A counterexample on a group partitioning problem

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Abstract

The *Ramsey number* $R_n(3)$ is the smallest positive integer such that colouring the edges of a complete graph on $R_n(3)$ vertices in n colours forces the appearance of a monochromatic triangle. We start with a proof that by partitioning the non-identity elements of a finite group into disjoint union of n symmetric product-free sets, we obtain a lower bound for the Ramsey number $R_n(3)$. Exact values of $R_n(3)$ are known for $n \leq 3$. The best known lower bound that $R_4(3) \geq 51$ was given by Chung. In 2006, Kramer gave over 100 pages proof that $R_4(3) \leq 62$. He then conjectured that $R_4(3) = 62$. In this paper, we say that the Ramsey number $R_n(3)$ is *solvable by group partitioning means* if there is a finite group G such that $|G| + 1 = R_n(3)$ and G^* can be partitioned as a disjoint union of n symmetric product-free sets. We show that $R_n(3)$ (for $n \leq 3$) are solvable by group paritioning means while $R_4(3)$ is not. Then conjecture that $R_3(5) \geq 257$ as well as raise the question of which Ramsey numbers are solvable by group partitioning means?

1 Introduction

Let G be a finite group, and S a non-empty subset of G. Then S is said to be *product-free* if $S \cap SS = \varnothing$. A *maximal product-free set* in G is a maximal by cardinality product-free set in G. Let $\lambda(G)$ denotes the cardinality of a maximal product-free set in G. Diananda and Yap [2] investigated $\lambda(G)$ when G is a finite abelian group, covering three cases: $|G|$ has at least one prime factor $p \equiv 2 \ mod\ 3$, no prime factor $p \equiv 2 \ mod\ 3$ but 3 is a factor of $|G|$, and lastly, where every prime factor of $|G|$ is a prime $p \equiv 1 \ mod\ 3$. Exact values were given in the first two cases whereas a bound was given in the third case, which was later completed by Green and Ruzsa in [6]. Not much is known about the structures and sizes of maximal product-free sets when the group is nonabelian. The *Ramsey number* $R_n(3)$ is the smallest positive integer such that colouring the edges of a complete graph on $R_n(3)$ vertices in n colours forces the appearance of a monochromatic triangle. Exact values of $R_n(3)$ are known for $n \leq 3$. The best known lower bound that $R_4(3) \geq 51$ was given by Chung [1] in 1973. Kramer [9], in 2006, after giving over 100 pages proof that $R_4(3) \leq 62$, conjectured that $R_4(3) = 62$. A symmetric product-free set is a product-free set S such that $S = S^{-1}$. For a finite group G, we start with a proof that if G^* (where $G^* = G \setminus \{1\}$) can be partitioned into disjoint union of m symmetric product-free sets, then $R_m(3) \geq |G| + 1$. This work shows that this group partition approach gives a sharp lower bound that coincides with the exact value of $R_m(3)$ for $m \leq 3$, but cannot be used to improve the known lower bound of $R_4(3)$ to r for $52 \leq r \leq 62$.

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2 Main results

The exact result for $R_1(3)$, $R_2(3)$ and $R_3(3)$ are known. We begin with new proofs of the small Ramsey numbers through partitioning non-identity elements of a finite group.

Theorem 1. *[1] If G is a finite group such that G^* can be partitioned into disjoint union of m symmetric product-free sets (where $m \geq 2$), then $R_m(3) \geq |G| + 1$.*

Proof. Suppose $G^* = S_1 \sqcup \cdots \sqcup S_m$ is a disjoint union of m symmetric product-free sets. We assign to the set S_i colour C_i for each $i \in \{1, \cdots, m\}$. Let $K_{|G|}$ be the complete graph on $|G|$ vertices: $v_1, v_2, \cdots, v_{|G|}$. [Note that the vertices of $K_{|G|}$ are the elements of G.] We m-colour $K_{|G|}$ as follows: colour the edge v_iv_j (from v_i to v_j) with colour C_k if $v_iv_j^{-1} \in S_k$. Since S_k is symmetric (i.e., $S_k = S_k^{-1}$), this induces a well-defined edge-colouring of the graph. Let v_a, v_b and v_c be any three vertices of $K_{|G|}$ and consider the triangle on these vertices. Suppose two of its edges say v_av_b and v_bv_c are coloured C_k. This means that $v_av_b^{-1}, v_bv_c^{-1} \in S_k$. Since S_k is product-free, we have that $(v_av_b^{-1})(v_bv_c^{-1}) = v_av_c^{-1} \notin S_k$. So v_av_c must be coloured C_l for $l \neq k$, and no monochromatic triangle is formed. Therefore $R_m(3) > |G|$. □

Proposition 2. *If $G^* = S$ is a symmetric product-free set in a finite group G, then $G \cong C_2$.*

Proof. Let T be a product-free set in G. For $x_1 \in T$, define $x_1T := \{x_1x_2 | x_2 \in T\}$. As $|x_1T| = |T|$ and $T \cup x_1T \subseteq G$, with $T \cap x_1T = \varnothing$, we have that $2|T| \leq |G|$; so $|T| \leq \frac{|G|}{2}$. This shows that the size of a product-free set in any finite group G is at most $\frac{|G|}{2}$. Now, as S is product-free, we have that $|S| \leq \frac{|G|}{2}$, and as $|G| = |S| + 1$, we conclude that $|G| \leq 2$. Indeed, $G \cong C_2$ as $C_2^* = \{x\}$ is the unique symmetric product-free set in C_2. □

Theorem 3. *If G is a finite group of even order such that $G^* = S_1 \sqcup S_2$, where S_1 and S_2 are symmetric product-free sets in G, then $|G| = 4$.*

Proof. Clearly $|S_1| \neq |S_2|$; otherwise $|G| = 2|S_1| + 1$ is odd. Without loss of generality, suppose $|S_1| < |S_2|$. We claim that $|S_2| = |S_1| + 1$. Suppose for contradiction that $|S_2| = |S_1| + m$, where $m \geq 3$ and odd. Then $|G| = 2|S_1| + m + 1$. So $\lambda(G) \leq \frac{|G|}{2} = |S_1| + \frac{m+1}{2} < |S_2|$; a contradiction. Hence, $|S_2| = |S_1| + 1$ as claimed, and we obtain that $|S_2| = \frac{|G|}{2}$. Now, $G^* = H^* \sqcup S_2$, where H is a maximal subgroup of index 2 in G. By Proposition 2, $H \cong C_2$. So $|S_1| = 1$, and we conclude that $|G| = 4$. □

Lemma 4. *Suppose G is a finite group of odd order such that $G^* = S_1 \sqcup S_2$, where S_1 and S_2 are symmetric product-free sets. Then: (i) $|S_1| = |S_2|$; (ii) $S_2 = S_1^2$; (iii) $G^* \subseteq S_1 \sqcup S_1S_1$.*

Proof. (i) As S_1 is product-free in G, we have that $S_1 \cap S_1S_1 = \varnothing$; so $(S_1S_1)^* \subseteq S_2$. Thus $|S_1| \leq |(S_1S_1)^*| \leq |S_2|$. Similarly, as S_2 is product-free, we obtain $|S_2| \leq |(S_2S_2)^*| \leq |S_1|$. Hence, $|S_1| = |S_2|$. Part (ii) follows from $S_1^2 \subseteq (S_1S_1)^* \subseteq S_2$ and $|S_1^2| = |S_1| = |S_2|$. For (iii), suppose for contradiction that $G^* \nsubseteq S_1 \sqcup S_1S_1$. Then there exists $y \in S_2$ such that $y \notin S_1S_1$. Let $g \in S_1$ be arbitrary. Then $g^{-1}y \notin S_1$ as $g(g^{-1}y) = y$. So either $g^{-1}y \in S_2$ or $g^{-1}y = 1$. The latter is not possible since $g^{-1}y = 1$ implies that $g = y \in S_2$. So we must have that $g^{-1}y \in S_2 \;\forall\; g \in S_1$. Thus, $S_1^{-1}y \subseteq S_2$. But $|S_1^{-1}y| = |S_1| = |S_2|$. So $S_1^{-1}y = S_2$. But $y \in S_2$. So $y = g^{-1}y$ for some $g \in S_1$. Therefore, $1 \in S_1$; a contradiction. Thus, $G^* \subseteq S_1 \sqcup S_1S_1$. □

Theorem 5. *Let G be a finite group of odd order. If G^* can be partitioned into disjoint union of two symmetric product-free sets, then G is either C_3 or C_5.*

Proof. Let G be a finite group of odd order such that $G^* = S_1 \sqcup S_2$, where S_1 and S_2 are symmetric product-free sets in G. Lemma 4(i) tells us that $|S_1| = |S_2|$; so $|G| = 2|S_1| + 1$. Since no product-free set in G can have size more than $|S_1|$, we note that S_1 is a maximal product-free set in G, and conclude that $\lambda(G) = \frac{|G|-1}{2} = |S_1|$. Let $g \in S_1$. Then $g^2 \in S_2$ and $g^4 \in S_1$. Suppose $g^3 \neq 1$. Clearly, $g^3 \notin S_1$; since $g, g^4 \in S_1$. So $g^3 \in S_2$. Now, $g^5 = g^3g^2 \in S_1 \cup \{1\}$. Also, $g^5 = gg^4 \in S_2 \cup \{1\}$. As $S_1 \cap S_2 = \varnothing$, we obtain that $g^5 = 1$. Thus, any non-identity element of G has order 3 or 5, and we conclude that the exponent of G is 3, 5 or 15. Suppose the exponent of G is 3. If $|S_1| = 1$, then $|G| = 3$; indeed, $C_3^* = \{x\} \sqcup \{x^2\}$. As $|G| = 3^i$, $i \geq 1$, we know that $|S_1| \neq 2, 3$. So, suppose $|S_1| \geq 4$. Let $x_1, x_2 \in S_1$, with $x_1 \neq x_2$. Lemma 3.3(ii) tells us that $S_2 = S_1^2$. So $x_1^2, x_2^2 \in S_2$. Observe that $1 \neq x_1x_2^2 \in S_1 \sqcup S_2$. If $x_1x_2^2 \in S_2$, then $\exists\ g \in S_2$ such that $x_1x_2^2 = g$. So $x_2^2 = x_1^2 g$; a contradiction! If $x_1x_2^2 \in S_1$, then $\exists\ g^* \in S_1$ such that $x_1x_2^2 = g^*$. So $x_1 = g^*x_2$; another contradiction! Therefore $x_1x_2^2 \notin G = S_1 \cup S_2 \cup \{1\}$, and we conclude that no such partition exists. Now, suppose the exponent of G is 5. Clearly, as $|G|$ is a power of 5, we know that $|S_1| \neq 1, 3, 4, 5, 6, 7, 8, 9, 10, 11$. If $|S_1| = 2$, then $|G| = 5$. Indeed, $C_5^* = \{x, x^4\} \sqcup \{x^2, x^3\}$. Now, suppose $|S_1| \geq 12$. Let $x_1, x_2 \in S_1$, with $x_1 \neq x_2$. Then $x_1^2, x_2^2 \in S_2$ and $x_1^4, x_2^4 \in S_1$. Observe that $1 \neq x_1x_2^4 \in S_1 \sqcup S_2$. As $x_1, x_2^4 \in S_1$ and S_1 is product-free, $x_1x_2^4 \notin S_1$. So $x_1x_2^4 \in S_2$. As S_2 is product-free, $(x_1x_2^4)(x_2^2) \notin S_2$. But $x_1x_2^4x_2^2 = x_1x_2 \in S_2$; a contradiction! Thus no such partition exists. Finally, suppose the exponent of G is 15. Then G has elements of order 3 and 5. Let $x_1, x_2 \in S_1$ be arbitrary. If $\circ(x_1) = \circ(x_2)$, then we get a similar conclusion as in the case where the exponent of G is 3 or 5, according as $\circ(x_1) = 3$ or $\circ(x_1) = 5$. Now, suppose $\circ(x_1) = 3 < 5 = \circ(x_2)$. Clearly, $1 \neq x_1^2x_2 \in S_1 \sqcup S_2$. Also, $x_1^2x_2 \notin S_1$; otherwise $x_1^2x_2 = g \in S_1$ shows that $x_2 = x_1g$, a fallacy! So $x_1^2x_2 \in S_2$. As S_2 is product-free, $(x_1^2)(x_1^2x_2) \notin S_2$. But $x_1^2x_1^2x_2 = x_1x_2 \in S_2$; a contradiction. Thus, no such partition exists. $\square$

Remark 6. Clearly, $R_1(3) = 3$. In the light of Theorems 3 and 5 therefore, $R_2(3) > 5$. We shall show that $R_2(3) = 6$. Suppose we 2-colour the edges of K_6 with colours blue and green. We label the vertices of K_6 as v_0, v_1, v_2, v_3, v_4 and v_5. Choose a vertex (say v_0) of K_6. By the Pigeonhole principle, at least three edges incident with v_0 must be coloured with the same colour (say blue). Without loss of generality, let those edges be v_0v_1, v_0v_2 and v_0v_3. If any of the edges v_1v_2, v_1v_3 or v_2v_3 is coloured blue, then we have a blue triangle. So suppose none of the three edges is coloured blue, then each of them is coloured green, and we obtain a green triangle. Thus, whenever we 2-colour the edges of K_6, we force the appearance of a monochromatic triangle. Therefore $R_2(3) = 6$.

In 1955, Greenwood and Gleason [7] proved that $R_{n+1}(3) \leq (n+1)(R_n(3) - 1) + 2$ for $n \geq 2$. We include a shorter proof for the reader's convenience. Let K_N be the complete graph on N vertices, where $N = (n+1)(R_n(3) - 1) + 2$. Suppose we $(n+1)$-colour the edges of K_N. Let the vertices of K_N be $v_0, v_1, \cdots, v_{N-1}$. Choose a vertex (say v_0) of K_N. By the Pigeonhole principle, at least $R_n(3)$ of the $(n+1)(R_n(3) - 1) + 1$ edges incident with v_0 must have the same colour (say blue). Let those edges be $v_0v_1, v_0v_2, \cdots, v_0v_m$, where $m = R_n(3)$. On the complete subgraph K_m with vertices $v_1, v_2, \cdots, v_m$, consider the edges v_iv_j (from v_i to v_j), where $1 \leq i < j \leq m$. If any of the edges of K_m is coloured blue, then we obtain a blue triangle with vertices v_0, v_i and v_j. If none of them is coloured blue, then they must be coloured with the other n colours. Therefore we have a monochromatic triangle in K_m, which by induction yields a monochromatic triangle in K_N. This completes the proof! This result of Greenwood and Gleason tells us that $R_3(3) \leq 17$. One can then use Theorem 1, with G given as C_2^4, $C_4 \times C_4$, $(C_4 \times C_2) \rtimes C_2$ or $C_2 \times D_8$ (see the table below) to show that $R_3(3) = 17$.

G	An example of a partition of G^* into disjoint union of 3 symmetric product-free sets
$C_2^4 = \langle x_1, x_2, x_3, x_4 \mid x_ix_j = x_jx_i,\ x_i^2 = 1 \text{ for } 1 \leq i, j \leq 4\rangle$	$\{x_1, x_2, x_3, x_4, x_1x_2x_3x_4\} \cup \{x_1x_2, x_1x_3, x_2x_4, x_1x_2x_3, x_1x_2x_4\} \cup \{x_1x_4, x_2x_3, x_3x_4, x_1x_3x_4, x_2x_3x_4\}$
$C_4 \times C_4 = \langle x, y \mid x^4 = 1 = y^4,\ xy = yx\rangle$	$\{x, x^3, y, y^3, x^2y^2\} \cup \{x^2, xy, x^3y^3, x^2y, x^2y^3\} \cup \{xy^3, x^3y, y^2, xy^2, x^3y^2\}$
$(C_4 \times C_2) \rtimes C_2 = \langle x, y \mid x^4 = 1 = y^2,\ (xyx)^2 = 1 = (yx^{-1})^4,\ (yxyx^{-1})^2 = 1\rangle$	$\{y, x, x^3, (xy)^2, x^3yx\} \cup \{yx, x^2, x^2y, x^3y, xyx\} \cup \{x^2yx, xy, yxy, x(xy)^2, x^2(xy)^2\}$
$C_2 \times D_8 = \langle x, y, z \mid x^2 = 1,\ y^2 = 1, z^2 = 1,\ (zx)^2 = 1,\ (zy)^2 = 1,\ (yx)^4 = 1\rangle$	$\{x, y, xz, (xy)^2, xyxz\} \cup \{xy, z, yx, xyx, yz\} \cup \{xyz, yxy, yxz, yxyz, (xy)^2z\}$

In 1967, Folkman [4] proved that $R_4(3) \leq 65$. Twenty-eight years later, Sánchez [11] improved that upper bound to 64. Sánchez's bound was improved by Kramer [8] (without computer) to 62 in the same year 1995, and later (with computer) in 2004 by Fettes, Kramer and Radziszowski [3]. On the lower bound direction, Whitehead in a 1973 published paper [12], established that $R_4(3) \geq 50$. In a paper published by Chung [1] in the same year, she proved that $R_{n+1}(3) \geq 3(R_n(3) - 1) + R_{n-2}(3)$ for $n \geq 3$. So $R_4(3) \geq 51$. Since 1995, the best known bound for $R_4(3)$ is that $51 \leq R_4(3) \leq 62$. In 2006, Kramer [9], after giving over 100 pages proof that $R_4(3) \leq 62$, expanding his earlier work [8], conjectured that $R_4(3) = 62$. Throughout this work, G stands for a finite group. We say G is *m-partitioned* if the non-identity elements of G can be partitioned into disjoint union of m symmetric product-free sets. A *locally maximal symmetric product-free set* (LMSPFS for short) in G is a symmetric product-free set which is not properly contained in any other symmetric product-free set in G. A natural question is whether Chung's lower bound for $R_4(3)$ can be improved to r for $52 \leq r \leq 62$. We shall use an algorithmic approach to show that the group partitioning approach cannot be used to improve Chung's lower bound to r for $52 \leq r \leq 62$. There are 56 groups of order from 51 up to 61. So we use algorithmic approach to assert that none of the 56 groups can be 4-partitioned.

Before we proceed, we introduce the term '*locally maximal product-free set*' here as a maximal by inclusion product-free set; i.e., a product-free set that is not contained in a strictly larger product-free set within the same group G. In a 2009 paper [5], Giudici and Hart gave a characterisation of locally maximal product-free sets (LMPFS for short) as follows:

Lemma 7. *[5, Lemma 3.1] Let S be a product-free set in a group G. Then S is locally maximal product-free if and only if $G = T(S) \cup \sqrt{S}$.*

Remark 8. Clearly, every (symmetric) product-free set is contained in a locally maximal product-free set which may or may not be symmetric. Suppose we cover $G^* = G \setminus \{1\}$ by m locally maximal product-free sets $L_1, L_2, \cdots, L_m$ which are not all symmetric, then to achieve the goal of Theorem 1 by partitioning G^* into disjoint union of m symmetric product-free sets $S_1, S_2, \cdots, S_m$ where $S_i \subseteq L_i$ for each $i \in [1, m]$, we remove from L_i an element whose inverse is not in L_i, and if an element and its inverse are in both L_i and L_j for $i < j$, then we discard them from L_j. Hence, to study the partitioning problem in Theorem 1, it is sufficient to consider a cover of G^* by locally maximal symmetric product-free sets (LMSPFS).

The next result in the sequel shows that the group partitioning approach into symmetric product-free sets cannot be used to prove the conjecture of Kramer that $R_4(3) = 62$.

Theorem 9. *The group of order* 61 *cannot be* 4*-partitioned.*

Proof. The LMSPFS in C_{61} are of sizes 12, 14, 16, 18 and 20, and there are 390, 1470, 435, 150 and 60 of them respectively. We aim to check whether the size of any union of four LMSPFS of sizes p, q, r and s gives 60, where $p, q, r, s \in \{12, 14, 16, 18, 20\}$. Suppose we 4-colour the edges of K_{61} with red, blue, green and yellow. Choose any vertex v_0 of K_{61}. By pigeonhole principle, at least 16 of the edges incident with v_0 must be coloured the same colour (say blue). Suppose we edge join v_0 with each of the vertices $v_1, v_2, \cdots, v_m$ respectively, where $m \geq 16$. Consider the complete graph K_m on those m vertices. If we colour any edge in K_m with colour blue, then we force the appearance of a blue triangle. So we only colour edges of K_m with any of the remaining three colours. As $R_3(3) = 17$, in order not to have a monochromatic triangle in K_m, we have that $m \leq 16$. This argument shows that the largest size of any symmetric product-free set involved in any 4-partition of C_{61} is 16. Thus, to perform a faster computation, we add conditions (i) and (ii) below. Condition (i) is that no sum of two numbers in $\{p, q, r, s\}$ is less than 28 in each trial. For instance, we can try LMSPFS of sizes $\{20, 20, 16, 12\}$ or $\{18, 18, 18, 14\}$, but cannot try $\{20, 18, 12, 12\}$ or $\{18, 16, 14, 12\}$. Condition (ii) is that at least two of p, q, r, s must be elements of $\{16, 18, 20\}$. For instance, we can try LMSPFS of sizes $\{18, 16, 16, 14\}$ or $\{18, 14, 16, 14\}$ but cannot try $\{14, 14, 14, 14\}$ or $\{20, 14, 14, 14\}$. So, we have 41 trials altogether. They are: $\{20, 20, 20, 20\}$, $\{20, 20, 20, 18\}$, $\{20, 20, 20, 16\}$, $\{20, 20, 20, 14\}$, $\{20, 20, 20, 12\}$, $\{20, 20, 18, 18\}$, $\{20, 20, 18, 16\}$, $\{20, 20, 18, 14\}$, $\{20, 20, 18, 12\}$, $\{20, 20, 16, 16\}$, $\{20, 2016, 14\}$, $\{20, 20, 16, 12\}$, $\{20, 20, 14, 14\}$, $\{20, 18, 18, 18\}$, $\{20, 18, 18, 16\}$, $\{20, 18, 18, 14\}$, $\{20, 18, 18, 12\}$, $\{20, 18, 16, 16\}$, $\{20, 18, 16, 14\}$, $\{20, 18, 16, 12\}$, $\{20, 18, 14, 14\}$, $\{20, 16, 16, 16\}$, $\{20, 16, 16, 14\}$, $\{20, 16, 16, 12\}$, $\{20, 16, 14, 14\}$, $\{18, 18, 18, 18\}$, $\{18, 18, 18, 16\}$, $\{18, 18, 18, 14\}$, $\{18, 18, 18, 12\}$, $\{18, 18, 16, 16\}$, $\{18, 18, 16, 14\}$, $\{18, 18, 16, 12\}$, $\{18, 18, 14, 14\}$, $\{18, 16, 16, 16\}$, $\{18, 16, 16, 14\}$, $\{18, 16, 16, 12\}$, $\{18, 16, 14, 14\}$, $\{16, 16, 16, 16\}$, $\{16, 16, 16, 14\}$, $\{16, 16, 16, 12\}$ and $\{16, 16, 14, 14\}$. We checked the 41 trials and could not find in any trial, four locally maximal symmetric product-free sets whose size of their union is 60. Therefore, C_{61} cannot be 4-partitioned. $\square$

Before we proceed, we state a result that will be useful in determining a largest possible size of our (locally maximal symmetric) product-free sets in finite abelian groups. We introduce the following definition of Diananda and Yap [2].

Definition. Let G be a finite abelian group. Then G is of *type I* if $|G|$ is divisible by a prime $p \equiv 2$ mod 3, and of *type II* if 3 is a factor of $|G|$ but $|G|$ has no prime factor $\equiv 2$ mod 3. Finally, G is of *type III* if every prime factor of $|G|$ is a prime $p \equiv 1$ *mod* 3.

Theorem 10 (Diananda, Yap, Green, Ruzsa)**.** *Let G be a finite abelian group. (i) If G is of type I, then $\lambda(G) = \frac{|G|}{3}(\frac{p+1}{p})$, where p is the least prime factor of $|G|$ such that $p \equiv 2$ mod 3. (ii) If G is of type II, then $\lambda(G) = \frac{|G|}{3}$. (iii) If G is of type III, then $\lambda(G) = \frac{|G|}{3}(\frac{m-1}{m})$, where m is the exponent of G.*

The maximum and minimum size of a LMSPFS in a finite group G will be denoted by M_G and N_G respectively. In this paragraph, we show that the group of other 51 cannot be 4-partitioned. We start by collecting the locally maximal symmetric product-free sets (LMSPFS) of all possible sizes in C_{51}. By Theorem 10(i), we know that $\lambda(C_{51}) = 18$; therefore $M_{C_{51}} \leq 18$. By Lemma 7, any LMSPFS S in C_{51} must satisfy $C_{51} = S \cup SS \cup \sqrt{S}$. Given any product-free set S in C_{51}, we know that $|\sqrt{S}| = |S|$ and $|SS| \leq \frac{|S|^2+|S|}{2}$; so $N_{C_{51}} \geq 8$. As any symmetric product-free set in C_{51} is of even order, we conclude that any LMSPFS in C_{51} can only have size m, where $m \in \{8, 10, 12, 14, 16, 18\}$. We tested all these and observed that no LMSPFS of size 8 exists; also that there are 16, 444, 112, 16 and 8 LMSPFS of sizes 10, 12, 14, 16 and 18 respectively. We set U to be a collection of all locally maximal symmetric product-free sets in C_{51}. So $|U| = 596$. As any symmetric product-free set in C_{51} is contained in a set in U, we search for four sets in U whose size of their union is 50. Our search shows that no such four LMSPFS exists. Therefore C_{51} is not 4-partitioned.

The argument for showing that a group of odd order $m \in \{53, 55, 57, 59, \underline{61}\}$ cannot be 4-partitioned is not very different. The general idea for the collection of locally maximal symmetric product-free sets in groups of odd order is to start by pairing each non-identity element of the group G with its inverse. Select only one element from each pair and add to a set, call this set K; so $|K| = \frac{|G|-1}{2}$. Get all the product-free sets of small sizes (say 1 up to v, where $v \geq 9$) using collection of elements of K. Then adjoin their inverses to get symmetric product-free sets (SPFS) of sizes 2 up to $2v$. Since the SPFS of the smallest size (i.e., of size 2 for groups of odd order) are very small, we use them with latter (larger) symmetric product-free sets to get all SPFS of sizes greater than $2v$ in G. Note that an upper bound for sizes of the SPFS is $\lambda(G)$, whose value is at most $\frac{|G|}{2}$ when $|G|$ is even and $< \frac{|G|}{2}$ when $|G|$ is odd. [Groups of odd orders from 51 up to 61 are cyclic (abelian), except for two groups $C_{11} \rtimes C_5$ and $C_{19} \rtimes C_3$ which are nonabelian. For an abelian group G of order 51, 53, 55, 57 or 59, we appeal to Theorem 10 for the value of $\lambda(G)$. For the two nonabelian groups of odd order, we apply a similar approach to proof of results of Diananda and Yap [2] (for instance see [2, Theorem 8] for $|G| = 57$) to obtain that $\lambda(C_{11} \rtimes C_5) = 22$ and $\lambda(C_{19} \rtimes C_3) = 19$. For groups of even order, we use the trivial upper bound that $\lambda(G) \leq \frac{|G|}{2}$.] We then check for local maximality using Lemma 7. Thus, we use this means to obtain all LMSPFS in any group G of odd order. The argument for groups of even order is similar. One of the differences is that we add all the involutions to K; which unlike the odd case yields locally maximal symmetric product-free sets of both even and odd sizes. A tool we used in sieving the LMSPFS in groups of even order is that all involutions in G must be in either S or SS. We have also verified that no group of order 52, 54, 56, 58 and 60 can be 4-partitioned.

We say that the Ramsey number $R_k(3)$ is *solvable by group partitioning means* if there is a finite group G such that $|G| + 1 = R_k(3)$ and G^* can be partitioned as a disjoint union of k symmetric product-free sets. We come to the end of this discussion with the following question:

Question 11. Which Ramsey numbers $R_k(3)$ are solvable by group partitioning means?

This discussion helps us know that $R_1(3)$, $R_2(3)$ and $R_3(3)$ are solvable by group partitioning means (GPM for short) whereas $R_4(3)$ is not solvable by group partitioning means. It will be interesting to know which Ramsey numbers $R_k(3)$ are solvable by GPM for $k \geq 5$. An interested reader may see [10, pp. 38-39] for bounds on $R_k(3)$ for some $k \geq 5$. We anticipate that $R_5(3)$ is solvable by GPM. However, we conjecture that $R_5(3) \geq 257$, and that the lower bound can be obtained by partitioning non-identity elements of a group of order 256 into disjoint union of 5 symmetric product-free sets.

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