



BIROn - Birkbeck Institutional Research Online

Chun, C. and Moffatt, I. and Noble, Steven and Rueckriemen, R. (2019) Matroids, delta-matroids and embedded graphs. *Journal of Combinatorial Theory, Series A* 167 , pp. 7-59. ISSN 0097-3165.

Downloaded from: <https://eprints.bbk.ac.uk/id/eprint/26891/>

Usage Guidelines:

Please refer to usage guidelines at <https://eprints.bbk.ac.uk/policies.html>
contact lib-eprints@bbk.ac.uk.

or alternatively

Matroids, delta-matroids and embedded graphs

Carolyn Chun^a, Iain Moffatt^b, Steven D. Noble^c, Ralf Rueckriemen^{d,1}

^a*Mathematics Department, United States Naval Academy, Chauvenet Hall, 572C Holloway Road, Annapolis, Maryland 21402-5002, United States of America*

^b*Department of Mathematics, Royal Holloway University of London, Egham, Surrey, TW20 0EX, United Kingdom*

^c*Department of Mathematics, Brunel University, Uxbridge, Middlesex, UB8 3PH, United Kingdom*

^d*Aschaffener Strasse 23, 10779, Berlin*

Abstract

Matroid theory is often thought of as a generalization of graph theory. In this paper we propose an analogous correspondence between embedded graphs and delta-matroids. We show that delta-matroids arise as the natural extension of graphic matroids to the setting of embedded graphs. We show that various basic ribbon graph operations and concepts have delta-matroid analogues, and illustrate how the connections between embedded graphs and delta-matroids can be exploited. Also, in direct analogy with the fact that the Tutte polynomial is matroidal, we show that several polynomials of embedded graphs from the literature, including the Las Vergnas, Bollabás-Riordan and Krushkal polynomials, are in fact delta-matroidal.

Keywords: matroid, delta-matroid, ribbon graph, quasi-tree, partial dual, topological graph polynomial

2010 MSC: 05B35, 05C10, 05C31, 05C83

1. Overview

Matroid theory is often thought of as a generalization of graph theory. Many results in graph theory turn out to be special cases of results in matroid theory. This is beneficial in two ways.

First, graph theory can serve as an excellent guide for studying matroids. As reported by Oxley, in [55], Tutte famously observed that, “If a theorem about graphs can be expressed in terms of edges and circuits alone it probably exemplifies a more general theorem about matroids.” Perhaps one of the most spectacular illustrations of the effect of graph theory on matroid theory

Email addresses: chun@usna.edu (Carolyn Chun), iain.moffatt@rhul.ac.uk (Iain Moffatt), steven.noble@brunel.ac.uk (Steven D. Noble), ralf@rueckriemen.de (Ralf Rueckriemen)

¹Ralf Rueckriemen was financed by the DFG through grant RU 1731/1-1.

10 can be found in Geelen, Gerards and Whittle’s recent and at the time of writ-
11 ing unpublished result that, for any finite field, the class of matroids that are
12 representable over that field is well-quasi-ordered by the minor relation. This
13 profound result is the matroid analogue of an equally profound result that came
14 out of Robertson and Seymour’s Graph Minors Project, in which, they proved
15 that graphs are well-quasi-ordered by the minor relation [57]. Rather than the
16 result itself, here we want to focus on the fact that, to quote a recent statement
17 of Whittle [36] about his work with Geelen and Gerards, “It would be incon-
18 ceivable to prove a structure theorem for matroids without the Graph Minors
19 Structure Theorem as a guide”.

20 Second, insights from matroid theory can lead to new results about graphs.
21 For example, Wu [68] established an upper bound for the number of edges of
22 a loopless 2-connected graph, which was an improvement on existing results
23 suggested by matroid duality. Graph theory and matroid theory are mutually
24 enriching, and this is the subject of [55] by Oxley.

25 The key purpose of this paper is to propose and study a similar correspon-
26 dence between embedded graphs and delta-matroids.

27 Delta-matroids, introduced by Bouchet [5], can be seen as a generalization of
28 matroids. Where a matroid has bases, a delta-matroid has feasible sets. These
29 satisfy a symmetric exchange axiom, but do not all have to be of the same size.
30 We give a formal definition in the next section. The greater generality of delta-
31 matroids allows us to capture not only information about a graph, but also
32 about its embedding in a surface. Bouchet was the first to observe a connection
33 between embedded graphs and delta-matroids in [6]. Our approach is more
34 direct than his and has the advantage that it enables us to exploit the theory
35 of ribbon graphs, much of which has developed since Bouchet did his work.

36 We will describe embedded graphs as ribbon graphs. The cycle matroid
37 of a connected graph is constructed by taking the collection of spanning trees
38 of the graph as its bases. In a connected ribbon graph, the spanning-trees
39 are precisely the genus-zero spanning ribbon subgraphs that have exactly one
40 boundary component. In the context of ribbon graphs, the genus-zero restriction
41 is artificial, and it is subgraphs with exactly one boundary component, called
42 *quasi-trees* that play the role of trees. It turns out that the edge set of a ribbon
43 graph together with its spanning quasi-trees form a delta-matroid.

44 Moreover, we will see that this delta-matroid arises as the natural extension
45 of a cycle matroid to the setting of embedded graphs, and that the delta-matroid
46 structure follows from basic properties of surfaces. We show that various con-
47 cepts related to cellularly embedded graphs are special cases of concepts for
48 delta-matroids. Because of this compatibility between the two structures, we
49 extend Bouchet’s initial ideas and propose that there is a correspondence be-
50 tween embedded graphs and delta-matroids that is analogous to the one between
51 graphs and matroids. We justify this proposition by illustrating how results
52 from topological graph theory can be used to guide the development of delta-
53 matroid theory, just as graph theory often guides matroid theory. We also see
54 that several polynomials of embedded graphs, including the Tutte, Las Vergnas,
55 Bollobás-Riordan and Krushkal polynomials, are in fact delta-matroidal objects,

56 just as many graph polynomials are matroidal.

57 The paper is structured as follows. In Section 2, we give an overview of
58 some relevant properties of matroids and delta-matroids. Section 3 contains
59 some background on cellularly embedded graphs. Most of the time, we will use
60 the language of ribbon graphs instead of cellularly embedded graphs. These
61 are equivalent concepts (see Figure 1), but ribbon graphs have the advantage of
62 being closed under the natural minor operations.

63 In Section 4, we describe how delta-matroids arise from ribbon graphs, em-
64 phasising that they arise as the natural extensions of various classes of matroids
65 associated with graphs. We show that some of these delta-matroids, albeit in a
66 different language, appeared in Bouchet’s foundational work in delta-matroids.
67 In Section 5 we discuss their connections with graphic matroids and describe
68 how basic properties of a ribbon graph are encoded in its delta-matroid. We
69 provide evidence of the basic compatibility between delta-matroids and ribbon
70 graphs. In particular, we prove that one of the most fundamental operations of
71 delta-matroids, the twist, is the delta-matroid analogue of a partial dual of a
72 ribbon graph, which turns out to be a key result in connecting the two areas.
73 We describe how to see edge structure and connectivity in a ribbon graph in
74 terms of its delta-matroid, and show how results on delta-matroid connectivity
75 inform ribbon graph theory. We also demonstrate that excluded minor char-
76 acterisations that have appeared in both the delta-matroid and ribbon graph
77 literature are translations of one another.

78 In Section 6, we discuss various polynomials. Some well-known graph poly-
79 nomials, and in particular the Tutte polynomial, are properly understood as
80 matroid polynomials, rather than graph polynomials. There has been consider-
81 able recent interest in extensions of the Tutte polynomial to graphs embedded
82 in surfaces. Three generalizations of the Tutte polynomial to embedded graphs
83 in the literature are the Las Vergnas polynomial, the Bollobás-Riordan poly-
84 nomial, and the Kruskal polynomial. We show that each of these generalizations is
85 determined by the delta-matroids of ribbon graphs, and that the ribbon graph
86 polynomials are special cases of more general delta-matroid polynomials. That
87 is, while the Tutte polynomial is properly a matroid polynomial, its topological
88 extensions are properly delta-matroid polynomials.

89 Our results here offer new perspectives on delta-matroids. We illustrate
90 here a fundamental interplay between ribbon graphs and delta-matroids, that is
91 analogous to the interplay between graphs and matroids. By doing so we offer
92 a new approach to delta-matroid theory.

93 2. Matroids and delta-matroids

94 Our terminology follows [5] and [56], except where explicitly stated.

95 2.1. Set systems and delta-matroids

96 A *set system* is a pair $D = (E, \mathcal{F})$ where E is a set, which we call the *ground*
97 *set*, and \mathcal{F} is a collection of subsets of E . The members of \mathcal{F} are called *feasible*

98 sets. A set system is *proper* if \mathcal{F} is not empty; it is *trivial* if E is empty. For
 99 a set system D we will often use $E(D)$ to denote its ground set and $\mathcal{F}(D)$ its
 100 collection of feasible sets. In this paper we will always assume that E is a finite
 101 set and will do so without further comment.

102 The *symmetric difference* of sets X and Y , denoted by $X \triangle Y$, is $(X \cup Y) -$
 103 $(X \cap Y)$.

104 A *delta-matroid* is a proper set system $D = (E, \mathcal{F})$ that satisfies the Sym-
 105 metric Exchange Axiom:

106 **Axiom 2.1** (Symmetric Exchange Axiom). For all (X, Y, u) with $X, Y \in \mathcal{F}$ and
 107 $u \in X \triangle Y$, there is an element $v \in X \triangle Y$ such that $X \triangle \{u, v\}$ is in \mathcal{F} .

108 Note that we allow $v = u$ in the Symmetric Exchange Axiom.

109 If the feasible sets of a delta-matroid are equicardinal, then the delta-matroid
 110 is a *matroid* and we refer to its feasible sets as its *bases*. If a set system forms
 111 a matroid M , then we usually denote M by (E, \mathcal{B}) , and often use $\mathcal{B}(M)$ to
 112 denote its collection of bases \mathcal{B} . It is not hard to see that the definition of a
 113 matroid given here is equivalent to the ‘usual’ definition of a matroid through
 114 bases given in, for example, [56, 67].

115 Throughout this paper, we will often omit the set brackets in the case of a
 116 single element set. For example, we write $E - e$ instead of $E - \{e\}$, or $F \cup e$
 117 instead of $F \cup \{e\}$.

118 2.2. Graphic matroids

119 For a graph $G = (V, E)$ with k connected components, let \mathcal{B} be the edge
 120 sets of the maximal spanning forests of G . \mathcal{B} is obviously non-empty, and its
 121 elements are equicardinal since each spanning forest of G has $|V| - k$ edges. It
 122 is not too hard to see that the Symmetric Exchange Axiom holds, and so the
 123 set system $M(G) = (E, \mathcal{B})$ is a matroid, which is called the *cycle matroid* of G .
 124 Any matroid that is the cycle matroid of a graph is a *graphic matroid*.

125 *Example 2.2.* If G is the graph shown in Figure 1(a), then $M(G) = (E, \mathcal{B})$ where
 126 $E = \{1, 2, 3, 4\}$ and $\mathcal{B} = \{\{1\}, \{2\}\}$.

127 2.3. Matroid rank

128 Let M be a matroid with ground set E . A subset I of E is an *independent*
 129 *set* of M if and only if it is a subset of a basis of M . A *rank function* is defined
 130 for all subsets of the ground set of a matroid. Its value on a subset A of E is
 131 the cardinality of the largest independent set contained in A . The rank of a set
 132 A is written $r_M(A)$, or just $r(A)$ if the matroid is clear from the context. Thus,
 133 $r_M(A) = \max\{|A \cap B| \mid B \in \mathcal{B}(M)\}$. We say that the *rank* of M , written $r(M)$,
 134 is equal to $r(E)$, which is equal to $|B|$, for any $B \in \mathcal{B}(M)$.

135 *Example 2.3.* For a graph $G = (V, E)$, the rank function of its cycle matroid
 136 $M = M(G)$ is given by $r(A) = |V| - k(A)$, where $k(A)$ is the number of
 137 connected components of the spanning subgraph (V, A) of G , and $A \subseteq E$.

138 *2.4. Width and evenness*

139 For a delta-matroid $D = (E, \mathcal{F})$, let $\mathcal{F}_{\max}(D)$ and $\mathcal{F}_{\min}(D)$ be the set of
 140 feasible sets with maximum and minimum cardinality, respectively. We will
 141 usually omit D when the context is clear. Let $D_{\max} := (E, \mathcal{F}_{\max})$ and let
 142 $D_{\min} := (E, \mathcal{F}_{\min})$. Then D_{\max} is the *upper matroid* and D_{\min} is the *lower ma-*
 143 *triod* of D . These matroids were defined by Bouchet in [6]. It is straightforward
 144 to show that the upper matroid and the lower matroid are indeed matroids.
 145 The *width* of D , denoted by $w(D)$, is defined by

$$w(D) := r(D_{\max}) - r(D_{\min}).$$

146 Thus the width of D is the difference between the sizes of its largest and smallest
 147 feasible sets.

148 If the sizes of the feasible sets of a delta-matroid all have the same parity,
 149 then we say that the delta-matroid is *even*. Otherwise, we say that the delta-
 150 matroid is *odd*. In particular, every matroid is an even delta-matroid. It is
 151 perhaps worth emphasising that an even delta-matroid need not have feasible
 152 sets of even cardinality.

153 It is convenient to record the following useful result here.

154 **Lemma 2.4.** *Let $D = (E, \mathcal{F})$ be a delta-matroid, let A be a subset of E and let*
 155 *$s_0 = \min\{|B \cap A| \mid B \in \mathcal{B}(D_{\min})\}$. Then for any $F \in \mathcal{F}$ we have $|F \cap A| \geq s_0$.*

156 *Proof.* We proceed by contradiction. If $s_0 = 0$, then there is nothing to prove,
 157 so we can assume that $s_0 > 0$. Suppose that $F \in \mathcal{F}$ and $|F \cap A| < s_0$. Choose
 158 $F' \in \mathcal{F}_{\min}$ with $|F' \cap A| = s_0$ and $|F' \cap F \cap A|$ as large as possible. Now there
 159 exists $x \in A \cap (F' - F)$ and so $x \in F' \triangle F$. Hence there exists y belonging to
 160 $F' \triangle F$ such that $F'' = F' \triangle \{x, y\} \in \mathcal{F}$. Because $F' \in \mathcal{F}_{\min}$, we have $y \in F - F'$.
 161 And because $|F' \cap A| = s_0$, we must have $y \in F \cap A$. But then $F'' \in \mathcal{F}_{\min}$,
 162 $|F'' \cap A| = s_0$ and $|F'' \cap F \cap A| > |F' \cap F \cap A|$, contradicting the choice of
 163 F' . \square

164 *2.5. Twists, duals, loops, coloops, and minors*

165 Twists, introduced by Bouchet in [5], are one of the fundamental operations
 166 of delta-matroid theory.

167 **Definition 2.5.** Let $D = (E, \mathcal{F})$ be a set system. For $A \subseteq E$, the *twist* of D
 168 with respect to A , denoted by $D * A$, is given by $(E, \{A \triangle X \mid X \in \mathcal{F}\})$. The
 169 *dual* of D , written D^* , is equal to $D * E$.

170 It follows easily from the identity $(F'_1 \triangle A) \triangle (F'_2 \triangle A) = F'_1 \triangle F'_2$ that
 171 the twist of a delta-matroid is also a delta-matroid. We restate this fact in the
 172 following lemma.

173 **Lemma 2.6** (Bouchet [5]). *Let D be a delta-matroid and let A be a subset of*
 174 *$E(D)$. Then $D * A$ is a delta-matroid.*

175 Although it is always a delta-matroid, a twist of a matroid $M = (E, \mathcal{B})$ need
 176 not be a matroid. (For example, if $M = (\{1, 2\}, \{\{1\}, \{2\}\})$ then $M * \{1\}$ has
 177 feasible sets $\{\emptyset, \{1, 2\}\}$ and so is not a matroid.) However, its dual $M^* = M * E$
 178 is always a matroid. The rank function of M^* is given by

$$r_{M^*}(A) = r_M(E - A) + |A| - r_M(E). \quad (2.1)$$

179 For a delta-matroid $D = (E, \mathcal{F})$, and $e \in E$, if e is in every feasible set of
 180 D , then we say that e is a *coloop* of D . If e is in no feasible set of D , then we
 181 say that e is a *loop* of D . Note that a coloop or loop of D is a loop or coloop,
 182 respectively, of $D * A$ for any subset A of E containing e .

183 If e is not a coloop, then, following Bouchet and Duchamp [11], we define D
 184 *delete* e , written $D \setminus e$, to be

$$D \setminus e := (E - e, \{F \mid F \in \mathcal{F} \text{ and } F \subseteq E - e\}).$$

185 If e is not a loop, then we define D *contract* e , written D/e , to be

$$D/e := (E - e, \{F - e \mid F \in \mathcal{F} \text{ and } e \in F\}).$$

186 If e is a loop or a coloop, then one of $D \setminus e$ and D/e has already been defined,
 187 so we can set $D/e = D \setminus e$.

188 Both $D \setminus e$ and D/e are delta-matroids (see [11]). Let D' be a delta-matroid
 189 obtained from D by a sequence of deletions and contractions. Then D' is inde-
 190 pendent of the order of the deletions and contractions used in its construction
 191 (see [11]) and D' is called a *minor* of D . If D' is formed from D by deleting the
 192 elements of X and contracting the elements of Y then we write $D' = D \setminus X/Y$.
 193 The *restriction* of D to a subset A of E , written $D|A$, is equal to $D \setminus (E - A)$.

194 Note that $D^* \setminus e = (D/e)^*$. The next result shows that deletion, contraction
 195 and twists are also related. It is a reformulation of Property 2.1 of [11].

196 **Lemma 2.7.** *For a delta-matroid D and distinct elements e and f of $E(D)$,*
 197 *we have*

- 198 1. $D \setminus e = ((D * f) \setminus e) * f$ and $D/e = ((D * f)/e) * f$;
- 199 2. $D \setminus e = (D * e)/e$ and $D/e = (D * e) \setminus e$.

200 Using Lemma 2.7 and induction we obtain the following.

201 **Proposition 2.8.** *Let D be a delta-matroid and let A, X , and Y be subsets of*
 202 *$E(D)$ with $X \cap Y = \emptyset$. Then*

$$(D * A) \setminus X/Y = (D \setminus ((X - A) \cup (Y \cap A)) / ((Y - A) \cup (X \cap A))) * (A - X - Y).$$

203 *In particular, $D \setminus X = (D^*/X)^*$ and, when A is the disjoint union of X and*
 204 *Y , we have*

$$(D * A) \setminus X/Y = D \setminus Y/X.$$

205 *2.6. Delta-matroid rank*

206 Bouchet defined an analogue of the rank function for delta-matroids in [4].
 207 For a delta-matroid $D = (E, \mathcal{F})$, it is denoted by ρ_D or simply ρ when D is
 208 clear from the context. Its value on a subset A of E is given by

$$\rho(A) := |E| - \min\{|A \triangle F| \mid F \in \mathcal{F}\}.$$

209 Note that the feasible sets of a delta-matroid can be recovered from its rank
 210 function.

211 An easy consequence of basic properties of the symmetric difference opera-
 212 tion is the following.

213 **Lemma 2.9.** *Let D be a delta-matroid and let A be a subset of $E(D)$. Then*
 214 $\rho_{D^*}(A) = \rho_D(E - A)$.

215 The next two results show how the rank function changes when an element
 216 is deleted or contracted.

217 **Lemma 2.10.** *Let $D = (E, \mathcal{F})$ be a delta-matroid and let e be an element in*
 218 *E , and X a subset of $E - e$. Then either e is a coloop or there exists $F \in \mathcal{F}$*
 219 *such that $\rho(X) = |E| - |X \triangle F|$ and $e \notin F$.*

220 *Proof.* Suppose e is not a coloop. Then there is a feasible set F avoiding e .
 221 Take $F' \in \mathcal{F}$ such that $\rho(X) = |E| - |X \triangle F'|$. If F' avoids e then the lemma
 222 holds, so we assume this is not the case. Then $e \in F' \triangle F$, so the Symmetric
 223 Exchange Axiom (Axiom 2.1) implies that there exists $f \in F' \triangle F$ such that
 224 $F'' = F' \triangle \{e, f\} \in \mathcal{F}$. If $f = e$, then $|X \triangle F''| = |X \triangle (F' - e)| < |X \triangle F'|$
 225 which is not possible, because $\rho(X) = |E| - |X \triangle F'|$. So $f \neq e$ and $X \triangle F'' =$
 226 $X \triangle (F' \triangle \{e, f\}) = X \triangle ((F' - e) \triangle f) = (X \triangle (F' - e)) \triangle f$, so we deduce
 227 that $|X \triangle F''| \leq |X \triangle F'|$. As F' was chosen from \mathcal{F} to minimize $|X \triangle F'|$, we
 228 deduce that $|X \triangle F''| = |X \triangle F'|$. Since $e \notin F''$, the lemma holds. \square

229 **Lemma 2.11.** *Let $D = (E, \mathcal{F})$ be a delta-matroid and let e be an element in*
 230 *E , and X a subset of $E - e$. Then*

$$\rho_{D \setminus e}(X) = \begin{cases} \rho_D(X), & \text{if } e \text{ is a coloop of } D \\ \rho_D(X) - 1, & \text{otherwise} \end{cases} \quad (2.2)$$

and

$$\rho_{D/e}(X) = \begin{cases} \rho_D(X \cup e), & \text{if } e \text{ is a loop of } D \\ \rho_D(X \cup e) - 1, & \text{otherwise.} \end{cases} \quad (2.3)$$

231 *Proof.* We first establish (2.2). Suppose that e is not a coloop. Lemma 2.10
 232 implies that there exists $F \in \mathcal{F}(D)$ such that $e \notin F$ and $\rho_D(X) = |E| - |X \triangle F|$.
 233 Thus $\rho_{D \setminus e}(X) \leq |E - e| - |X \triangle F| = |E| - |X \triangle F| - 1 = \rho_D(X) - 1$. Moreover

234 every feasible set of $D \setminus e$ is a feasible set of D . Hence $\rho_D(X) \leq \rho_{D \setminus e}(X) + 1$.
 235 Combining these two inequalities gives the result.

236 Suppose that e is a coloop of D . Let A be a subset of $E - e$. Then A is
 237 a feasible set of $D \setminus e$ if and only if $A \cup e$ is a feasible set of D . Furthermore,
 238 $|X \triangle A| = |X \triangle (A \cup e)| - 1$. Take $F \in \mathcal{F}(D \setminus e)$ such that $\rho_{D \setminus e}(X) = |E| - |X \triangle F|$.
 239 Then $F \cup e$ is in \mathcal{F} and has smallest symmetric difference with X of all feasible
 240 sets in \mathcal{F} . Thus $\rho_D(X) = |E| - |X \triangle (F \cup e)| = |E| - |X \triangle F| - 1 = |E - e| -$
 241 $|X \triangle F| = \rho_{D \setminus e}(X)$.

242 Now Equation (2.3) is obtained by using duality. Lemma 2.9 implies that
 243 $\rho_{D/e}(X) = \rho_{(D/e)^*}(E - e - X) = \rho_{D^* \setminus e}(E - e - X)$. Using Equation (2.2), we
 244 obtain

$$\rho_{D^* \setminus e}(E - e - X) = \begin{cases} \rho_{D^*}(E - e - X), & \text{if } e \text{ is a coloop of } D^* \\ \rho_{D^*}(E - e - X) - 1, & \text{otherwise.} \end{cases}$$

245 The result follows by applying duality again and noting that e is a coloop of D^*
 246 if and only if it is a loop of D . \square

247 3. Ribbon graphs

248 We are concerned here with connections between cellularly embedded graphs
 249 and delta-matroids. As it is much more convenient for our purposes, we real-
 250 ize cellularly embedded graphs as ribbon graphs. This section provides a brief
 251 overview of ribbon graphs, as well as standard ribbon graph notation and con-
 252 structions. A more thorough treatment of the topics covered in this section can
 253 be found in, for example, [32].

254 3.1. Cellularly embedded graphs and ribbon graphs

255 3.1.1. Ribbon graphs

256 A *cellularly embedded graph* $G \subset \Sigma$ is a graph drawn on a closed compact
 257 surface Σ in such a way that edges only intersect at their ends, and such that
 258 each connected component of $\Sigma - G$ is homeomorphic to a disc. Note that each
 259 connected component of G must be embedded in a different component of the
 260 surface.

261 Two cellularly embedded graphs $G \subset \Sigma$ and $G' \subset \Sigma'$ are *equivalent* if there is
 262 a homeomorphism, $\varphi : \Sigma \rightarrow \Sigma'$, which is orientation-preserving if Σ is orientable,
 263 and has the property that $\varphi|_G : G \rightarrow G'$ is a graph isomorphism. We consider
 264 cellularly embedded graphs up to equivalence.

265 Ribbon graphs provide an alternative, and more natural for the present
 266 setting, description of cellularly embedded graphs.

267 **Definition 3.1.** A *ribbon graph* $G = (V(G), E(G))$ is a surface with boundary,
 268 represented as the union of two sets of discs: a set $V(G)$ of *vertices* and a set
 269 of *edges* $E(G)$ with the following properties.

- 270 1. The vertices and edges intersect in disjoint line segments.

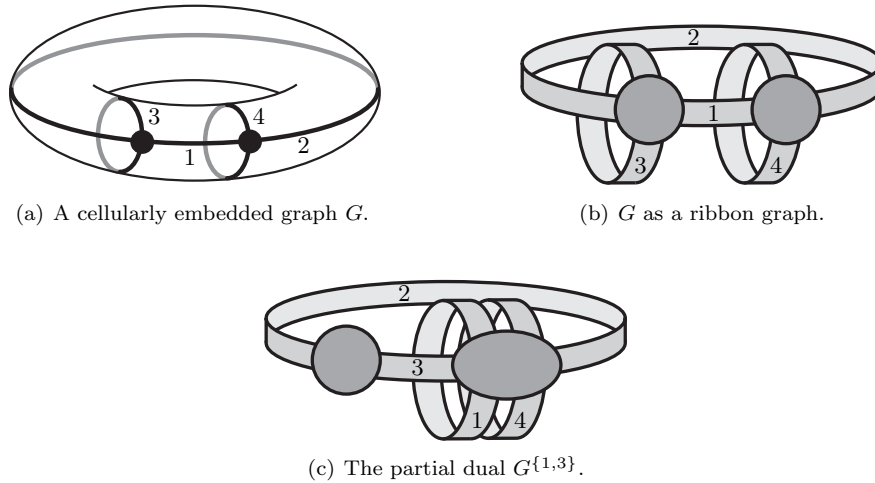


Figure 1: Embedded graphs and ribbon graphs.

- 271 2. Each such line segment lies on the boundary of precisely one vertex and
 272 precisely one edge. In particular, no two vertices intersect, and no two
 273 edges intersect.
 274 3. Every edge contains exactly two such line segments.

275 It is well-known that ribbon graphs are just descriptions of cellularly embed-
 276 ded graphs (see for example [39]). If G is a cellularly embedded graph,
 277 then a ribbon graph representation results from taking a small neighbourhood
 278 of the cellularly embedded graph G , and deleting its complement. On the other
 279 hand, if G is a ribbon graph, then, topologically, it is a surface with boundary.
 280 Capping off the holes, that is, ‘filling in’ each hole by identifying its boundary
 281 component with the boundary of a disc, results in a ribbon graph embedded in a
 282 closed surface from which a graph embedded in the surface is readily obtained.
 283 Figure 1 shows an embedded graph described as both a cellularly embedded
 284 graph and a ribbon graph. We say that two ribbon graphs are *equivalent* if they
 285 define equivalent cellularly embedded graphs, and we consider ribbon graphs
 286 up to equivalence. This means that ribbon graphs are considered up to homeo-
 287 morphisms that preserve the graph structure of the ribbon graph and the cyclic
 288 order of half-edges at each of its vertices.

289 3.1.2. Ribbon subgraphs and edge deletion

290 Let $G = (V, E)$ be a ribbon graph. Then a ribbon graph H is a *ribbon*
 291 *subgraph* of G if it can be obtained by removing vertices and edges of G . If
 292 $V(H) = V(G)$ then H is a *spanning ribbon subgraph* of G . Note that every
 293 subset A of E uniquely determines a spanning ribbon subgraph (V, A) of G .

294 If e is an edge of G , then G *delete* e , written $G \setminus e$, is defined to be the
 295 ribbon subgraph $(V, E - e)$ of G . Similarly, for $A \subseteq E$, $G \setminus A$ is defined to be

296 $(V, E - A)$. Table 1 shows the local effect of deleting an edge of a ribbon graph.

297 An important observation about ribbon subgraphs is that if a ribbon graph
298 G is realised as a graph cellularly embedded in a surface Σ , and $G \setminus e$, or a
299 ribbon subgraph H of G , is realised as a graph cellularly embedded in a surface
300 Σ' , then Σ and Σ' need not be homeomorphic.

301 3.1.3. Standard parameters

302 A ribbon graph is a graph with additional structure and so standard graph
303 terminology carries over to ribbon graphs. If G is a ribbon graph, then $v(G)$
304 and $e(G)$ denote $|V(G)|$ and $|E(G)|$, respectively. Furthermore, $k(G)$ denotes
305 the number of connected components in G , and $f(G)$ is the number of boundary
306 components of the surface defining the ribbon graph. For example, the ribbon
307 graph G of Figure 1(b) has $f(G) = 2$. Note that, if G is realised as a cellularly
308 embedded graph, then $f(G)$ is the number of its faces. The *rank* of G , denoted
309 by $r(G)$, is defined to be $v(G) - k(G)$, and the *nullity* of G , denoted by $n(G)$,
310 is defined to be $e(G) - r(G)$.

311 A ribbon graph G is *orientable* if it is orientable when regarded as a surface.
312 We define a ribbon graph parameter t by setting $t(G) = 1$ if G is non-orientable,
313 and $t(G) = 0$ otherwise.

314 The *genus* of a ribbon graph G is its genus when regarded as a surface. If
315 G is realised as a graph cellularly embedded in Σ , then its genus is exactly the
316 genus of Σ , and G is orientable if and only if Σ is. The *Euler genus*, $\gamma(G)$, of G
317 is the genus of G if G is non-orientable, and is twice its genus if G is orientable.
318 Euler's formula gives $\gamma(G) = 2k(G) - v(G) + e(G) - f(G)$. We say that a ribbon
319 graph G is *plane* if $\gamma(G) = 0$. Note that we allow plane graphs to have more
320 than one connected component. Plane ribbon graphs correspond to graphs that
321 can be cellularly embedded in some disjoint union of spheres.

322 For each subset A of E , we let $r(A)$, $k(A)$, $n(A)$, $f(A)$, $t(A)$, and $\gamma(A)$ each
323 refer to the spanning ribbon subgraph (V, A) of G , where G is given by context.
324 When the choice of G is not clear from the context, we write $r_G(A)$, $k_G(A)$, etc..
325 Observe that the function r on E defined here coincides with the rank function
326 of the cycle matroid $M(G)$ of G .

327 The following result is an obvious, but useful, consequence of the fact that
328 each edge of a ribbon graph meets one or two boundary components.

329 **Proposition 3.2.** *If G is ribbon graph, $A \subseteq E(G)$ and $e \in E(G)$, then $f(A)$
330 and $f(A \triangle e)$ differ by at most one.*

331 3.1.4. Loops and bridges

332 An edge e of a ribbon graph G is a *bridge* if $k(G \setminus e) > k(G)$. The edge e
333 is a *loop* if it is incident with exactly one vertex. We will abuse notation and
334 also use the term loop to describe the ribbon subgraph of G consisting of e and
335 its incident vertex. In ribbon graphs, loops can have various properties. A loop
336 or cycle is said to be *non-orientable* if it is homeomorphic to a Möbius band.
337 Otherwise it is *orientable*. Two cycles C_1 and C_2 in G are said to be *interlaced*
338 if there is a vertex v such that $V(C_1) \cap V(C_2) = \{v\}$, and C_1 and C_2 are met in

339 the cyclic order $C_1 C_2 C_1 C_2$ when travelling around the boundary of the vertex
 340 v . A loop is *non-trivial* if it is interlaced with some cycle in G , otherwise it is
 341 *trivial*.

342 3.1.5. Ribbon graph minors

343 For a ribbon graph G with an edge e recall that $G \setminus e$ is obtained by removing
 344 e from G . Similarly, if v is a vertex of G , then the *vertex deletion* $G \setminus v$ is defined
 345 to be the ribbon graph obtained from G by removing the vertex v together with
 346 all its incident edges.

347 The definition of edge contraction, introduced in [3, 22], is a little more
 348 involved than that of edge deletion.

349 **Definition 3.3.** Let G be a ribbon graph. Let $e \in E(G)$ and u and v be
 350 its incident vertices, which are not necessarily distinct. Then G/e denotes the
 351 ribbon graph obtained as follows. Consider the boundary component(s) of $e \cup$
 352 $u \cup v$ as curves on G . For each resulting curve, attach a disc, which will form
 353 a vertex of G/e , by identifying its boundary component with the curve. Delete
 354 e , u and v from the resulting complex. We say that G/e is obtained from G by
 355 *contracting e* .

356 A ribbon graph H is a *minor* of a ribbon graph G if H is obtained from G
 357 by a sequence of edge deletions, vertex deletions, and edge contractions.

358 The local effect of contracting an edge of a ribbon graph is shown in Ta-
 359 ble 1. Observe that contracting an edge may change the number of vertices
 360 or orientability of a ribbon graph. Since deletion and contraction are local
 361 operations, if some edges in a ribbon graph are deleted and some others are
 362 contracted, then the same ribbon graph will be produced regardless of the order
 363 of operations.

364 The definition of edge contraction might be a little surprising at first. How-
 365 ever, the reader should see that it is natural upon observing that Definition 3.3
 366 is just an expression of the obvious idea of contraction as the ‘identification of
 367 e and its incident vertices into a single vertex’ in a way that allows it to be ap-
 368 plied to loops. (See also the discussion in [32] on this topic.) Unlike for graphs,
 369 when working with ribbon graph minors it is necessary to be able to contract
 370 loops as otherwise the set of ribbon graphs will contain infinite anti-chains when
 371 quasi-ordered using the minor relation (see [53]).

372 3.1.6. Separability

373 For a ribbon graph G and non-trivial ribbon subgraphs P and Q of G , we
 374 write $G = P \sqcup Q$ when G is the *disjoint union* of P and Q , that is, when
 375 $G = P \cup Q$ and $P \cap Q = \emptyset$. A vertex v of G is a *separating vertex* if there
 376 are non-trivial ribbon subgraphs P and Q of G such that $G = P \cup Q$ and
 377 $P \cap Q = \{v\}$. In this case we write $G = P \oplus Q$.

378 We write $G = P \curlywedge Q$, if $G = P \oplus Q$ and no cycle in P is interlaced with
 379 a cycle in Q . Observe it is possible that $G = P \curlywedge Q$ and $G' = P \curlyvee Q$, for
 380 non-equivalent ribbon graphs G and G' .

	non-loop	non-orientable loop	orientable loop
G			
$G \setminus e$			
G/e $= G^e \setminus e$			
G^e			

Table 1: Operations on an edge e (highlighted in bold) of a ribbon graph. The ribbon graphs are identical outside of the region shown.

381 (We remark that here there is a close relationship with the join operation, \vee ,
382 on ribbon graphs: $G = P \vee Q$ if and only if $P = G_1 \vee \cdots \vee G_i$, $Q = G_{i+1} \vee \cdots \vee G_n$,
383 and, for some permutation σ , $G = G_{\sigma(1)} \vee \cdots \vee G_{\sigma(n)}$, where each join occurs
384 at the same vertex. We refer the reader to [51, 52] for a fuller discussion of
385 separability for ribbon graphs.)

386 3.2. Geometric duals and partial duals

387 The construction of the *geometric dual*, G^* , of a cellularly embedded graph
388 G is well known: $V(G^*)$ is obtained by placing one vertex in each face of G , and
389 $E(G^*)$ is obtained by embedding an edge of G^* between two vertices whenever
390 the faces of G in which they lie are adjacent. Geometric duality has a particu-
391 larly neat description when translated to the language of ribbon graphs. Let
392 $G = (V(G), E(G))$ be a ribbon graph. Recalling that, topologically, a ribbon
393 graph is a surface with boundary, we cap off the holes using a set of discs, de-
394 noted by $V(G^*)$, to obtain a surface without boundary. The *geometric dual* of
395 G is the ribbon graph $G^* = (V(G^*), E(G))$. Observe that, for ribbon graphs,
396 the edges of G and G^* are identical. The only change is which arcs on their
397 boundaries do and do not intersect vertices. This allows us to consider a subset
398 A of edges of G as also being a subset of edges of G^* and vice versa. We adopt
399 this convention. Although it is common to distinguish the two sets by writing
400 A and A^* , doing so proves to be notationally difficult in the current setting.

401 Chmutov, in [22], introduced a far-reaching generalization of geometric dual-
402 ity, called partial duality. Roughly speaking, a partial dual of a ribbon graph
403 is obtained by forming the geometric dual with respect to only a subset of its edges.
404 Partial duality arises as a natural operation in knot theory, topological graph

405 theory, graph polynomials, and quantum field theory. We will see later that it is
 406 also an analogue of a fundamental operation on delta-matroids. Here we define
 407 partial duals directly on ribbon graphs. We refer the reader to [22, 31, 50] or
 408 the exposition [32] for alternative constructions and other perspectives of partial
 409 duals.

410 Let $G = (V, E)$ be a ribbon graph and $A \subseteq E$. The partial dual G^A of G
 411 is obtained by forming the geometric dual of G as described above but ignoring
 412 the edges not in A as follows. Regard the boundary components of the spanning
 413 ribbon subgraph (V, A) of G as curves on the surface of G . Glue a disc to G
 414 along each connected component of this curve and remove the interior of all
 415 vertices of G . The resulting ribbon graph is the *partial dual* G^A .

416 We identify the edges of G with those of G^A using the natural correspon-
 417 dence. Table 1 shows the local effect of partial duality on an edge e (highlighted
 418 in bold) of a ribbon graph G . The ribbon graphs are identical outside of the
 419 regions shown. In fact Table 1 serves as a perfectly adequate definition of partial
 420 duality for this paper.

421 Observe from Table 1 that e is a bridge of G if and only if e is a trivial
 422 orientable loop in G^e ; e is a non-loop non-bridge edge of G if and only if e is a
 423 non-trivial orientable loop in G^e ; and e is a (non-)trivial non-orientable loop in
 424 G if and only if e is a (non-)trivial non-orientable loop in G^e . We also record
 425 the following basic properties of partial duality for use later.

426 **Proposition 3.4** (Chmutov [22]). *Let G be a ribbon graph and $A, B \subseteq E(G)$.
 427 Then*

- 428 1. $G^{E(G)} = G^*$ and $G^\emptyset = G$;
- 429 2. $(G^A)^B = G^{A \Delta B}$;
- 430 3. $G/e = G^e \setminus e$;
- 431 4. G is orientable if and only if G^A is orientable.

432 Note that it follows from the proposition that partial duals may be formed
 433 one edge at a time. Also note that the form of Item 3 of the proposition is very
 434 similar to that of the second part of Lemma 2.7. We will return to this later.

435 3.3. Quasi-trees

436 Quasi-trees are one of our fundamental objects of study. They are the ana-
 437 logue of trees for ribbon graphs, and our terminology reflects this. A *quasi-tree*
 438 Q is a connected ribbon graph with exactly one boundary component. If G is
 439 a connected ribbon graph, a *spanning quasi-tree* Q of G is a spanning ribbon
 440 subgraph with exactly one boundary component. For disconnected graphs, we
 441 abuse notation by saying that Q is a *spanning quasi-tree* of G if $k(Q) = k(G)$
 442 and the connected components of Q are spanning quasi-trees of the connected
 443 components of G .

444 We record the following basic facts about quasi-trees for reference later.
 445 For (3), recall that, for ribbon graphs, $E(G) = E(G^*)$.

446 **Lemma 3.5.** *Let G be a ribbon graph, and Q be a spanning quasi-tree of G .
 447 Then the following hold.*

- 448 1. $0 \leq \gamma(Q) \leq \gamma(G)$.
449 2. $\gamma(Q) = 0$ if and only if Q is a maximal spanning forest of G .
450 3. $(V(G), A)$ is a spanning quasi-tree of G of Euler genus γ if and only if
451 $(V(G^*), A^c)$ is a spanning quasi-tree of G^* of Euler genus $\gamma(G) - \gamma$.
452 4. If $Q = (V(G), A)$ then $\gamma(Q) = \gamma(G)$ if and only if $(V(G^*), A^c)$ is a maxi-
453 mal spanning forest of G^* .

454 *Proof.* Items (1) and (2) follow easily from Euler's formula. Item 4 is an immedi-
455 ate consequence of (2) and (3). It remains to prove (3). For this first assume that
456 G is connected. Consider the intermediate step of the formation of G^* from G ,
457 as described in Section 3.2, in which the holes of G have been capped off with ele-
458 ments of $V(G^*)$ giving a surface $\Sigma := V(G) \cup V(G^*) \cup E(G)$. For each $A \subseteq E(G)$,
459 observe that $V(G) \cup A = (\Sigma \setminus V(G^*)) \setminus A^c$ and $V(G^*) \cup A^c = (\Sigma \setminus V(G)) \setminus A$ have
460 the same boundary components. Thus $Q := (V(G), A)$ is a spanning quasi-tree
461 of G if and only if $Q' := (V(G^*), A^c)$ is a spanning quasi-tree of G^* . Sup-
462 pose that Q and Q' are both spanning quasi-trees. Then each of Q and Q' has
463 one boundary component and is connected. Moreover $v(Q') = v(G^*) = f(G)$.
464 Euler's formula gives $\gamma(Q) = 2k(Q) - v(Q) + e(Q) - f(Q) = 1 - v(G) + |A|$
465 and $\gamma(Q') = 2k(Q') - v(Q') + e(Q') - f(Q') = 1 - f(G) + e(G) - |A|$. Thus
466 $\gamma(Q) + \gamma(Q') = 2 - v(G) + e(G) - f(G) = \gamma(G)$. Extending the result to dis-
467 connected graphs is straightforward because each of the parameters v , e , f and
468 k is additive over connected components, and the geometric dual of a discon-
469 nected ribbon graph is the disjoint union of the geometric duals of its connected
470 components. \square

471 4. Delta-matroids from ribbon graphs

472 4.1. Defining the delta-matroids

473 Consider a connected ribbon graph $G = (V, E)$. We start by considering
474 some standard ways that G gives rise to a matroid. The most fundamental
475 matroid associated with G is its cycle matroid $M(G) = (E, \mathcal{B})$, where \mathcal{B} consists
476 of the edge sets of the spanning trees of G . The matroid $M(G)$ contains no
477 information about the topological structure of G , only its graphical structure.
478 This is because trees always have genus zero and therefore cannot depend upon
479 the embedding of G . Our aim here is to find the matroidal analogue of an
480 embedded graph, and to do this we clearly need to adapt the definitions of
481 $M(G)$. By thinking of the the construction of $M(G)$ in terms of ribbon graphs
482 it becomes obvious how this should be done: spanning trees are genus-zero
483 spanning ribbon subgraphs with exactly one boundary component, so to retain
484 topological information, we drop the genus zero condition, consider quasi-trees
485 instead of trees, and obtain the set system (E, \mathcal{F}) , where \mathcal{F} consists of the edge
486 sets of the spanning quasi-trees of G .

487 There is a natural variation of the construction of a cycle matroid obtained
488 by choosing $n \in \mathbb{N}_0$, taking E as the ground set and \mathcal{B} to be either the edge sets
489 formed by deleting n edges from each spanning tree, or the edge sets formed

490 by adding n edges to each spanning tree. In the former case, \mathcal{B} consists of the
491 edge sets of spanning forests of G having exactly $n + 1$ connected components
492 and (E, \mathcal{B}) is shown to be a matroid by noting that it is the n th truncation
493 of $M(G)$, see [56]. In the latter case, (E, \mathcal{B}) is the dual of the n th truncation
494 of $M(G)^*$. Consider this construction in terms of quasi-trees of ribbon graphs:
495 the number of boundary components is not determined by the number of edges
496 added or removed and can be anywhere between 1 and $n + 1$, if n edges are
497 added or removed. In the quasi-tree setting it no longer makes sense to make the
498 distinction between adding and removing edges, as we did in the case of matroids
499 and spanning trees. These ribbon graph extensions of matroids naturally lead
500 us to the make the following definition.

501 **Definition 4.1.** Let $G = (V, E)$ be a ribbon graph with $k(G)$ connected com-
502 ponents, and let $n \in \mathbb{N}_0$. Then we define

- 503 1. $\mathcal{F}_{\leq n}(G) := \{A \subseteq E \mid f(A) \leq k(G) + n\}$, and
504 2. $\mathcal{F}_n(G) := \{A \subseteq E \mid f(A) = k(G) + n\}$.

505 For a connected ribbon graph, $\mathcal{F}_n(G)$ is the collection of all edge sets that
506 determine a spanning ribbon subgraph of G with exactly $n + 1$ boundary com-
507 ponents, and $\mathcal{F}_{\leq n}(G)$ is the collection of all edge sets that determine a span-
508 ning ribbon graph of G with at most $n + 1$ boundary components. Note that
509 $\mathcal{F}_{\leq 0}(G) = \mathcal{F}_0(G)$. This set will be particularly important to us here, and later
510 we will denote it by just $\mathcal{F}(G)$. Note that $\mathcal{F}_n(G)$ may be empty.

Example 4.2. For the ribbon graph G of Figure 1(b),

$$\begin{aligned} \mathcal{F}_0(G) &= \mathcal{F}_{\leq 0}(G) = \{\{1\}, \{2\}, \{1, 2, 3\}, \{1, 2, 4\}\}, \\ \mathcal{F}_1(G) &= \{\emptyset, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{1, 2, 3, 4\}\}, \\ \mathcal{F}_2(G) &= \{\{3\}, \{4\}, \{1, 3, 4\}, \{2, 3, 4\}\}, \\ \mathcal{F}_3(G) &= \{\{3, 4\}\}, \text{ and} \\ \mathcal{F}_n(G) &= \emptyset, \text{ for } n > 3. \end{aligned}$$

511 Then $\mathcal{F}_{\leq n}(G)$ can be found easily from these.

512 **Definition 4.3.** For a ribbon graph $G = (V, E)$ and a non-negative integer
513 n , let $D_{\leq n}(G)$ denote the set system $(E, \mathcal{F}_{\leq n}(G))$, and $D_n(G)$ denote the set
514 system $(E, \mathcal{F}_n(G))$.

515 **Theorem 4.4.** Let $G = (V, E)$ be a ribbon graph, and $n \in \mathbb{N}_0$. Then

- 516 1. $D_{\leq n}(G) = (E, \mathcal{F}_{\leq n}(G))$ is a delta-matroid, and
517 2. $D_1(G) = (E, \mathcal{F}_1(G))$ is a delta-matroid, if G is non-empty and orientable.

518 The proof of Theorem 4.4 follows from the next lemma. For the next two
519 proofs we use G_A to denote the spanning ribbon subgraph (V, A) of G . Note
520 that G_A does *not* denote the induced ribbon subgraph $G|_A$.

521 **Lemma 4.5.** Suppose $A \in \mathcal{F}_n(G)$, $B \in \mathcal{F}_{\leq n}(G)$, $e \in A \triangle B$, and $A \triangle e \notin$
522 $\mathcal{F}_{\leq n}(G)$. Then there exists $f \in A \triangle B$ such that $A \triangle \{e, f\} \in \mathcal{F}_n(G)$.

523 *Proof.* The ribbon graph G_A has $n + k(G)$ boundary components and G_B has
524 at most $n + k(G)$ boundary components. By Proposition 3.2, $f(A \triangle e)$ and $f(A)$
525 differ by at most one. Thus $G_{A \triangle e}$ has $n + k(G) + 1$ boundary components (as
526 $A \triangle e \notin \mathcal{F}_{\leq n}(G)$). We think of $G_{A \triangle e}$ as a ribbon subgraph of $G_{A \cup B}$. We can
527 then consider how the edges in $(A \triangle B) \setminus e$ meet the boundary components of
528 $G_{A \triangle e}$.

529 If there is an edge $f \in (B \setminus A) \setminus e$ that intersects two distinct boundary
530 components of $G_{A \triangle e}$, then adding this edge to $G_{A \triangle e}$ will give a ribbon subgraph
531 with one fewer boundary component, and so $A \triangle \{e, f\} \in \mathcal{F}_n$. If there is an
532 edge $f \in (A \setminus B) \setminus e$ that meets two distinct boundary components of $G_{A \triangle e}$,
533 then removing this edge from $G_{A \triangle e}$ results in a ribbon subgraph with one fewer
534 boundary component, and so $A \triangle \{e, f\} \in \mathcal{F}_n(G)$.

535 All that remains is the case in which each edge in $(A \triangle B) \setminus e$ intersects
536 exactly one boundary component of $G_{A \triangle e}$. We shall show that this case cannot
537 happen.

538 To see why, observe that G_B can be obtained from $G_{A \triangle e}$ by first deleting
539 the edges in $(A \setminus B) \setminus e$ and then adding the edges in $(B \setminus A) \setminus e$, one by one.
540 Colour the boundary components of $G_{A \triangle e}$ so that each one receives a different
541 colour. Whenever an edge is added or deleted, the only boundary components
542 that change are those intersecting an edge that is deleted or those intersecting
543 the two line segments forming the ends of an edge that is added. At each step
544 the number of boundary components may stay the same, or increase or decrease
545 by one. After a step where the number of boundary components increases
546 by one, the two new boundary components are given the same colour as the
547 one they replace. We claim that when the number of boundary components
548 decreases by one, the two boundary components being replaced have the same
549 colour. The single boundary component replacing them may then be given this
550 common colour. Suppose that the claim is not true and consider the first time
551 that an edge f is added or deleted in such a way that the number of boundary
552 components decreases and the two boundary components C_1 and C_2 that are
553 changed by the edge addition or deletion have different colours. Let G' denote
554 the ribbon graph obtained just before f is added or deleted. Both C_1 and
555 C_2 contain a line segment that is removed from the boundary of G' after the
556 addition or deletion of f . Let L_1 and L_2 denote these line segments. Then
557 L_1 and L_2 are part of the boundary of each ribbon graph in the process up to
558 the current step, including $G_{A \triangle e}$. Although the boundary components to which
559 these line segments belong may change, their colours do not. As $f \in (A \triangle B) \setminus e$,
560 it intersects exactly one boundary component of $G_{A \triangle e}$. Therefore L_1 and L_2
561 have the same colour in $G_{A \triangle e}$, and consequently in G' . Thus the claim follows
562 and moreover all the original colours used to colour the boundary components of
563 $G_{A \triangle e}$ are used to colour the boundary components of G_B . Therefore G_B has at
564 least as many boundary components as $G_{A \triangle e}$. This contradicts our hypotheses
565 from the statement of the lemma that $B \in \mathcal{F}_{\leq n}(G)$ and $A \triangle e \notin \mathcal{F}_{\leq n}(G)$. \square

566 *Proof of Theorem 4.4.* In each case it is enough to show that the given families
567 of feasible sets satisfy the Symmetric Exchange Axiom.

568 For Item 1, let $A, B \in \mathcal{F}_{\leq n}(G)$ and $e \in A \triangle B$. If $A \triangle e \in \mathcal{F}_{\leq n}(G)$,
569 then taking $f = e$ gives $A \triangle \{e, f\} \in \mathcal{F}_{\leq n}(G)$, as desired. In the exceptional
570 case, $A \triangle e \notin \mathcal{F}_{\leq n}(G)$, so it follows from Proposition 3.2 that $A \in \mathcal{F}_n(G)$.
571 Then Lemma 4.5 guarantees that there is an element $f \in A \triangle B$ such that
572 $A \triangle \{e, f\} \in \mathcal{F}_{\leq n}(G)$.

573 For Item 2, we first observe that it follows easily from Euler's formula that
574 the parity of $f(A) - f(B)$ is the same as the parity of $e(A) - e(B)$. In particular,
575 the sizes of all spanning quasi-trees of G have the same parity, and the sizes
576 of all members of \mathcal{F}_1 have the opposite parity. By Proposition 3.2, we have
577 $|f(A \triangle e) - f(A)| \leq 1$. Thus, if $A \in \mathcal{F}_0(G)$ and $e \in E$, then $A \triangle e \in \mathcal{F}_1(G)$,
578 so D_1 is a proper set system. Let A, B be members of $\mathcal{F}_1(G)$ and $e \in A \triangle B$.
579 If $A \triangle e \notin \mathcal{F}_{\leq 1}(G)$, then by Lemma 4.5, there exists $f \in A \triangle e$ such that
580 $A \triangle \{e, f\} \in \mathcal{F}_1(G)$. It remains to consider what happens if $A \triangle e \in \mathcal{F}_0(G)$. As
581 $|A|$ and $|B|$ have the same parity, there exists $f \in (A \triangle B) - e$. Now, by our
582 earlier observation, $(A \triangle e) \triangle f \in \mathcal{F}_1(G)$. Hence $D_1(G)$ is a delta-matroid. \square

583 In general, the set system $D_n(G)$ is not a delta-matroid. For example, if G
584 is the plane graph obtained by taking a triangle with edges 1, 2, 3 and adding
585 an edge 4 in parallel with edge 3, then $\mathcal{F}_2(G) = \{\emptyset, \{3, 4\}, \{1, 2, 3, 4\}\}$ and it
586 is readily seen that $D_2(G)$ is not a delta-matroid. Also, if G is non-orientable
587 $D_1(G)$ may not be a delta-matroid. Consider, for example, the ribbon graph
588 G of Euler genus 2 obtained by adding an interlaced non-orientable loop to a
589 plane 2-cycle.

590 4.2. Ribbon-graphic delta-matroids

591 One of the main purposes of this article is to illustrate that the delta-matroid
592 $D_0(G) = D_{\leq 0}(G)$ plays a role in delta-matroid theory analogous to the role
593 graphic matroids play in matroid theory. In this subsection we set up some ad-
594 ditional terminology for these delta-matroids and show that they have appeared
595 in the literature in other guises.

596 **Definition 4.6.** Let $G = (V, E)$ be a ribbon graph. We use $\mathcal{F}(G)$ to denote
597 the set $\mathcal{F}_0(G) = \mathcal{F}_{\leq 0}(G)$, so that

$$\mathcal{F}(G) := \{F \subseteq E(G) \mid F \text{ is the edge set of a spanning quasi-tree of } G\},$$

598 and $D(G) = (E, \mathcal{F})$ to denote the delta-matroid $D_0(G) = D_{\leq 0}(G)$. We say that
599 $D(G)$ is a *ribbon-graphic* delta-matroid.

600 *Example 4.7.* For the ribbon graph G of Figure 1(b),

$$D(G) = (\{1, 2, 3\}, \{\{1\}, \{2\}, \{1, 2, 3\}, \{1, 2, 4\}\}).$$

601 To relate the delta-matroid $D(G)$ to the literature, particularly to Bouchet's
602 foundational work on delta-matroids, we take what may appear to be a detour
603 into transition systems. Let $F = (V, E)$ be a 4-regular graph. Each vertex
604 v of F is incident with exactly four half-edges. A *transition* τ_v at a vertex v

605 is a partition of the half-edges at v into two pairs, and a *transition system*,
 606 $\tau := \{\tau_v \mid v \in V\}$ of F is a choice of transition at each of its vertices.

607 For the purposes of this section, we allow graphs to include *free loops*, that
 608 is edges which are not incident with any vertex. We think of a free loop as a
 609 circular edge or as a cycle on zero vertices. Given a transition system τ of F ,
 610 we can obtain a set of free loops as follows. If (u, v) and (w, v) are two non-loop
 611 edges whose half edges are paired at the vertex v , then we replace these two
 612 edges with a single edge (u, w) . In the case of a loop, we temporarily imagine
 613 an extra vertex of degree two on the loop, carry out the operation, and then
 614 suppress the temporary vertex. Doing this replacement for each pair of half
 615 edges paired together in the transition system τ results in a set of free loops,
 616 that we denote by $F(\tau)$ and call a *graph state*.

617 Since F is 4-regular, at each vertex there are three transitions. Choose ex-
 618 actly two transitions τ_v and τ'_v at each vertex, and consider the set \mathcal{T} consisting
 619 of all transition systems of F in which the transition at each vertex v is one of
 620 the distinguished transitions, τ_v or τ'_v . An element of \mathcal{T} is called an *allowable*
 621 *transversal*. Fix some allowable transversal $T \in \mathcal{T}$, and let

$$D(F, \mathcal{T}, T) = (T, \{\tau \cap T \mid \tau \in \mathcal{T} \text{ and } |F(\tau)| = k(F)\}).$$

622 Kotzig's Theorem [42] implies that $D(F, \mathcal{T}, T)$ is a proper set system. Bouchet
 623 showed in [5] that $D(F, \mathcal{T}, T)$ is a delta-matroid. A delta-matroid that can be
 624 obtained in this way is called an *Eulerian delta-matroid*. (Note that although
 625 Bouchet never uses the term "Eulerian delta-matroid" in [5], it is implied that
 626 this is the intended definition by his later work, such as [8].)

627 Bouchet showed that $D(G)$ is a delta-matroid, albeit using a different lan-
 628 guage. Following [6], let G be a connected graph cellularly embedded in a surface
 629 Σ , and let G^* be its geometric dual. Consider the natural immersion of $G \cup G^*$
 630 in Σ . For each $B \subseteq E(G)$ let B^* denote the corresponding set in $E(G^*)$. A set
 631 $B \subseteq E(G)$ is said to be a *base* if $\Sigma - \text{cl}(B \cup (B^c)^*)$ is connected, where cl denotes
 632 the topological closure operator. Let $\mathcal{F}_b(G)$ denote the collection of all bases of
 633 G . Bouchet showed that $\mathcal{F}_b(G)$ satisfies the Symmetric Exchange Axiom, and
 634 so the pair $D_{\text{cell}}(G) = (E, \mathcal{F}_b(G))$ is a delta-matroid.

635 By changing from the language of cellularly embedded graph to ribbon
 636 graphs we can see that $D(G)$ and $D_{\text{cell}}(G)$ are identical objects. To see this
 637 consider $G \subset \Sigma$ and $G^* \subset \Sigma$ as ribbon graphs G' and G'^* respectively. Then
 638 $\Sigma = V(G') \cup V(G'^*) \cup E(G)$ as described in Section 3.2. It is not hard to see
 639 that the number of components of $\Sigma - \text{cl}(B \cup (B^c)^*)$ is exactly the number of
 640 boundary components of $G' \setminus B^c$. It follows that B defines a base of $G \subset \Sigma$ if
 641 and only $(V(G'), B)$ is a spanning quasi-tree of G' . Thus $D(G)$ and $D_{\text{cell}}(G)$
 642 coincide.

643 Bouchet did not use the language of quasi-trees to show that $D_{\text{cell}}(G)$ is a
 644 delta-matroid, but rather transition systems and Eulerian delta-matroids, iden-
 645 tifying it with a construction from [5]. For this, again let G be a connected
 646 graph cellularly embedded in a surface. Its *medial graph*, G_m , is the embedded
 647 graph constructed by placing a vertex on each edge of G , and then drawing the

648 edges of the medial graph by following the face boundaries of G (so each vertex
649 of G_m is of degree 4). The medial graph of an isolated vertex is a free loop.
650 The vertices of G_m are 4-valent and correspond to the edges of G . Every medial
651 graph has a *canonical face 2-colouring* given by colouring faces corresponding
652 to a vertex of G black, and the remaining faces white. We can use the canoni-
653 cal face 2-colouring to distinguish among the three types of vertex transitions.
654 We call a vertex transition *white* if it pairs half-edges that share a white face,
655 *black* if it pairs half-edges that share a black face, and *crossing* otherwise. If \mathcal{T}_m
656 consists of all the transition systems that have only white or black transitions
657 at each vertex, and W consists only of the white transitions, then it is not hard
658 to see that $D(G) = D(G_m, \mathcal{T}_m, W)$.

659 This discussion shows that every ribbon-graphic delta-matroid is Eulerian.
660 In fact, ribbon-graphic delta-matroids are exactly Eulerian delta-matroids.

661 **Theorem 4.8** (Bouchet [6]). *A delta-matroid D is Eulerian if and only if $D \cong$
662 $D(G)$, for some ribbon graph G .*

663 *Sketch of proof.* If D is Eulerian then, by definition, we can obtain it as some
664 $D(F, \mathcal{T}, T)$. We need to find a ribbon graph G such that $D = D(G_m, \mathcal{T}_m, W)$.
665 But such a ribbon graph can be obtained as a cycle family graph of F , from [31].
666 (The cycle family graphs of F are precisely the embedded graphs that have a
667 medial graph isomorphic to F .) The six choices at each vertex in the construc-
668 tion of a cycle family graph correspond to the six choices of the white and black
669 transitions of G_m (c.f. the proof of Theorem 4.12 of [31]). \square

670 We have just seen that the delta-matroids of ribbon graphs considered here
671 appeared in a rather different framework as Eulerian delta-matroids in Bouchet's
672 initial work on delta-matroids. Here, we are proposing that for many purposes,
673 the class of Eulerian delta-matroids, and delta-matroid theory in general, is best
674 thought of as extensions of ribbon graph theory. (Saying this, of course there are
675 certainly situations where it is most helpful to think of Eulerian delta-matroids
676 as generalisations of transition systems.) As we will demonstrate here, this is
677 because there is a natural and fundamental compatibility between ribbon graph
678 theory and delta-matroid theory, with many constructions, results, and proofs
679 in the two areas being translations of one another.

680 From the perspective of Eulerian delta-matroids, $D(G_m, \mathcal{T}_m, W)$ is signifi-
681 cant since the transition systems of G_m arise canonically. Another setting in
682 which canonical transition systems arise is in digraphs. Suppose that \vec{F} is a
683 4-regular digraph with two incoming and two outgoing half-edges at each ver-
684 tex. At each of its vertices there are two natural transitions that are consistent
685 with the direction of the half-edges of the digraph. We take $\vec{\mathcal{T}}$ to be the set
686 of all transition systems that arise from these choices. Then for each $\vec{T} \in \vec{\mathcal{T}}$,
687 $D(\vec{F}, \vec{\mathcal{T}}, \vec{T})$ is a delta-matroid. We call a delta-matroid arising in this way a
688 *directed Eulerian delta-matroid*.

689 **Theorem 4.9** (Bouchet [6]). *A delta-matroid D is directed Eulerian if and only
690 if $D = D(G)$, for some orientable ribbon graph G .*

691 *Sketch of proof.* First suppose that $D = D(G)$, for some orientable ribbon graph
 692 G . Arbitrarily orient (the surface) G and draw its canonically face 2-coloured
 693 medial graph G_m on it. Direct each edge of G_m so that it is consistent with the
 694 orientation of the black face it bounds.

695 Conversely, suppose that D is directed Eulerian, arising from a digraph \vec{F} .
 696 By the proof Theorem 4.8, we know $D = D(G_m, \mathcal{T}_m, W)$ for some ribbon graph
 697 G , where the underlying graphs of G_m and \vec{F} are isomorphic. The direction
 698 of \vec{F} induces a direction of G_m . Furthermore, by forming the twisted duals
 699 (see [31]) $G^{\tau(e)}$ or $G^{\tau\delta(e)}$, if necessary, we may assume that the transitions
 700 that are consistent with the directions of \vec{F} coincide with the black and white
 701 transitions of G_m . These directions induce an orientation on each black face of
 702 G_m , and hence of each vertex and half-edge of G . Since the black and white
 703 transitions of G_m are consistent with transitions coming from the directions of
 704 \vec{F} , these orientations of vertices must be consistent and so G is orientable. \square

705 Combining Theorems 4.8 and 4.9, and using the fact from Proposition 5.3
 706 that $D(G)$ is even if and only if G is orientable, immediately gives the following.

707 **Corollary 4.10** (Bouchet [6]). *A delta-matroid D is directed Eulerian if and*
 708 *only if it is Eulerian and even.*

709 In recent papers, Traldi introduced the transition matroid of an abstract
 710 four-regular graph [62] and the isotropic matroid of a symmetric binary matrix [61].
 711 These two matroids have almost identical definitions: both are binary matroids
 712 described by a representation, with the only difference being a permutation of some
 713 of the columns labels. Moreover, both are relevant to ribbon graphs. We have
 714 described the fundamental relationship between a ribbon graph and its medial graph,
 715 which is an embedded four-regular graph; in Section 5.7 we describe how a ribbon
 716 graph with one vertex may be represented by a symmetric binary matrix. In [16]
 717 Brijder and Traldi describe the construction of the transition matroid of a ribbon
 718 graph. We now describe the almost identical construction of the isotropic matroid
 719 of a ribbon graph, and discuss the extent to which it determines the ribbon graph.
 720

721 Let $G = (V, E)$ be a connected ribbon graph and G_m be its canonically face
 722 2-coloured medial graph. Let T be a transition system in \mathcal{T}_m with $|G_m(T)| = 1$.
 723 In other words, T defines an Eulerian circuit $C(T)$ in G_m with no crossing
 724 transitions. Apply an orientation to the edges of G_m , so that $C(T)$ is now a
 725 directed Eulerian cycle.

726 We say that two vertices u and v of G_m are *interlaced* with respect to T if
 727 they are met in the cyclic order $u v u v$ when travelling round $C(T)$. Let $A(G, T)$
 728 denote the binary $|E|$ by $|E|$ matrix whose rows and columns are indexed by the
 729 elements of E . The (e, e) -entry of $A(G, T)$ is zero if and only if in G_m ,
 730 opposite edges at the vertex corresponding to e have inconsistent orientations in
 731 $C(T)$. For $e \neq f$, the (e, f) -entry is one if and only the vertices corresponding
 732 to e and f in G_m are interlaced with respect to T .

733 We now let $IAS(G, T)$ be the $|E| \times 3|E|$ matrix

$$(I \mid A(G, T) \mid I + A(G, T)).$$

734 The *isotropic matroid* of G is the binary matroid $M[IAS(G, T)]$ with representation
735 $IAS(G, T)$. Each edge of G indexes three columns of $IAS(G, T)$, one in
736 each of the three blocks, with the order of the indices consistent with the indices
737 of $A(G, T)$. Following Traldi, we use e_ϕ , e_χ and e_ψ to denote the columns
738 of $IAS(G, T)$ corresponding to e in I , $A(G, T)$ and $I + A(G, T)$ respectively.
739 For $\nu \in \{\phi, \chi, \psi\}$, let $E_\nu = \{e_\nu \mid e \in E\}$. A basis of $M[IAS(G, T)]$ is called
740 *transverse* if for each $e \in E$, it contains precisely one of e_ϕ , e_χ and e_ψ .

741 The isotropic matroid itself does not determine $D(G)$, because knowledge of
742 T is required. Let T_w denote the edges of G where, at the corresponding vertex
743 of G_m , T takes the white transition. Then from the discussion above T_w is a
744 feasible set of $D(G)$. We claim that $T_w \triangle F$ is a feasible set of $D(G)$ if and
745 only if the principal submatrix of $A(G, T)$ corresponding to the edges of F is
746 non-singular. This is easily verified when $|F| \leq 2$, by considering the effect of
747 switching the transitions of T from black to white or vice versa at the vertices of
748 G_m corresponding to edges in F . Results of Bouchet presented as Lemmas 5.40
749 and 5.42, and Theorem 5.44 in Section 5.7 show that this is enough to verify
750 the claim. Thus there is a bijection between transverse bases of $M[IAS(G, T)]$
751 which do not intersect E_ψ and feasible sets of $D(G)$ associating a basis B with
752 the feasible set $(B \cap E_\chi) \triangle T_w$.

753 In [61], Traldi introduces the isotropic matroid of a symmetric binary matrix
754 A , which has a representation of the same form as above, that is

$$(I \mid A \mid I + A).$$

755 In particular in [61, Theorem 7] he describes exactly when two binary symmetric
756 matrices have isomorphic isotropic matroids. To translate this result to ribbon
757 graphs, requires the notion of twisted duality from [31]. Two ribbon graphs are
758 twisted duals of each other if and only if their medial graphs are isomorphic as
759 abstract graphs. Given a connected ribbon graph G and a spanning quasi-tree Q
760 of G , let $T(Q)$ denote the transition system of G_m taking the white transition at
761 vertices of G_m corresponding to edges of Q and the black transition otherwise.

762 **Theorem 4.11.** *Let G_1 and G_2 be connected ribbon graphs and let Q_1 and Q_2*
763 *be spanning quasi-trees of G_1 and G_2 respectively. Then $IAS(G_1, T(Q_1)) \simeq$*
764 *$IAS(G_2, T(Q_2))$ if and only if $D(G_1) \simeq D(G_3)$ for some twisted dual G_3 of G_2 .*

765 In Section 5.7 we discuss binary delta-matroids, which arise from binary
766 symmetric matrices. Further results from [61] describe how any binary delta-
767 matroid can be viewed as an isotropic matroid.

768 4.3. The spread of a delta-matroid

769 In Section 4.1 we associated a family of delta-matroids to a ribbon graph.
770 In this section we introduce an operation on delta-matroids that enables us to
771 relate $D_{\leq n}(G)$ to $D(G)$.

772 **Definition 4.12.** Let $D = (E, \mathcal{F})$ be a delta-matroid and n be a non-negative
773 integer. Then we define $\mathcal{F}_{\leq n}$ by

$$\mathcal{F}_{\leq n} := \{F \triangle A \mid F \in \mathcal{F} \text{ and } A \subseteq E \text{ and } |A| \leq n\}.$$

774 We say that the set system $D_{\leq n} := (E, \mathcal{F}_{\leq n})$ is the n -spread of D .

775 Note that $D_{\leq 0} = D$. In order to show that $D_{\leq n}$ is a delta-matroid, we will
 776 define delta-matroid sum. This sum is not the same concept as the direct sum,
 777 which we define later. We will only refer to the sum in this section, so confusion
 778 should not arise. If $D = (E, \mathcal{F})$ and $D' = (E, \mathcal{F}')$ are proper set-systems then
 779 their *sum* is the set system $(E, \mathcal{F} \triangle \mathcal{F}')$ where

$$\mathcal{F} \triangle \mathcal{F}' := \{F \triangle F' \mid F \in \mathcal{F} \text{ and } F' \in \mathcal{F}'\}.$$

780 Bouchet and Schwärzler [12] attribute the following result to Duchamp. A proof
 781 of the corresponding result for jump systems may be found in [10] and it is easy
 782 to translate this proof to delta-matroids.

783 **Theorem 4.13.** *If D and D' are delta-matroids, then their sum is also a delta-*
 784 *matroid.*

785 **Proposition 4.14.** *If $D = (E, \mathcal{F})$ is a delta-matroid and n a non-negative*
 786 *integer, then $D_{\leq n}$ is a delta-matroid.*

787 *Proof.* Let $O_n = (E, \{\emptyset\}_{\leq n})$. Then it is clear that O_n is a delta-matroid and
 788 that $D_{\leq n}$ is the sum of D and O_n . The result follows from Theorem 4.13. \square

789 *Remark 4.15.* Theorem 4.13 can be used to generate interesting families of delta-
 790 matroids. The *uniform matroid*, denoted by $U_{r,m}$, is a matroid with m elements
 791 in the ground set and rank r , such that every subset of the ground set with r
 792 elements is a basis. An interesting family of delta-matroids may be constructed
 793 by taking the sum of a delta-matroid D with the uniform matroid of rank r
 794 defined on the ground set of D . This gives a delta-matroid in which a set F is
 795 feasible if and only if there is a feasible set F' of D with $|F \triangle F'| = r$.

796 The following is an easy observation concerning spreads.

797 **Proposition 4.16.** *If $D = (E, \mathcal{F})$ is a delta-matroid, n is a non-negative integer*
 798 *and A is a subset of E then $(D * A)_{\leq n} = D_{\leq n} * A$.*

799 *Proof.* A set is feasible in the n -spread of D if and only if it is feasible in $D * X$
 800 for some X with $|X| \leq n$. That is,

$$\mathcal{F}_{\leq n} = \bigcup_{\substack{X \subseteq E \\ |X| \leq n}} \mathcal{F}(D * X).$$

801 Thus

$$\mathcal{F}(D_{\leq n} * A) = \bigcup_{\substack{X \subseteq E \\ |X| \leq n}} \mathcal{F}((D * X) * A) = \bigcup_{\substack{X \subseteq E \\ |X| \leq n}} \mathcal{F}((D * A) * X) = \mathcal{F}((D * A)_{\leq n}).$$

802 \square

803 **Definition 4.17.** Let $D = (E, \mathcal{F})$ be a delta-matroid and n a non-negative
804 number. Then we define $\mathcal{F}_{\Delta n}$ as $\mathcal{F}_{\leq n} - \mathcal{F}_{\leq n-1}$. The n -toggle of D , which is
805 denoted by $D_{\Delta n}$, is defined to be $(E, \mathcal{F}_{\Delta n})$.

806 Note that $D_{\Delta 0} = D$.

807 **Proposition 4.18.** Let $D = (E, \mathcal{F})$ be an even delta-matroid with $E \neq \emptyset$. Then
808 $D_{\Delta 1}$ is a delta-matroid.

809 *Proof.* Take A and B in $\mathcal{F}_{\Delta 1}$ and x in $A \Delta B$. Then A and B are in $\mathcal{F}_{\leq 1}$. By
810 Proposition 4.14, there is an element y in $A \Delta B$ such that $A \Delta \{x, y\} \in \mathcal{F}_{\leq 1}$.
811 If $y \neq x$ then $A \Delta \{x, y\} \in \mathcal{F}_{\Delta 1}$, so we may assume that $y = x$. In this case we
812 must have $A \Delta x \in \mathcal{F}$. Now $|A \Delta B| \geq 2$, because D is even, so we may choose
813 $z \in (A \Delta B) - x$. Clearly $A \Delta \{x, z\} = (A \Delta x) \Delta z$ is in $\mathcal{F}_{\Delta 1}$. \square

814 The following theorem shows that $D_{\leq n}(G)$ and $D_n(G)$ can be obtained from
815 $D(G)$ by n -spreads and n -toggles.

816 **Theorem 4.19.** Let $G = (V, E)$ be a ribbon graph and n a non-negative number.
817 Then

- 818 1. $D_{\leq n}(G)$ is the n -spread of $D(G)$, that is $D_{\leq n}(G) = D(G)_{\leq n}$; and
- 819 2. $D_n(G) = D(G)_{\Delta n}$.

820 *Proof.* Item (2) follows directly from (1), since $\mathcal{F}_n(G) = \mathcal{F}_{\leq n}(G) - \mathcal{F}_{\leq n-1}(G)$,
821 and $\mathcal{F}(D(G)_{\Delta n}) = \mathcal{F}(D(G)_{\leq n}) - \mathcal{F}(D(G)_{\leq n-1})$. Thus it suffices to show that
822 (1) holds.

823 We will show that

824 **4.19.1.** $\mathcal{F}_{\leq n}(G)$ is contained in the feasible sets of the n -spread of $D(G)$.

825 We proceed using induction on n . Clearly the result is true when $n = 0$.
826 Take $F \in \mathcal{F}_{\leq n}(G)$. Suppose there is an edge $e \in F$ such that e is incident with
827 two boundary components of (V, F) . Then e is not a bridge, so $f(F \Delta e) - k(G) =$
828 $(f(F) - 1) - k(G)$. Hence $F \Delta e$ is in $\mathcal{F}_{\leq n-1}$. By induction, we know that $F \Delta e$
829 is a feasible set in the $(n-1)$ -spread of $D(G)$. Hence $F \Delta e = F' \Delta A$, where
830 $F' \in \mathcal{F}(G)$ and $|A| \leq n-1$. Then $F = (F' \Delta A) \Delta e = F' \Delta (A \Delta e)$, so F is
831 in the n -spread of $D(G)$. So we may assume that each connected component
832 of (V, F) has exactly one boundary component. If $k(F) \neq k(G)$ then there is
833 an edge e of G which is not in F , joining two connected components of (V, F) .
834 Thus $f(F \Delta e) - k(G) = (f(F) - 1) - k(G)$ and the result follows in a similar
835 way. If $k(F) = k(G)$ then F is a spanning quasi-tree in G . Thus F is in $\mathcal{F}(G)$,
836 which is itself contained in the n -spread of $D(G)$ and 4.19.1 holds.

837 We conclude this proof by showing that

838 **4.19.2.** the feasible sets in the n -spread of $D(G)$ are contained in $\mathcal{F}_{\leq n}(G)$.

839 Again we proceed using induction on n . Clearly the result is true when $n = 0$.
840 Take F in the n -spread of $D(G)$. Then there is a spanning quasi-tree F' of G and
841 a set A with $|A| \leq n$ such that $F = F' \Delta A$. If A is empty, then there is nothing

842 to prove, so let $a \in A$. Now $F \triangle a = F' \triangle (A - a)$ is in the $(n-1)$ -spread of $D(G)$
843 and, by induction, is contained in $\mathcal{F}_{\leq n-1}(G)$. Thus $f(F \triangle a) - k(G) \leq n - 1$.
844 But, by Proposition 3.2, the number of boundary components of F and $F \triangle a$
845 differ by at most one. Hence $f(F) - k(G) \leq f(F \triangle a) - k(G) + 1 \leq n$, so
846 $F \in \mathcal{F}_{\leq n}(G)$. Thus 4.19.2 holds. \square

847 Two natural questions arise from the preceding results. Is D_n a delta-
848 matroid for $n \geq 2$? Can the evenness condition be dropped from Proposi-
849 tion 4.18? Both questions have negative answers. We saw at the end of Sec-
850 tion 4.1 an example that showed that in general $D_2(G)$, which equals $D(G)_{\Delta 2}$, is
851 not a delta-matroid. Also the example given there showing that $D_1(G)$, which
852 equals $D(G)_{\Delta 1}$, may not be a delta-matroid shows that evenness cannot be
853 dropped. (We will shortly see (Proposition 5.3) that $D(G)$ is even if and only if
854 G is orientable.) The class of delta-matroids whose 1-toggle is a delta-matroid
855 may be a nice class. It would be interesting to have a characterisation of it.

856 5. Delta-matroids and ribbon graphs: geometric interplay

857 5.1. Duals, partial duals and twists

858 Recall from Section 2 that, if $D = (E, \mathcal{F})$ is a delta-matroid and $A \subseteq E$,
859 then the *twist* of D with respect to A , is the delta-matroid $D * A := (E, \{A \triangle X \mid$
860 $X \in \mathcal{F}\})$. In particular, the dual D^* of D is equal to $D * E$. Thus we may regard
861 a twist $D * A$ as being a ‘partial dual’ of a delta-matroid in the sense that the
862 dual is ‘formed with respect to only the elements in A ’. The following theorem
863 shows that this notion of partial duality corresponds exactly to partial duality
864 of ribbon graphs (see Section 3.2). That is, on the delta-matroid level, twisting
865 and partial duality are equivalent. Although this is a fairly simple result, it will
866 prove to be extremely useful and important in what follows.

867 **Theorem 5.1.** *Let $G = (V, E)$ be a ribbon graph, $A \subseteq E$ and $e \in E$. Then*
868 $D_{\leq k}(G^A) = D_{\leq k}(G) * A$ *and, in particular, $D(G^A) = D(G) * A$. Furthermore,*
869 *if G is orientable, then $D_1(G^A) = D_1(G) * A$.*

870 *Proof.* We will first prove the statement for $D(G)$. It is enough to prove it for
871 $A = \{e\}$. We need to show for each $Q \subseteq E$ that $(V(G), Q)$ is a spanning quasi-
872 tree of G if and only if $(V(G^e), Q \triangle e)$ is a spanning quasi-tree of G^e . But this
873 follows immediately upon observing that in Table 1, in all cases, G and $G^e \setminus e$,
874 as well as $G \setminus e$ and G^e have the same number of boundary components.

875 The general statement follows directly from the facts that $D_{\leq k}(G)$ is the
876 k -spread of $D(G)$ and the k -spread and twisting commute. These facts are
877 established by Theorem 4.19 and by Proposition 4.16, respectively. \square

878 For matroids $M(G^*) = M(G)^*$ when G is a plane graph. However, this
879 identity does not hold for non-plane graphs. The following corollary, which is
880 obtained by taking $A = E(G)$ in Theorem 5.1, explains why this is. It shows
881 that geometric duality is a delta-matroidal property, rather than a matroidal
882 property. The duality identity $M(G^*) = M(G)^*$ holds only for plane graphs
883 because it is only in this case that $M(G)$ and $D(G)$ coincide.

884 **Corollary 5.2.** *Let G be a ribbon graph. Then $D_{\leq k}(G^*) = D_{\leq k}(G)^*$ and, in*
 885 *particular, $D(G^*) = D(G)^*$. Furthermore, if G is orientable, then $D_1(G^*) =$*
 886 *$D_1(G)^*$.*

887 *5.2. Seeing ribbon graph structures in a delta-matroid*

888 Next we show that basic topological information about G can be recovered
 889 from its delta-matroid. Because of the connection with Bouchet's work that we
 890 have established, we could derive Item 4 in the following proposition from [6,
 891 Theorem 5.3], but we instead give a direct proof for completeness. We also give
 892 a short proof for Item 3, although it follows from [6, Theorem 4.1(iv)].

893 **Proposition 5.3.** *Let G be a ribbon graph and let $D = D(G)$.*

- 894 1. *The feasible sets of D with cardinality m are in 1-1 correspondence with*
 895 *the spanning quasi-trees of G with Euler genus $m - v(G) + k(G)$.*
- 896 2. *The rank of D_{\min} is equal to the size of a maximal spanning forest, that*
 897 *is, $r(D_{\min}) = v(G) - k(G)$.*
- 898 3. *The width of a ribbon-graphic delta-matroid is equal to the Euler genus of*
 899 *the underlying ribbon graph, that is, $\gamma(G) = w(D)$.*
- 900 4. *The delta-matroid D is even if and only if G is orientable.*

901 *Proof.* The one-to-one correspondence in (1) follows immediately from the defi-
 902 nition of D . Take $F \in \mathcal{F}(D)$ and let Q be the corresponding spanning quasi-tree.
 903 Let $m = |F|$. Then $e(Q) = m$. Furthermore, $v(Q) = v(G)$ and $f(Q) = k(G)$.
 904 Euler's formula gives $\gamma(Q) = m - v(G) + k(G)$. This completes the proof of (1).

905 Now (1) implies that m is minimized (respectively maximized) whenever
 906 $\gamma(Q)$ is minimized (respectively maximized). Thus, if $m = r(D_{\min})$, then by
 907 applying Lemma 3.5 we obtain $\gamma(Q) = 0$ and that Q is a maximal spanning
 908 forest of G . Moreover $v(G) - k(G) = r(D_{\min})$. Thus (2) holds.

909 On the other hand, if $m = r(D_{\max})$ then $\gamma(Q)$ is maximized, so by applying
 910 Lemma 3.5 again we deduce that $\gamma(Q) = \gamma(G)$. Thus (3) holds.

911 Finally, we show that (4) holds. Suppose that G is orientable. Then every
 912 ribbon subgraph is orientable and so $\gamma(Q)$ is even for each spanning quasi-tree
 913 Q of G . It follows from (1) that $|F| - r(D_{\min})$ is even for each feasible set F ,
 914 and so the size of each feasible set has the same parity and D is even.

915 If G is non-orientable then it contains a non-orientable cycle C . Let e be
 916 an edge of C . Then $C - e$ may be extended to a maximal spanning forest F not
 917 containing e . But $F \cup e$ is also a spanning quasi-tree of G . Thus D has feasible
 918 sets with cardinalities of both parities, so it is odd. \square

919 Recall that, if D is a delta-matroid, then D_{\min} and D_{\max} are matroids. The
 920 properties from Proposition 5.3 allow us to recognise $D(G)_{\min}$ and $D(G)_{\max}$
 921 in terms of cycle matroids associated with G . The following corollary can be
 922 recovered from [6], but we give an independent proof here for completeness.

923 **Corollary 5.4.** *Let G be a ribbon graph. Then*

- 924 1. $D(G)_{\min} = M(G)$;

- 925 2. $D(G)_{\max} = (M(G^*))^*$;
 926 3. $D(G) = M(G)$ if and only if G is a plane ribbon graph, otherwise $D(G)$
 927 is not a matroid.

928 *Proof.* By Proposition 5.3, the feasible sets of $D(G)_{\min}$ are exactly the edge sets
 929 of the genus-zero spanning quasi-trees of G . By Lemma 3.5, these are the edge
 930 sets of the maximal spanning forests of G . Thus they are exactly the bases of
 931 $M(G)$. Thus (1) holds.

932 Next we prove (2). Proposition 5.3 implies that F is a feasible set in $D(G)_{\max}$
 933 if and only if $(V(G), F)$ is a spanning quasi-tree of G of genus $\gamma(G)$, which by
 934 Lemma 3.5(3) occurs if and only if $(V(G^*), F^c)$ is a spanning tree of G^* . Then
 935 (1) implies that this holds exactly when F^c is a feasible set in $D(G^*)_{\min} =$
 936 $M(G^*)$. The result follows.

937 Finally, we consider (3). The ribbon graph G is plane if and only if $\gamma(G) =$
 938 0. By Proposition 5.3(3), this occurs exactly when $w(D(G)) = 0$. But if
 939 $w(D(G)) = 0$, then $D(G) = D(G)_{\min} = M(G)$. If $w(D(G)) > 0$, then $D(G)$ has
 940 feasible sets of different sizes and cannot be a matroid. \square

941 A consequence of Corollary 5.4 is that, for a ribbon graph G , the span-
 942 ning quasi-trees of minimal genus, and of maximal genus, both give rise to
 943 matroids. It is natural to ask if the edge sets of spanning quasi-trees of any
 944 fixed genus form the bases of a matroid. Although these sets are equicardinal,
 945 it is not hard to see that this is not the case in general. For example,
 946 while $(\{1, 2, 3, 4\}, \{\emptyset, \{1, 2\}, \{3, 4\}, \{1, 2, 3, 4\}\})$ is a delta-matroid, the set sys-
 947 tem $(\{1, 2, 3, 4\}, \{\{1, 2\}, \{3, 4\}\})$ is not a matroid.

948 We now consider when some other classes of delta-matroids that we have
 949 defined in terms of ribbon graphs are matroids.

950 **Proposition 5.5.** *Let $G = (V, E)$ be a connected ribbon graph.*

- 951 1. *If G is orientable, then $D_1(G)$ is a matroid if and only if one of the*
 952 *following occurs:*
 953 (a) *G is a tree, hence $D_1(G) \cong U_{|E|-1, |E|}$; or*
 954 (b) *G is a collection of trivial orientable loops on one vertex, hence*
 955 *$D_1(G) \cong U_{1, |E|}$; or*
 956 (c) *G is a pair of interlaced orientable loops on one vertex, so $D_1(G) \cong$*
 957 *$U_{1, 2}$.*
 958 2. *If $D_{\leq k}(G)$ is a matroid for some integer $k \geq 1$, then G comprises a single*
 959 *vertex and no edges.*

960 *Proof.* For (1), suppose there exists F in $\mathcal{F}(G)$ such that $F \neq \emptyset$ and $F \neq E$.
 961 Then, as G is orientable, Theorem 4.19(2) and Proposition 5.3(4) imply that
 962 $D_1(G)$ has feasible sets of size $|F| - 1$ and of size $|F| + 1$. Hence $D_1(G)$ is not a
 963 matroid in the case that $\mathcal{F}(G)$ is not contained in $\{\emptyset, E\}$. If $\mathcal{F}(G) = \{\emptyset\}$, then
 964 G is a collection of trivial orientable loops all connected to the same vertex,
 965 embedded in the sphere. In this case $D_1(G)$ is the uniform matroid of rank one,
 966 namely $U_{1, |E|}$. If $\mathcal{F}(G) = \{E\}$, then G is a tree and $D_1(G) = U_{|E|-1, |E|}$ is the
 967 uniform matroid of rank $|E| - 1$. Finally, suppose that $\mathcal{F}(G) = \{\emptyset, E\}$. For

968 $e \in E$, Axiom 2.1 implies that there is an element $f \in E$ such that $\emptyset \triangle \{e, f\}$
 969 is feasible. Hence $E = \{e, f\}$. As, G is orientable, we must have $f \neq e$, so G
 970 consists of a pair of interlaced orientable loops and $D_1(G) = \{\{e, f\}, \{\{e\}, \{f\}\}\}$
 971 is a matroid isomorphic to $U_{1,2}$. The reverse implication is easily checked. Hence
 972 (1) holds.

973 Now we show that (2) holds. Suppose that (V, F) is a spanning tree of G
 974 and $e \in E$. Then $(V, F \triangle e)$ has at most two boundary components, so both F
 975 and $F \triangle e$ are in $\mathcal{F}_{\leq k}(G)$. Therefore if $E \neq \emptyset$ then $D_{\leq k}(G)$ is not a matroid.
 976 As we assumed that G is connected, it must comprise a single vertex and no
 977 edges. \square

978 5.3. Loops, coloops, and ribbon loops

979 For a graph G , it is well-known that an element e is a loop or coloop in
 980 $M(G)$ if and only if e is loop or bridge, respectively, in G . One would expect
 981 such a relation to hold for ribbon graphs and their delta-matroids, and the
 982 following proposition shows that indeed it does. However, while coloops in
 983 $D(G)$ correspond directly to bridges in a ribbon graph G , one has to be a little
 984 more careful in the case of loops. The difficulty is that, unlike graphs, ribbon
 985 graphs have different types of loops, orientable or non-orientable, and trivial
 986 or non-trivial. Loops in $D(G)$ do not correspond to loops in G in general, but
 987 rather to trivial orientable loops in G .

988 **Lemma 5.6.** *Let G be a ribbon graph, $D(G) = (E, \mathcal{F})$, and $e \in E(G)$. Then*

- 989 1. *e is a coloop in $D(G)$ if and only if e is a bridge in G ; and*
- 990 2. *e is a loop in $D(G)$ if and only if e is a trivial orientable loop in G .*

991 *Proof.* For the first item, if e is a bridge of G , then any ribbon subgraph of G
 992 not containing e has more connected components than G and therefore has more
 993 than $k(G)$ boundary components and is not a spanning quasi-tree. Thus if e is
 994 a bridge it appears in every feasible set of $D(G)$ and so is a coloop. Conversely,
 995 if e is a coloop in $D(G)$ then it appears in every spanning quasi-tree of G . In
 996 particular, it appears in every spanning tree of G and is therefore a bridge.

997 For the second item, e is a trivial orientable loop in G if and only if e is a
 998 bridge in G^* . Corollary 5.2 and item 1 imply that this occurs if and only if e is
 999 a coloop in $D(G^*) = D(G)^*$. This holds if and only if e is a loop in $D(G)$. \square

1000 We have seen that loops in ribbon graphs can be classified into several types.
 1001 It turns out that that this classification may be usefully extended to elements
 1002 of delta-matroids in general.

1003 **Definition 5.7.** Let $D = (E, \mathcal{F})$ be a delta-matroid.

- 1004 1. An element e of E is a *ribbon loop* if e is a loop in D_{\min} .
- 1005 2. A ribbon loop e is *non-orientable* if e is a ribbon loop in $D * e$ and is
 1006 *orientable* otherwise.
- 1007 3. An orientable ribbon loop e is *trivial* if e is in no feasible set of D and is
 1008 *non-trivial* otherwise.

1009 4. A non-orientable ribbon loop e is *trivial* if $F \triangle e$ is in \mathcal{F} for every feasible
 1010 set $F \in \mathcal{F}$ and is *non-trivial* otherwise.

1011 If e is a loop in D then it is a ribbon loop of D , but the converse is not true
 1012 in general. In fact, e is a loop in D if and only if it is a trivial orientable ribbon
 1013 loop of D .

1014 We now show that the various types of loops in a ribbon graph G correspond
 1015 to the various types of ribbon loops in the delta-matroid $D(G)$.

1016 **Proposition 5.8.** *Let G be a ribbon graph, $D = D(G) = (E, \mathcal{F})$, and $e \in E(G)$.
 1017 Then*

- 1018 1. e is a loop in G if and only if e is a ribbon loop in $D(G)$;
- 1019 2. e is an orientable loop in G if and only if e is an orientable ribbon loop in
 1020 $D(G)$;
- 1021 3. e is a trivial loop in G if and only if e is a trivial ribbon loop in $D(G)$.

1022 *Proof.* We prove (1) first. An edge e is a loop of G if and only if e is an edge of
 1023 no spanning tree of G . This holds if and only if e appears in no feasible set of
 1024 D_{\min} .

1025 Next we consider (2). From Table 1 we see that a loop e of G is orientable
 1026 if and only if it is not a loop of G^e . By (1), e is not a loop of G^e if and only
 1027 if it is not a ribbon loop of $D(G^e)$. The result follows since $D(G^e) = D * e$, by
 1028 Theorem 5.1. Thus (2) holds.

1029 For (3), by Lemma 5.6, e is a trivial orientable loop of G if and only if e is
 1030 a loop of D if and only if e is a trivial orientable ribbon loop of D . It remains
 1031 to deal with trivial non-orientable loops.

1032 Suppose first that e is a trivial non-orientable loop of G . Take $F \in \mathcal{F}$. A
 1033 trivial ribbon loop is not interlaced with any cycle of G so $F \triangle e \in \mathcal{F}$. Hence e
 1034 is a trivial non-orientable ribbon loop in D .

1035 Suppose finally that e is a trivial non-orientable ribbon loop in D . It is
 1036 enough to show that e is trivial in G . Suppose that this is not the case. Take
 1037 C to be a cycle interlaced with e and take $f \in C$. We may extend $C - f$ to a
 1038 maximal spanning forest F' of G . As F' contains no cycle, we know that $e \notin F'$
 1039 and $f \notin F'$. Now exactly one of $F' \cup f$ and $(F' \cup f) \triangle e$ is a spanning quasi-tree
 1040 of G , depending on whether or not C is orientable, a contradiction. Thus e is a
 1041 non-trivial non-orientable loop of G and (3) holds. \square

1042 **Lemma 5.9.** *Let D be a delta-matroid and e an element of D . Then e is neither
 1043 a coloop nor a ribbon loop in D if and only if e is a non-trivial orientable ribbon
 1044 loop in $D * e$.*

1045 *Proof.* Suppose that e is neither a coloop nor a ribbon loop of D . Then e belongs
 1046 to some basis of D_{\min} , so no basis of $(D * e)_{\min}$ contains e . Thus e is a ribbon
 1047 loop of $D * e$. Moreover e is an orientable ribbon loop of $D * e$ because it is not
 1048 a ribbon loop of $(D * e) * e = D$ and it is non-trivial because it is not a coloop
 1049 of $(D * e) * e = D$.

1050 On the other hand, if e is a non-trivial orientable ribbon loop of $D * e$, then,
 1051 by Definition 5.7(2), e is not a ribbon loop of $(D * e) * e = D$. Furthermore, e
 1052 is not a loop of $D * e$ (as it is not a trivial orientable ribbon loop), so it is not
 1053 a coloop of D . \square

1054 Another illustration of how ribbon graphs can inform delta-matroids is as
 1055 follows. Suppose that G is a ribbon graph with a non-orientable loop e . If Q
 1056 is a maximal spanning forest of G then $Q \cup e$ is a spanning quasi-tree. The
 1057 following lemma shows that this property holds for delta-matroids in general.

1058 **Lemma 5.10.** *Let $D = (E, \mathcal{F})$ be a delta-matroid with $r(D_{\min}) = r$ and suppose*
 1059 *that e is a non-orientable ribbon loop of D . Then a subset F of $E - e$ is a basis*
 1060 *of D_{\min} if and only if $F \cup e$ is a feasible set of D with cardinality $r + 1$.*

1061 *Proof.* Let $F \subseteq E - e$ with $|F| = r$ and $F \cup e \in \mathcal{F}$. Suppose for contradiction that
 1062 $F \notin \mathcal{F}$. Let $A = E - (F \cup e)$. Since e is a ribbon loop of D , every minimum sized
 1063 feasible set of D contains an element of A . By applying Lemma 2.4 we see that
 1064 every feasible set F' of D must satisfy $|F' \cap A| \geq 1$. However, $|(F \cup e) \cap A| = 0$,
 1065 a contradiction. Thus $F \in \mathcal{F}$.

1066 By Definition 5.7(2), e is non-orientable ribbon loop of $D * e$, and so $r((D * e)_{\min}) = r$. Thus, by applying the previous argument to $D * e$, we see that if
 1067 $F \subseteq E - e$ with $|F| = r$ and $F \cup e \in \mathcal{F}(D * e)$ then $F \in \mathcal{F}(D * e)$. So if $F \subseteq E - e$
 1068 with $|F| = r$ and $F \in \mathcal{F}(D)$ then $F \cup e \in \mathcal{F}(D)$. \square

1070 5.4. Deletion, contraction, and minors

1071 Deletion and contraction for ribbon graphs and for delta-matroids are com-
 1072 patible operations.

1073 **Proposition 5.11.** *Let G be a ribbon graph, and $e \in E(G)$. Then*

- 1074 1. $D(G \setminus e) = D(G) \setminus e$;
- 1075 2. $D(G/e) = D(G)/e$.

1076 *Proof.* If e is a bridge of G then it belongs to every spanning quasi-tree of G .
 1077 Moreover, a subset F of $E - e$ is a spanning quasi-tree of $G \setminus e$ if and only if
 1078 $F \cup e$ is a spanning quasi-tree of G . By Lemma 5.6, e is a coloop of $D(G)$ and
 1079 so the first part follows in this case.

1080 On the other hand if e is not a bridge of G , then G and $G \setminus e$ have the same
 1081 number of connected components. Thus the spanning quasi-trees of $G \setminus e$ are
 1082 precisely the spanning quasi-trees of G that do not contain e . By Lemma 5.6
 1083 again, e is not a coloop of $D(G)$ and so the first part also follows in this case.

1084 Using Proposition 3.4(3), we have $D(G/e) = D(G^e \setminus e)$, which by Theo-
 1085 rem 5.1 and the first part of this proposition is the same as $(D(G) * e) \setminus e$. Using
 1086 Lemma 2.7, $(D(G) * e) \setminus e = D(G)/e$. \square

1087 *Remark 5.12.* $D_{\leq k}(G \setminus e) \neq (D_{\leq k}(G)) \setminus e$ and $D_{\leq k}(G/e) \neq (D_{\leq k}(G))/e$, in
 1088 general. To construct an example illustrating the former, take a path with $k + 1$
 1089 edges, attach a non-orientable loop to one of the vertices and let e be one of the

1090 edges in the path. An example illustrating the latter can then be constructed
 1091 by taking the dual. The examples with $k = 1$ also illustrate that in general
 1092 $D_1(G \setminus e) \neq (D_1(G)) \setminus e$ and $D_1(G/e) \neq (D_1(G))/e$.

1093 The next corollary follows immediately from Proposition 5.11.

1094 **Corollary 5.13.** *Let G and H be ribbon graphs. If H is a minor of G then*
 1095 *$D(H)$ is a minor of $D(G)$.*

1096 The reverse inclusion is not true, because non-isomorphic ribbon graphs may
 1097 have isomorphic ribbon-graphic delta-matroids.

1098 We will refer to a “ D -minor” to mean a “minor isomorphic to D ”. A class \mathcal{C}
 1099 of delta-matroids or ribbon graphs is said to be *minor-closed* if, for each $X \in \mathcal{C}$,
 1100 every minor of X is also in \mathcal{C} . An *excluded minor* for a minor-closed class \mathcal{C} of
 1101 delta-matroids or ribbon graphs is a delta-matroid or ribbon graph, respectively,
 1102 that is not in \mathcal{C} but has each of its proper minors in \mathcal{C} .

1103 As a first illustration of the fact that ribbon graph intuition can lead to
 1104 results about delta-matroids, we consider even delta-matroids. Recall that an
 1105 even delta-matroid is one whose feasible sets all have the same parity. Being
 1106 even is preserved under taking minors, hence it may be characterised by a set of
 1107 excluded minors. Our aim is to find the set of excluded minors for even delta-
 1108 matroids. Consider the corresponding problem for ribbon graphs. A ribbon
 1109 graph is non-orientable if and only if it contains a non-orientable cycle. Edges
 1110 in a cycle can be contracted to give a loop, and it follows that a ribbon graph
 1111 is orientable if and only if it has no G_0 -minor, where G_0 is the ribbon graph
 1112 consisting of a single non-orientable loop. Recalling from Proposition 5.3(4),
 1113 that a ribbon graph G is orientable if and only if $D(G)$ is even, we deduce that
 1114 $D(G)$ is even if and only if it contains no $D(G_0)$ -minor. This leads us to posit
 1115 that a delta-matroid D is even if and only if it contains no X_0 -minor, where
 1116 $X_0 = D(G_0) = (\{a\}, \{\emptyset, \{a\}\})$. This turns out to be a slight reformulation of a
 1117 result of Bouchet.

1118 **Theorem 5.14** (Bouchet [6]). *Let $X_0 = (\{a\}, \{\emptyset, \{a\}\})$. A delta-matroid $D =$
 1119 (E, \mathcal{F}) is even if and only if it has no X_0 -minor.*

1120 *Proof.* If D is even, then it clearly does not have X_0 as a minor, as any minor
 1121 of D is even. By Bouchet’s result, [6, Lemma 5.4], a delta-matroid is odd if and
 1122 only if it has a feasible set F and an element $e \notin F$ such that $F \cup e$ is feasible. In
 1123 this case $D/F \setminus (E - (F \cup e))$ is isomorphic to X_0 , hence the result follows. \square

1124 *Remark 5.15.* As a further illustration of the interactions between ribbon graphs
 1125 and delta-matroids, it is interesting to note that Bouchet’s characterisation of
 1126 odd delta-matroids given in the proof of Theorem 5.14 is the direct analogue of
 1127 the ribbon graph result that G is non-orientable if and only if it has a spanning
 1128 quasi-tree Q and an edge e not in Q such that $Q \cup e$ is a spanning quasi-tree.

1129 An excellent illustration of the compatibility between delta-matroid and rib-
 1130 bon graph theory is found by considering twists of matroids. As the class of

1131 matroids is not closed under twists but every matroid is a delta-matroid, twist-
 1132 ing provides a way to construct delta-matroids from matroids. Delta-matroids
 1133 arising from twists of matroids are of interest since they are an intermediate step
 1134 between delta-matroid theory in general and the much better developed field of
 1135 matroid theory. Suppose we are faced with the problem of characterising the
 1136 class of delta-matroids that arise as twists of matroids. How can we use the
 1137 insights of ribbon graphs to tackle this problem?

1138 Suppose that $G = (V, E)$ is a ribbon graph with ribbon-graphic delta-
 1139 matroid $D = D(G)$. We wish to understand when D is the twist of a matroid,
 1140 that is, we want to determine if $D = M * A$ for some matroid M and for some
 1141 $A \subseteq E$. As twists are involutory, we can reformulate this problem as one of
 1142 determining if $D * B = M$ for some matroid M and some $B \subseteq E$. By Theo-
 1143 rem 5.1, $D * B = D(G) * B = D(G^B)$, but, by Corollary 5.4(3), $D(G^B)$ is a
 1144 matroid if and only if G^B is a plane graph. Thus D is a twist of a matroid
 1145 if and only if G is the partial dual of a plane graph. Given our principle that
 1146 embedded graphs inform us about delta-matroids, to characterize the class of
 1147 delta-matroids that are twists of matroids, we should look for characterizations
 1148 of the class of ribbon graphs that arise as partial duals of plane graphs. For-
 1149 tunately, due to connections with knot theory (see [51]), this class of ribbon
 1150 graphs has been characterised. Let G_0 be the ribbon graph consisting of a sin-
 1151 gle non-orientable loop; G_1 be the orientable ribbon graph given by vertex set
 1152 $\{1, 2\}$, edge set $\{a, b, c\}$ with the incident edges at each vertex having the cyclic
 1153 order abc , with respect to some orientation of G_1 ; and let G_2 be the orientable
 1154 ribbon graph given by vertex set $\{1\}$, edge set $\{a, b, c\}$ with the cyclic order
 1155 $abcabc$ at the vertex. Then the following holds.

1156 **Theorem 5.16** (Moffatt [53]). *G is a partial dual of a plane graph if and only*
 1157 *if it has no minors equivalent to G_0 , G_1 , or G_2 .*

1158 The discussion above and our principle that ribbon graphs inform us about
 1159 delta-matroids lead us to the conjecture that a delta-matroid D is the twist of
 1160 a matroid if and only if it does not have a minor isomorphic to $D(G_0)$, $D(G_1)$,
 1161 or $D(G_2)$. Indeed this result is true and is readily derived from work of A.
 1162 Duchamp (see [25] for details of the derivation).

1163 **Theorem 5.17** (Duchamp [28]). *A delta-matroid D is the twist of a matroid if*
 1164 *and only if it does not have a minor isomorphic to $D(G_0)$, $D(G_1)$, or $D(G_2)$.*

1165 In this example ribbon graph theory led to a result obtainable from the
 1166 literature, but below we will see examples where ribbon graph theory leads to
 1167 genuinely new structural delta-matroid theory.

1168 5.5. Separability and connectivity for delta-matroids

1169 If v is a separating vertex of a graph G , with P and Q being the subgraphs
 1170 that intersect in v , then knowledge of P , Q and v gives complete knowledge of
 1171 G . However, if G is a ribbon graph this is no longer the case. For example,
 1172 suppose that P and Q are orientable loops. Then G has genus zero or one,

1173 depending on whether or not P and Q are interlaced. Thus separability is
 1174 a much more subtle concept for ribbon graphs than for graphs. Given our
 1175 principle that graphs are matroidal, while ribbon graphs are delta-matroidal,
 1176 we should expect ‘connectivity’ for delta-matroids to be more subtle than for
 1177 matroids. In this section, we define notions of connectivity and separability of
 1178 delta-matroids that reflect the corresponding concepts for ribbon graphs defined
 1179 in Section 3.1.6.

1180 For matroids $M_1 = (E_1, \mathcal{B}_1)$ and $M_2 = (E_2, \mathcal{B}_2)$, where E_1 is disjoint from
 1181 E_2 , the *direct sum* of M_1 and M_2 , written $M_1 \oplus M_2$, is constructed as follows.

$$M_1 \oplus M_2 := (E_1 \cup E_2, \{B_1 \cup B_2 \mid B_1 \in \mathcal{B}_1 \text{ and } B_2 \in \mathcal{B}_2\}).$$

1182 If $M = M_1 \oplus M_2$, for non-trivial M_1 and M_2 , then we say that M is *disconnected*
 1183 and that E_1 and E_2 are each *separating*. We say that M is *connected* if it
 1184 is not disconnected. The connectivity of cycle matroids is closely linked to
 1185 the connectivity of the underlying graph. A graph is *2-connected* if it has a
 1186 single connected component and no separating vertex. The following is well-
 1187 known [56].

1188 **Proposition 5.18.** *Let G be a graph. Then $M(G)$ is connected if and only*
 1189 *if G is 2-connected. Moreover if $M(G) = M_1 \oplus M_2$, for non-trivial M_1 and*
 1190 *M_2 , then $M_1 = M(G_1)$ and $M_2 = M(G_2)$ for some graphs $G_1 = (V_1, E_1)$ and*
 1191 *$G_2 = (V_2, E_2)$ such that $G = (V_1 \cup V_2, E_1 \cup E_2)$, and E_1 and E_2 are disjoint,*
 1192 *and V_1 and V_2 are either disjoint or intersect in a single vertex.*

1193 Motivated by separability for ribbon graphs, we generalize this concept to
 1194 delta-matroids in two slightly different ways. The second definition is from [37].

1195 **Definition 5.19.** Let $D = (E, \mathcal{F})$ be a delta-matroid. Then D is *separable* if
 1196 D_{\min} is disconnected.

1197 **Definition 5.20.** For delta-matroids $D = (E, \mathcal{F})$ and $\tilde{D} = (\tilde{E}, \tilde{\mathcal{F}})$ with $E \cap \tilde{E} =$
 1198 \emptyset , the *direct sum* of D and \tilde{D} is written $D \oplus \tilde{D}$ and is the delta-matroid defined
 1199 as

$$D \oplus \tilde{D} := (E \cup \tilde{E}, \{F \cup \tilde{F} \mid F \in \mathcal{F} \text{ and } \tilde{F} \in \tilde{\mathcal{F}}\}).$$

1200 If a delta-matroid can be written as $D \oplus \tilde{D}$ for some non-trivial delta-matroids
 1201 D and \tilde{D} , then we say it is *disconnected*. A delta-matroid is *connected* if it is
 1202 not disconnected.

1203 We defined separability and connectivity for delta-matroids so that they are
 1204 compatible with the corresponding concepts for ribbon graphs, as in the follow-
 1205 ing propositions, the first of which follows immediately from Proposition 5.18.

1206 **Proposition 5.21.** *Let G be a ribbon graph. Then $D(G)$ is separable if and*
 1207 *only if there exist non-trivial ribbon graphs G_1 and G_2 such that $G = G_1 \sqcup G_2$*
 1208 *or $G = G_1 \oplus G_2$.*

1209 *Moreover if $D(G)_{\min} = M_1 \oplus M_2$, for some non-trivial M_1 and M_2 , then*
 1210 *there exist non-trivial ribbon graphs G_1 and G_2 such that $D(G)_{\min} = M(G_1) \oplus$*
 1211 *$M(G_2)$ and $G = G_1 \sqcup G_2$ or $G = G_1 \oplus G_2$.*

1212 **Proposition 5.22.** *Let G be a ribbon graph. Then $D(G)$ is disconnected if*
 1213 *and only if there exist non-trivial ribbon graphs G_1 and G_2 such that either*
 1214 *$G = G_1 \sqcup G_2$ or $G = G_1 \vee G_2$.*

1215 *Proof.* If $G = G_1 \sqcup G_2$ or $G = G_1 \vee G_2$ then it is easy to see that $D(G)$ is
 1216 disconnected.

1217 Suppose now that $D(G)$ is disconnected. Then $D(G)$ is separable. The
 1218 previous proposition implies that this is only possible if $G = G_1 \sqcup G_2$ or $G =$
 1219 $G_1 \oplus G_2$ for some non-trivial G_1 and G_2 . Moreover if $D(G) = D \oplus D'$ then
 1220 $D_{\min} = D(G_1)_{\min}$ and $D'_{\min} = D(G_2)_{\min}$ for non-trivial ribbon graphs G_1 and
 1221 G_2 such that $G = G_1 \sqcup G_2$ or $G = G_1 \oplus G_2$.

1222 It remains to show that if $G = G_1 \oplus G_2$, but $G \neq G_1 \vee G_2$ then $D(G) \neq$
 1223 $D(G_1) \oplus D(G_2)$. If $G = G_1 \oplus G_2$, but $G \neq G_1 \vee G_2$ then there are two interlaced
 1224 cycles C_1 and C_2 of G_1 and G_2 , respectively, intersecting in G at a vertex v . Let
 1225 $e_1 \in E(C_1)$ and let F_1 be a maximal forest of G_1 with $C_1 - \{e_1\} \subseteq F_1$. Define
 1226 F_2 similarly. Now $F_i \cup \{e_i\} \in \mathcal{F}(D(G_i))$ if and only if C_i is non-orientable.
 1227 However $(F_1 \cup \{e_1\}) \cup (F_2 \cup \{e_2\}) \in \mathcal{F}(D(G))$ except when both C_1 and C_2 are
 1228 non-orientable. Consequently $D(G) \neq D(G_1) \oplus D(G_2)$. \square

1229 We emphasize the unfortunate clash between ribbon graph and delta-matroid
 1230 notation that while $D(G_1 \vee G_2) = D(G_1) \oplus D(G_2)$, in general, $D(G_1 \oplus G_2) \neq$
 1231 $D(G_1) \oplus D(G_2)$.

1232 For another illustration of how ribbon graphs inform delta-matroids we re-
 1233 turn to the problem of characterising twists of matroids from the end of Sec-
 1234 tion 5.4. In that section we saw how ribbon graph theory led to an excluded
 1235 minor characterisation of twists of matroids. We will now see how they lead to
 1236 a rough structure theorem for twists of matroids.

1237 As before the ribbon graph analogue of a twist of a matroid is a partial
 1238 dual of a plane graph. Motivated by knot theory, in [51] (see also [52]), Moffatt
 1239 gave a rough structure theorem for the class of partial duals of plane graphs.
 1240 This rough structure theorem ensures that every such ribbon graph admits a
 1241 particular decomposition into plane ribbon graphs.

1242 **Theorem 5.23** (Moffatt [51]). *Let G be a ribbon graph and $A \subseteq E(G)$. Then*
 1243 *the partial dual G^A is a plane graph if and only if all of the connected components*
 1244 *of $G|_A$ and $G|_{A^c}$ are plane and every vertex of G that is in both $G|_A$ and $G|_{A^c}$*
 1245 *is a separating vertex of G .*

1246 We now translate this into delta-matroids. If $D = D(G)$ then “ G^A is a plane
 1247 graph” becomes “ $D * A$ is a matroid”, and “ $G|_A$ and $G|_{A^c}$ are plane” becomes
 1248 “ $D \setminus A^c$ and $D \setminus A$ are both matroids”. By Proposition 5.21, $D(G)$ is separable
 1249 if and only if there exist ribbon graphs G_1 and G_2 such that $G = G_1 \sqcup G_2$ or
 1250 $G = G_1 \oplus G_2$. Thus the condition that every vertex of G that is incident with
 1251 edges in A and edges in $E(G) - A$ is a separating vertex of G becomes A is
 1252 separating in D_{\min} . Thus we have deduced the following theorem for ribbon-
 1253 graphic delta-matroids. Our principle that ribbon graphs inform us about delta-
 1254 matroids led us to conjecture that it holds for delta-matroids in general, and we
 1255 showed that this is indeed the case.

1256 **Theorem 5.24** (Chun et al [25]). *Let D be a delta-matroid and A be a non-*
 1257 *empty proper subset of $E(D)$. Then $D * A$ is a matroid if and only if the following*
 1258 *two conditions hold:*

- 1259 1. *A is separating in D_{\min} , and*
- 1260 2. *$D \setminus A$ and $D \setminus A^c$ are both matroids.*

1261 We emphasise that the ribbon graph theory genuinely led us to the formu-
 1262 lation of Theorem 5.24. We probably would not have found the result without
 1263 the insights and guidance of ribbon graphs.

1264 We have just given an example of how ribbon graphs inform delta-matroids.
 1265 We now give an example of delta-matroid theory giving a result about ribbon
 1266 graphs.

1267 The following inductive tools have been fundamental in the development of
 1268 matroid theory.

1269 **Theorem 5.25** (Tutte [63]). *Let M be a connected matroid. If $e \in E(M)$, then*
 1270 *$M \setminus e$ or M/e is connected.*

1271 **Theorem 5.26** (Brylawski [17], Seymour [58]). *Let M be a connected matroid*
 1272 *with a connected minor N . If $e \in E(M) - E(N)$, then $M \setminus e$ or M/e is connected*
 1273 *with N as a minor.*

1274 Bouchet generalized Theorem 5.25 to the context of delta-matroids in [9].
 1275 The actual result that he proved is for an even more general object, called a
 1276 multimatroid, but we state a special case of his result here in terms of delta-
 1277 matroids.

1278 **Theorem 5.27** (Bouchet [9]). *Let D be a connected even delta-matroid. If*
 1279 *$e \in E(D)$, then $D \setminus e$ or D/e is connected.*

1280 By exploiting results of Brijder and Hooeboom [15], Chun, Chun, and Noble
 1281 in [24] derived another consequence of Bouchet’s result, extending Theorem 5.27
 1282 to the class of vf-safe delta-matroids. These were introduced by Brijder and
 1283 Hooeboom in [14], and include ribbon-graphic delta-matroids and binary delta-
 1284 matroids, which are discussed in Section 5.7. Given a vf-safe delta-matroid D
 1285 and a subset A of its ground set, the delta-matroid $D + A$ has ground set $E(D)$
 1286 and F is feasible if and only if D has an odd number of feasible sets F' satisfying
 1287 $F - A \subseteq F' \subseteq F$. For more details on the delta-matroid $D + A$, including how
 1288 the $+$ operation interacts with other delta-matroid operations such as twisting,
 1289 see [13, 14].

1290 **Theorem 5.28** (Chun et al. [24]). *Let D be a connected vf-safe delta-matroid.*
 1291 *If $e \in E(D)$, then at least two of $D \setminus e$, D/e and $(D + e)/e$ are connected.*

1292 For a ribbon graph G and subset A of its edges, informally we define $G + A$
 1293 to be the ribbon graph formed from G by adding a “half-twist” to the edges
 1294 in A (see [31] for a formal definition of the partial Petrial and Petrie dual). It
 1295 is shown in [25] that $D(G + A) = D(G) + A$. A ribbon graph is said to be

1296 2-connected if $G \neq P \sqcup Q$ and $G \neq P \vee Q$, for any non-trivial ribbon graphs P
 1297 and Q . The next theorem, also proved by Chun, Chun, and Noble [24] follows
 1298 immediately from the two preceding theorems and Proposition 5.22.

1299 **Theorem 5.29** (Chun et al. [24]). *Let G be a 2-connected ribbon graph. Then*
 1300 *at least two of $G \setminus e$, G/e and $(G + e)/e$ are 2-connected.*

1301 In [24], Chun, Chun, and Noble generalized Theorem 5.26 to multimatroids.
 1302 We state two special cases of the result here in terms of delta-matroids.

1303 **Theorem 5.30** (Chun et al. [24]). *Let D be a connected even delta-matroid with*
 1304 *a connected minor D' . If $e \in E(D) - E(D')$, then $D \setminus e$ or D/e is connected*
 1305 *with D' as a minor.*

1306 To state the second special case, we need to concept of a 3-minor in a vf-safe
 1307 delta-matroid. We say that D' is a 3-minor of a vf-safe delta-matroid D , if
 1308 $D' = ((D \setminus X/Y) + Z)/Z$ for disjoint subsets X , Y and Z of $E(D)$. It is not
 1309 difficult to establish that the three operations used in forming a 3-minor have
 1310 the desirable property that they may be applied element by element in any order
 1311 without changing the result.

1312 **Theorem 5.31** (Chun et al. [24]). *Let D be a connected vf-safe delta-matroid*
 1313 *with a connected 3-minor D' . If $e \in E(D) - E(D')$, then $D \setminus e$, D/e or $(D + e)/e$*
 1314 *is connected with D' as a 3-minor.*

1315 The next two results follow immediately from the previous two.

1316 **Theorem 5.32** (Chun et al. [24]). *Let G be a 2-connected, orientable ribbon*
 1317 *graph. If H is a 2-connected minor of G and $e \in E(G) - E(H)$, then $G \setminus e$ or*
 1318 *G/e is 2-connected with H as a minor.*

1319 A 3-minor in a ribbon graph in an analogous way to which it is defined in a
 1320 vf-safe delta-matroid.

1321 **Theorem 5.33** (Chun et al. [24]). *Let G be a 2-connected ribbon graph. If H is*
 1322 *a 2-connected 3-minor of G and $e \in E(G) - E(H)$, then $G \setminus e$, G/e or $(G + e)/e$*
 1323 *is 2-connected with H as a 3-minor.*

1324 As we mentioned above, this result is a nice example of delta-matroids pro-
 1325 viding insight into ribbon-graphs. It is extremely unlikely that we would have
 1326 established Theorem 5.33 without the intuition provided by delta-matroids.

1327 5.6. Rank functions

1328 In this section we examine delta-matroid rank and its connections to ribbon
 1329 graph structures. Let $G = (V, E)$ be a graph, $M = M(G)$ be its cycle matroid,
 1330 and $A \subseteq E$. It is well-known that the rank function of M can be expressed in
 1331 terms of graph parameters: $r_M(A) = v(G) - k_G(A)$. In this section we express
 1332 the rank function of a ribbon-graphic delta-matroid in terms of ribbon graph
 1333 parameters.

1334 For our next proof, we need a new piece of terminology. Let H and K be
1335 distinct spanning ribbon subgraphs of G . Then we say that K is obtained from
1336 H by an *edge-toggle* if $E(H) = E(K) \triangle e$ for some edge $e \in E(G)$. Recall
1337 that, for ribbon graph G and $A \subseteq E(G)$, functions such as $\rho_{D(G)}(A)$, $e(A)$, and
1338 $f(A)$ refer to $\rho_{D(G)}((V(G), A))$, $e((V(G), A))$, and $f((V(G), A))$, respectively,
1339 as defined in Section 3.1.2.

1340 **Theorem 5.34.** *Let $G = (V, E)$ be a ribbon graph and $A \subseteq E$. Then*

$$\rho_{D(G)}(A) = e(G) - f(A) + k(G).$$

1341 *Proof.* To prove the theorem it is enough to show that for a ribbon graph $G =$
1342 (V, E) with $D(G) = (E, \mathcal{F})$ we have $\min\{|A \triangle F| \mid F \in \mathcal{F}\} = f(A) - k(G)$. To
1343 do this, set

$$q(A) := \min\{|X| + |Y| \mid X, Y \subseteq E, \text{ and } (V, (A - X) \cup Y) \text{ is a spanning quasi-tree}\}.$$

1344 Then $q(A)$ is the smallest number of edge-toggles needed to transform (V, A)
1345 into a spanning quasi-tree. Clearly $q(A) = \min\{|A \triangle F| \mid F \in \mathcal{F}\}$, and so we
1346 need to show that $q(A) = f(A) - k(G)$.

1347 First observe that $q(A) \geq f(A) - k(G)$ since an edge-toggle can decrease the
1348 number of boundary components by at most one.

1349 To show that $q(A) \leq f(A) - k(G)$ we argue by induction on $f(A)$. If $f(A) =$
1350 $k(G)$, then (V, A) is a spanning quasi-tree and $q(A) = 0 = f(A) - k(G)$. For
1351 the inductive hypothesis, suppose that $q(A) \leq f(A) - k(G)$ for all A with
1352 $f(A) < r$. Now suppose that $f(A) = r > k(G)$. There are two cases to
1353 consider: $k(A) > k(G)$ and $k(A) = k(G)$.

1354 If $k(A) > k(G)$, then G has an edge $e \notin A$ such that $k(A \cup e) = k(A) - 1$.
1355 Then we must also have $f(A \cup e) = f(A) - 1$. The inductive hypothesis then
1356 gives $q(A \cup e) \leq f(A \cup e) - k(G) = f(A) - k(G) - 1$. So a sequence of at most
1357 $f(A) - k(G) - 1$ edge-toggles transforms $(V, A \cup e)$ to a spanning quasi-tree.
1358 Placing ‘add e ’ at the start of this sequence of edge-toggles gives a sequence of
1359 at most $f(A) - k(G)$ edge-toggles that transforms (V, A) to a spanning quasi-
1360 tree. Thus $q(A) \leq f(A) - k(G)$.

1361 If $k(A) = k(G)$, then, since (V, A) has more than $k(G)$ boundary compo-
1362 nents, $A \neq \emptyset$. Each edge of (V, A) intersects either one or two boundary compo-
1363 nents of (V, A) . There must be some edge $e \in A$ that intersects two boundary
1364 components since $f(A) > k(G)$ and $k(A) = k(G)$. Then $f(A - e) = f(A) - 1$.
1365 The inductive hypothesis then gives $q(A - e) \leq f(A - e) - k(G) = f(A) - k(G) - 1$.
1366 So, proceeding as in the case where $k(A) > k(G)$, a sequence of at most
1367 $f(A) - k(G) - 1$ edge-toggles transforms $(V, A - e)$ to a spanning quasi-tree.
1368 Placing ‘subtract e ’ at the start of this sequence of edge-toggles gives a sequence
1369 of at most $f(A) - k(G)$ edge-toggles that transforms (V, A) to a spanning quasi-
1370 tree. Thus $q(A) \leq f(A) - k(G)$. This completes the proof of the theorem. \square

1371 *Remark 5.35.* This theorem can be seen as a corollary of the extended Cohn-
1372 Lempel equality from [60]. One associates to the ribbon graph its medial graph,

1373 which is 4-regular. Next, one translates the ribbon graph parameters $e(G), f(A)$
1374 and $k(G)$ into parameters depending on the medial graph. One can also con-
1375 struct the delta-matroid of the ribbon graph from the medial graph, see [6] and
1376 the proof of Theorem 4.8. Once the definition of the rank of the delta-matroid is
1377 translated to the medial graph, Theorem 5.34 follows from the extended Cohn-
1378 Lempel equality.

1379 The theorem above immediately provides us with the following interpretation
1380 of ρ for ribbon-graphic delta-matroids.

1381 **Corollary 5.36.** *Let $G = (V, E)$ be a ribbon graph, $A \subseteq E(G)$, and $D = D(G)$.
1382 Then $|E| - \rho_D(A)$ is equal to the minimum number of edge-toggles required to
1383 transform (V, A) into a spanning quasi-tree of G .*

1384 The ribbon graph interpretation of $\rho_{D(G)}$ can be used to discover results
1385 about ρ_D for a general delta-matroid D . For example, recall from the proof of
1386 Lemma 3.5 that the boundary components of $G \setminus A^c$ and $G^* \setminus A$ coincide and
1387 so $f_G(A) = f_{G^*}(A^c)$. Thus, for ribbon-graphic delta-matroids, it follows that
1388 $\rho_{D^*}(A) = \rho_D(E - A)$. This identity holds for delta-matroids in general, as we
1389 saw earlier in Lemma 2.9.

1390 For reference later, we record the following basic facts about rank functions.

1391 **Corollary 5.37.** *Let $G = (V, E)$ be a ribbon graph. Then*

- 1392 1. $r_{M(G)}(A) = r_{D(G)_{\min}}(A)$;
- 1393 2. $r_{M(G^*)}(A) = r_{(D(G)_{\max})^*}(A) = r_{(D(G)_{\max})}(A^c) + |A| - r_{(D(G)_{\max})}(E)$;
- 1394 3. $\rho_{D(G)}(A) = \rho_{D(G^*)}(E - A)$.

1395 *Proof.* The first part follows immediately from the fact that $M(G) = D(G)_{\min}$.
1396 For the second part, first note that Corollary 5.4(2) implies that $M(G^*) =$
1397 $(D(G)_{\max})^*$. Thus $r_{M(G^*)}(A) = r_{(D(G)_{\max})^*}(A)$. Equation (2.1) implies that
1398 this is equal to $r_{(D(G)_{\max})}(A^c) + |A| - r_{(D(G)_{\max})}(E)$. Thus (2) holds. As
1399 $(D(G))^* = D(G^*)$ by Corollary 5.2, the third part follows from Lemma 2.9. \square

To motivate some delta-matroid results, consider a ribbon graph G and a
set $A \subseteq E(G)$. Then $r(A) = v(G) - k(A)$ and $\rho(A) = e(G) - f(A) + k(G)$.
Euler's formula and Proposition 5.3(3) give

$$\begin{aligned} \rho(A) - r(A) - n(G) + n(A) &= (e(G) - f(A) + k(G)) - (v(G) - k(A)) \\ &\quad - (e(G) - v(G) + k(G)) + n(A) \\ &= k(A) - f(A) + n(A) = \gamma(A) \\ &= w(D(G \setminus A^c)) = w(D(G) \setminus A^c) = w(D(G)|A). \end{aligned}$$

1400 This identity holds more generally for delta-matroids, which we will show
1401 after we state the following lemma, the simple proof of which we omit.

1402 **Lemma 5.38.** *Let $D = (E, \mathcal{F})$ be a delta-matroid. Then $r(D_{\max}) = \rho_D(E)$
1403 and $r(D_{\min}) = |E| - \rho_D(\emptyset)$.*

1404 **Proposition 5.39.** *Let $D = (E, \mathcal{F})$ be a delta-matroid and let $A \subseteq E$. Then*

- 1405 1. $r((D|A)_{\min}) = r_{D_{\min}}(A)$;
- 1406 2. $r((D|A)_{\max}) = \rho_D(A) - n_{D_{\min}}(E) + n_{D_{\min}}(A)$;
- 1407 3. $w(D|A) = \rho_D(A) - r_{D_{\min}}(A) - n_{D_{\min}}(E) + n_{D_{\min}}(A)$.

1408 *Proof.* Let F_0 be a feasible set of D having smallest possible intersection with
 1409 A^c . By Lemma 2.4, we may assume that $F_0 \in \mathcal{F}(D_{\min})$. Let $Y = F_0 \cap A^c$ and
 1410 $Z = A^c - Y$. If the elements of Z are deleted one by one from D , then no coloop
 1411 is deleted because there is a feasible set F_0 missing Z . However every element
 1412 of Y is a coloop of $D \setminus Z$. Thus $D|A = D \setminus Z/Y$. We have

$$\mathcal{F}(D|A) = \{F - Y \mid F \in \mathcal{F}(D), F \cap A^c = Y\}.$$

1413 Therefore

$$r((D|A)_{\min}) = |F_0| - |Y| = \max_{F \in \mathcal{F}(D_{\min})} \{|F \cap A|\} = r_{D_{\min}}(A), \quad (5.1)$$

1414 establishing the first part.

1415 Applying Lemma 2.11 $|A^c|$ times to delete first the elements of Z and then
 1416 those of Y implies that

$$\rho_{D|A}(A) = \rho_D(A) - |E| + |A| + |Y|. \quad (5.2)$$

1417 By applying Lemma 5.38 to $D|A$ and Equation (5.2), we obtain

$$r((D|A)_{\max}) = \rho_{D|A}(A) = \rho_D(A) + |A| + |Y| - |E|. \quad (5.3)$$

1418 Now $n_{D_{\min}}(E) = |E| - |F_0|$ and, by Equation (5.1), $n_{D_{\min}}(A) = |A| - (|F_0| - |Y|)$.
 1419 Substituting into Equation (5.3) yields the second part.

1420 The final part follows immediately by subtracting the equation in the first
 1421 part from that in the second part. \square

1422 5.7. Representability

1423 Let \mathbb{K} be a finite field. For a finite set E , let C be a skew-symmetric $|E|$ by
 1424 $|E|$ matrix over \mathbb{K} , with rows and columns indexed, in the same order, by the
 1425 elements of E . Note that we only allow the diagonal of C to be non-zero when
 1426 \mathbb{K} has characteristic two. Let $C[A]$ be the principal submatrix of C induced by
 1427 the set $A \subseteq E$.

1428 We define the delta-matroid $D(C) = (E, \mathcal{F})$, where $A \in \mathcal{F}$ if and only
 1429 if $C[A]$ is non-singular over \mathbb{K} . By convention $C[\emptyset]$ is non-singular. Bouchet
 1430 showed in [4] that $D(C)$ is indeed a delta-matroid. Observe that $\emptyset \in \mathcal{F}(D(C))$,
 1431 for every C .

1432 A delta-matroid is called *representable over \mathbb{K}* if it has a twist that is iso-
 1433 morphic to $D(C)$ for some matrix C .

1434 **Lemma 5.40** (Bouchet [4]). *Suppose that a delta-matroid D is representable*
 1435 *over a field \mathbb{K} . Let F be any feasible set of D . Then $D * F = D(C)$ for some*
 1436 *skew-symmetric matrix C over \mathbb{K} .*

1437 Suppose that M is a matroid representable over \mathbb{K} and that B is a basis of
 1438 M . Then M has a representation of the form $(I|A)$ where I is a $|B|$ by $|B|$
 1439 identity matrix and the columns of I correspond to the elements of B . It is not
 1440 difficult to see that if

$$C = \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix},$$

1441 then $M * B = D(C)$. Thus we have the following result.

1442 **Proposition 5.41** (Bouchet [4]). *A matroid representable over a field \mathbb{K} is also*
 1443 *representable over \mathbb{K} as a delta-matroid.*

1444 A delta-matroid representable over the field with two elements is called *bi-*
 1445 *binary*. If $D = D(C)$ is a binary delta-matroid, then its feasible sets of all sizes
 1446 are determined by its feasible sets of size at most two. By combining this ob-
 1447 servation with Lemma 5.40, we obtain the following.

1448 **Lemma 5.42** (Bouchet and Duchamp [11]). *Let F be a feasible set of a binary*
 1449 *delta-matroid D . Then the feasible sets of D are determined by $\{X \mid |F \triangle X| \leq$*
 1450 *2 and $X \in \mathcal{F}(D)\}$.*

1451 Bouchet and Duchamp gave an excluded-minor characterisation of binary
 1452 delta-matroids.

1453 **Theorem 5.43** (Bouchet and Duchamp [11]). *A delta-matroid is a binary delta-*
 1454 *matroid if and only if it has no minor isomorphic to a twist of $S_1, S_2, S_3, S_4,$ or*
 1455 *S_5 , where*

- 1456 1. $S_1 = (\{1, 2, 3\}, \{\emptyset, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\})$,
- 1457 2. $S_2 = (\{1, 2, 3\}, \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\})$,
- 1458 3. $S_3 = (\{1, 2, 3\}, \{\emptyset, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\})$,
- 1459 4. $S_4 = (\{1, 2, 3, 4\}, \{\emptyset, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\})$,
- 1460 5. $S_5 = (\{1, 2, 3, 4\}, \{\emptyset, \{1, 2\}, \{1, 4\}, \{2, 3\}, \{3, 4\}, \{1, 2, 3, 4\}\})$.

1461 It is easy to check that no twist of S_1, S_2, S_3 or S_4 is a matroid and that
 1462 the uniform matroid $U_{2,4}$ is the only twist of S_5 that is a matroid. Note that
 1463 this result implies Tutte's characterization of binary matroids [65] because $U_{2,4}$
 1464 is the unique excluded minor for the class of binary matroids.

1465 It is well known that graphic matroids are representable over every field. An
 1466 analogous result holds for ribbon graphic delta-matroids. Let D be a ribbon-
 1467 graphic delta-matroid. It is readily verified that S_1, \dots, S_5 do not arise as the
 1468 delta-matroids of any ribbon graph. Consequently D has no twist of any delta-
 1469 matroid in $\{S_1, \dots, S_5\}$ as a minor. So Theorem 5.43 implies that D is binary.

1470 **Theorem 5.44** (Bouchet [4]). *Every ribbon-graphic delta-matroid is a binary*
 1471 *delta-matroid.*

1472 Knowing that $D(G)$ is binary, it is straightforward to write down a binary
 1473 representation for the delta-matroid of a ribbon graph $G = (V, E)$ that has a

1474 single vertex. Let $C = (c_{e,f} \mid e, f \in E)$ be the binary matrix representing $D(G)$.
 1475 Let $c_{e,e}$ be one if e is non-orientable and let $c_{e,e}$ be zero otherwise. Let both
 1476 $c_{e,f}$ and $c_{f,e}$ be one if e and f are interlaced; otherwise they are both zero.

1477 If G is connected and has more than one vertex, then a binary representation
 1478 for $D(G)$ can be found by forming the partial dual G^Q , where Q is the edge set
 1479 of a spanning quasi-tree, then forming a matrix C as above using G^Q .

1480 Bouchet's proof of Theorem 5.44 predates Theorem 5.43, and is more in-
 1481 volved. The difficulty is showing that $D(G) = D(C)$. He extended Theorem 5.44
 1482 to other fields as follows.

1483 **Theorem 5.45** (Bouchet [4]). *An even ribbon-graphic delta-matroid is repre-*
 1484 *sentable over any field.*

1485 As even ribbon-graphic delta-matroids correspond precisely to the delta-
 1486 matroids formed from orientable ribbon graphs, the following is obvious.

1487 **Corollary 5.46** (Bouchet [4]). *The delta-matroids of orientable ribbon graphs*
 1488 *are representable over any field.*

1489 *Remark 5.47.* If \mathbb{K} is a field with a characteristic different from two, any non-
 1490 singular skew-symmetric matrix is of even size. Hence any delta-matroid that
 1491 is representable over a field of characteristic different from two has to be even.
 1492 Thus the delta-matroid of any non-orientable ribbon graph is not representable
 1493 over any field with characteristic different from two.

1494 *Remark 5.48.* Not all binary delta-matroids are ribbon-graphic. The matroid
 1495 $M(K_5)$ is a binary matroid and hence by Proposition 5.41 it is a binary delta-
 1496 matroid. However, it is not ribbon-graphic. If $M(K_5)$ is isomorphic to $D(G)$
 1497 for some graph G , then by Corollary 5.4(3), G must be planar, and then $D(G)$
 1498 and $M(G)$ are isomorphic. This is impossible because $M(K_5)$ is not isomorphic
 1499 to the cycle matroid of any other graph, and G is planar but K_5 is not.

1500 5.8. Characterising ribbon-graphic delta-matroids

1501 Just as not all matroids are graphic, not all delta-matroids are ribbon-
 1502 graphic. It is natural to ask for a characterisation of ribbon-graphic delta-
 1503 matroids, and such a characterisation can be recovered from work of Geelen
 1504 and Oum. In [38] Geelen and Oum built on the work of Bouchet [7] in the
 1505 area of circle graphs and found pivot-minor-minimal non-circle-graphs. As an
 1506 application of this they obtained the excluded minors for ribbon-graphic delta-
 1507 matroids.

1508 **Theorem 5.49** (Geelen and Oum [38]). *A delta-matroid is ribbon-graphic if*
 1509 *and only if it does not contain a minor isomorphic to a twist of a delta-matroid*
 1510 *in $\{S_1, S_2, \dots, S_5\}$, where S_1, S_2, \dots, S_5 are as in Theorem 5.43, or in the set*
 1511 *of 166 binary delta-matroids found by the authors of [38].*

1512 **6. Topological analogues of the Tutte polynomial**

The *Tutte polynomial*, $T(G; x, y)$, of a graph or ribbon graph $G = (V, E)$ can be defined as the state sum

$$T(G; x, y) = \sum_{A \subseteq E(G)} (x - 1)^{r(G) - r(A)} (y - 1)^{n(A)}.$$

1513 The Tutte polynomial is perhaps the most studied of all graph polynomials be-
 1514 cause of the vast range of its specializations, including graph invariants from
 1515 statistical physics and knot theory, and because of its interplay with other key
 1516 graph polynomials such as the interlace polynomial, Penrose polynomial, chro-
 1517 matic polynomial and flow polynomial. Tutte introduced his eponymous poly-
 1518 nomial in [64]. A good recent survey is [29]. More details on specializations can
 1519 be found in [67] and [18], and historical background can be found in [35].

1520 We think of the Tutte polynomial as a polynomial over the ring of integers,
 1521 $T(G; x, y) \in \mathbb{Z}[x, y]$. Both it and all the other polynomials in this section can
 1522 also be defined over an arbitrary commutative unitary ring, but, for simplicity
 1523 of exposition, we will work over \mathbb{Z} .

It is well-known that the Tutte polynomial is matroidal, in the sense that all of its parameters depend only on the cycle matroid $M(G)$ of G , rather than the graph itself. It is defined for all matroids by replacing G with M in the definition above. The Tutte polynomial can readily be extended to delta-matroids by setting

$$T(D; x, y) := T(D_{\min}; x, y) = \sum_{A \subseteq E(D)} (x - 1)^{r_{D_{\min}}(D) - r_{D_{\min}}(A)} (y - 1)^{n_{D_{\min}}(A)}.$$

1524 Since $D(G)_{\min} = M(G)$, we have $T(D(G); x, y) = T(G; x, y)$.

1525 There has been much recent interest in extensions of the Tutte polynomial
 1526 to embedded graphs and ribbon graphs. By the term ‘extension’ here we mean
 1527 that the polynomial should include the Tutte polynomial as a specialization,
 1528 and that it should encode topological information about the embedding of the
 1529 graph in some way. We refer to such polynomials loosely as ‘topological Tutte
 1530 polynomials’. The Tutte polynomial itself clearly does not depend upon the
 1531 embedding.

1532 Here we are concerned with three such polynomials: the Las Vergnas poly-
 1533 nomial, the ribbon graph polynomial of Bollobás and Riordan, and the Krushkal
 1534 polynomial. We show that, while the Tutte polynomial is matroidal, the topo-
 1535 logical Tutte polynomials are delta-matroidal, that is, they depend only on the
 1536 delta-matroid of a ribbon graph, and they are well-defined for delta-matroids.

1537 Why should we expect this to be the case? Above we defined the Tutte
 1538 polynomial in terms of a sum over spanning subgraphs of G . The Tutte poly-
 1539 nomial was originally defined (see [64]) as a sum over the set of maximal spanning
 1540 forests of G . It was recently shown that each of the three topological Tutte poly-
 1541 nomials mentioned above can be expressed as a sum over the set of spanning
 1542 quasi-trees of a ribbon graph. See [20, 27, 66] for the ribbon graph polynomial,

1543 and [19] for the Krushkal and Las Vergnas polynomials. Given that $T(G)$ is
 1544 determined by $M(G)$, which is in turn determined by the set of maximal span-
 1545 ning forests of G , and the topological Tutte polynomials are determined by their
 1546 spanning quasi-trees which also determine $D(G)$, it seems reasonable to expect,
 1547 and it is indeed the case, that the topological Tutte polynomials are determined
 1548 by $D(G)$.

We consider the three polynomials in their chronological order, and so start
 with the Las Vergnas polynomial $L(G; x, y, z)$ from [45, 46, 47]. The Las Vergnas
 polynomial arose as a special case of Las Vergnas' Tutte polynomial of a mor-
 phism of matroids of [48], and can be defined in terms of the cycle matroid
 $M(G)$ of an embedded graph G and the *bond matroid* $B(G^*) := (M(G^*))^*$ of
 its geometric dual G^* . The *Las Vergnas polynomial*, $L(G; x, y, z) \in \mathbb{Z}[x, y, z]$, of
 an embedded graph or ribbon graph G is defined by

$$L(G; x, y, z) := \sum_{A \subseteq E(G)} (x-1)^{r_{M(G)}(E) - r_{M(G)}(A)} \cdot (y-1)^{n_{B(G^*)}(A)} z^{r_{B(G^*)}(E) - r_{M(G)}(E) - (r_{B(G^*)}(A) - r_{M(G)}(A))}.$$

1549 Observe that when G is a plane graph, then $B(G^*) = (M(G^*))^* = M(G)$ and
 1550 so $L(G; x, y, z) = T(G; x, y)$. Las Vergnas [46] proved that for any embedded
 1551 graph G ,

$$(y-1)^{\gamma(G)} L(G; x, y, 1/(y-1)) = T(G; x, y).$$

1552 Recalling from Corollary 5.4(2) that $D(G)_{\min} = M(G)$ and $D(G)_{\max} =$
 1553 $(M(G^*))^* = B(G^*)$, it is clear how to extend $L(G; x, y, z)$ to delta-matroids.

Definition 6.1. Let $D = (E, \mathcal{F})$ be a delta-matroid. Then the Las Vergnas
 polynomial $L(D; x, y, z)$ is given by

$$L(D; x, y, z) := \sum_{A \subseteq E} (x-1)^{r_{D_{\min}}(E) - r_{D_{\min}}(A)} \cdot (y-1)^{n_{D_{\max}}(A)} z^{r_{D_{\max}}(E) - r_{D_{\min}}(E) - (r_{D_{\max}}(A) - r_{D_{\min}}(A))}.$$

1554 It is immediate from the definition that the ribbon graph and delta-matroid
 1555 versions of $L(G)$ coincide.

1556 **Theorem 6.2.** *Let G be a connected ribbon graph. Then*

$$L(G; x, y, z) = L(D(G); x, y, z).$$

1557 Just as with the ribbon graph version, $L(D; x, y, z) = T(D; x, y)$ when D is
 1558 a matroid, and for any delta-matroid D we have

$$(y-1)^{w(D)} L(D; x, y, 1/(y-1)) = T(D; x, y).$$

1559 To see why this identity holds, expand and simplify the exponents of $(y -$
 1560 $1)^{w(D)} L(D; x, y, 1/(y-1))$, noting that $w(D) = r_{D_{\max}}(E) - r_{D_{\min}}(E)$.

1561 The chronologically second and most studied of the three topological graph
 1562 polynomials in this section is Bollobás and Riordan’s ribbon graph polynomial
 1563 of [2, 3]. Let $G = (V, E)$ be a ribbon graph. Then the *ribbon graph poly-*
 1564 *nomial* or the *Bollobás-Riordan polynomial* of G , denoted by $R(G; x, y, z, w) \in$
 1565 $\mathbb{Z}[x, y, z, w]/\langle w^2 - w \rangle$, is defined by

$$R(G; x, y, z, w) = \sum_{A \subseteq E} (x - 1)^{r(E) - r(A)} y^{n(A)} z^{\gamma(A)} w^{t(A)}. \quad (6.1)$$

1566 To extend this polynomial to delta-matroids $D = (E, \mathcal{F})$, first, for $A \subseteq E$, define
 1567 $t(A)$ by setting $t(A) = 0$ if $D|A$ is even, and $t(A) = 1$ otherwise. Next observe
 1568 that, by Lemma 5.3(3) and Proposition 5.11, we have $\gamma(A) = \gamma(G \setminus A^c) =$
 1569 $w(D(G \setminus A^c)) = w(D(G)|A)$. To simplify notation a little, we let $w_D(A) :=$
 1570 $w(D|A)$.

1571 **Definition 6.3.** Let $D = (E, \mathcal{F})$ be a delta-matroid. Then the *Bollobás-*
 1572 *Riordan polynomial* $R(D; x, y, z, w) \in \mathbb{Z}[x, y, z, w]/\langle w^2 - w \rangle$, of D is

$$R(D; x, y, z, w) := \sum_{A \subseteq E} (x - 1)^{r_{D_{\min}}(E) - r_{D_{\min}}(A)} y^{n_{D_{\min}}(A)} z^{w_D(A)} w^{t(A)}.$$

1573 By construction, the ribbon graph and delta-matroid versions of $R(G)$ coin-
 1574 cide, that is, Bollobás and Riordan’s ribbon graph polynomial is delta-matroidal.

1575 **Theorem 6.4.** *Let G be a ribbon graph. Then*

$$R(G; x, y, z, w) = R(D(G); x, y, z, w).$$

1576 Recall from Section 4.2 that the isotropic matroid of a ribbon graph G
 1577 is defined in terms of G and a quasi-tree Q of G . In [62], Traldi, working
 1578 in the language of transition matroids, showed that $R(G)$ can be determined
 1579 from $k(G)$, the isotropic matroid of G and the quasi-tree Q . By the discussion
 1580 following Corollary 4.10, the isotropic matroid and a quasi-tree determine $D(G)$,
 1581 and so it can be deduced from Theorem 6.4 that knowledge of $k(G)$ is not
 1582 needed: $R(G)$ is determined entirely by information in the isotropic matroid
 1583 and the quasi-tree Q .

1584 *Remark 6.5.* The observation that the Bollobás–Riordan polynomial is delta-
 1585 matroidal helps to explain the form of the deletion–contraction identity for the
 1586 Bollobás–Riordan polynomial. More precisely it helps to explain why there is
 1587 generally no known deletion–contraction identity when the edge being removed
 1588 is a loop. The exponents of x and y depend on the rank function of the lower
 1589 matroid. An orientable non-trivial loop e of a ribbon graph G is not a loop of
 1590 $D(G)$ but is a loop of $D(G)_{\min}$. This means that $(D(G)/e)_{\min}$ is not generally
 1591 the same as $(D(G)_{\min})/e$ and moreover $(D(G)_{\min})/e$ cannot always be recovered
 1592 from $(D(G)/e)_{\min}$.

1593 Most of the results on the Bollobás–Riordan polynomial in the literature (for
 1594 example, [3, 21, 26, 31, 33, 30, 41]) hold not for the full four-variable polynomial

1595 but for the normalised two-variable version $x^{\gamma(G)/2}R_G(x+1, y, 1/\sqrt{xy}, 1)$. This
 1596 two-variable version of the polynomial has a particularly natural form when
 1597 expressed in terms of delta-matroids. Define a function σ on delta-matroids
 1598 by $\sigma(D) := \frac{1}{2}(r(D_{\max}) + r(D_{\min}))$, and for $A \subseteq E(D)$, $\sigma_D(A) := \sigma(D|A)$,
 1599 omitting the subscript D whenever the context is clear. We define the *two-*
 1600 *variable Bollobás-Riordan polynomial* of a delta-matroid to be

$$\tilde{R}(D; x, y) := \sum_{A \subseteq E} (x-1)^{\sigma(E)-\sigma(A)} (y-1)^{|A|-\sigma(A)}. \quad (6.2)$$

1601 One immediately notices from (6.2) that if D is a matroid with rank function
 1602 r , then $\sigma(A) = r(A)$, so $\tilde{R}(D; x, y)$ is exactly the Tutte polynomial $T(D; x, y)$.
 1603 It is also readily verified, using Proposition 5.39(1), that $\tilde{R}(D; x+1, y+1) =$
 1604 $x^{w(D)/2}R(D; x+1, y, 1/\sqrt{xy}, 1)$.

1605 It is well-known that the Tutte polynomial of a graph or matroid has a re-
 1606 cursive deletion-contraction definition that expresses $T(M)$ as a $\mathbb{Z}[x, y]$ -linear
 1607 combination of Tutte polynomials. Analogously, the two-variable Bollobás-
 1608 Riordan polynomial was shown to have a recursive deletion-contraction defi-
 1609 nition in [25], given in terms of $R(D; x+1, y, 1/\sqrt{xy}, 1)$, and in [43], given in
 1610 terms of $\tilde{R}(D; x, y)$. The difference in the two forms is due to the factor $x^{w(E)/2}$.
 1611 Moreover, Krajewski, Moffatt, and Tanasa showed in [43] that $\tilde{R}(D; x, y)$ is the
 1612 graph polynomial canonically associated with a natural Hopf algebra generated
 1613 by delta-matroid deletion and contraction, just as the Tutte polynomial is the
 1614 polynomial canonically associated with a Hopf algebra generated by matroid
 1615 deletion and contraction. Furthermore, $\tilde{R}(D)$ encodes fundamental combinato-
 1616 rial information about D .

1617 **Theorem 6.6.** *For any delta-matroid D , the following hold.*

1618 1. $\tilde{R}(D; u/v + 1, uv + 1)$ gives the bivariate generating function of D with
 1619 respect to number of feasible sets of each size and rank:

$$v^{\sigma(D)}u^{-w(D)/2}\tilde{R}(D; u/v + 1, uv + 1) = \sum_{A \subseteq E(D)} v^{|A|}u^{|E(D)|-\rho_D(A)};$$

- 1620 2. $\tilde{R}(D^*; x, y) = \tilde{R}(D; y, x)$;
 1621 3. $\tilde{R}(D; 1, 1) = 0$ unless D is a matroid, in which case it equals the number
 1622 of bases of D ;
 1623 4. $\tilde{R}(D; 1, 2)$ is the number of independent sets in D_{\min} ;
 1624 5. $\tilde{R}(D; 2, 1)$ is the number of spanning sets in D_{\max} ;
 1625 6. $\tilde{R}(D; 2, 2) = 2^{|E(D)|}$.

1626 *Proof.* Part (1) follows easily from the definition of $\tilde{R}(D)$ and Proposition 5.39.

1627 Let $E = E(D)$. Then (2) follows by applying Proposition 5.39 to show that
 1628 for any subset A of E , the difference $\sigma_{D^*}(E) - \sigma_{D^*}(A) = |E - A| - \sigma_D(E - A)$.

1629 For (3), $\tilde{R}(D; 1, 1) = \sum_{A \subseteq E} 0^{\sigma(E)-\sigma(A)} 0^{|A|-\sigma(A)}$. A term in the sum is non-
 1630 zero if and only if $\sigma(E) - \sigma(A) = |A| - \sigma(A) = 0$. We have $\sigma(E) = \sigma(A)$ if and

1631 only if $r(D_{\max}) - r((D|A)_{\max}) + r(D_{\min}) - r((D|A)_{\min}) = 0$, which occurs if
 1632 and only if $r(D_{\max}) = r((D|A)_{\max})$ and $r(D_{\min}) = r((D|A)_{\min})$. On the other
 1633 hand, $|A| - \sigma(A) = 0$ if and only if $r((D|A)_{\max}) = r((D|A)_{\min}) = |A|$.
 1634 Therefore $\sigma(E) - \sigma(A) = |A| - \sigma(A) = 0$ if and only if

$$r(D_{\max}) = r((D|A)_{\max}) = r(D_{\min}) = r((D|A)_{\min}) = |A|,$$

1635 which occurs if and only if D is a matroid and A is a basis of D .

1636 For (4), $\tilde{R}(D; 2, 1) = \sum_{A \subseteq E} 0^{|A| - \sigma(A)}$. It follows from above that a term
 1637 in the sum is non-zero if and only if $r((D|A)_{\max}) = r((D|A)_{\min}) = |A|$. If
 1638 $r((D|A)_{\min}) = |A|$ then, by Proposition 5.39, $r_{D_{\min}}(A) = |A|$. Consequently
 1639 A is independent in D_{\min} . On the other hand, if A is independent in D_{\min} ,
 1640 then $r((D|A)_{\min}) = |A|$, the only feasible set of $D|A$ is A , so $r((D|A)_{\max}) =$
 1641 $r((D|A)_{\min}) = |A|$.

1642 Recall that a spanning set A of a matroid M , is a subset of $E(M)$ such that
 1643 $r(A) = r(M)$. Part (5) follows from Parts (2) and (4), because the complement
 1644 of an independent set of a matroid is a spanning set of its dual.

1645 Part (6) is obvious. □

1646 The final polynomial we consider in this section is the Krushkal polynomial
 1647 of [44]. This polynomial generalizes the Bollobás-Riordan polynomial by adding
 1648 a parameter that records some information about the geometric dual. Although
 1649 the Krushkal polynomial is also defined for non-cellularly embedded graphs, here
 1650 we restrict to cellularly embedded graphs, or, equivalently, ribbon graphs. The
 1651 *Krushkal polynomial* of G , denoted by $K(G; x, y, a, b) \in \mathbb{Z}[x, y, a, b]$, is defined
 1652 by

$$K(G; x, y, a, b) := \sum_{A \subseteq E(G)} (x-1)^{r_G(E) - r_G(A)} y^{r_{G^*}(E) - r_{G^*}(A^c)} a^{\gamma_G(A)} b^{\gamma_{G^*}(A^c)}. \quad (6.3)$$

1653 We note that the exponent of a is usually written as $k(A) - f(A) + n(A)$,
 1654 which is equal to $\gamma(A)$ by Euler's formula, and similarly for the b exponent. (An
 1655 analogous comment holds for the z exponent of the Bollobás-Riordan poly-
 1656 nomial.) Also note, for comparison with the literature, that the exponents of a
 1657 and b here are given by the Euler genus, rather than one-half of the Euler genus
 1658 as in [44].

1659 We showed that $\gamma(A) = w_D(A)$ in Proposition 5.3(3). Using Corollary 5.2,
 1660 we have $\gamma_{G^*}(A^c) = \gamma(G^* \setminus A) = w(D(G^* \setminus A)) = w(D(G^*) \setminus A) = w(D(G)^* \setminus A) =$
 1661 $w_{D(G)^*}(A^c)$.

Definition 6.7. Let $D = (E, \mathcal{F})$ be a delta-matroid. Then the *Krushkal poly-*
nomial $K(D; x, y, a, b) \in \mathbb{Z}[x, y, a, b]$, of D is

$$K(D; x, y, a, b) := \sum_{A \subseteq E} (x-1)^{r_{D_{\min}}(E) - r_{D_{\min}}(A)} y^{r_{(D^*)_{\min}}(E) - r_{(D^*)_{\min}}(A^c)} a^{w_D(A)} b^{w_{D^*}(A^c)}.$$

1662 We immediately have that the Krushkal polynomial of a ribbon graph is
 1663 delta-matroidal.

1664 **Theorem 6.8.** *Let G be a ribbon graph. Then*

$$K(G; x, y, a, b) = K(D(G); x, y, a, b).$$

1665 Krushkal observed in [44] that, when G is a plane graph, $T(G; x, y) =$
 1666 $K(G; x, y - 1, a, b)$. The analogous result holds for delta-matroids. Using Equa-
 1667 tion (2.1), for any matroid $M = (E, \mathcal{B})$ and subset A of E , we have

$$|A| - r_M(A) = r(M^*) - r_{M^*}(E - A). \quad (6.4)$$

1668 When D is a matroid, this equation together with the fact that $w_D(A) =$
 1669 $w_{D^*}(A^c) = 0$ implies that $T(D; x, y) = K(D; x, y - 1, a, b)$.

1670 For non-plane graphs, the Tutte polynomial can still be recovered from the
 1671 Krushkal polynomial using the identity

$$T(G; x, y + 1) = y^{\gamma(G)/2} K(G; x, y, y^{1/2}, y^{-1/2})$$

1672 (see [19, 44]). The Krushkal polynomial, however, contains not only the Tutte
 1673 polynomial as a specialization, but also the Bollobás-Riordan polynomial at
 1674 $w = 1$ (see [44]), and the Las Vergnas polynomial (see [1, 19]). Each of these
 1675 results holds in the delta-matroid setting.

1676 **Theorem 6.9.** *Let D be a delta-matroid. Then*

- 1677 1. $T(D; x, y + 1) = y^{w(D)/2} K(D; x, y, y^{1/2}, y^{-1/2});$
- 1678 2. $L(D; x, y, z) = z^{w(D)/2} K(D; x, y - 1, z^{-1/2}, z^{1/2});$
- 1679 3. $R(D; x, y, z, 1) = y^{w(D)/2} K(D; x, y, zy^{1/2}, y^{-1/2}).$

1680 *Proof.* The first item follows from the third item upon noting that $T(D; x, y +$
 1681 $1) = R(D; x, y, 1, 1)$.

For the second item, the exponents of x in each summand on the left-hand
 and right-hand side agree. Using Equation (6.4) and $(D_{\max})^* = (D^*)_{\min}$, we
 see that the exponents of $y - 1$ in each summand on both sides agree. For the z
 term, the z exponent of each summand of $z^{w(D)/2} K(D; x, y - 1, z^{-1/2}, z^{1/2})$ is
 $\frac{1}{2}(w(D) - w_D(A) + w_{D^*}(A^c))$. By Proposition 5.39,

$$\begin{aligned} & w(D) - w_D(A) + w_{D^*}(A^c) \\ &= r_{D_{\max}}(E) - r_{D_{\min}}(E) - \rho_D(A) + r_{D_{\min}}(A) + |E| - r_{D_{\min}}(E) - |A| \\ & \quad + r_{D_{\min}}(A) + \rho_{D^*}(A^c) - r_{D_{\min}^*}(A^c) - |E| + r_{D_{\min}^*}(E) + |A^c| - r_{D_{\min}^*}(A^c). \end{aligned} \quad (6.5)$$

1682 By Lemma 2.9, $\rho_D(A) = \rho_{D^*}(A^c)$. Additionally using Equation (2.1) and
 1683 $(D_{\max})^* = (D^*)_{\min}$ we obtain

$$r_{D_{\min}^*}(A) = r_{(D_{\max})^*}(A) = |A| + r_{D_{\max}}(A^c) - r_{D_{\max}}(E).$$

Substituting these into Equation (6.5) allows us to rewrite it as

$$\begin{aligned}
w(D) - w_D(A) + w_{D^*}(A^c) &= r_{D_{\max}}(E) - 2r_{D_{\min}}(E) + 2r_{D_{\min}}(A) - |A| \\
&\quad - 2|A^c| - 2r_{D_{\max}}(A) + 2r_{D_{\max}}(E) + |E| - r_{D_{\max}}(E) + |A^c| \\
&= 2(r_{D_{\max}}(E) - r_{D_{\min}}(E) - r_{D_{\max}}(A) + r_{D_{\min}}(A)).
\end{aligned}$$

1684 But this is just twice the z exponent of the summands of $L(D; x, y, z)$.

1685 For the third item, it is easy to see that the exponents of x and z on the
1686 left-hand and right-hand sides agree. The exponent of y on the right-hand side
1687 is

$$\frac{1}{2}(w(D) + 2r_{(D^*)_{\min}}(E) - 2r_{(D^*)_{\min}}(A^c) + w_D(A) - w_{D^*}(A^c)).$$

1688 Using Proposition 5.39 and $(D_{\max})^* = (D^*)_{\min}$, it is straightforward to show
1689 that this is equal to $n_{D_{\min}}(A)$, as required. \square

1690 The ribbon graph versions of Theorem 6.9 from [1, 19, 44] can be recovered
1691 by taking D to be $D(G)$.

1692 Since the Tutte polynomial can be defined in terms of matroid rank functions,
1693 it is interesting to observe that, by Proposition 5.39, we can express $K(D)$, and
1694 therefore $R(D)$, entirely in terms of rank functions associated with D . Let
1695 $E = E(D)$. For $A \subseteq E(D)$, let

- 1696 • $K_x(D, A) = r_{D_{\min}}(E) - r_{D_{\min}}(A)$;
- 1697 • $K_y(D, A) = r_{(D^*)_{\min}}(E) - r_{(D^*)_{\min}}(A^c)$;
- 1698 • $K_a(D, A) = \rho_D(A) - r_{D_{\min}}(A) - n_{D_{\min}}(E) + n_{D_{\min}}(A)$;
- 1699 • $K_b(D, A) = \rho_{D^*}(A^c) - r_{(D^*)_{\min}}(A^c) - n_{(D^*)_{\min}}(E) + n_{(D^*)_{\min}}(A^c)$.

1700 Then

$$K(D; x, y, a, b) = \sum_{A \subseteq E} (x-1)^{K_x(D,A)} y^{K_y(D,A)} a^{K_a(D,A)} b^{K_b(D,A)}. \quad (6.6)$$

1701 The Tutte polynomial of a plane graph satisfies the duality relation:

$$T(G; x, y) = T(G^*; y, x).$$

1702 This identity is actually matroidal as $T(M; x, y) = T(M^*; y, x)$, and the result
1703 for graphs follows since $M(G^*) = M(G)^*$ when G is a plane graph. Similar
1704 duality identities were shown for the Bollobás-Riordan polynomial in [34, 49],
1705 the Krushkal polynomial in [44], and the Las Vergnas polynomial in [45]. The
1706 following theorem shows that each of these duality relations holds on the level
1707 of delta-matroids.

1708 **Theorem 6.10.** *Let D be a delta-matroid. Then*

- 1709 1. $K(D; x, y-1, a, b) = K(D^*; y, x-1, b, a)$;

- 1710 2. $x^{w(D)/2}R(D; x + 1, y, 1/\sqrt{xy}, 1) = y^{w(D^*)/2}R(D^*; y + 1, x, 1/\sqrt{xy}, 1);$
 1711 3. $L(D; x, y, z) = z^{w(D)}L(D^*; y, x, z^{-1}).$

1712 *Proof.* The first part can be proven by writing down the sums for the two sides of
 1713 the equation and observing that summing over all $A \subseteq E$ is the same as summing
 1714 over all $A^c \subseteq E$. The second and third parts then follow by Theorem 6.9. \square

1715 The corresponding duality relations for the ribbon graph versions of the
 1716 polynomials from [34, 44, 45, 49] follow from the theorem as $D(G^*) = D(G)^*$.

1717 We conclude with an application to knot theory. There is a well-known
 1718 way to associate a plane graph G_L to an alternating link diagram L such that
 1719 the Kauffman bracket $\langle L \rangle$, or Jones polynomial (if the writhe of the link is
 1720 known), of L can be recovered from the Tutte polynomial of G_L together with
 1721 knowledge of $k(G_L)$ (see [59]). Recently, Dasbach, Futer, Kalfagianni, Lin and
 1722 Stoltzfus, in [26], extended this result to all link diagrams (including those that
 1723 are not alternating) by describing how a ribbon graph \mathbb{A}_L can be associated with
 1724 any link diagram L . It was also shown in [26] that the Kauffman bracket and
 1725 Jones polynomial (again provided the writhe of the link is known) of L can be
 1726 recovered from the Bollobás-Riordan polynomial of \mathbb{A}_L together with knowledge
 1727 of $k(\mathbb{A}_L)$.

1728 If a link diagram is split, then we can use a sequence of Reidemeister II
 1729 moves to obtain an equivalent non-split diagram. If we construct \mathbb{A}_L from this
 1730 diagram then we know it is a connected ribbon graph, so we no longer need
 1731 knowledge of $k(\mathbb{A}_L)$.

1732 Recalling that the Tutte polynomial of G_L can be recovered from its cycle
 1733 matroid $M(G_L)$, this means that the Kauffman bracket of an *alternating* link is
 1734 matroidal in the sense that it can be recovered from a matroid associated with
 1735 any of its non-split diagrams. Since the Bollobás-Riordan polynomial of \mathbb{A}_K is
 1736 determined by $D(\mathbb{A}_K)$, this means that in general, the Kauffman bracket can
 1737 be regarded as a delta-matroidal object.

1738 **Theorem 6.11.** *The Kauffman bracket of a link is delta-matroidal, in the sense*
 1739 *that it is determined by delta-matroids associated with non-split link diagrams.*

1740 This result also holds for virtual link diagrams by [23] and for links in real
 1741 projective space by [54], and, if we know the writhe of the link diagram we can
 1742 extend both results to the Jones polynomial. Finally, using [49], we can show
 1743 the homfly-pt polynomial of a class of links is delta-matroidal.

1744 Acknowledgements

1745 We thank the anonymous referees for a very careful reading. Their comments
 1746 have significantly improved the presentation of this paper. We also thank Tony
 1747 Nixon, Sang-il Oum, and Lorenzo Traldi for many helpful suggestions and com-
 1748 ments.

1749 **References**

- 1750 [1] R. Askanazi, S. Chmutov, C. Estill, J. Michel, and P. Stollenwerk, Polyno-
1751 mial invariants of graphs on surfaces, *Quantum Topol.* **4** (2013) 77–90.
- 1752 [2] B. Bollobás, and O. Riordan, A polynomial for graphs on orientable sur-
1753 faces, *Proc. Lond. Math. Soc.* **83** (2001) 513–531.
- 1754 [3] B. Bollobás, and O. Riordan, A polynomial of graphs on surfaces, *Math.*
1755 *Ann.* **323** (2002) 81–96.
- 1756 [4] A. Bouchet, Representability of Δ -matroids, in *Proceedings of the 6th Hun-*
1757 *garian Colloquium of Combinatorics*, Colloq. Math. Soc. János Bolyai,
1758 (1987) 167–182.
- 1759 [5] A. Bouchet, Greedy algorithm and symmetric matroids, *Math. Program.*
1760 **38** (1987) 147–159.
- 1761 [6] A. Bouchet, Maps and delta-matroids, *Discrete Math.* **78** (1989) 59–71.
- 1762 [7] A. Bouchet, Circle graph obstructions, *J. Combin. Theory Ser. B* **60** (1994)
1763 107–144.
- 1764 [8] A. Bouchet, Coverings and delta-coverings, in: E. Balas, J. Clausen (Eds.),
1765 Integer Programming and Combinatorial Optimization, Lecture Notes in
1766 Computer Science, **920**, Springer, Berlin, 1995, 228–243.
- 1767 [9] A. Bouchet, Multimatroids III. Tightness and fundamental graphs, *Europ.*
1768 *J. Combin.* **22** (2001) 657–677.
- 1769 [10] A. Bouchet and W. Cunningham, Delta-matroids, jump systems, and
1770 bisubmodular polyhedra, *SIAM J. Discrete Math.* **8** (1995) 17–32.
- 1771 [11] A. Bouchet and A. Duchamp, Representability of delta-matroids over
1772 $GF(2)$, *Linear Algebra Appl.* **146** (1991) 67–78.
- 1773 [12] A. Bouchet and W. Schwärzler, The delta-sum of matching delta-matroids,
1774 *Discrete Appl. Math.* **181** (1998) 53–63.
- 1775 [13] R. Brijder H. Hoogeboom, The group structure of pivot and loop com-
1776 plementation on graphs and set systems, *European J. Combin.* **32** (2011)
1777 1353–1367.
- 1778 [14] R. Brijder and H. Hoogeboom, Nullity and loop complementation for delta-
1779 matroids, *SIAM J. Discrete Math.* **27** (2013), 492–506.
- 1780 [15] R. Brijder and H. Hoogeboom, Interlace polynomials for multimatroids and
1781 delta-matroids, *European J. Combin.* **40** (2014) 142–167.
- 1782 [16] R. Brijder, L. Traldi, Isotropic matroids. II. Circle graphs, *Electron. J.*
1783 *Combin.*, **23** (2016) Paper 4.2.

- 1784 [17] T. Brylawski, A decomposition for combinatorial geometries, *J. Combin.*
1785 *Theory Ser. B* **171** (1972) 235–282.
- 1786 [18] T. Brylawski and J. Oxley, The Tutte polynomial and its applications,
1787 in *Matroid applications*, Encyclopedia Math. Appl. **40**, Cambridge Univ.
1788 Press, Cambridge, (1992) 123–225.
- 1789 [19] C. Butler, A quasi-tree expansion of the Krushkal polynomial, *Adv. in Appl.*
1790 *Math.* **94** (2018), 3–22.
- 1791 [20] A. Champanerkar, I. Kofman, and N. Stoltzfus, Quasi-tree expansion for
1792 the Bollobás-Riordan-Tutte polynomial, *Bull. Lond. Math. Soc.* **43** (2011)
1793 972–984.
- 1794 [21] S. Chmutov and I. Pak, The Kauffman bracket of virtual links and the
1795 Bollobás-Riordan polynomial, *Mosc. Math. J.* **7** (2007) 409–418.
- 1796 [22] S. Chmutov, Generalized duality for graphs on surfaces and the signed
1797 Bollobás-Riordan polynomial, *J. Combin. Theory Ser. B* **99** (2009) 617–
1798 638.
- 1799 [23] S. Chmutov and J. Voltz, Thistlethwaite’s theorem for virtual links, *J. Knot*
1800 *Theory Ramifications* **17** (2008) 1189–1198.
- 1801 [24] C. Chun, D. Chun, S. D. Noble, Inductive tools for connected delta-
1802 matroids and multimatroids, *European J. Combin.* **63** (2017) 59–69.
- 1803 [25] C. Chun, I. Moffatt, S. D. Noble, and R. Rueckriemen, On the interplay
1804 between embedded graphs and delta-matroids, preprint.
- 1805 [26] O. Dasbach, D. Futer, E. Kalfagianni, X.-S. Lin and N. Stoltzfus, Alternat-
1806 ing sum formulae for the determinant and other link invariants, *Journal of*
1807 *Knot Theory and its Ramifications* **19** (2010) 765–782.
- 1808 [27] E. Dewey, A quasitree expansion of the Bollobás-Riordan polynomial, un-
1809 published work, available at <http://www.math.wisc.edu/dewey/>
- 1810 [28] A. Duchamp, Delta-matroids whose fundamental graphs are bipartite. *Lin-*
1811 *ear Algebra Appl.* **160** (1992) 99–112.
- 1812 [29] J. Ellis-Monaghan, and C. Merino, Graph polynomials and their applica-
1813 tions I: The Tutte polynomial, in *Structural Analysis of Complex Networks*,
1814 Birkhäuser, (2011) 219–255.
- 1815 [30] J. Ellis-Monaghan and I. Sarmiento, A recipe theorem for the topological
1816 Tutte polynomial of Bollobás and Riordan, *European J. Combin.* **32** (2011)
1817 782–794.
- 1818 [31] J. Ellis-Monaghan and I. Moffatt, Twisted duality for embedded graphs,
1819 *Trans. Amer. Math. Soc.* **364** (2012) 1529–1569.

- 1820 [32] J. Ellis-Monaghan and I. Moffatt, *Graphs on surfaces: Dualities, Polyno-*
1821 *mials, and Knots*, Springer, (2013).
- 1822 [33] J. Ellis-Monaghan and I. Moffatt, Evaluations of topological Tutte polyno-
1823 mials, *Combin. Probab. Comput.* **24** (2015) 556–583.
- 1824 [34] J. Ellis-Monaghan, and I. Sarmiento, A recipe theorem for the topological
1825 Tutte polynomial of Bollobás and Riordan, *European J. Combin.* **32** (2011)
1826 782–794.
- 1827 [35] G. Farr, Tutte–Whitney polynomials: some history and generalizations, in
1828 G. R. Grimmett and C. J. H. McDiarmid (eds.), *Combinatorics, Complexity*
1829 *and Chance: A Tribute to Dominic Welsh*, Oxford University Press, (2007)
1830 28–52.
- 1831 [36] J. Geelen, B. Gerards and G. Whittle, Structure in minor-closed classes of
1832 matroids, in S. R. Blackburn, S. Gerke and M. Wildon (eds.), *Surveys in*
1833 *Combinatorics 2013*, London Math. Soc. Lecture Notes **409**, Cambridge
1834 University Press (2013) 327–362.
- 1835 [37] J. Geelen, S. Iwata, and K. Murota, The linear delta-matroid parity prob-
1836 lem, *J. Combin. Theory Ser. B* **88** (2003) 377–398.
- 1837 [38] J. Geelen, S. Oum, Circle graph obstructions under pivoting. *J. Graph*
1838 *Theory* **61** (2009) 1–11.
- 1839 [39] J. Gross and T. Tucker, Topological graph theory, Wiley-interscience pub-
1840 lication, (1987).
- 1841 [40] V. Jones, A polynomial invariant for knots via von Neumann algebra, *Bull.*
1842 *Amer. Math. Soc.* **12** (1985) 103–111.
- 1843 [41] M. Korn and I. Pak, Combinatorial evaluations of the Tutte polynomial,
1844 preprint.
- 1845 [42] A. Kotzig, Eulerian lines in finite 4-valent graphs and their transformations,
1846 in: *Theory of Graphs (Proc. Colloq. Tihany, 1966)*, Academic Press, New
1847 York, (1968) 219–230.
- 1848 [43] T. Krajewski, I. Moffatt, A. Tanasa, Hopf algebras and Tutte polynomials,
1849 *Adv. in Appl. Math.* **95** (2018) 271–330.
- 1850 [44] S. Krushkal, Graphs, links, and duality on surfaces, *Combin. Probab. Com-*
1851 *put.* **20** (2011) 267–287.
- 1852 [45] M. Las Vergnas, Eulerian circuits of 4-valent graphs imbedded in surfaces.
1853 Algebraic methods in graph theory, Vol. I, II (Szeged, 1978), *Colloq. Math.*
1854 *Soc. János Bolyai* **25**, North-Holland, Amsterdam–New York, (1981) 451–
1855 477.

- 1856 [46] M. Las Vergnas, On the Tutte polynomial of a morphism of matroids,
1857 *Annals Discrete Mathematics* **8** (1980) 7–20.
- 1858 [47] M. Las Vergnas, Sur les activités des orientations d’une géométrie combi-
1859 natoire, Colloque Mathématiques Discrètes: Codes et Hypergraphes (Brus-
1860 sels, 1978), *Cahiers Centre Études Rech. Opér.* **20** (1978) 293–300.
- 1861 [48] M. Las Vergnas, Extensions normales d’un matroïde, polynôme de Tutte
1862 d’un morphisme, *C. R. Acad. Sci. Paris Sér. A-B* **22** (1975) 1479–1482.
- 1863 [49] I. Moffatt, Knot invariants and the Bollobás-Riordan polynomial of embed-
1864 ded graphs, *European J. Combin.* **29** (2008) 95–107.
- 1865 [50] I. Moffatt, A characterization of partially dual graphs, *J. Graph Theory* **67**
1866 (2011) 198–217.
- 1867 [51] I. Moffatt, Partial duals of plane graphs, separability and the graphs of
1868 knots, *Algebr. Geom. Topol.* **12** (2012) 1099–1136.
- 1869 [52] I. Moffatt, Separability and the genus of a partial dual, *European J. Com-
1870 bin.* **34** (2013) 355–378.
- 1871 [53] I. Moffatt, Excluded minors and the ribbon graphs of knots, *J. Graph
1872 Theory*, **81** (2016) 329–341.
- 1873 [54] I. Moffatt, J. Strömberg, On the ribbon graphs of links in real projective
1874 space, *Involve*, **9** (2016) 133–153.
- 1875 [55] J. Oxley, On the interplay between graphs and matroids, in J. W. P.
1876 Hirschfeld (ed.), *Surveys in Combinatorics, 2001*, London Math. Soc. Lec-
1877 ture Notes **288**, Cambridge University Press, Cambridge, (2001) 199–239.
- 1878 [56] J. Oxley, *Matroid theory*, Second edition, Oxford University Press, New
1879 York, (2011).
- 1880 [57] N. Robertson, P. Seymour, Graph Minors XX, Wagner’s conjecture, *J.
1881 Combin. Theory Ser. B* **92** (2004) 325–357.
- 1882 [58] P. Seymour, A note on the production of matroid minors, *J. Combin. The-
1883 ory Ser. B* **22** (1977) 289–295.
- 1884 [59] M. B. Thistlethwaite, A spanning tree expansion of the Jones polynomial,
1885 *Topology* **26** (1987) 297–309.
- 1886 [60] L. Traldi, Binary nullity, Euler circuits and interlace polynomials, *European
1887 J. Combin.* **32** (2011) 944–950.
- 1888 [61] L. Traldi, Binary matroids and local complementation, *European J. Com-
1889 bin.* **45** (2015) 21–40.
- 1890 [62] L. Traldi, The transition matroid of a 4-regular graph: an introduction,
1891 *European J. Combin.* **50** (2015) 180–207.

- 1892 [63] W. Tutte, Connectivity in matroids, *Canad. J. Math.* **18** (1966) 1301–1324.
- 1893 [64] W. Tutte, A ring in graph theory, *Mathematical Proceedings of the Cam-*
1894 *bridge Philosophical Society* **43** (1947) 26–40.
- 1895 [65] W. Tutte, Lectures on matroids, *Journal of Research of the National Bureau*
1896 *of Standards Section B* **69B** (1965), 1-47.
- 1897 [66] F. Vignes-Tourneret, Non-orientable quasi-trees for the Bollobás-Riordan
1898 polynomial, *European J. Combin.* **32** (2011) 510–532.
- 1899 [67] D. J. A. Welsh, *Complexity: Knots, Colouring and Counting*, Cambridge
1900 University Press (1993).
- 1901 [68] P.-L. Wu, An upper bound on the number of edges of a 2-connected graph,
1902 *Combin. Probab. Comput.* **6** (1997) 107–113.