The Excluded 3-minors for Vf-safe Delta-matroids

By

Joseph E. Bonin, Carolyn Chun and Steven D. Noble

Birkbeck Pure Mathematics Preprint Series | Preprint Number 45

www.bbk.ac.uk/ems/research/pure/preprints

THE EXCLUDED 3-MINORS FOR VF-SAFE DELTA-MATROIDS

JOSEPH E. BONIN, CAROLYN CHUN, AND STEVEN D. NOBLE

ABSTRACT. Vf-safe delta-matroids have the desirable property of behaving well under certain duality operations. Several important classes of delta-matroids are known to be vf-safe, including the class of ribbon-graphic delta-matroids, which is related to the class of ribbon graphs or embedded graphs in the same way that graphic matroids correspond to graphs. In this paper, we characterize vf-safe delta-matroids and ribbon-graphic delta-matroids by finding the minimal obstructions, called 3-minors, to belonging to the class. We find the unique (up to twisted duality) excluded 3-minor within the class of set systems for the class of vf-safe delta-matroids. Geelen and Oum [17] found the 166 (up to twists) excluded minors for ribbon-graphic delta-matroids. By translating Bouchet's characterization of circle graphs to the language of 3-minors, we show that this class can also be characterized amongst delta-matroids by a set of three excluded 3-minors up to twisted duality.

1. INTRODUCTION

A *set system* is a pair $S = (E, \mathcal{F})$, where E, or $E(S)$, is a set, called the *ground set*, and $\mathcal{F}$, or $\mathcal{F}(S)$, is a collection of subsets of E. (All set systems in this paper have finite ground sets.) The members of $\mathcal{F}$ are the *feasible sets*. We say that S is *proper* if $\mathcal{F} \neq \emptyset$.

A matroid M has many associated set systems with $E = E(M)$. The most important of these from the perspective of this paper has $\mathcal{F} = \mathcal{B}(M)$, the set of bases of M. Recall that the bases of a matroid satisfy the following exchange property: for any $B_1, B_2 \in \mathcal{B}(M)$ and for each element $x \in B_1 - B_2$, there is a $y \in B_2 - B_1$ for which $B_1 \triangle \{x, y\} \in \mathcal{B}(M)$. To get the definition of a delta-matroid, replace set differences by symmetric differences. Thus, as introduced by Bouchet in [2], a *delta-matroid* is a proper set system $D = (E, \mathcal{F})$ for which $\mathcal{F}$ satisfies the *delta-matroid symmetric exchange axiom*:

(SE) for all triples (X, Y, u) with X and Y in $\mathcal{F}$ and $u \in X \triangle Y$, there is a $v \in X \triangle Y$ (perhaps u itself) such that $X \triangle \{u, v\}$ is in $\mathcal{F}$.

Clearly every matroid $(E(M), \mathcal{B}(M))$ is a delta-matroid.

Just as there is a mutually-enriching interplay between matroid theory and graph theory, the theory of delta-matroids has substantial connections with the theory of embedded graphs or equivalently ribbon graphs; see [13, 14]. Brijder and Hoogeboom [9, 10, 11] introduced the operation of loop complementation, which we define in the next paragraph. This operation is natural for the class of binary delta-matroids and its subclass of ribbon-graphic delta-matroids. These classes are closed under loop complementation, but that is not true for the class of all delta-matroids. We investigate when loop complementation of a delta-matroid yields a delta-matroid.

For a set system $S = (E, \mathcal{F})$ and $e \in E$, we define $S + e$ to be the set system

$$S + e = (E, \mathcal{F} \triangle \{F \cup e : e \notin F \in \mathcal{F}\}). \tag{1.1}$$

Date: July 3, 2018.

(As in matroid theory, we often omit set braces from singletons.) Note that $(S+e)+e = S$ and that $S + e$ is proper if and only if S is proper. It is straightforward to check that if $e_1, e_2 \in E$ then $(S + e_1) + e_2 = (S + e_2) + e_1$. Thus if $X = \{e_1, \dots , e_n\}$ is a subset of E, then the set system $S + X$ is unambiguously defined by

$$S + X = ((S + e_1) + \cdots) + e_n. \tag{1.2}$$

This operation is called the *loop complementation of S on X*. There is a natural operation of embedded graphs, namely *partial Petriality*, to which loop complementation corresponds. More precisely if two embedded graphs are partial Petrials of each other then their ribbon graphic delta-matroids are related by a loop complementation [14, Section 4].

For a delta-matroid D and element $e \in E(D)$, the set system $D + e$ need not be a delta-matroid. Consider, for example, the delta-matroid $D_3 = (\{a, b, c\}, 2^{\{a,b,c\}} - \{\{a, b, c\}\})$. Then $D_3 + \{a, b, c\}$ is the set system $(\{a, b, c\}, \{\emptyset, \{a, b, c\}\})$. This is not a delta-matroid. In fact, it is an excluded minor for the class of delta-matroids [1].

Another operation on delta-matroids is the twist. For $A \subseteq E$, the *twist of S on A*, which is also called the *partial dual of S with respect to A*, denoted $S * A$, is given by

$$S * A = (E, \{F \triangle A \,:\, F \in \mathcal{F}\}).$$

Note that $(S * A) * A = S$. The *dual* S^* of S is $S * E$. In contrast with loop complementation, each twist of a delta-matroid is a delta-matroid. Apart from the dual, the twists of a matroid $(E(M), \mathcal{B}(M))$ are generally not matroids, as discussed in [14, Theorem 3.4].

Two set systems are said to be *twisted duals* of one another if one may be obtained from the other by a sequence of twists and loop complementations. Following [11], a delta-matroid is said to be *vf-safe* if all of its twisted duals are delta-matroids. (The term vf-safe is short for 'vertex-flip safe'. Both of the terms vf-safe and loop complementation are named for operations on graphs representing binary delta-matroids [9], which we discuss in Section 5.)

Delta-matroids belonging to certain important classes are known to be vf-safe. In fact, every twisted dual of a ribbon-graphic delta-matroid is a ribbon-graphic delta-matroid [14, Theorem 2.1,Theorem 4.1], and every twisted dual of a binary delta-matroid is a binary delta-matroid [11, Theorem 8.2]. Brijder and Hoogeboom showed that quaternary matroids are vf-safe [12], although, as mentioned earlier, matroids are not closed under twists.

In the main result of this paper, Theorem 4.4, we identify D_3 as essentially the unique obstacle for a delta-matroid to be vf-safe. More precisely, we show that the excluded 3-minors for the class of vf-safe delta-matroids within the class of set systems comprise the 28 set systems that are the twisted duals of D_3. These set systems are given in Tables 2–7. In the final section of the paper, we relate 3-minors to other minor operations that have appeared in the literature, and we apply Theorem 4.4 to recast some known results using short lists of excluded 3-minors.

2. Background

Let $S = (E, \mathcal{F})$ be a proper set system. An element $e \in E$ is a *loop* of S if no set in $\mathcal{F}$ contains e. If e is in every set in $\mathcal{F}$, then e is a *coloop*. If e is not a loop, then the *contraction of e from S*, written S/e, is given by

$$S/e = (E - e, \{F - e : e \in F \in \mathcal{F}\}).$$

If e is not a coloop, then the *deletion of e from S*, written $S\backslash e$, is given by

$$S\backslash e = (E - e, \{F \subseteq E - e : F \in \mathcal{F}\}).$$

If e is a loop or a coloop, then one of S/e and $S\backslash e$ has already been defined, so we can set $S/e = S\backslash e$. Any sequence of deletions and contractions, starting from S, gives another set system S', called a *minor* of S. Each minor of S is a proper set system.

The order in which elements are deleted or contracted can matter. See [1] for an example. However, for disjoint subsets X and Y of E, if some set in $\mathcal{F}$ is disjoint from X and contains Y, then the deletions and contractions in $S\backslash X/Y$ can be done in any order, and

$$S\backslash X/Y = (E - (X \cup Y), \{F - Y \,:\, F \in \mathcal{F} \text{ and } Y \subseteq F \subseteq E - X\}).$$

The following lemma, which is [1, Lemma 2.1], shows that all minors of a proper set system are of this type.

Lemma 2.1. *For any minor S' of a proper set system $S = (E, \mathcal{F})$, there are disjoint subsets X and Y of E such that*

$$S' = S\backslash X/Y = (E - (X \cup Y), \{F - Y \,:\, F \in \mathcal{F} \text{ and } Y \subseteq F \subseteq E - X\}).$$

Bouchet and Duchamp [3] showed that, if S is a delta-matroid and $S' = S\backslash X/Y$, then S' is a delta-matroid and S' is independent of the order of the deletions and contractions.

The following definition from [9] is equivalent to that given in equations (1.1)–(1.2). Equivalence can be shown by a routine induction on $|A|$.

Definition 2.2. *If $S = (E, \mathcal{F})$ is a set system and A is a subset of E, then the* loop complementation *of S* by *A, denoted by $S+A$, is the set system on E such that F is feasible in $S + A$ if and only if S has an odd number of feasible sets F' with $F - A \subseteq F' \subseteq F$.*

Note that if $A = \{e\}$, then the feasible sets of $S + e$ that do not contain e are the same as those of S, and a set F containing e is feasible in $S + e$ if and only if one but not both of F and $F - e$ is feasible in S. That is, so long as e is not a loop or coloop,

$$\mathcal{F}(S + e) = \mathcal{F}(S\backslash e) \cup \{F \cup e : F \in \mathcal{F}(S\backslash e) \bigtriangleup \mathcal{F}(S/e)\}.$$

If e is a loop, then $\mathcal{F}(S + e) = \mathcal{F} \cup \{F \cup e : F \in \mathcal{F}\}$. If e is a coloop, then $S + e = S$.

The twist and loop complementation operations interact in the following way. If A and B are disjoint subsets of E then $(S + A) * B = (S * B) + A$ (a two-element case check and routine induction suffice to verify this), but in general $(S * A) + A \neq (S + A) * A$. However $((S + A) * A) + A = ((S * A) + A) * A$ (see [9]). It follows that there are at most six twisted duals of S with respect to a fixed set A. These relations ensure that any twisted dual of S is of the form $((S * X) + Y) * Z$ for suitably chosen subsets X, Y and Z of E with $X \subseteq Z$. The first relation is used in the proof of the following result.

Lemma 2.3. *Let $S = (E, \mathcal{F})$ be a proper set system, and let a and b be distinct elements of E. Then*

(i) $S + a\backslash a = S\backslash a$,
(ii) $S + a\backslash b = S\backslash b + a$, *and*
(iii) $S + a/b = S/b + a$.

Proof. If a is a coloop of S, then $S + a = S$, so statement (i) follows. Also, a is not a coloop of S if and only if it is not a coloop of $S+a$, in which case the feasible sets avoiding a are the same in S and $S + a$ by the definition.

For statement (ii), observe that b is a coloop of $S + a$ if and only if it is a coloop of S. When b is not a coloop of S, statement (ii) holds since for each side, the feasible sets are the sets F with $b \notin F$ for which an odd number of the sets X with $F - a \subseteq X \subseteq F$ are in $\mathcal{F}$. When b is a coloop of S, we need to show that $S + a/b = S/b + a$. This holds since for each side, the feasible sets are the sets F with $b \notin F$ for which an odd number of the sets X with $(F - a) \cup b \subseteq X \subseteq F \cup b$ are in $\mathcal{F}$.

It is easy to check that $S'/e = S' * e \backslash e$, so, using statement (ii), we get statement (iii):

$$S + a/b = ((S + a) * b)\backslash b = ((S * b) + a)\backslash b = ((S * b)\backslash b) + a = S/b + a. \qquad \square$$

The counterpart, for contractions, of statement (i) is false, as taking $S = D_3$ shows.

3. 3-MINORS

We introduce a third minor operation on set systems. For a proper set system S, we define $S \ddagger e$ to be the set system $(S + e)/e$. This minor operation was first defined by Ellis–Monaghan and Moffatt [15] for ribbon graphs and in an equivalent way by Brijder and Hoogeboom [10] for delta-matroids. The notation $\ddagger$ is new, but it seems appropriate to shorten the unwieldy $+e/e$ notation. Motivation for this definition comes from two directions. First, one way to define the Penrose polynomial of a ribbon graph is by specifying a recursive relation analogous to the deletion-contraction recurrence of the chromatic polynomial with this minor operation replacing contraction. For this reason, following a suggestion of Iain Moffatt [18], we propose calling the operation *Penrose contraction*. Second, there is a class of combinatorial objects called multimatroids [6, 7, 8], of which tight 3-matroids are a particular subclass. Brijder and Hoogeboom [10] showed that tight 3-matroids are equivalent (in a sense that we do not make precise here) to vf-safe delta-matroids. Tight 3-matroids have three minor operations corresponding to deletion, contraction, and Penrose contraction in vf-safe delta-matroids.

Any sequence of the three minor operations, starting from S, gives another proper set system S', called a 3-*minor* of S. A collection $\mathcal{C}$ of proper set systems is 3-*minor closed* if every 3-minor of every member of $\mathcal{C}$ is in $\mathcal{C}$. Given such a collection $\mathcal{C}$, a proper set system S is an *excluded* 3-*minor* for $\mathcal{C}$ if $S \notin \mathcal{C}$ and all other 3-minors of S are in $\mathcal{C}$. A proper set system belongs to $\mathcal{C}$ if and only if none of its 3-minors is an excluded 3-minor for $\mathcal{C}$. Thus, the excluded 3-minors determine $\mathcal{C}$; they are the 3-minor-minimal obstructions to membership in $\mathcal{C}$.

For a given class $\mathcal{C}$, there may be substantially fewer excluded 3-minors than excluded minors. For instance, in [17], Geelen and Oum found 166 delta-matroids that, up to twists, are the excluded minors for ribbon-graphic delta-matroids within the class of binary delta-matroids. In contrast, in Theorem 5.8, we show that every binary matroid that does not have a twisted dual of one of three delta-matroids as a 3-minor is ribbon-graphic.

An element e is called a *pseudo-loop* of S if e is a loop of $S + e$. The next lemma characterizes these elements.

Lemma 3.1. *For an element e in a proper set system S, the following statements are equivalent:*

(i) *e is a loop of $S + e$, that is, a pseudo-loop of S,*
(ii) *$F \cup e \in \mathcal{F}(S)$ if and only if $F \in \mathcal{F}(S)$, and*
(iii) *$S * e = S$.*

Pseudo-loops of S are neither loops nor coloops of S. Furthermore, $S \ddagger e = S \setminus e = S/e$ if and only if e is a loop, a coloop, or a pseudo-loop of S.

Proof. The equivalence of statements (i)–(iii) is immediate from the definitions. Statement (ii) implies that pseudo-loops are neither loops nor coloops. If e is a loop of S, then $S \ddagger e = S \setminus e$ since $\mathcal{F}(S + e) = \mathcal{F}(S) \cup \{F \cup e : F \in \mathcal{F}(S)\}$; also, $S \setminus e = S/e$ by definition. If e is a coloop of S, then $S \ddagger e = S/e$ since $S + e = S$; also, $S \setminus e = S/e$ by definition. If e is a pseudo-loop of S, then statements (i) and (ii) gives the equality. If e is not a loop, a coloop, or a pseudo-loop of S, then $S \setminus e \neq S/e$ by the failure of statement (ii) and the fact that some, but not all, sets in $\mathcal{F}(S)$ contain e. □

The following two results show that one may choose the operations used to form a 3-minor in such a way that they commute.

Lemma 3.2. *Let $S = (E, \mathcal{F})$ be a proper set system, and let X, Y, and Z be pairwise disjoint subsets of E. If there is a set F with*

(3.1) $F \subseteq E - (X \cup Y \cup Z)$ *and* $|\mathcal{F} \cap \{F' : F \cup Y \subseteq F' \subseteq F \cup Y \cup Z\}|$ *is odd,*

then the minor operations in $S \setminus X/Y \ddagger Z$ can be done in any order and a set F is feasible in $S \setminus X/Y \ddagger Z$ if and only if it satisfies Condition (3.1).

Proof. A set F meets Condition (3.1) if and only if $F \subseteq E - (X \cup Y \cup Z)$ and $F \cup Y \cup Z$ is in $\mathcal{F}(S + Z)$. If there is at least one set satisfying Condition (3.1), the remarks preceding Lemma 2.1 imply that the deletions and contractions in forming $(S + Z) \setminus X/(Y \cup Z)$ from $S + Z$ may be done in any order and a set F is feasible in $(S + Z) \setminus X/(Y \cup Z)$ if and only if it satisfies Condition (3.1). Lemma 2.3 implies that we may defer taking a loop complementation of an element in Z until just before it is contracted. The result follows. □

We next show that for every 3-minor of a proper set system, there are pairwise disjoint sets X, Y and Z satisfying Condition (3.1).

Lemma 3.3. *Let S' be a 3-minor of a proper set system $S = (E, \mathcal{F})$. Then there are pairwise disjoint subsets X, Y, and Z of E such that $S' = S \setminus X/Y \ddagger Z$ and there is a set F satisfying Condition (3.1).*

Proof. Suppose we get S' from S by, for each of $e_1, e_2, \dots, e_k$ in turn, performing one the three minor operations, giving the sequence of minors $S_0 = S, S_1, \dots, S_k = S'$. Let X be the set of elements e_i in $\{e_1, \dots, e_k\}$ that satisfy at least one of the following conditions:

(1) e_i is a loop or a pseudo-loop of S_{i-1}, so $S_i = S_{i-1} \setminus e_i$, or
(2) e_i is not a coloop of S_{i-1} and $S_i = S_{i-1} \setminus e_i$.

Let Y be the set of elements e_i in $\{e_1, \dots, e_k\} - X$ such that e_i is either a coloop of S_{i-1} or $S_i = S_{i-1}/e_i$. Note that if $e_i \in Y$ then it is not a loop in S_{i-1}. Finally let $Z = \{e_1, \dots, e_k\} - (X \cup Y)$, so that Z comprises the elements e_i in $\{e_1, \dots, e_k\}$ for which $S_i = S_{i-1} \ddagger e_i$ but e_i is not a loop, pseudo-loop or coloop. Then there is always at least one set F satisfying Condition (3.1). □

Table 1 shows the result of applying one of the three minor operations that remove e after taking one of the six twisted duals, with respect to e, of a proper set system. If instead the minor operation removes a different element from that used for the twisted dual, then these operations commute.

We next show that any 3-minor of a twisted dual of a proper set system S is a twisted dual of some 3-minor of S. It is easy to see that the converse is also true.

Lemma 3.4. *Suppose S is a proper set system and S' is a twisted dual of S. If S'' is a 3-minor of S', then S has a 3-minor that is a twisted dual of S''.*

Proof. There are subsets A and B of $E(S)$ such that we obtain S'' from S by first forming a twisted dual for each element of A and then performing one of the three minor operations for each element of B. The remarks before this lemma imply that one may reorder these

	$/e$	$\backslash e$	$\ddagger e$
S	S/e	$S\backslash e$	$S \ddagger e$
$S*e$	$S\backslash e$	S/e	$S \ddagger e$
$S+e$	$S \ddagger e$	$S\backslash e$	S/e
$(S+e)*e$	$S\backslash e$	$S \ddagger e$	S/e
$(S*e)+e$	$S \ddagger e$	S/e	$S\backslash e$
$((S*e)+e)*e$	S/e	$S \ddagger e$	$S\backslash e$

TABLE 1. Interaction of minor operations and twisted duality.

operations to first deal with the elements of $A \cap B$, one by one, forming a twisted dual for an element and then a 3-minor before moving on to the next element. According to Table 1 each of these pairs of operations may be replaced by a single 3-minor operation. Next a 3-minor is formed for each element of $B - A$. The resulting set system is a twisted dual of S'' with respect to the elements of $A - B$. □

4. CHARACTERIZATIONS BY EXCLUDED 3-MINORS

Brijder and Hoogeboom [11] showed that the class of vf-safe delta-matroids is minor-closed. A computer search for excluded minors for this class turns up many examples with apparently little structure. The class of vf-safe delta-matroids is defined using both the twist and loop complementation operations, so it is natural to try to characterize this class using 3-minors. By Lemma 4.1 below, the class of vf-safe delta-matroids is closed under Penrose contraction, so, with the result in [11], it is closed under 3-minors. The main result of this section, Theorem 4.4, gives the excluded 3-minors for the class of vf-safe delta-matroids within the class of set systems.

Lemma 4.1. *If S is vf-safe and $e \in E(S)$, then $S \ddagger e$ is vf-safe.*

Proof. If S is vf-safe, then all of its twisted duals are vf-safe by definition, so $S + e$ is vf-safe. Theorem 8.3 in [11] states that every minor of a vf-safe delta-matroid is vf-safe. Thus $S\ddagger e = S + e/e$ is vf-safe. □

Let

$$S_i = (\{e_1, e_2, \dots, e_i\}, \{\emptyset, \{e_1, e_2, \dots, e_i\}\}).$$

Let $\mathcal{S}$ be the set of all twists of the set systems in $\{S_3, S_4, \dots\}$. Let

- $T_1 = (\{a,b,c\}, \{\emptyset, \{a,b\}, \{a,b,c\}\});$
- $T_2 = (\{a,b,c\}, \{\emptyset, \{a,b\}, \{a,c\}, \{a,b,c\}\});$
- $T_3 = (\{a,b,c\}, \{\emptyset, \{a\}, \{a,b\}, \{a,b,c\}\});$
- $T_4 = (\{a,b,c\}, \{\emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}\});$
- $T_5 = (\{a,b,c,d\}, \{\emptyset, \{a,b\}, \{a,b,c,d\}\});$
- $T_6 = (\{a,b,c,d\}, \{\emptyset, \{a,b\}, \{a,c\}, \{a,b,c,d\}\});$
- $T_7 = (\{a,b,c,d\}, \{\emptyset, \{a,b\}, \{a,c\}, \{a,d\}, \{a,b,c,d\}\});$
- $T_8 = (\{a,b,c,d\}, \{\emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,b,c,d\}\}).$

Let $\mathcal{T}$ be the set of all twists of the set systems in $\{T_1, T_2, \dots, T_8\}$. By the following result from [1, Theorem 5.1], these are all of the excluded minors for delta-matroids within the class of set systems.

Theorem 4.2. *A proper set system S is a delta-matroid if and only if S has no minor isomorphic to a set system in $\mathcal{S} \cup \mathcal{T}$.*

The following lemma is key for finding the excluded 3-minors for vf-safe delta-matroids within the class of set systems.

Lemma 4.3. *Let S be an excluded 3-minor for the class of vf-safe delta-matroids. Then S has a twisted dual that is isomorphic to a set system in $\mathcal{S} \cup \mathcal{T}$.*

Proof. Such an excluded 3-minor S either is not a delta-matroid and all of its other minors are delta-matroids, or it is a delta-matroid and has a twisted dual S' that is not a delta-matroid. In the former case S is isomorphic to a set system in $\mathcal{S} \cup \mathcal{T}$ and the lemma holds. In the latter case S' has a minor S'' isomorphic to a member of $\mathcal{S} \cup \mathcal{T}$. By Lemma 3.4, S has a 3-minor S''' that is a twisted dual of S''. Therefore S''' is not a vf-safe delta-matroid. The only 3-minor of S that is not a vf-safe delta-matroid is S itself. Hence $S = S'''$ and the lemma holds. □

To connect the next result with the remarks in Section 1, note that $D_3 + \{a, b, c\} = S_3$.

Theorem 4.4. *A proper set system is a vf-safe delta-matroid if and only if it has no 3-minor that is isomorphic to a twisted dual of S_3.*

Proof. All proper set systems with two elements are delta-matroids, and therefore each one is vf-safe, so the twisted duals of S_3 are excluded 3-minors for the class of vf-safe delta-matroids. By Lemma 4.3 every excluded 3-minor for the class of vf-safe delta-matroids must be a twisted dual of a set system in $\mathcal{S} \cup \mathcal{T}$. We first consider the set systems with three-element ground sets. We have $T_1^* + c = S_3$ and $T_2^* + \{b, c\} \simeq T_3 + a = T_1$ and $T_4 + a = T_2$, so every excluded 3-minor of size three is a twisted dual of S_3.

Lastly, we show that no other set system in $\mathcal{S} \cup \mathcal{T}$ is an excluded 3-minor. Lemma 3.4 implies that each twisted dual of an excluded 3-minor is an excluded 3-minor, so it suffices to show that each of T_5, T_6, T_7, T_8, and S_n, for $n \geq 4$, has a smaller member of $\mathcal{S} \cup \mathcal{T}$ as a 3-minor. Indeed, $S_n \ddagger e_n = S_{n-1}$, for $n \geq 4$, $T_5 \ddagger d = T_1$, $T_6 \ddagger d = T_8 \ddagger d = T_2$, and $T_7 \ddagger d = T_4$. □

As stated in the introduction, there are 28 twisted duals of S_3, up to isomorphism. These excluded 3-minors are listed in Tables 2–7.

5. 3-MINORS AND VERTEX MINORS

We now explain the link between 3-minors and vertex minors in binary delta-matroids. Vertex minors are well-studied, but are only defined for binary delta-matroids. In contrast, the concept of a 3-minor is relatively unstudied, but is important due to its direct correlation with ribbon-graph operations and its applicability beyond binary delta-matroids. For this reason, and for completeness, we give a full explanation here. Although the key ideas presented here are not new, the link between vertex minors and 3-minors has not previously been fully described.

A delta-matroid is *normal* if the empty set is feasible. A delta-matroid is *even* if for every pair F_1 and F_2 of its feasible sets $|F_1 \triangle F_2|$ is even. Equivalently, the sizes of all its feasible sets are of the same parity. Let M denote a symmetric binary matrix with rows and columns indexed by $[n] = \{1, \ldots, n\}$. Take $E = [n]$ and $\mathcal{F}$ to be the collection of subsets S of $[n]$ for which the principal submatrix of M comprising the rows and columns indexed by elements of S is non-singular. Bouchet [3] showed that $D(M) = (E, \mathcal{F})$ is a delta-matroid. We call such delta-matroids *basic binary*. (In the literature, what we have called basic binary delta-matroids are often called graphic, but we prefer to avoid this term to prevent confusion with ribbon-graphic delta-matroids.) A delta-matroid is *binary* [3] if it is a twist of a basic binary delta-matroid.

It follows immediately from the definition that every basic binary delta-matroid is normal and that a basic binary delta-matroid is uniquely determined by its feasible sets of size at most two. A well-known result of linear algebra states that a symmetric matrix with an odd number of rows (and columns) and zero diagonal is singular. Consequently a basic binary delta-matroid is even if and only if it has no feasible sets of size one.

Let A be a matrix over an arbitrary field with rows and columns indexed by $[n]$, and let X be a subset of $[n]$ such that the principal sub-matrix $P = A[X]$ is non-singular. Suppose without loss of generality that $A = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$. Then the matrix $A * X$ is defined by

$$A * X = \begin{pmatrix} P^{-1} & -P^{-1}Q \\ RP^{-1} & S - RP^{-1}Q \end{pmatrix}.$$

Note that if A is a symmetric binary matrix then $A * X$ is symmetric. The following result is due to Tucker [20].

Theorem 5.1. *Let A be a matrix over an arbitrary field with rows and columns indexed by $[n]$, and let X be a subset of $[n]$ such that the principal sub-matrix $P = A[X]$ is non-singular. Then for every subset Y of $[n]$, we have*

$$\det\big((A * X)[Y]\big) = \frac{\det(A[X \triangle Y])}{\det(A[X])}.$$

In particular for any subset Y of $[n]$, the principal submatrix $(A * X)[Y]$ is non-singular if and only if the principal submatrix $A[X \triangle Y]$ is non-singular.

The following corollary is immediate.

Corollary 5.2. *Suppose that A is a binary matrix, and X is a feasible set of $D(A)$. Then $D(A) * X = D(A * X)$.*

See [3] for an alternative proof of this result that holds for arbitrary fields. A consequence of this corollary is that every normal twist of a basic binary delta-matroid is basic binary.

A *looped simple graph* is a graph without parallel edges but in which each vertex is allowed to have up to one loop. Much of the time we will forbid loops; we call such graphs *loopless simple graphs*. Recall that basic binary delta-matroids are completely determined by their feasible sets with size two or fewer. Clearly basic binary delta-matroids on the set $[n]$ are in one-to-one correspondence with looped simple graphs with vertex set $[n]$; likewise, even basic binary delta-matroids on $[n]$ are in one-to-one correspondence with loopless simple graphs with vertex set $[n]$.

We take adjacency matrices to always be binary. Given a looped simple graph G and its adjacency matrix A, we let $D(G)$ denote the basic binary delta-matroid $D(A)$. If X is a subset of the edges of G, then X labels a subset of the rows and columns of A, and we define $G * X$ to be the looped simple graph with adjacency matrix $A * X$.

We now consider various transformations that may be applied to G and their effect on $D(G)$.

The loop complementation operation of Brijder and Hoogeboom was first defined in terms of basic binary delta-matroids. If G is a looped simple graph and v is a vertex of G, then the loop complementation $G + v$ is formed by toggling the loop at v, that is, removing a loop if there is one at v and adding one at v if there is no loop there.

The following lemma from [9] is straightforward.

Lemma 5.3. *Let G be a looped simple graph with vertex v. Then $D(G + v) = D(G) + v$.*

Our next operation is local complementation. There are several variations in the definition of local complementation: see, for instance, [19]. We will only require certain cases of what is defined there. For a looped simple graph G with vertex v, let $N_G(v)$ denote the *open neighbourhood* of v, that is, the set of vertices, excluding v, that are adjacent to v in G. We will generally write N instead of N_G when there is no possibility of confusion. The *local complementation* of G at v, denoted by G^v, is formed by toggling the adjacencies between pairs of neighbours of v, that is, for every distinct pair x, y from the neighbourhood of v, delete edge xy if x and y are adjacent in G and add edge xy if x and y are not adjacent in G. Additionally, if there is a loop at v, then the loop status of every vertex in the open neighbourhood of v is toggled. In both cases, adjacencies involving one or more non-neighbours of v or v itself are unchanged and the presence or not of a loop at v is unaffected. To distinguish the two cases, it will be helpful to refer to local complementation at v as *simple local complementation* when v is loopless, and *non-simple local complementation* when there is a loop at v.

For delta-matroid D and subset $A \subseteq E(D)$, let $D\bar{*}A$ denote the *dual pivot on* A, which is equal to $D + A * A + A$. The following result is implicit in the results of [19], but is not expressed in this form.

Proposition 5.4. *Let G be a loopless simple graph with vertex v. Then $D(G^v) = (D(G)\bar{*}v)+ N(v)$.*

Proof. Let A be the adjacency matrix of G. Then A is symmetric and all of its diagonal entries are zero. Notice that the simple local complementation G^v can be formed by adding a loop at v, performing the non-simple local complementation at v and then removing the loops added at v and all of its neighbours.

We have $D(G + v) = D(G) + v$. Assume without loss of generality that $v = 1$ and let $Z = [n] - 1$. Then the adjacency matrix of $G + v$ is $\begin{pmatrix} 1 & c \\ c^t & A[Z] \end{pmatrix}$ for some vector c. Then it follows from Corollary 5.2 that $(D(G) + v) * v = D((G + v) * v) = D(A')$ where $A' = \begin{pmatrix} 1 & c \\ c^t & A[Z] + c^t c \end{pmatrix}$.

A diagonal entry of $c^t c$ is non-zero if it corresponds to a neighbour of v and an off-diagonal entry of $c^t c$ is non-zero if both the row and column correspond to neighbours of v. Thus $(D(G) + v) * v = D(G')$ where G' is formed from G by adding a loop at v and performing the non-simple local complementation at v. Now G' has loops at v and at all neighbours of v, so

$$D(G^v) = D(G' + v + N(v)) = D(G') + v + N(v) = (D(G)\bar{*}v) + N(v). \qquad \square$$

The corollary below is well-known and follows from the previous result.

Corollary 5.5. *Let G be a loopless simple graph with adjacent vertices v and w. Then $D(((G^v)^w)^v) = D(G) * \{v, w\}$.*

Proof. We have

$$D(((G^v)^w)^v) = ((D(G)\bar{*}v + N(v))\bar{*}w + N_{G^v}(w))\bar{*}v + N_{(G^v)^w}(v).$$

It follows from the discussion before Lemma 2.3 that one may reorder the loop complement and twist operations so that those involving a particular vertex of G are done consecutively. The result follows by considering the effect of the operations involving each vertex of G separately and noting that

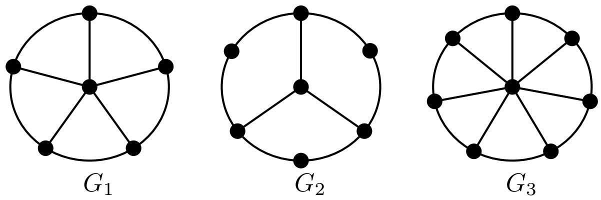

FIGURE 1. A complete set of circle graph obstructions.

(1) a common neighbour of v and w in G is a neighbour of v but not w in both G^v and $(G^v)^w$,
(2) a vertex other than w that is a neighbour of v but not w in G is a neighbour of both v and w in G^v and of w but not v in $(G^v)^w$, and
(3) a vertex other than v that is a neighbour of w but not v in G is a neighbour of both v and w in $(G^v)^w$ and of w but not v in G^v. □

A *vertex minor* of a looped simple graph G is formed from G by a sequence of local complementations and deletions of vertices. It is easy to check that if v and w are different vertices of an unlooped simple graph, then $(G^v) \setminus w = (G \setminus w)^v$ and thus one may assume that all the local complementations are done first.

Perhaps the most important use of vertex minors is Bouchet's characterization of circle graphs. A *chord diagram* is a collection of chords of a circle. Chord diagrams are in one-to-one correspondence with orientable ribbon graphs with one vertex. (For more information on ribbon graphs, see [16] or [14].) To see this think of the circle and its interior as the vertex of a ribbon graph and for each chord attach a ribbon to the vertex at the points corresponding to the endpoints of the chord. Clearly two chords intersect if and only if the corresponding ribbons e_1 and e_2 are interlaced, that is, as one traverses the vertex one meets an end of e_1, then an end of e_2, then the other end of e_1, and finally the other end of e_2. An unlooped simple graph is a *circle graph* if it is the intersection graph of the chords in a chord diagram, that is, there is a vertex corresponding to each chord and they are adjacent if and only if the chords cross. Equivalently a circle graph is the interlacement graph of an orientable ribbon graph with one vertex: it has a vertex for each ribbon and two vertices are adjacent if the corresponding ribbons are interlaced. Bouchet established the following result [5].

Theorem 5.6. *An unlooped simple graph is a circle graph if and only if it has no vertex minor isomorphic to the graphs G_1, G_2 or G_3 depicted in Figure 1.*

We are now ready to state the link between 3-minors and vertex minors.

Theorem 5.7. (1) *Let G be a unlooped simple graph and let H be a vertex minor of G. Then $D(H)$ is a 3-minor of $D(G)$.*
(2) *Let D be a twisted dual of a basic binary delta-matroid and let D' be a 3-minor of D. Then there are graphs G and G' such that $D(G)$ and $D(G')$ are twisted duals of D and D' respectively, and G' is a vertex minor of G.*

Proof. For part (1), note that a vertex minor of an unlooped simple graph is obtained by a sequence of local complementations and vertex deletions. The result follows from Proposition 5.4 and the fact that if v is a vertex of G then $D(G \setminus v) = D(G) \setminus v$.

For part (2), let F be a feasible set of D and let

$$B = \{e \in E(D) : \{e\} \in \mathcal{F}(D * F)\}.$$

The remark following Corollary 5.2 implies that $D*F$ is basic binary, so $(D*F)+B$ is an even basic binary delta-matroid, so $(D*F)+B = D(G)$ for some unlooped simple graph G. It follows from Lemma 3.4 that there is a delta-matroid D'' that is a 3-minor of $D(G)$ and a twisted dual of D'. We shall prove by induction on k that if G is an unlooped simple graph and D'' is a 3-minor of $D(G)$ with k fewer elements, then there is an unlooped simple graph G' that is a vertex minor of G and such that $D(G')$ is a twisted dual of D''. The result then follows.

If $k=0$, then take $G'=G$. Otherwise D'' is obtained from $D(G)$ by a sequence of k deletions, contractions and Penrose contractions. Suppose that the first operation is the deletion of v. Then take $G''=G\setminus v$, which is a vertex minor of G. Furthermore $D(G)\setminus v = D(G'')$ and D'' is a 3-minor of $D(G'')$ with $k-1$ fewer edges. Hence, by induction, there is an unlooped simple graph G' that is a vertex minor of G'' and hence of G, and such that $D(G')$ is a twisted dual of D''. Suppose next that the first operation is the Penrose contraction of v. Then take $G''=(G^v)\setminus v$. We have

$$\begin{aligned} D(G'') &= D(G^v \setminus v)\\ &= ((((D(G)+v)*v)+v)+N(v))\setminus v\\ &= ((((D(G)*v)+v)*v)\setminus v)+N(v)\\ &= (((D(G)*v)+v)/v)+N(v)\\ &= (D(G)\ddagger v)+N(v). \end{aligned}$$

(The last equality uses Table 1.) Now $D(G'')$ is a twisted dual of $D(G)\ddagger v$, so it has a 3-minor D''' with $k-1$ fewer elements that is a twisted dual of D''. Hence, by induction, there is an unlooped simple graph G' that is a vertex minor of G'' such that $D(G')$ is a twisted dual of D''' and consequently of D''. In the final case the first operation is the contraction of v. If v is an isolated vertex of G then v appears in no feasible set of $D(G)$ of size at most two and consequently in no feasible set of $D(G)$ of any size. Thus v is a loop of $D(G)$ and $D(G)/v = D(G)\setminus v = D(G\setminus v)$. If v is not an isolated vertex of v then let w be a neighbour of v. We have

$$\begin{aligned} D(((G^v)^w)^v\setminus v) &= D(((G^v)^w)^v)\setminus v\\ &= (D(G)*\{v,w\})\setminus v\\ &= (D(G)/v)*w. \end{aligned}$$

The proof of this case is completed in a similar way to the case of Penrose contraction. □

From the preceding result we obtain the following restatement of Bouchet's result, determining the three binary delta-matroids that are the excluded 3-minors for ribbon-graphic delta-matroids.

Theorem 5.8. *A binary delta-matroid is ribbon-graphic if and only if it has no 3-minor that is a twisted dual of $D(G_1)$, $D(G_2)$ or $D(G_3)$.*

Proof. If D is a binary delta-matroid and v is an element of D then D is ribbon-graphic if and only if $D+v$ is ribbon graphic, because it follows from Theorem 4.1 of [14] that if R is a ribbon graph with $D=D(R)$ then $D+v$ is the delta-matroid corresponding to the ribbon graph formed from R by applying a half-twist to v. Let

$$B=\{e\in E(D):\{e\}\in\mathcal{F}(D)\}.$$

Then $D+B$ is even and, furthermore, $D+B$ is ribbon-graphic if and only if D is ribbon-graphic. Now $D+B=D(G)$ where G is an unlooped simple graph. Bouchet's Theorem 5.6 states that G is a circle graph if and only if G has no vertex minor isomorphic to G_1, G_2 or G_3. Equivalently $D+B$ is ribbon-graphic if and only if it has no 3-minor that is a twisted dual of $D(G_1)$, $D(G_2)$ or $D(G_3)$. As $D+B$ is a twisted dual of D, the result follows. □

We close by presenting excluded 3-minor results for the classes of binary delta-matroids and ribbon graphic delta-matroids that follow from Theorem 4.4. Bouchet [4] proved that every minor of a binary delta-matroid is binary and gave the following excluded-minor characterization of binary delta-matroids.

Theorem 5.9. *A delta-matroid is binary if and only if it does not have a minor isomorphic to any of the following five delta-matroids or their twists.*

(1) $B_1=(\{a,b,c\},\{\emptyset,\{a,b\},\{a,c\},\{b,c\},\{a,b,c\}\})$;
(2) $B_2=S_3+\{a,b,c\}$;
(3) $B_3=(\{a,b,c\},\{\emptyset,\{b\},\{c\},\{a,b\},\{a,c\},\{a,b,c\}\})$;
(4) $B_4=(\{a,b,c,d\},\{\emptyset,\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{c,d\}\})$;
(5) $B_5=(\{a,b,c,d\},\{\emptyset,\{a,b\},\{a,d\},\{b,c\},\{c,d\},\{a,b,c,d\}\})$.

We obtain corollaries of this result characterizing binary delta-matroids in terms excluded 3-minors.

Corollary 5.10. *A vf-safe delta-matroid is binary if and only if it has no* 3*-minor that is a twisted dual of* B_1.

Proof. Theorem 8.2 of [11] states that every twisted dual of a binary delta-matroid is a binary delta-matroid. In particular every binary delta-matroid is vf-safe. Moreover, every 3-minor of a binary delta-matroid is binary. The delta-matroid B_1 is vf-safe and all of its 3-minors are binary. Thus all of its twisted duals are excluded 3-minors for the class of binary delta-matroids.

Suppose that D is a vf-safe delta-matroid that is not binary. Then Theorem 5.9 implies that D has a minor isomorphic to a twist of B_i for $1\le i\le 5$. The delta-matroid B_2 is not vf-safe and $B_4 \ddagger d=B_2$, so D has no minor isomorphic to a twist of B_2 or of B_4. Furthermore $(B_3+a)^*=B_1$, and $B_5 \ddagger d\simeq B_3$. Thus D has a 3-minor that is isomorphic to a twisted dual of B_1. □

By combining this result with Theorem 4.4, we obtain the following.

Corollary 5.11. *A proper set system is a binary delta-matroid if and only if it has no* 3*-minor that is a twisted dual of* B_1 *or* S_3.

Finally we combine the last two results with Theorem 5.8.

Corollary 5.12. *A proper set system is a ribbon graphic delta-matroid if and only if it has no* 3*-minor that is a twisted dual of* B_1, S_3, $D(G_1)$, $D(G_2)$ *or* $D(G_3)$.

6. APPENDIX: THE TWISTED DUALS OF S_3

As proved in Theorem 4.4, these twisted duals of S_3 are the excluded 3-minors for vf-safe delta-matroids.

S_3	$\emptyset$, $\{a,b,c\}$
$S_3 * \{a\}$	$\{a\}$, $\{b,c\}$

TABLE 2. All twists of S_3 up to isomorphism.

$\emptyset$, $\{a\}$, $\{a,b,c\}$	$S_3+\{a\}$	$\emptyset$, $\{b,c\}$, $\{a,b,c\}$	$(S_3+\{a\})^*$
$\emptyset$, $\{a\}$, $\{b,c\}$	$(S_3+\{a\})*\{a\}$	$\{a\}$, $\{b,c\}$, $\{a,b,c\}$	$(S_3+\{a\})*\{b,c\}$
$\{b\}$, $\{a,b\}$, $\{a,c\}$	$(S_3+\{a\})*\{b\}$	$\{b\}$, $\{c\}$, $\{a,c\}$	$(S_3+\{a\})*\{a,c\}$

TABLE 3. All twists of $S_3+\{a\}$ up to isomorphism. Dual pairs are side by side.

$\emptyset$, $\{a\}$, $\{b\}$, $\{a,b\}$, $\{a,b,c\}$	$S_3+\{a,b\}$	$\emptyset$, $\{c\}$, $\{a,c\}$, $\{b,c\}$, $\{a,b,c\}$	$(S_3+\{a,b\})^*$
$\emptyset$, $\{a\}$, $\{b\}$, $\{a,b\}$, $\{b,c\}$	$(S_3+\{a,b\})*\{a\}$	$\{a\}$, $\{c\}$, $\{a,c\}$, $\{b,c\}$, $\{a,b,c\}$	$(S_3+\{a,b\})*\{b,c\}$
$\{c\}$, $\{a,b\}$, $\{a,c\}$, $\{b,c\}$, $\{a,b,c\}$	$(S_3+\{a,b\})*\{c\}$	$\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a,b\}$	$(S_3+\{a,b\})*\{a,b\}$

TABLE 4. All twists of $S_3+\{a,b\}$ up to isomorphism. Dual pairs are side by side.

$\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a,b\}$, $\{a,c\}$, $\{b,c\}$	$S_3+\{a,b,c\}$	$\{a\}$, $\{b\}$, $\{c\}$, $\{a,b\}$, $\{a,c\}$, $\{b,c\}$, $\{a,b,c\}$	$(S_3+\{a,b,c\})^*$
$\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a,b\}$, $\{a,c\}$, $\{a,b,c\}$	$S_3+\{a,b,c\}*\{a\}$	$\emptyset$, $\{b\}$, $\{c\}$, $\{a,b\}$, $\{a,c\}$, $\{b,c\}$, $\{a,b,c\}$	$S_3+\{a,b,c\}*\{b,c\}$

TABLE 5. All twists of $S_3+\{a,b,c\}$ up to isomorphism. Dual pairs are side by side.

$(S_3 * \{a\}) + \{a,b\}$	$((S_3 * \{a\}) + \{a,b\})^*$
$\{a\}$ $\{a,b\}$, $\{b,c\}$ $\{a,b,c\}$	$\emptyset$ $\{a\}$, $\{c\}$ $\{b,c\}$

$((S_3 * \{a\}) + \{a,b\}) * \{a\}$
$\emptyset$ $\{b\}$ $\{b,c\}$ $\{a,b,c\}$

$((S_3 * \{a\}) + \{a,b\}) * \{b\}$
$\{a\}$, $\{c\}$ $\{a,b\}$, $\{a,c\}$

TABLE 6. All twists of $(S_3 * \{a\}) + \{a,b\}$ up to isomorphism. Dual pairs are side by side.

$(S_3 * \{a\}) + \{a,b,c\}$	$((S_3 * \{a\}) + \{a,b,c\})^*$
$\{a\}$ $\{a,b\}$, $\{a,c\}$, $\{b,c\}$	$\{a\}$, $\{b\}$, $\{c\}$ $\{b,c\}$

$((S_3 * \{a\}) + \{a,b,c\}) * \{a\}$	$((S_3 * \{a\}) + \{a,b,c\}) * \{b,c\}$
$\emptyset$ $\{b\}$, $\{c\}$ $\{a,b,c\}$	$\emptyset$ $\{a,b\}$, $\{a,c\}$ $\{a,b,c\}$

$((S_3 * \{a\}) + \{a,b,c\}) * \{b\}$	$((S_3 * \{a\}) + \{a,b,c\}) * \{a,c\}$
$\{a\}$, $\{c\}$ $\{a,b\}$ $\{a,b,c\}$	$\emptyset$ $\{c\}$ $\{a,b\}$, $\{b,c\}$

TABLE 7. All twists of $(S_3 * \{a\}) + \{a,b,c\}$ up to isomorphism. Dual pairs are side by side.

REFERENCES

[1] J. Bonin, C. Chun, and S. Noble, Delta-matroids as subsystems of sequences of Higgs lifts. Preprint.

[2] A. Bouchet, Greedy algorithm and symmetric matroids, *Math. Program.* **38** (1987) 147–159.

[3] A. Bouchet. Representability of Δ-matroids. Combinatorics (Eger, 1987), Colloq. Math. Soc. János Bolyai, **52**, North-Holland, Amsterdam, (1988) 167–182.

[4] A. Bouchet. Representability of delta-matroids over $GF(2)$. *Linear Algebra and Its Applications*, **78**, (1991) 67–78.

[5] A. Bouchet. Circle graph obstructions. *Journal of Combinatorial Theory Series B*, **60** (1994) 107–144.

[6] A. Bouchet, Multimatroids I. Coverings by independent sets, *SIAM J. Discrete Math.* **10** (1997) 626–646.

[7] A. Bouchet, Multimatroids II. Orthogonality, minors and connectivity, *Electron. J. Combin.* **8** (1998) R8.

[8] A. Bouchet, Multimatroids III. Tightness and fundamental graphs, *Europ. J. Combin.* **22** (2001) 657–677.

[9] R. Brijder and H. Hoogeboom. The group structure of pivot and loop complementation on graphs and set systems. *European Journal of Combinatorics*, **32** (2011) 1353–1367.

[10] R. Brijder and H. Hoogeboom. Interlace polynomials for multimatroids and delta-matroids. *European Journal of Combinatorics*, **40** (2014) 142–167.

[11] R. Brijder and H. Hoogeboom. Nullity and loop complementation for delta-matroids. *SIAM Journal on Discrete Mathematics*, **27** (2013) 492–506.

[12] R. Brijder, and H. Hoogeboom, Quaternary matroids are vf-safe. Preprint, `arXiv:1302.4415v2`.

[13] C. Chun, I. Moffatt, S. D. Noble, and R. Rueckriemen, Embedded graphs and delta-matroids. Preprint, `arXiv:1403.0920v2`.

[14] C. Chun, I. Moffatt, S. D. Noble and R. Rueckriemen. On the interplay between embedded graphs and delta-matroids. Preprint, `arXiv:1602.01306`.

[15] J. A. Ellis-Monaghan and I. Moffatt. A Penrose polynomial for embedded graphs. *European Journal of Combinatorics*, **34** (2013) 424–445.

[16] J. Ellis-Monaghan and I. Moffatt, *Graphs on surfaces: Dualities, Polynomials, and Knots*, Springer, (2013).

[17] J. Geelen, S. Oum, Circle graph obstructions under pivoting. *J. Graph Theory* **61** (2009) 1–11.

[18] I. Moffatt. Private communication, (2017).

[19] L. Traldi. Binary matroids and local complementation, *European Journal of Combinatorics*, **45** (2015) 21–40.

[20] A. W. Tucker. A combinatorial equivalence of matrices. *Combinatorial Analysis, Proc. Symposia Appl. Math., vol. X*, American Mathematical Society, Providence (1960) 129–140.

(J. Bonin) DEPARTMENT OF MATHEMATICS, THE GEORGE WASHINGTON UNIVERSITY, WASHINGTON, D.C. 20052, USA

E-mail address, J. Bonin: `jbonin@gwu.edu`

(C. Chun) DEPARTMENT OF MATHEMATICS, UNITED STATES NAVAL ACADEMY, ANNAPOLIS, MD, 21402, USA

E-mail address, C. Chun: `chun@usna.edu`

(S. Noble) DEPARTMENT OF ECONOMICS, MATHEMATICS AND STATISTICS, BIRKBECK, UNIVERSITY OF LONDON, LONDON WC1E 7HX, UNITED KINGDOM

E-mail address, S. Noble: `s.noble@bbk.ac.uk`