Three questions of Bertram on locally maximal sum-free sets II

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Abstract

We recently answered the three questions of Bertram in the finite abelian case. In this paper, we answer the nonabelian analogues of the questions of Bertram on locally maximal sum-free sets.

Key words and phrases: Sum-free sets, locally maximal, finite groups.

1 Introduction

A non-empty subset S of a group G is sum-free if there is no solution to the equation $xy = z$ for $x, y, z \in S$; equivalently, if $S \cap SS = \varnothing$, where $SS = \{xy|\ x, y \in S\}$. For a finite group G, a locally maximal sum-free set in G is a maximal by inclusion sum-free set in G; i.e., a sum-free subset S of G such that given any other sum-free set T in G with $S \subseteq T$, then $S = T$. For insight to works on locally maximal sum-free sets, the reader may see [10, 5, 6, 2]. A locally maximal sum-free set in a group G can be characterised as a sum-free set S in G satisfying

$$G = S \cup SS \cup SS^{-1} \cup S^{-1}S \cup \sqrt{S}, \tag{1.1}$$

where $SS = \{xy|\ x, y \in S\}$, $SS^{-1} = \{xy^{-1}|\ x, y \in S\}$, $S^{-1}S = \{x^{-1}y|\ x, y \in S\}$ and $\sqrt{S} = \{x \in G|\ x^2 \in S\}$ (see [6, Lemma 3.1]). To better understand locally maximal sum-free sets, Bertram [3, Section 5] asked the following questions:

Question 1. *Does S locally maximal sum-free in a finite (abelian) group imply $|\sqrt{S}| \leq 2|S|$?*

Question 2. *Does there exists a sequence of finite (abelian) groups G and locally maximal sum-free sets $S \subset G$ such that $\frac{|SS|}{|S|} \to \infty$ as $|G| \to \infty$?*

Question 3. *Does there exists a sequence of finite (abelian) groups G and locally maximal sum-free sets $S \subset G$ such that $|S| < c|G|^{\frac{1}{2}}$ as $|G| \to \infty$, where c is a constant?*

These questions were recently answered in the finite abelian case (see [1]). The goal of this paper is to answer the nonabelian analogues of the three questions.

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2 Main results

We give answers to Bertram type questions in the nonabelian groups case. On the first type question on whether every locally maximal sum-free set S in a finite nonabelian group satisfy $|\sqrt{S}| \leq 2|S|$? The answer is 'no'. A first example is the Quaternion group Q_8 of order 8, where the locally maximal sum-free set S is the singleton set containing the unique involution. It satisfies $|\sqrt{S}| = 6 > 2 = 2|S|$. This example was given in the classification of finite groups containing locally maximal sum-free set of size 1 (see [6, Theorem 4.1]). But not much is known since then. Unlike in the finite abelian case where we couldn't find a sequence of locally maximal sum-free set S satisfying $|\sqrt{S}| > 2|S|$, we show that such exists in the finite nonabelian case (see Theorem 2.3 below).

Notation. We write $C_2^n = \langle x_1, \cdots, x_n \mid x_i^2 = 1, x_ix_j = x_jx_i, 1 \leq i, j \leq n\rangle$ for elementary abelian 2-groups of finite rank n, and $\delta(G)$ for the minimal size of locally maximal sum-free set in G.

Proposition 2.1. *$\delta(Q_8 \times C_2^n) \leq \delta(C_2^{n+1})$ for $n \geq 0$.*

Proof. Let the unique involution in Q_8 be denoted by z. In the light of equation (1.1), the set $S = \{z\}$ is locally maximal sum-free in Q_8, and the singleton set consisting of the non-identity element of C_2 is locally maximal sum-free in C_2. This verifies the result for $n = 0$. Suppose $n \geq 1$. Let the generating elements of C_2^n be $x_1, \cdots, x_n$ and generating elements of C_2^{n+1} be $x_1, \cdots, x_n, x_{n+1}$. Take a locally maximal sum-free set (say S) in C_2^{n+1} whose size yields $\delta(C_2^{n+1})$ such that S contains x_{n+1}. Substitute z for x_{n+1} in S, and call the resulting set T. Clearly, T is sum-free in $Q_8 \times C_2^n$ (i.e., $T \cap TT = \varnothing$), and $|T \cup TT| = |S \cup SS| = 2^{n+1}$. Observe that T consists only of involutions. Now, equation (1.1) tells us that the set T is locally maximal sum-free if and only if $Q_8 \times C_2^n = T \cup TT \cup \sqrt{T}$. As $\sqrt{T}$ consists only of elements of order 4, we have that $T \cap \sqrt{T} = \varnothing$. But any non-identity element of TT is an involution; whence $TT \cap \sqrt{T} = \varnothing$. Thus, any two sets chosen from $\{T, TT, \sqrt{T}\}$ are disjoint. As $|\sqrt{T}| = |\sqrt{\{z\}}| = 3(2^{n+1})$, we conclude that $Q_8 \times C_2^n = T \cup TT \cup \sqrt{T}$; whence T is locally maximal sum-free in $Q_8 \times C_2^n$. □

An observation of [1] is the following:

Observation 2.2. *Let w_{ij} denote the set of all words of length i in C_2^j. For $n \geq 4$, let $G = C_2^n = C_2^qC_2^r$, where $q + r = n$ and $q = r + 1$ or $q = r + 2$ according as n being odd or even. With the generators of C_2^q and C_2^r given as $\{x_1, \cdots, x_q\}$ and $\{x_{q+1}, \cdots, x_{q+r}\}$ respectively, we have that*

$$V := \{x_2, \cdots, x_n\} \cup \{x_1x_{q+1}, \cdots, x_1x_{q+r}\} \cup \bigcup_{i=2}^{r} \left(w_{ir}x_i \cup w_{ir}x_1x_i\right) \cup \bigcup_{\substack{i \geq 3 \\ \text{and odd}}} w_{iq}$$

is a locally maximal sum-free set of size $2^{k+1} - 3$ in C_2^{2k} and size $3(2^k) - 3$ in C_2^{2k+1} for $k \geq 2$.

Theorem 2.3. *For $n \geq 4$, there exists a locally maximal sum-free subset W of $Q_8 \times C_2^{n-1}$ such that $|\sqrt{W}| > 2|W|$.*

Proof. As noted in Observation 2.2, the locally maximal sum-free set V is of size $2^{k+1} - 3$ in C_2^{2k} and size $3(2^k) - 3$ in C_2^{2k+1} for $k \geq 2$. Let $G = Q_8 \times C_2^{n-1}$ for $n \geq 4$. We denote by W the locally maximal sum-free set in G obtained by replacing one generating element of C_2^n found in V with the unique involution in Q_8 (idea from the construction of the set T in the proof of Proposition 2.1). For $n = 2k$, we have that $|\sqrt{W}| = \frac{3}{4}|G| = 3(2^{2k}) > 2^{k+2} - 6 = 2|W|$, and for $n = 2k + 1$, we have that $|\sqrt{W}| = \frac{3}{4}|G| = 3(2^{2k+1}) > 3(2^{k+1}) - 6 = 2|W|$. □

Remark 2.4. Bertram's third type question is whether there exists a sequence of nonabelian groups G and locally maximal sum-free sets $S \subset G$ such that $|S| < c|G|^{\frac{1}{2}}$ as $|G| \to \infty$, where c is a constant. As a consequence of the proof of Theorem 2.3, we answer this question as follows (with $c = 1$). Take $G = Q_8 \times C_2^{n-1}$, where $n = 2k$ for $k \geq 2$. The locally maximal sum-free set W in the proof of Theorem 2.3 satisfies $|W| = 2^{k+1} - 3 < 2^{k+1} = |G|^{\frac{1}{2}}$ as $|G| \to \infty$.

Bertram's second type question asks whether there exists a sequence of finite non-abelian groups G and locally maximal sum-free sets $S \subset G$ such that $\frac{|SS|}{|S|} \to \infty$ as $|G| \to \infty$. In the light of Observation 2.2, we ascertain that the locally maximal sum-free set W in the proof of Theorem 2.3 yields a set of size $2^{k+1} - 3$ in $Q_8 \times C_2^{2k-1}$ and size $3(2^k) - 3$ in $Q_8 \times C_2^{2k}$ for $k \geq 2$. In the first case,

$$\frac{|WW|}{|W|} = \frac{2^{2k} - 2^{k+1} + 3}{2^{k+1} - 3} > 2^{k-1} - 1 \to \infty \text{ as } k \to \infty,$$

and for the latter case, we have

$$\frac{|WW|}{|W|} = \frac{2^{2k+1} - 3(2^k) + 3}{3(2^k) - 3} > \frac{2^{k+1} - 3}{3} \to \infty \text{ as } k \to \infty.$$

We give Theorem 2.5 below whose proof motivated Remark 2.6 which clearly gives another answer to the second type question of Bertram. Before that, we discuss a bit about non-symmetric complete sum-free sets. A complete sum-free set in a finite group G is a sum-free set S satisfying $G = S \cup SS$. Any sum-free set S with the property that $S = S^{-1}$ is said to be symmetric; otherwise S is non-symmetric. A sum-free set that is both non-symmetric and complete is called a non-symmetric complete sum-free set. Cameron [4] asked whether there exists a natural number n_0 such that for all $|G| \geq n_0$, there is a non-symmetric complete sum-free set in G, where $G = \mathbb{Z}_n$. Payne [9] answered with $n_0 = 890,626$, with an emphasis that a smaller value of n_0 is possible. In the proof of Theorem 2.5 below, we give a construction of non-symmetric complete sum-free subset S of the Alternating group A_m of degree m for $m \geq 4$. This allows us to also show that the smallest value of n_0 is 12 in Cameron's type question for $G = A_m$. We note here that the (two) sum-free sets in A_3 have size 1; so none of them is complete. Hence, Theorem 2.5 below also helps us conclude that the smallest value of n_0 in Cameron's type question for $G = A_m$ is 12.

Theorem 2.5. *A_n contains a non-symmetric complete sum-free set for $n \geq 4$.*

Proof. Let $n \geq 4$. We define a subset S of A_n as

$$S := (1, 2, n)A_{n-1}. \tag{2.1}$$

Our claim is that S is a non-symmetric complete sum-free set in A_n. As S is a non-trivial coset of A_{n-1} in A_n, we know that S is sum-free in A_n. Clearly, $(1, 2, n) \in S$ but $(1, 2, n)^{-1} = (1, 2, n)^2 \notin S$. Hence, S is non-symmetric. It remains to show that S is complete. This suffices to show that each element of A_n is either an element of S or SS. For $n = 4$, taking elements of A_4 as $\{1, (1, 2, 3),$ $(1, 3, 2)$, $(1, 2, 4)$, $(1, 4, 2)$, $(1, 3, 4)$, $(1, 4, 3)$, $(2, 3, 4)$, $(2, 4, 3)$, $(1, 2)(3, 4)$, $(1, 3)(2, 4)$, $(1, 4)(2, 3)\}$ and $S = \{(1, 2, 4), (1, 4)(2, 3), (1, 3, 4)\} \subset A_4$, we see immediately that $SS = A_4 \backslash S$. Now, suppose $n \geq 5$. We partition A_n into n distinct cosets as follows:

$$A_n = A_{n-1} \dot{\cup} (1, 2, n)A_{n-1} \dot{\cup} (1, n, 2)A_{n-1}, (1, n, 3)A_{n-1} \dot{\cup} \cdots \dot{\cup} (1, n, n-1)A_{n-1}. \tag{2.2}$$

We leave it as an easy exercise for the reader to verify that each member of the set

$$U = \{(1, 2, n)A_{n-1}, (1, n, 2)A_{n-1}, (1, n, 3)A_{n-1}, \cdots, (1, n, n-1)A_{n-1}\} \tag{2.3}$$

is a non-trivial coset of A_{n-1} in A_n, and that any two members of U are disjoint. Let $x \in A_n$ be arbitrary. If $x \in A_{n-1}$, then $x = [(1,2,n)(1,2)(3,4)][(1,2,n)(1,2)(3,4)x] \in SS$. If $x \in (1,2,n)A_{n-1}$, then $x \in S$ by definition. If $x \in (1,n,2)A_{n-1}$, then as $(1,n,2)A_{n-1} = [(1,2,n)][(1,2,n)A_{n-1}]$, we deduce that $x \in SS$. Finally, suppose $x \in (1,n,k)A_{n-1}$ for $3 \leq k \leq n-1$. Then $x = (1,n,k)y$ for some $y \in A_{n-1}$. Note that $(1,n,k) = [(1,2,n)(1,k,2)][(1,2,n)(1,k,2)]$. Therefore

$$(1,n,k)y = [(1,2,n)(1,k,2)][(1,2,n)(1,k,2)y] \in SS;$$

whence $(1,n,k)A_{n-1} \subseteq SS$ for $3 \leq k \leq n-1$. This completes the proof. □

Remark 2.6. The locally maximal sum-free set S constructed in the proof of Theorem 2.5 satisfies

$$\frac{|SS|}{|S|} = \frac{|A_n| - |A_{n-1}|}{|A_{n-1}|} = n - 1 \to \infty \text{ as } n \to \infty.$$

Concluding Remarks. (i) We note here that the constructed non-symmetric complete sum-free set S in (2.1) satisfies $S^{-1}S = S^{-1}(1,2,n) = A_{n-1}$ and $SS^{-1} = S(1,2,n)^{-1} \cong A_{n-1}$. In particular, $|SS^{-1}|$ attains its minimum possible value, and an unusual property that

$$\frac{|SS^{-1}|}{|SS|} = \frac{1}{n-1} \to 0 \text{ as } n \to \infty$$

is also satisfied.

(ii) On a different note, we emphasize that Proposition 2.1 leads us to ask the question: among all finite groups of a given order, which group(s) does its locally maximal sum-free sets yield the smallest possible size? In attempt to answer this, we pose the conjecture below (which can easily be proved when G is a finite abelian group or a 2-group of coclass 1).

Conjecture 2.7. Given $n \geq 0$, $\delta(Q_8 \times C_2^n) \leq \delta(G)$ for all finite groups G of order 2^{n+3}.

Following the classification of finite groups containing locally maximal sum-free sets of sizes 1 and 2 given in Section 4 of [6], the conjecture above is clearly true for $n = 0$ and 1. We have also verified by computational means that it is true for $n = 2$ and 3.

(iii) A final note for this study is that the three questions of Bertram can be answered in the nonabelian case using the group $Q_8 \times C_2^n$ for some defined $n \in \mathbb{N} \cup \{0\}$.

As connection to other works, we note here that the Cayley index of any finite group G is at most 16. This maximum value 16 is attained in Q_8 (see [7, Lemma 2.6]) and $Q_8 \times C_2$ (see [8, Section 4]). The Cayley index of the group $Q_8 \times C_2^n$ is 8 for $n \geq 2$ (see [8, Proposition 4.7]).

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