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By

Chimere S. Anabanti

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Three questions of Bertram on locally maximal sum-free sets

C. S. Anabanti*
c.anabanti@mail.bbk.ac.uk

Abstract

Let G be a finite group, and S a sum-free subset of G. The set S is locally maximal in G if S is not properly contained in any other sum-free set in G. If S is a locally maximal sum-free set in a finite abelian group G, then $G = S \cup SS \cup SS^{-1} \cup \sqrt{S}$, where $SS = \{xy|\ x, y \in S\}$, $SS^{-1} = \{xy^{-1}|\ x, y \in S\}$ and $\sqrt{S} = \{x \in G|\ x^2 \in S\}$. Each set S in a finite group of odd order satisfies $|\sqrt{S}| = |S|$. No such result is known for finite abelian groups of even order in general. In view to understanding locally maximal sum-free sets, Bertram asked the following questions:
(i) Does S locally maximal sum-free in a finite abelian group imply $|\sqrt{S}| \leq 2|S|$?
(ii) Does there exists a sequence of finite abelian groups G and locally maximal sum-free sets $S \subset G$ such that $\frac{|SS|}{|S|} \to \infty$ as $|G| \to \infty$?
(iii) Does there exists a sequence of abelian groups G and locally maximal sum-free sets $S \subset G$ such that $|S| < c|G|^{\frac{1}{2}}$ as $|G| \to \infty$, where c is a constant?
In this paper, we answer question (i) in the negation, then (ii) and (iii) in affirmation.

Key words and phrases: Sum-free sets, locally maximal, maximal, finite abelian groups.

1 Preliminaries

A non-empty subset S of a group G is sum-free if there is no solution to the equation $xy = z$ for $x, y, z \in S$; equivalently, if $S \cap SS = \varnothing$, where $SS = \{xy|\ x, y \in S\}$. Let S be a sum-free set in a finite group G, and $x \in S$. As $S \cap xS = \varnothing$ and $S \cup xS \subseteq G$, we obtain that $2|S| \leq |G|$; this tells us that a sum-free set in G has size at most $\frac{|G|}{2}$. Sizes of maximal by cardinality sum-free sets in finite abelian groups were studied (among others) by Erdős [10], Yap [20], Diananda and Yap [9], Rhemtula and Street [17], Babai and Sós [5], and Green and Ruzsa [14]. On the other hand, not much is known about the structures and sizes of maximal by inclusion sum-free sets. For a finite group G, a locally maximal sum-free set in G is a maximal by inclusion sum-free set in G; i.e., a sum-free subset S of G such that given any other sum-free set T in G with $S \subseteq T$, then $S = T$. Since every sum-free set in a finite group G is contained in a locally maximal sum-free set in G, we can gain information about sum-free sets in a group by studying its locally maximal sum-free sets. In connection with Group Ramsey Theory, Street and Whitehead [18] noted that every partition of a finite group G (or in fact, of $G^* = G \setminus \{1\}$) into sum-free sets can be embedded into a covering by locally maximal sum-free sets, and hence to find such partitions, it is useful to understand locally maximal sum-free sets. Among other results, they calculated locally maximal sum-free sets in groups of small orders, up to 16 in [18, 19] as well as a few higher sizes. Going in another direction, Giudici and Hart [13] started the classification of finite groups containing locally maximal sum-free sets. They classified all finite groups containing locally maximal sum-free sets of

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sizes 1 and 2, as well as some of size 3. The size 3 problem was resolved by Anabanti and Hart [3]. Except for a few finite groups containing locally maximal sum-free sets of size 4 classified in [1, 4], the classification problem is open for size $k \geq 4$. A locally maximal sum-free set in an abelian group G can be characterised as a sum-free set S in G satisfying

$$G = S \cup SS \cup SS^{-1} \cup \sqrt{S}, \tag{1.1}$$

where $SS = \{xy|\ x, y \in S\}$, $SS^{-1} = \{xy^{-1}|\ x, y \in S\}$ and $\sqrt{S} = \{x \in G|\ x^2 \in S\}$ (see [13, Lemma 3.1]). Each (locally maximal sum-free) set S in a finite (abelian) group of odd order satisfies $|\sqrt{S}| = |S|$. No such result is known for finite abelian groups of even order in general. Bertram [6, p.41] showed that there are some examples of locally maximal sum-free sets S in abelian groups of even order satisfying $|\sqrt{S}| = 2|S|$. His examples were in the cyclic group $C_{4n} = \langle x|\ x^{4n} = 1\rangle$ of order $4n$ with the locally maximal sum-free set S given as $\{x^2, x^6, x^{10}, x^{14}, \cdots, x^{4n-2}\}$, as well as the multiplicative group $C_4^2 = \langle x_1, x_2|\ x_1^4 = 1 = x_2^4,\ x_1x_2 = x_2x_1\rangle$, with $S = \{x_1^2, x_1^2x_2^2, x_1^2x_2^3, x_1^2x_2\}$. He remarked that there is ample evidence that every locally maximal sum-free set S in an abelian group of even order satisfies $|\sqrt{S}| \leq 2|S|$. While giving example with $\{x_1^2, x_1^2x_2^2, x_1^2x_2^3\}$ in C_4^2, he emphasized that his assertion is not necessarily true for sum-free sets which are not locally maximal. To better understand locally maximal sum-free sets, Bertram [6, Section 5] asked the following questions:

Question 1. *Does every locally maximal sum-free set S in a finite abelian group satisfy $|\sqrt{S}| \leq 2|S|$?*

Question 2. *Does there exists a sequence of finite abelian groups G and locally maximal sum-free sets $S \subset G$ such that $\frac{|SS|}{|S|} \to \infty$ as $|G| \to \infty$?*

Question 3. *Does there exists a sequence of finite abelian groups G and locally maximal sum-free sets $S \subset G$ such that $|S| < c|G|^{\frac{1}{2}}$ as $|G| \to \infty$, where c is a constant?*

This paper is aimed at answering these questions. In the next section, we answer the first question in the negation, and the other two questions in affirmation.

2 Main results

Suppose S is a locally maximal sum-free set in a finite abelian group G satisfying $|\sqrt{S}| > 2|S|$. As each element of a finite group of odd order has exactly one square root, $|G|$ must be even. Now,

$$\frac{-1 + \sqrt{12|G| - 23}}{6} \leq |S| < \frac{|G|}{4}. \tag{2.1}$$

The first inequality of (2.1) follows from Theorem 4(iii) of [6] which can be proved from the observation that $|SS| \leq \frac{|S|(|S|+1)}{2}$, $|SS^{-1}| \leq |S|^2 - |S| + 1$ and $|\sqrt{S}| \leq \frac{|G|}{2}$. We note that $|\sqrt{S}| \leq \frac{|G|}{2}$ follows from the fact that $\sqrt{S}$ is sum-free in an abelian group whenever S is sum-free, and that a sum-free set in a finite group G has size at most $\frac{|G|}{2}$. The latter inequality of (2.1) follows from the hypothesis that $2|S| < |\sqrt{S}|$ as well as $|\sqrt{S}| \leq \frac{|G|}{2}$. Guided by (2.1), we wrote a series of programs in GAP[12] to check for locally maximal sum-free sets S in abelian groups G of even order less than or equal to 52 such that $|\sqrt{S}| > 2|S|$. For faster computation in [12], we exempt the following groups all of whose locally maximal sum-free sets S clearly satisfy $|\sqrt{S}| \leq 2|S|$: finite cyclic groups, elementary abelian 2-groups and all groups of odd order. Among abelian groups of even order up to 52, only in two groups of order 40 ($C_2 \times C_4 \times C_5$ and $C_2^3 \times C_5$), a group of order 44 ($C_2^2 \times C_{11}$) and two groups of order 48 ($C_2^4 \times C_3$ and $C_4^2 \times C_3$) that we found locally maximal sum-free sets S satisfying $|\sqrt{S}| > 2|S|$. We note here that the locally maximal sum-free sets S satisfying $|\sqrt{S}| > 2|S|$ in the listed groups of order less than 52 are all of size 7. However, a group of order 60 (viz. $C_2^2 \times C_3 \times C_5$) contains locally maximal sum-free sets S of sizes 7 and 9 satisfying $|\sqrt{S}| > 2|S|$. We are thereby moved by these experimental results to answer Question 1 in the negation (see Theorem 2.1 below).

Theorem 2.1. *There exists a locally maximal sum-free set S in the group $C_2^3 \times C_5$ of order 40 such that $|\sqrt{S}| > 2|S|$.*

Proof. Let $G = C_2^3 \times C_5$, where $C_2^3 \times C_5 = \langle x_1, x_2, x_3, x_4 |\ x_1^2 = 1 = x_2^2,\ x_3^2 = 1 = x_4^5,\ x_ix_j = x_jx_i$ for $1 \leq i, j \leq 4\rangle$. We define a subset S of G as $S := \{x_3,\ x_1x_2,\ x_2x_3,\ x_4^2,\ x_1x_4^2,\ x_4^3,\ x_1x_4^3\}$. Our claim is that S is locally maximal sum-free in G, and $|\sqrt{S}| > 2|S|$. The sum-free property of S is easy to verify. For the local maximality condition, as $S = S^{-1}$, in the light of Equation (1.1), we only show that $G = S \cup SS \cup \sqrt{S}$. Now, $SS = \{1, x_1,\ x_2, x_4,\ x_1x_3,\ x_1x_4,\ x_1x_2x_3,\ x_2x_4^2,\ x_3x_4^2,\ x_1x_2x_4^2,\ x_1x_3x_4^2,\ x_2x_3x_4^2,\ x_2x_4^3,\ x_3x_4^3,\ x_4^4,\ x_1x_2x_3x_4^2,\ x_1x_2x_4^3,\ x_1x_3x_4^3,\ x_1x_4^4,\ x_2x_3x_4^3,\ x_1x_2x_3x_4^3\}$ and $\sqrt{S} = \{x_4,\ x_4^4,\ x_3x_4,\ x_3x_4^4,\ x_2x_4,\ x_2x_4^4,\ x_2x_3x_4,\ x_2x_3x_4^4,\ x_1x_4,\ x_1x_4^4,\ x_1x_3x_4,\ x_1x_3x_4^4,\ x_1x_2x_4,\ x_1x_2x_4^4,\ x_1x_2x_3x_4,\ x_1x_2x_3x_4^4\}$. Thus, $S \cup SS \cup \sqrt{S} = G$ and we conclude that S is locally maximal. Our calculation shows that $|\sqrt{S}| = 16 > 14 = 2|S|$. This completes the proof! □

It will also be interesting to determine whether or not there exists a sequence of finite abelian groups G and locally maximal sum-free sets $U \subset G$ such that $|\sqrt{U}| > 2|U|$. At the moment, we haven't been able to obtain such a sequence. For the rest of the section, we focus on answering Questions 2 and 3 of Section 1. Suppose $S = \{x_1, x_2, \cdots, x_m\}$ is a locally maximal sum-free set in a finite abelian group G. As $SS \subseteq \{x_1x_1, \cdots, x_1x_m\} \cup \{x_2x_2, \cdots, x_2x_m\} \cup \cdots \cup \{x_{m-1}x_{m-1}, x_{m-1}x_m\} \cup \{x_mx_m\}$, we have that $|SS| \leq m + (m-1) + \cdots + 2 + 1 = \frac{m(m+1)}{2}$. If $|SS| \approx \frac{|S|(|S|+1)}{2}$, then $\frac{|SS|}{|S|} \approx \frac{|S|+1}{2}$. So there could be a possibility of answering Question 2 in affirmation. We think of a possible group whose elements are either in S or SS for a locally maximal sum-free set S so that $|S|$ will be as small as possible. From the study of groups with similar properties [18, 4, 2], the kind of groups that come to mind are the elementary abelian 2-groups since if S is a locally maximal sum-free set in an elementary abelian 2-group G, then $SS = SS^{-1}$ and $\sqrt{S} = \varnothing$; so equation (1.1) yields $G = S \cup SS$. But $|SS| \leq \frac{|S|(|S|+1)}{2} - |S| + 1$ because $|S^2| = \#\{x^2 |\ x \in S\} = 1$; so $|G| \leq \frac{|S|^2+|S|+2}{2}$. Thus, if an elementary abelian 2-group G contains a locally maximal sum-free set S, then $|S| \geq \frac{-1+\sqrt{8|G|-7}}{2}$. This bound is tight since the set $\{x_1, x_2, x_3, x_4, x_1x_2x_3x_4\}$ is locally maximal sum-free in $C_2^4 = \langle x_1, x_2, x_3, x_4 \mid x_i^2 = 1,\ x_ix_j = x_jx_j$ for $1 \leq i, j \leq 4\rangle$. We are now faced with the question of what possibly the minimal size of a locally maximal sum-free set in such groups can be? To the best of our knowledge, the problem of obtaining minimal sizes of locally maximal sum-free sets in finite groups was first raised by Street and Whitehead [18, p. 226], and subsequently by Babai and Sós [5, p. 111]. This problem is also of great interest to finite geometers who study the packing problem: determination of minimal size of a complete cap in PG$(n-1, 2)$. The projective space of dimension n over GF(q) is denoted by PG(n, q). A k**-cap** in PG(n, q) is a set of k points, no three of which are collinear. A k-cap (see [11]) is called complete if it is not contained in a $(k+1)$-cap of the same projective space. Complete caps in PG$(n-1, 2)$ are synonymous to locally maximal sum-free sets in C_2^n. Klopsch and Lev [16, Section 3] described its connection with Coding theory. A number of researchers (for instance, [7, 8, 15]) have proved some bounds for the minimal sizes of locally maximal sum-free sets in elementary abelian 2-groups. An interested reader may see [8] for analogue of the best known bound on the minimal sizes of locally maximal sum-free sets in elementary abelian 2-groups. A direct analogue of the results of [7] gave rise to Theorem 2.2 below.

Notation. We write $C_2^n = \langle x_1, \cdots, x_n \mid x_i^2 = 1, x_i x_j = x_j x_i, 1 \leq i, j \leq n \rangle$ for the elementary abelian 2-group of finite rank n. In C_2^n, we call the identity element the unique word of length 0, elements with single letter are called words of length 1, elements with double letters (example $x_i x_j$, $i \neq j$) are called words of length 2, and so on. We denote the length of a word w by $l(w)$, and write w_{ij} for words of length i in C_2^j; i.e., $w_{ij} := \{w \in C_2^j | \; l(w) = i\}$. Finally, we write $\delta(G)$ for the minimal size of a locally maximal sum-free set in G.

Theorem 2.2. *For $t \geq 2$, $\delta(C_2^{2t}) \leq 2^{t+1} - 3$ and $\delta(C_2^{2t+1}) \leq 3(2^t) - 3$.*

Proof. The result follows from Claims 2.0.1 and 2.0.2 below.

Claim 2.0.1. *For $n \geq 4$, let $G = C_2^n = C_2^q C_2^r$, where $q + r = n$ and $q = r + 1$ or $q = r + 2$ according as n being odd or even. With the generators of C_2^q and C_2^r given as $\{x_1, \cdots, x_q\}$ and $\{x_{q+1}, \cdots, x_{q+r}\}$ respectively, the set*

$$V := \{x_2, \cdots, x_n\} \cup \{x_1 x_{q+1}, \cdots, x_1 x_{q+r}\} \cup \bigcup_{i=2}^{r} \big(w_{ir} x_i \cup w_{ir} x_1 x_i\big) \cup \bigcup_{\substack{i \geq 3 \\ \text{and odd}}} w_{iq}$$

is locally maximal sum-free in G.

Claim 2.0.2. *The locally maximal sum-free set V constructed above attains the defined upper bound, with $r = t$ or $t - 1$ according as n being odd or even.*

□

We now answer Questions 2 and 3 respectively (in affirmation) in Observations 2.3 and 2.4 below.

Observation 2.3. *Theorem 2.2 guarantees the existence of a locally maximal sum-free set (example with the locally maximal sum-free set V in the proof of Theorem 2.2) of size $2^{n+1} - 3$ in C_2^{2n} and size $3(2^n) - 3$ in C_2^{2n+1} for $n \geq 2$. In the first case,*

$$\frac{|VV|}{|V|} = \frac{2^{2n} - 2^{n+1} + 3}{2^{n+1} - 3} > 2^{n-1} - 1 \to \infty \text{ as } n \to \infty,$$

and for the latter case, we have

$$\frac{|VV|}{|V|} = \frac{2^{2n+1} - 3(2^n) + 3}{3(2^n) - 3} > \frac{2^{n+1} - 3}{3} \to \infty \text{ as } n \to \infty.$$

Observation 2.4. *Let G be an elementary abelian 2-group of finite rank $2n$ for $n \geq 2$. Theorem 2.2 guarantees the existence of a locally maximal sum-free set (example with the locally maximal sum-free set V in the proof of Theorem 2.2) of size $2^{n+1} - 3$ in G. Indeed, V satisfies the condition of Question 3 as*

$$|V| = 2^{n+1} - 3 < 2^{n+1} = 2(|G|^{\frac{1}{2}}) \text{ as } |G| \to \infty,$$

with $c = 2$.

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