already shown that $k = 0$ is the largest value). The best known values of k are listed in Table 1. We do not claim that these values are best possible, merely lower bounds on the correct value, and it is likely that these may be improved upon. The author hopes to push these computations further in the future. Furthermore, the author is happy to provide details of the computations done on request.

H	k	H	k	H	k	H	k	H	k
A_5	0	$L_2(13)$	4	$L_3(3)$	14	$L_2(27)$	12	$U_4(2)$	6
$L_2(7)$	2	$L_2(17)$	6	$U_3(3)$	6	$L_2(29)$	12	${}^2B_2(8)$	52
A_6	2	A_7	14	$L_2(23)$	2	$L_2(31)$	14		
$L_2(8)$	4	$L_2(19)$	6	$L_2(25)$	10	A_8	18		
$L_2(11)$	4	$L_2(16)$	2	M_{11}	0	$L_3(4)$	4		

Table 1 Values of k such that every a simple group H with $|H| < 30000$ the group H^r is a strongly real Beauville group for every $r \leq k$.

5 Almost Simple Groups

We first recall the definition of almost simple groups.

Definition 6. Let G be a group. Recall that we say G is almost simple if there exists a simple group S such that $S \leq G \leq \mathrm{Aut}(S)$.

For example, any simple group is almost simple, as are the symmetric groups.

In [26] Fuertes and González-Diez considered which of the symmetric groups are strongly real Beauville groups. The first place the more general question of which almost simple groups are (strongly real) Beauville groups was the author's discussion given in [22, Section 5] where the following conjecture is asserted.

Conjecture 4. A split extension of a simple group is a Beauville group if, and only if, it is a strongly real Beauville group.

There are multiple 'warning shots' to be fired here — there are infinitely many almost simple groups that are not even 2-generated, let alone Beauville groups, the smallest example being $\mathrm{PSL}_4(9)$ whose outer automorphism group is $2 \times D_8$ (and more generally, if p is an odd prime and r is an even positive integer then $Aut(PSL_4(p^r))$ is not 2-generated). Even among the almost simple groups that are 2-generated, many are not Beauville groups — for example the almost simple groups ${}^2B_2(2^{2n+1}) : 3$ where $n \equiv 1 \pmod 3$ are never Beauville groups since for any $x, y \in {}^2B_2(2^{2n+1}) : 3$ we have that $\Sigma(x,y)$ contains elements from the only class of elements of order 3.

6 The Groups $G:2 \times G:2$

In [22, Lemma 7] the author proves that for $n \geq 5$ the groups $S_n \times S_n$ are strongly real. Note that since the abelianisation of $S_n \times S_n \times S_n$ for any integer $n > 1$ is elementary abelian of order eight, which is clearly not 2-generated, it follows that $S_n \times S_n \times S_n$ cannot be 2-generated. Since similar remarks apply to the groups $G:2 \times G:2 \times G:2$ for any simple groups G with an outer automorphism of order 2 (of which there are infinitely many besides the symmetric groups), it is natural to consider groups in which G is not necessarily the alternating group.

Question 4. For which simple groups G is the group $G:2 \times G:2$ a strongly real Beauville group.

It is easy to see that if G is a sporadic simple group with a non-trivial outer automorphism (namely one of the groups M_{12}, M_{22}, J_2, HS, J_3, McL, He, Suz, O'N, Fi_{22}, HN and Fi_{24}) then the strongly real Beauville structures obtained for the groups $G:2$ in [22, Section 5] provide further examples of groups of this kind. As far as the author is aware this class of groups has not been investigated elsewhere in the literature.

7 Abelian and Nilpotent Groups

Recall that the abelian Beauville groups were classified in Theorem 1 and that an immediate corollary of this result is that every abelain Beauville group is strongly real.

Theorem 1 has been put to great use by González-Diez, Jones and Torres-Teigell in [34] where several structural results concerning the surfaces defined by abelian Beauville groups are proved. For each abelian Beauville group they describe all the surfaces arising from that group, enumerate them up to isomorphism and impose constraints on their automorphism groups. As a consequence they show that all such surfaces are defined over $\mathbb{Q}$.

After the abelian groups, the next most natural class of finite groups to consider are the nilpotent groups. In [2, Lemma 1.3] Barker, Boston and the author note the following easy Lemma.

Lemma 2. *Let G and G' be Beauville groups and let $\{\{x_1,y_1,\},\{x_2,y_2\}\}$ and $\{\{x'_1,y'_1,\},\{x'_2,y'_2\}\}$ be their respective Beauville structures. Suppose that*

$$gcd(o(x_i),o(x'_i)) = gcd(o(y_i),o(y'_i)) = 1$$

for $i = 1,2$. Then $\{\{(x_1,x'_1,),(y_1,y'_1)\},\{(x_2,x'_2),(y_2,y'_2)\}\}$ is a Beauville structure for the group $G\times G'$.

Recall that a finite group is nilpotent if, and only if, it is isomorphic to the direct product of its Sylow subgroups. It thus follows that Lemma 2, and its easy to prove converse, reduces the study of nilpotent Beauville groups to that of Beauville p-groups. Note that Theorem 1 gives us infinitely many examples of Beauville p-groups for every prime $p > 3$: simply let n be any power of p. Early examples of Beauville 2-groups and 3-groups were constructed by Fuertes, González-Diez and Jaikin-Zapirain in [27] where a Beauville group of order 2^{12} and another of order 3^{12} were constructed. Even earlier than this, two (mixed) Beauville 2-groups of order 2^8 arose as part of a classification due to Bauer, Catanese and Grunewald in [9] of certain classes of surfaces of general type, which give rise to examples of (unmixed) Beauville 2-groups of order 2^7.

More recently, in [2] Barker, Boston and the author classified the Beauville p-groups of order at most p^4 and made substantial progress on the cases of groups of order p^5 and p^6. More recently still in [50] Stix and Vdovina have constructed infinite series of Beauville p-groups. In particular this gives the first examples of non-abelian Beauville p-groups of arbitrarily large order and any prime $p \geq 5$. To do this they make use of the theory of pro-p groups and in doing so provide generalisations of examples from [2]. Other recent work on Beauville p-groups has been conducted by Barker, Boston, Peyerimhoff and Vdovina in [3, 4, 5].

Up until now, however, the only example of Beauville p-groups that have been explicitly shown to be strongly real have been the abelian ones. Below we give what is, as far as the author is aware, the very first example of non-abelian strongly real Beauville p-group.

Lemma 3. *There exist strongly real Beauville 2-groups.*

Proof. Consider the group

$$G = \langle u,v \,|\, (u^iv^j)^4\, i,j = 0,1,2,3\rangle.$$

Straightforward computations verify that $|G| = 2^{14}$ and that $\{\{u,v\},\{uvu,vuv\}\}$ is a Beauville structure. Moreover, the function mapping $u \leftrightarrow u^{-1}$ and $v \leftrightarrow v^{-1}$ is an automorphism since it simply permutes the relations appearing the above presentation. This automorphism of this group clearly inverts all of the elements in the above Beauville structure so we have a strongly real Beauville structure and thus a strongly real Beauville group.

As far as the author is aware, the above is an isolated example — replacing 4 with a higher power of 2 or replacing 2 with a larger prime does not appear to produce a finite group. The utility of the above group stems from the fact that it admits an usually easy to write down presentation and in particular a presentation that makes writing down a useful automorphism so explicitly unusually easy. Clearly much work on the following question remains to be done.

Question 5.

(a) Are there infinitely many strongly real Beauville p-groups?
(b) What proportion of the 2-generated p-groups that are Beauville groups are strongly real?

We briefly remark that, as far as the author is aware, the following remains a pressing open problem in the study of Beauville p-groups more generally as mentioned by the author in [21, Problem 4.8].

Problem 1. Construct infinitely many Beauville 3-groups.

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