Birkbeck Working Papers in Economics & Finance

ISSN 1745-8587

School of Economics, Mathematics and Statistics

BWPEF 0712

Variance Dispersion and Correlation Swaps

Antoine Jacquier
Birkbeck, University of London

Saad Slaoui
AXA IM, Paris

July 2007

▪ Birkbeck, University of London ▪ Malet Street ▪ London ▪ WC1E 7HX ▪

Variance Dispersion and Correlation Swaps

Antoine Jacquier*
Birkbeck College, University of London
Zeliade Systems

Saad Slaoui
AXA IM, Paris

July 2007

Abstract In the recent years, banks have sold structured products such as Worst-of options, Everest and Himalayas, resulting in a short correlation exposure. They have hence become interested in offsetting part of this exposure, namely buying back correlation. Two ways have been proposed for such a strategy : either pure correlation swaps or dispersion trades, taking position in an index option and the opposite position in the components options. These dispersion trades have been traded using calls, puts, straddles, and they now trade variance swaps as well as third generation volatility products, namely gamma swaps and barrier variance swaps. When considering a dispersion trade via variance swaps, one immediately sees that it gives a correlation exposure. But it has empirically been showed that the implied correlation - in such a dispersion trade - was not equal to the strike of a correlation swap with the same maturity. Indeed, the implied correlation tends to be around 10 points higher. The purpose of this paper is to theoretically explain such a spread. In fact, we prove that the P&L of a dispersion trade is equal to the sum of the spread between implied and realised correlation - multiplied by an average variance of the components - and a volatility part. Furthermore, this volatility part is of second order, and, more precisely, is of Volga order. Thus the observed correlation spread can be totally explained by the Volga of the dispersion trade. This result is to be reviewed when considering different weighting schemes for the dispersion trade.

*Corresponding author : a.jacquier@ems.bbk.ac.uk

Contents

1 Introduction

For some years now, volatility has become a traded asset, showing great liquidity, both on equity and index markets. It has become that important that some types of options on it have been created and traded in huge quantities. Indeed, variance swaps are very liquid nowadays for many stocks, and options on variance and on volatility have been the subject of several research, both from Academics and Professionals. Furthermore, these products have given birth to positions on correlation, which had to be hedged. Hence, products such as correlation swaps have been proposed to answer these needs. Our purpose here is to compare the fair correlation priced in a correlation swap and the implied correlation of a dispersion trade. Indeed, a dispersion trade can be built upon Variance or Gamma swaps, hence creating an almost pure exposition to correlation, independent of the level of the stock. Furthermore, we will try to find an explanation of the observed spread between these two correlations in terms of the second-order derivatives of such dispersion trades. We indeed believe that the moves in volatility, both of the index and of its components, have a real impact on the implied correlation.

2 Products

2.1 European Call and Put options

We just remind the Greeks of standard European call and put options, written on a stock S, with strike K, maturity T (at time $t = T - \tau$), risk-free rate r and volatility σ. Below, the superscript C denotes a Call option and P a put option. We also separate the variance vega (derivative wrt the variance) and the vol vega (derivative wrt the volatility), as both are now used in papers. N denotes the cumulative standard Gaussian probability function and ϕ its density. $\Psi = \frac{\partial^2 V}{\partial S \partial \sigma}$ is called the Vanna, Υ_σ is the Vega of the option (wrt the volatility), Υ_v the Vega wrt the variance, Ω the derivative wrt to the interest rate, and Λ_σ or Λ_v the derivative of the vega wrt the volatility or the variance (the Volga).

Greek	Call	Put
Δ	$N\left(d_1\right)$	$N\left(d_1\right)-1$
Γ	$\frac{\phi(d_1)}{S\sigma\sqrt{\tau}}$	$\frac{\phi(d_1)}{S\sigma\sqrt{\tau}}$
Υ_σ	$S\sqrt{\tau}\phi\left(d_1\right)$	$S\sqrt{\tau}\phi\left(d_1\right)$
Υ_v	$\frac{S}{2\sigma\sqrt{\tau}}\phi\left(d_1\right)$	$\frac{S}{2\sigma\sqrt{\tau}}\phi\left(d_1\right)$
Θ	$-\frac{S\sigma\phi(d_1)}{2\sqrt{\tau}}-rKe^{-r\tau}N\left(d_2\right)$	$-\frac{S\sigma\phi(d_1)}{2\sqrt{\tau}}+rKe^{-r\tau}N\left(-d_2\right)$
Ω	$K\tau e^{-r\tau}N\left(d_2\right)$	$-K\tau e^{-r\tau}N\left(-d_2\right)$
Ψ	$-\frac{d_2}{\sigma}\phi\left(d_1\right)$	$-\frac{d_2}{\sigma}\phi\left(d_1\right)$
Λ	$S\sqrt{\tau}\frac{d_1 d_2}{\sigma}\phi\left(d_1\right)$	$S\sqrt{\tau}\frac{d_1 d_2}{\sigma}\phi\left(d_1\right)$

2.2 Straddle

A straddle consists in being long a call and a put with the same characteristics (strike, stock, maturity). The price of it is also, within the Black-Scholes framework :

$$\Pi_t = S_t\left[2N\left(d_1\right)-1\right]-Ke^{-r\tau}\left[N\left(d_2\right)-1\right]$$

with

$$\begin{cases} d_1 = \frac{ln\left(\frac{S_t}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} \\ \\ d_2 = d_1-\sigma\sqrt{\tau} \end{cases}$$

The Greeks are therefore the following :

$$\left[\begin{array}{lll} \Delta = 2N\left(d_1\right)-1 & \Gamma = \frac{2}{S\sigma\sqrt{\tau}}\phi\left(d_1\right) & \Omega = \tau Ke^{-r\tau}\left[2N\left(d_2\right)-1\right] \\ \\ \Upsilon_v = \frac{S\sqrt{\tau}}{\sigma}\phi\left(d_1\right) & \Upsilon_\sigma = 2S_t\sqrt{\tau}\phi\left(d_1\right) & \Theta = \frac{\sigma S}{\sqrt{\tau}}\phi\left(d_1\right)-rKe^{-r\sqrt{\tau}}\left[2N\left(d_2\right)-1\right] \\ \\ \Psi = -2\frac{d_2}{\sigma}\phi\left(d_1\right) & \Lambda = 2\frac{d_1 d_2}{\sigma}S\sqrt{\tau}\phi\left(d_1\right) & \end{array}\right.$$

2.3 Variance Swap

A variance swap has the following payoff :

$$\mathbb{V} = N\left(\frac{1}{T}\int_0^T \sigma_t^2 dt - K_{\mathbb{V}}\right)$$

Referring to the paper by Demeterfi, Derman, Kamal and Zhou, the fair price $K_{\mathbb{V}}^{0,T}$ of the variance swap is equal to :

$$K_{\mathbb{V}} = \frac{2}{T}\left[rT - \left(\frac{S_0}{S_*}e^{rT} - 1\right) - ln\left(\frac{S_*}{S_0}\right) + e^{rT}\int_0^{S_*}\frac{1}{K^2}P\left(S_0, K, T\right)dK + e^{rT}\int_{S_*}^{+\infty}\frac{1}{K^2}C\left(S_0, K, T\right)dK\right]$$

where S_* represents the option liquidity threshold. Hence, the variance swap is fully replicable by an infinite number of European calls and puts. We have the following greeks, at time $t = T - \tau$:

$$\left[\begin{array}{ccc} \varDelta = \frac{2}{T}\left(\frac{1}{S_*} - \frac{1}{S_t}\right) & \Gamma = \frac{2}{S_t^2 T} & \Theta = -\frac{\sigma^2}{T} \\ \Upsilon_{\sigma^2} = \frac{\tau}{T} & \frac{\partial \Gamma}{\partial S} = -\frac{4}{S_t^3 T} & \frac{\partial V_{\sigma^2}}{\partial S} = 0 \\ \frac{\partial \Gamma}{\partial \tau} = 0 & \frac{\partial V_{\sigma^2}}{\partial \tau} = \frac{1}{T} & \Upsilon_{\sigma} = 2\sigma\frac{\tau}{T} \end{array}\right.$$

Moreover, if we take $S_* = S_0 e^{rT}$, that such that the liquidity threshold is equal to the forward value of the stock price, the above formula simplifies to

$$K_{\mathbb{V}}^{0,T} = \frac{2}{T}e^{rT}\left[\int_0^{S_*}\frac{1}{K^2}P\left(S_0, K, T\right)dK + \int_{S_*}^{+\infty}\frac{1}{K^2}C\left(S_0, K, T\right)dK\right]$$

A variance swap is interesting in terms of both trading and risk management as :

- it provides a one-direction position on the volatility / variance.
- it allows one to speculate on the difference between the realised and the implied volatility. Hence, if one expects a rise in volatility, then he should go long a variance swap, and vice-versa.
- As the correlation between the stock price and its volatility has proven to be negative, variance swaps are also a means to hedge equity positions.

From a mark-to-market point of view, the value at time $0 \leq t \leq T$ ($\tau = T - t$) of the variance swap strike with maturity T is

$$\begin{aligned} \Pi_t^T &= e^{-r\tau}\mathbb{E}_t\left[\frac{1}{T}\int_0^T \sigma_u^2 du - K_{\mathbb{V}}^{0,T}\right] \\ &= e^{-r\tau}\left[\frac{1}{t}\frac{t}{T}\int_0^t \sigma_u^2 du - K_{\mathbb{V}}^{0,T} + \frac{1}{\tau}\frac{\tau}{T}\mathbb{E}_t\left[\int_t^T \sigma_u^2 du\right]\right] \\ &= e^{-r\tau}\left[\frac{t}{T}\left(\frac{1}{t}\int_0^t \sigma_u^2 du - K_{\mathbb{V}}^{0,T}\right) + \frac{\tau}{T}\mathbb{E}_t\left(\frac{1}{\tau}\int_t^T \sigma_u^2 du - K_{\mathbb{V}}^{0,T}\right)\right] \end{aligned}$$

But $K_{\mathbb{V}}^{t,T} = \mathbb{E}_t \left(\frac{1}{\tau} \int_t^T \sigma_u^2 du \right)$ and hence

$$\Pi_t^T = e^{-r\tau} \left[\frac{1}{T} \int_0^t \sigma_u^2 du - K_{\mathbb{V}}^{0,T} + \frac{\tau}{T} K_{\mathbb{V}}^{t,T} \right]$$

Hence, we just need to calculate the new strike of the variance swap with the remaining maturity $\tau = T - t$.

2.4 Gamma Swap

A Gamma swap looks like a Variance swap, but weighs the daily square returns by the price level. Formally speaking, its payoff is

$$\mathbb{V} = N \left(\frac{1}{T} \int_0^T \sigma_t^2 \frac{S_t}{S_0} dt - K_{\Gamma}^{0,T} \right)$$

The replication for this option is based on both the Itô formula and the Carr-Madan formula [CM]. Let us consider the following function

$$f(F_t) = e^{rt} \left[F_t ln \left(\frac{F_t}{F_0} \right) - F_t + F_0 \right]$$

where F_t represents the futures price of a stock S_t. We suppose that $\left(dF_t^2 \right) = \sigma_t^2 F_t^2 dt$. Using Itô formula, we have :

$$f(F_T) = \int_0^T \frac{\partial f}{\partial t} dt + \int_0^T \frac{\partial f}{\partial F} dF_t + \int_0^T \frac{\partial^2 f}{\partial F^2} \sigma_t^2 F_t^2 dt$$

$$= r \int_0^T e^{rt} \left[F_t ln \left(\frac{F_t}{F_0} \right) - F_t + F_0 \right] dt + \int_0^T e^{rt} ln \left(\frac{F_t}{F_0} \right) dF_t + \frac{1}{2} \int_0^T e^{rt} \sigma_t^2 F_t dt$$

Hence, the floating leg of the Gamma swap can be written as :

$$\frac{1}{T} \int_0^T \sigma_t^2 \frac{S_t}{S_0} dt = \frac{2}{TS_0} \left[f(F_T) - r \int_0^T e^{rt} \left[F_t ln \left(\frac{F_t}{F_0} \right) - F_t + F_0 \right] dt - \int_0^T e^{rt} ln \left(\frac{F_t}{F_0} \right) dF_t \right]$$

And we used the fact that $F_t e^{rt} = S_t$. Now, Carr and Madan proved that for any function ϕ on the futures price can be split as :

$$\phi(F_T) = \phi(\kappa) + \phi'(\kappa) \left[(F_T - \kappa)_+ - (\kappa - F_T)_+ \right] + \int_0^\kappa \phi''(K) (F_T - K)_+ dK + \int_\kappa^{+\infty} \phi''(K) (K - F_T)_+ dK$$

where κ represents a threshold (for example a liquidity threshold in the case of a variance or a gamma swap). We now consider the function $\phi = e^{-rT} f$, where

f is the above function. We obtain, taking $\kappa = F_0$ (the at-the-money forward spot price.

$$\phi(F_T) = \int_0^{F_0} \frac{1}{K}(K - F_T)_+ \, dK + \int_{F_0}^{+\infty} \frac{1}{K}(F_T - K)_+ \, dK$$

Inputing this equation into the above floating leg of the Gamma swap, we eventually have

$$\frac{1}{T}\int_0^T \sigma_t^2 \frac{S_t}{S_0} dt = \frac{2}{TS_0} e^{rT}\left[\int_0^{F_0} \frac{1}{K}(K - F_T)_+ \, dK + \int_{F_0}^{+\infty} \frac{1}{K}(F_T - K)_+ \, dK\right]$$

$$- \frac{2}{TS_0}\left[r\int_0^T e^{rt}\left[F_t ln\left(\frac{F_t}{F_0}\right) - F_t + F_0\right] dt + \int_0^T e^{rt} ln\left(\frac{F_t}{F_0}\right) dF_t\right]$$

Hence, a long position in a Gamma Swap consists in :

- Being long a continuum of calls and puts weighted by the inverse of the strike
- Rolling a futures position $-2ln\left(\frac{F_t}{F_0}\right)$
- Holding a zero-coupon bond, which is worth $-2r\left[F_t ln\left(\frac{F_t}{F_0}\right) - F_t + F_0\right]$ at time t.

At time $t = 0$, the fair value of the Gamma swap is hence :

$$K_\Gamma^{0,T} = \mathbb{E}_0\left(\frac{1}{T}\int_0^T \sigma_t^2 \frac{S_t}{S_0} dt\right) = \frac{2e^{2rT}}{TS_0}\left[\int_0^{F_0} \frac{1}{K} P(S_0, K, T) \, dK + \int_{F_0}^{+\infty} \frac{1}{K} C(S_0, K, T) \, dK\right]$$

Where $C(S_0, K, T)$ and $P(S_0, K, T)$ represent European Calls and Puts, written on the stock S, with strike K and maturity T, where $K_\Gamma^{0,T}$ is the strike for a gamma swaps created at time 0 with maturity T. We can also calculate the price of the Gamma Swap at time $t = T - \tau$

$$\frac{1}{T}\int_0^t \sigma_u^2 \frac{S_u}{S_0} du - K_\Gamma^{0,T} = \frac{1}{T}\int_0^t \sigma_u^2 \frac{S_u}{S_0} du + \frac{1}{T}\int_t^T \sigma_u^2 \frac{S_u}{S_0} du - K_\Gamma^{0,T}$$

The first term of the right side of the equation is past, and the two other terms are strikes. Hence :

$$\mathbb{E}_t\left(\frac{1}{T}\int_0^T \sigma_u^2 \frac{S_u}{S_0} du - K_\Gamma^{0,T}\right) = \frac{1}{T}\int_0^t \sigma_u^2 \frac{S_u}{S_0} du + \frac{\tau}{T} K_\Gamma^{t,T} - K_\Gamma^{0,T}$$

As proved in the appendix, we have the following Greeks for a Gamma Swap :

$$\begin{cases} \Upsilon_\sigma = 2\frac{\tau\sigma}{S_0 T}e^{r(T+\tau)}S_t \\ \\ \Psi = 2\frac{\tau\sigma}{S_0 T}e^{r(T+\tau)} \\ \\ \Lambda_\sigma = 2\frac{\tau}{S_0 T}e^{r(T+\tau)}S_t \\ \\ \Gamma = \frac{e^{r(T+\tau)}}{S_0 T}\frac{2}{S_t} \end{cases}$$

Both variance and gamma swaps provide exposure to volatility. However, one of the main difference, from a management point of view, is that variance swaps offer a constant cash Gamma, whereas Gamma Swaps provide a constant share Gamma, and hence does not require a dynamic reallocation. Furthermore, as Gamma Swaps is weighted by the performance of the underlying stock, it takes into account jumps in it, hence traders do not need to cap it, as it is the case for Variance Swaps (Conditional variance swaps, Up variance swaps, corridor variance swaps,. . .).

3 Correlation trading

3.1 Implied correlation

Consider an index (i.e. a basket) with n stocks. σ_i represents the volatility of the ith stock, w_i its weight within the index, and ρ_i the correlation between stocks i and j. If we replicate the index, we constitue a basket with the following volatility :

$$\sigma_I^2 = \sum_{i=1}^{n} w_i^2\sigma_i^2 + \sum_{i=1,j\neq i}^{n} w_i w_j \sigma_i \sigma_j \rho_{ij}$$

We can then define the implied correlation in this portfolio, namely an average level of correlation, as follows :

$$\rho_{imp} = \frac{\sigma_I^2 - \sum_{i=1}^{n} w_i^2\sigma_i^2}{\sum_{i=1,j\neq i}^{n} w_i w_j \sigma_i \sigma_j}$$

Where σ_I represents the volatility of the index. We can also write the above formula as :

$$\rho_{imp} = \sum_{i=1}^{n}\sum_{j>i}\rho_{ij}\frac{w_i w_j \sigma_i \sigma_j}{\sum_{i=1}^{n}\sum_{j>i}\sigma_i\sigma_j}$$

Bossu ([Bossu1]) assumed that, under some reasonable conditions, the term $\sum_{i=1}^{n} w_i\sigma_i^2$ is close to zero and hence, a good proxy for the implied correlation is

$$\rho_{imp} = \frac{\sigma_I^2}{\left(\sum_{i=1}^{n} w_i\sigma_i\right)^2}$$

3.2 Correlation Swaps

A correlation swap is an instrument similar to a variance swap, that pays at maturity the notional multiplied by the difference between the realised correlation and a strike. Mathematically speaking, the payoff of such an option is

$$\Pi = \frac{\sum_{1\leq i<j\leq n} w_i w_j \rho_{ij}}{\sum_{1\leq i<j\leq n} w_i w_j} - K$$

As for implied correlation, the realised correlation ρ above can be approximated as :

$$\rho = \frac{\sigma_I^2}{\left(\sum_{i=1}^n w_i \sigma_i\right)^2} \approx \frac{\sigma_I^2}{\sum_{i=1}^n w_i \sigma_i^2}$$

where the $(\sigma_I, \sigma_1, \ldots, \sigma_n)$ account for realised volatilities. We refer to [Bossu1] and the corresponding presentation for the details of this approximation (which is, in fact, a lower bound, thanks to Jensen's inequality). Hence, the realised correlation can be seen as the ratio between two traded products, through variance swaps, or variance dispersion trades. Based on this proxy, Bossu proves the following two points :

- The correlation swap can be dynamically quasi-replicated by a variance dispersion trade

- The P&L of a variance dispersion trading is worth $\sum_{i=1}^n w_i \sigma_i^2 (1-\rho)$, where ρ represents the realised correlation.

Though this result is really nice, several points need to be pointed out :

- Liquidity is not enough on all markets for variance swaps, neither for every index and its components.

- This model does not specify the form of the volatility. Indeed, it does not take into account the possible random moves in the volatility, namely a vol of vol parameter.

4 Dispersion Trading

4.1 P&L of a delta-hedged portfolio, with constant volatility

We here consider an option V_t written on an asset S_t. The hedged portfolio consists in being short the option, long the stock price, resulting in a certain amount of cash. Namely, the P&L variation of the portfolio Π at time t is worth

$$\Delta\Pi_t = \Delta V_t - \delta \Delta S_t + (\delta S_t - V_t)\, r \Delta t$$

The first part corresponds to the price variation of the option, the second one to the stock price move, of which we hold δ units, and the third part is the risk-free return of the amount of cash to make the portfolio have zero value. Now, the Taylor expansion of the option price has the following form :

$$\Delta V_t = \delta \Delta S + \frac{1}{2}\Gamma \left(\Delta S\right)^2 + \theta \Delta t$$

Hence, the P&L variation is now

$$\Delta \Pi_t = \delta \Delta S + \frac{1}{2}\Gamma \left(\Delta S\right)^2 + \theta \Delta t - \delta \Delta S_t + \left(\delta S_t - V_t\right) r \Delta t$$

Moreover, as the option price follows the Black & Scholes differential equation :

$$\theta + r S_t \delta + \frac{\sigma^2}{2} S_t^2 \Gamma = r V_t$$

We thus obtain the final P&L for the portfolio on $[t, t+dt]$:

$$P\&L_{[t,t+dt]} = \frac{1}{2}\Gamma S_t^2 \left[\left(\frac{dS_t}{S_t}\right)^2 - \sigma_t^2 dt \right] \tag{1}$$

4.2 P&L of a delta-hedged portfolio, with time-running volatility

We here consider the P&L of a trader who holds an option and delta-hedges it with the underlying stock. As we do wish to analyse the volatility risk, we stay in this incomplete market, as opposed to traditional stochastic volatility option pricing framework. The dynamics for the stock is now $dS_t = \mu dt + \sigma_t dW_t$. For the volatility, we assume a general type of dynamics : $d\sigma_t = \mu_{\sigma,t} dt + \xi \sigma_t dW_t^\sigma$ (with $d < W, W^\sigma >_t = \rho dt$). As before, the P&L of the trader on the period $[t, t+dt]$ is

$$\Delta \Pi_t = \Delta V_t - \delta \Delta S_t + \left(\delta S_t - V_t\right) r \Delta t$$

We now use a Taylor expansion of ΔV with respect to the time, the stock and the volatility

$$\Delta V_t = \theta dt + \frac{\partial V}{\partial S}\Delta S_t + \frac{\partial V}{\partial \sigma}\Delta\sigma + \frac{1}{2}\left[\frac{\partial^2 V}{\partial S^2}\left(\Delta S\right)^2 + \frac{\partial^2 V}{\partial \sigma^2}\left(\Delta\sigma\right)^2 + 2\frac{\partial^2 V}{\partial S \partial \sigma}\Delta S \Delta \sigma\right]$$

where σ represents the time-running volatility. Now, in the P&L formula, we can replace the $rV_t dt$ term by its value given in the Black-Scholes PDE, calculated with the implied volatility. Indeed, this is the very volatility that had to be input to determine the amount of cash to lock the position. Hence, we obtain :

$$\begin{aligned} P\&L = &\theta dt + \frac{\partial V}{\partial S} dS_t + \frac{\partial V}{\partial \sigma} d\sigma + \frac{1}{2}\left[\frac{\partial^2 V}{\partial S^2}\left(dS\right)^2 + \frac{\partial^2 V}{\partial \sigma^2}\left(d\sigma\right)^2 + 2\frac{\partial^2 V}{\partial S \partial \sigma} dS \Delta\sigma\right] \\ &- \delta S_t + r\delta S_t dt - \left(\theta + \frac{1}{2}\sigma_{\text{impt}}^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + r S_t \frac{\partial V}{\partial S}\right) dt \end{aligned}$$

Which is then worth

$$P\&L = \frac{1}{2}\Gamma S_t^2 \left[\left(\frac{dS_t}{S_t} \right)^2 - \sigma^2_{\mathrm{imp}t} dt \right] + \frac{\partial V}{\partial \sigma} d\sigma + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2} (d\sigma)^2 + \frac{\partial^2 V}{\partial S \partial \sigma} S_t \sigma_t dW_t d\sigma$$

In trading terms, this can be expressed as :

$$\boxed{P\&L = \frac{1}{2}\Gamma S_t^2 \left[\left(\frac{dS_t}{S_t} \right)^2 - \sigma^2_{\mathrm{imp},t} dt \right] + \text{Vega } d\sigma + \frac{1}{2}\text{Volga } \xi^2 \sigma_t^2 dt + \text{Vanna } \sigma_t S_t \rho \xi dt} \tag{2}$$

Where

$$\begin{cases} Vega = \frac{\partial V}{\partial \sigma} \\ \\ Volga = \frac{\partial^2 V}{\partial \sigma^2} \\ \\ Vanna = \frac{\partial^2 V}{\partial \sigma \partial S} \\ \\ \rho : \text{Correlation between the stock price and the volatility} \\ \xi : \text{Volatility of volatility} \end{cases}$$

4.3 Delta-hedged dispersion trades with $d\sigma = 0$

We consider the dispersion trade as being short the index option and long the stock options. We also consider it delta-hedged.The P&L of a delta-hedged option Π in the Black-Scholes framework is

$$P\&L = \theta \left[\left(\frac{dS}{S\sigma\sqrt{dt}} \right)^2 - 1 \right]$$

The term $n = \frac{dS}{S\sigma\sqrt{dt}}$ represents the standardised move of the underlying on the considered period. Let us now consider an index I composed by n stocks $(S_i)_{i=1,\dots,n}$. We first develop the P&L of a long position in the Index, in terms of its constituents, then decompose it into idiosyncratic risk and systematic risk. We will use the following notations :

$$\begin{cases} n_i = \frac{dS_i}{S_i \sigma_i \sqrt{dt}} : \text{standardised move of the } i\text{th stock} \\ \\ n_I = \frac{dS_I}{S_I \sigma_I \sqrt{dt}} : \text{standardised move of the index} \\ \\ p_i : \text{number of shares } i \text{ in the index} \\ w_i : \text{weight of share } i \text{ in the index} \\ \sigma_i : \text{volatility of stock } i \\ \sigma_I : \text{volatility of the index} \\ \rho_{ij} : \text{correlation between stocks } i \text{ and } j \\ \theta_i : \text{theta of the option written on stock } i \\ \theta_I : \text{theta of the option written on the index} \end{cases}$$

The P&L of the index can hence be written as :

$$\begin{aligned}
P\&L &= \theta_I \left(n_I^2 - 1\right) \\
&= \theta_I \left[\left(\sum_{i=1}^{n} w_i n_i \frac{\sigma_i}{\sigma_I} \right)^2 - 1 \right] \\
&= \theta_I \left[\sum_{i=1}^{n} \left(w_i n_i \frac{\sigma_i}{\sigma_I} \right)^2 + \sum_{i \neq j} \frac{w_i w_j \sigma_i \sigma_j}{\sigma_I} n_i n_j - 1 \right] \\
&= \theta_I \sum_{i}^{n} \frac{w_i^2 \sigma_i^2}{\sigma_I^2} \left(n_i^2 - 1\right) + \theta_I \sum_{i \neq j} \frac{w_i w_j \sigma_i \sigma_j}{\sigma_I^2} \left(n_i n_j - \rho_{ij}\right)
\end{aligned}$$

We here above used the following equalities :

$$\begin{cases}
n_I & = \frac{dI}{I \sigma_I \sqrt{dt}} \\
& = \frac{1}{\sigma_I \sqrt{dt}} \frac{\sum_{i=1}^{n} p_i dS_i}{\sum_{j=1}^{n} p_j S_j} \\
& = \frac{1}{\sigma_I} \sum_{i=1}^{n} \frac{p_i S_i}{\sum_{j=1}^{n} p_j S_j} \sigma_i \frac{dS_i}{\sigma_i S_i \sqrt{dt}} \\
& = \sum_{i=1}^{n} w_i \frac{\sigma_i}{\sigma_I} n_i
\end{cases}$$

and

$$\sigma_I^2 = \sum_{i=1}^{n} \left(w_i \sigma_i\right)^2 n_i^2 + \sum_{i \neq j} w_i w_j \sigma_i \sigma_j \rho_{ij}$$

Hence, the dispersion trade, namely shorting the Index option and being long the options on the stocks has the following P&L :

$$\begin{aligned}
P\&L &= \sum_{i=1}^{n} P\&L_i - P\&L_I \\
&= \sum_{i=1}^{n} \theta_i \left(n_i^2 - 1\right) + \theta_I \left(n_I^2 - 1\right)
\end{aligned}$$

The short and long position in the options will be reflected in the sign of the $(\theta_1, \ldots, \theta_n, \theta_I)$. More precisely, a long position will mean a positive θ whereas a short position will have a negative θ.

4.4 Weighting schemes for dispersion trading

Here above, when considering the weights of the stocks in the index, we did not specify what they precisely were. In fact, when building a dispersion trade, one faces two problems : first, which stocks to pick ? Then, how to weight them ? Indeed, as there may lack liquidity on some stocks, the trader will not take into account all the components of the index. Thus, he'd rather select those that show great characteristics and liquidity. From his point of view, he can build several weighting strategies :

- Vega-hedging weighting

The trader will build his dispersion such that the vega of the index equals the sum of the vegas of the constituents. Hence, this will immune him against short moves in the volatility.

- Gamma-hedging weighting

The Gamma of the index is worth the sum of the Gammas of the components. As the portfolio is already delta-hedged, this weighting scheme protects the trader against any move in the stocks, but leaves him with a Vega position.

- Theta-hedging weighting

This strategy is rather different from the previous two, as it will result in both a short Vega as well as a short Gamma position.

5 Correlation Swaps VS Dispersion Trades

We here focus of the core topic of our paper, namely the difference between the strike of a correlation swap and the implied correlation obtained through Dispersion Trading. Empirical proofs do observe a spread - approximately 10 points - between the strike of a correlation swap and the Dispersion implied correlation (see [Parilla2] and [Parilla3]). We first write down the relation between a dispersion trade - through variance swaps - and a correlation swap ; thanks to this relation, we analyse the influence of the dynamics of the volatility on this very spread. In the whole section, we will consider that the nominal of the Index Variance swap is equal to 1.

5.1 Analytical formula for the spread

We keep the previous notations, namely an index I, with implied volatility σ_I, realized volatility $\hat{\sigma_I}$ composed with n stocks with characteristics $(\sigma_i, \hat{\sigma_i}, w_i)_{i=1,...n}$. The implied and the realized correlation are obtained as previously :

$$\sigma_I^2 = \sum_{i=1}^{n} w_i \sigma_i^2 + \sum_{i \neq j} w_i w_j \rho \sigma_i \sigma_j$$

and

$$\hat{\sigma}_I^2 = \sum_{i=1}^{n} w_i \hat{\sigma}_i^2 + \sum_{i \neq j} w_i w_j \hat{\rho} \hat{\sigma}_i \hat{\sigma}_j$$

If we subtract these two equalities, we obtain :

$$\begin{aligned}
\hat{\sigma}_I^2 - \sigma_I^2 &= \sum_{i=1}^{n} w_i^2 \left(\hat{\sigma}_I^2 - \sigma_i^2\right) + \sum_{i \neq j} w_i w_j \left[\hat{\sigma}_i \hat{\sigma}_j \hat{\rho} - \sigma_i \sigma_j \rho\right] \\
&= \sum_{i=1}^{n} w_i^2 \left(\hat{\sigma}_I^2 - \sigma_i^2\right) + \sum_{i \neq j} w_i w_j \left[\sigma_i \sigma_j \left(\hat{\rho} - \rho\right) + \left(\hat{\sigma_i} \hat{\sigma_j} - \sigma_i \sigma_j\right) \hat{\rho}\right] \\
&= \sum_{i=1}^{n} w_i^2 \left(\hat{\sigma}_I^2 - \sigma_i^2\right) + \sum_{i \neq j} w_i w_j \sigma_i \sigma_j \left(\hat{\rho} - \rho\right) + \sum_{i \neq j} \hat{\rho} \left(\hat{\sigma_i} \hat{\sigma_j} - \sigma_i \sigma_j\right)
\end{aligned}$$

Hence, we obtain :

$$\left[\sum_{i=1}^{n} w_i^2 \left(\hat{\sigma}_I^2 - \sigma_i^2\right) - \left(\hat{\sigma}_I^2 - \sigma_I^2\right)\right] - \sum_{i \neq j} w_i w_j \sigma_i \sigma_j \left(\hat{\rho} - \rho\right) = \sum_{i \neq j} \hat{\rho} \left(\hat{\sigma_i} \hat{\sigma_j} - \sigma_i \sigma_j\right)$$

From a financial point of view, the above formula evaluates the P&L of a position consisting of being short a Dispersion Trade through Variance Swaps (Short the Index Variance Swap and Long the Components' Variance Swaps) and long a Correlation Swap. The right member, the P&L, is the spread we are considering.

5.2 Gamma P&L of the Dispersion Trade

We here only consider the Gamma part of the P&L of the variance swap of the index. We have (as in section 4.3)

$$P\&L_I^\Gamma = \frac{1}{2}\Gamma_I I_t^2 \left[\left(\frac{dI_t}{I_t}\right)^2 - \sigma_I^2 dt\right]$$

$$= \frac{1}{2}\Gamma_I I_t^2 \left[\left(\sum_{i=1}^n w_i \frac{dS_i}{S_i}\right)^2 - \left(\sum_{i=1}^n w_i^2\sigma_i^2 + \sum_{i\neq j} w_i w_j \sigma_i\sigma_j\rho_{ij}\right)dt\right]$$

$$= \frac{1}{2}\Gamma_I I_t^2 \left[\sum_{i=1}^n w_i^2 \left(\frac{dS_i}{S_i}\right)^2 + \sum_{i\neq j} w_i w_j \frac{dS_i dS_j}{S_i S_j} - \sum_{i=1}^n w_i^2\sigma_i^2 - \sum_{i\neq j} w_i w_j \sigma_i\sigma_j\rho_{ij}\right]$$

$$= \frac{1}{2}\Gamma_I I_t^2 \left[\sum_{i=1}^n w_i^2 \left[\left(\frac{dS_i}{S_i}\right)^2 - \sigma_i^2 dt\right] + \sum_{i\neq j} w_i w_j \sigma_i\sigma_j \left[\frac{dS_i dS_j}{S_i S_j \sigma_i\sigma_j} - \rho_{ij} dt\right]\right]$$

Let us make a break to analyse this formula. As before, in the context of Dispersion, we assume that the correlations ρ_{ij} between the components are all equal to an average one ρ. Furthermore, as this correlation is the one that makes the implied variance of the index and the implied variance of the weighted sum of the components equal, it exactly represents the implied correlation. Then $\frac{dS_i dS_j}{S_i S_j}$ is the instantaneous realized covariance between the two stocks S_i and S_j, and hence $\frac{dS_i dS_j}{S_i S_j \sigma_i \sigma_j}$ is precisely the instantaneous realized correlation between the two stocks. Again, we assume it is the same for all pairs of stocks, and we note it $\hat{\rho}$. Then, we can replace the weights $w_i = \frac{p_i S_i}{I}$. We therefore obtain :

$$P\&L_I^\Gamma = \frac{1}{2}\Gamma_I \left[\sum_{i=1}^n p_i^2 S_i^2 \left[\left(\frac{dS_i}{S_i}\right) - \sigma_i^2 dt\right] + I_t^2 \sum_{i\neq j} w_i w_j \sigma_i\sigma_j \left(\hat{\rho} - \rho\right) dt\right]$$

Hence, suppose we consider a position in a dispersion trade with variance swaps (α_i represents the proportion of variance swaps for the ith stock), the Gamma P&L is then worth

$$\sum_{i=1}^n \alpha_i P\&L_i^\Gamma - P\&L_I^\Gamma = \sum_{i=1}^n \frac{1}{2} S_i^2 \left[\left(\frac{dS_i}{S_i}\right) - \sigma_i^2 dt\right]\left(\alpha_i\Gamma_i - p_i^2\Gamma_I\right) + \frac{1}{2}\Gamma_I \sum_{i\neq j} p_i p_j \sigma_i\sigma_j S_i S_j \left(\rho - \hat{\rho}\right) dt$$

The P&L of the dispersion trade is hence equal to the sum of a spread between the implied and the realized correlation over a period of time $[t, t+dt]$ (pure correlation exposure) and a volatility exposure. Now, we recall that the Gamma of a variance swap for a maturity T is : $\Gamma = \frac{2}{TS^2}$. Hence, we can rewrite the Gamma P&L for the Index Variance Swap as :

$$P\&L_I^\Gamma = \sum_{i=1}^{n} \frac{1}{T} p_i^2 \frac{S_i^2}{I_t^2} \left[\left(\frac{dS_i}{S_i} \right) - \sigma_i^2 dt \right] + \frac{1}{T I_t^2} \sum_{i \neq j} p_i p_j \sigma_i \sigma_j S_i S_j \left(\hat{\rho} - \rho \right) dt$$

And hence

$$\sum_{i=1}^{n} \alpha_i P\&L_i^\Gamma - P\&L_I^\Gamma = \frac{1}{T} \sum_{i=1}^{n} \left[\left(\frac{dS_i}{S_i} \right) - \sigma_i^2 dt \right] \left(\alpha_i - p_i^2 \frac{S_i^2}{I_t^2} \right) + \frac{1}{T} \sum_{i \neq j} w_i w_j \sigma_i \sigma_j \left(\rho - \hat{\rho} \right) dt$$

which is

$$\sum_{i=1}^{n} \alpha_i P\&L_i^\Gamma - P\&L_I^\Gamma = \frac{1}{T} \sum_{i=1}^{n} \left[\left(\frac{dS_i}{S_i} \right)^2 - \sigma_i^2 dt \right] \left(\alpha_i - w_i^2 \right) + \frac{1}{T} \sum_{i \neq j} w_i w_j \sigma_i \sigma_j \left(\rho - \hat{\rho} \right) dt$$

The sum that multiplies the correlation spread does not depend on the correlation, but only on the components of the index. Hence, we can note $\beta^V = \frac{1}{T} \sum_{i \neq j} w_i w_j \sigma_i \sigma_j$ and eventually write

$$P\&L_{\text{Disp}}^\Gamma = \frac{1}{T} \sum_{i=1}^{n} \left[\left(\frac{dS_i}{S_i} \right)^2 - \sigma_i^2 dt \right] \left(\alpha_i - w_i^2 \right) + \beta^V \left(\rho - \hat{\rho} \right) dt \tag{3}$$

If we take $\alpha_i = w_i^2$, then we see that the Gamma P&L of the dispersion Trade is exactly the spread between implied and realized correlation, multiplied by a factor β which corresponds to a weighted average variance of the components of the index :

$$\boxed{P\&L_{\text{Disp}}^\Gamma = \beta^V \left(\rho - \hat{\rho} \right) dt} \tag{4}$$

In fact, as we will analyse it later, this weighting scheme is not used. However, this approximation (considering that the Gamma P&L is pure correlation exposure) is quite fair, and we will measure the induced error further in this paper.

5.3 Total P&L of the Dispersion Trade

In the previous subsection, we proved that the Gamma P&L of a Dispersion Trade is exactly a correlation P&L. Hence, the observed difference between the implied correlation of a Dispersion Trade and the strike of the correlation swap with the same characteristics (about 10 points) is precisely due to the volatility terms, namely the combined effects of the Vega, the Volga (Vomma) and Vanna. Using (2) and (4), we can now write :

$$P\&L_{\text{Disp}} = P\&L_{\text{Disp}}^\Gamma + P\&L_{\text{Disp}}^{\text{Vol}}$$

Where the $P\&L^\Gamma$ contains the correlation exposure, and the $P\&L^{\text{Vol}}$ contains all the Vegas, Volgas and Vannas. More precisely :

$$P\&L_{\text{Disp}}^{\text{Vol}} = \sum_{i=1}^{n} \alpha_i \left[\text{Vega}_i d\sigma_i + \frac{1}{2}\text{Volga}_i \xi_i^2 \sigma_i^2 dt + \text{Vanna}_i \sigma_i S_i \rho_i \xi_i dt \right]$$
$$- \left[\text{Vega}_I d\sigma_I + \frac{1}{2}\text{Volga}_i \xi_I^2 \sigma_I^2 dt + \text{Vanna}_I \sigma_I I \rho_I \xi_I dt \right]$$

When replacing the Greeks by their values for a variance swap, we obtain (the Vanna being null) :

$$P\&L_{\text{Disp}} = P\&L_{\text{Disp}}^\Gamma + 2\frac{\tau}{T}\left[\left(\sum_{i=1}^{n} \alpha_i \sigma_i^{\text{imp}} d\sigma_i \right) - \sigma^{\text{imp}} d\sigma \right] dt + \frac{\tau}{T}\left[\left(\sum_{i=1}^{n} \alpha_i \xi_i^2 \sigma_{i,t}^2 \right) - \xi_I^2 \sigma_I^2 \right] dt$$

5.4 P&L with different weighting schemes

In the following weighting schemes strategy, we will consider that the Gamma P&L of the Dispersion Trade is pure correlation exposure, hence respects (4). Concerning the notations, α_i is still the proportion of variance swaps of the ith stock ($\alpha_i = \frac{N_i}{N_I}$), and we consider $N_I = 1$, w_i the weight of stock i in the index and N_i represents the notional of the ith Variance swap. We also do not write the negative signs for the Greeks ; therefore, when writing the Greek of a product, one has to bear in mind that its sign depend on the position the trader has on this very product.

Vega flat Strategy
In this strategy, the Vega Notional of the Index Variance Swap is equal to the sum of the Vega notionals of the components :

$$N_i \Upsilon_{\sigma,i} = N_I \Upsilon_{\sigma,I} w_i$$

and hence

$$\alpha_i = \frac{\sigma_I}{\sigma_i} w_i$$

The Vegas of the P&L hence disappear and we are left

$$P\&L_{\text{Disp}} = P\&L_{\text{Disp}}^\Gamma + \frac{\tau}{T}\left[\left(\sum_{i=1}^{n} \alpha_i \xi_i^2 \sigma_{i,t}^2 \right) - \xi_I^2 \sigma_I^2 \right] dt$$

with the above mentioned approximation, we therefore have :

$$P\&L_{\text{Disp}} = \beta^V (\rho - \hat{\rho})\, dt + \frac{\tau}{T}\left[\left(\sum_{i=1}^{n} \frac{\sigma_I}{\sigma_i} w_i \xi_i^2 \sigma_{i,t}^2 \right) - \xi_I^2 \sigma_I^2 \right] dt$$

Now, the error due to the approximation in the Gamma P&L is worth $\sum_{i=1}^{n} \left[\left(\frac{dS_i}{S_i} \right)^2 - \sigma_i^2 dt \right] \left(\alpha_i - w_i^2 \right)$. We focus on the $\left(\alpha_i - w_i^2 \right)$ part. Here we

have :
$$\alpha_i - w_i^2 = w_i \frac{\sigma_I}{\sigma_i} - w_i^2 = w_i^2 \left(\frac{\sigma_I}{w_i \sigma_i} - 1 \right)$$
which is indeed very close to 0. From a very theoretical point of view, this formula tells us that the observed difference between the strike of a correlation swap and the implied correlation via VarSwap dispersion trades can be simply explained by the Volga of the dispersion trade, hence, by the vol of vol terms.

Vega weighted flat Strategy
In this strategy, we have :
$$N_i \Upsilon_{\sigma,i} = N_I \Upsilon_{\sigma,I} w_i \frac{\sigma_I}{\sum_{j=1}^n w_j \sigma_j}$$
and hence
$$\alpha_i = \frac{\sigma_I}{\sigma_i} w_i \frac{\sigma_I}{\sum_{j=1}^n w_j \sigma_j}$$
We are left with
$$P\&L_{\text{Disp}} = P\&L^{\Gamma}_{\text{Disp}} + \frac{\tau}{T} \left[\left(\sum_{i=1}^n \alpha_i \xi_i^2 \sigma_{i,t}^2 \right) - \xi_I^2 \sigma_I^2 \right] dt + \frac{\tau}{T} \left[\left(\sum_{i=1}^n \alpha_i \xi_i \sigma_{i,t}^2 \right) - \xi_I^2 \sigma_I^2 \right] dt$$
with the above mentioned approximation, we therefore have :
$$P\&L_{\text{Disp}} = \beta^V (\rho - \hat{\rho})\, dt + \frac{\tau}{T} \left[\left(\sum_{i=1}^n \frac{\sigma_I}{\sigma_i} w_i \frac{\sigma_I}{\sum_{j=1}^n w_j \sigma_j} \xi_i^2 \sigma_{i,t}^2 \right) - \xi_I^2 \sigma_I^2 \right] dt$$
$$+ \frac{\tau}{T} \left[\left(\sum_{i=1}^n \frac{\sigma_I}{\sigma_i} w_i \frac{\sigma_I}{\sum_{j=1}^n w_j \sigma_j} \xi_i^2 \sigma_{i,t}^2 \right) - \xi_I^2 \sigma_I^2 \right] dt$$

The error due to the approximation in the Gamma P&L is then worth
$$\alpha_i - w_i^2 = w_i \frac{\sigma_I}{\sigma_i} \frac{\sigma_I}{\sum_{j=1}^n w_j \sigma_j} - w_i^2 = w_i^2 \left(\frac{\sigma_I}{w_i \sigma_i} \frac{\sigma_I}{\sum_{j=1}^n w_j \sigma_j} - 1 \right)$$
which is indeed very close to 0. From a very theoretical point of view, this formula tells us that the observed difference between the strike of a correlation swap and the implied correlation via VarSwap dispersion trades can be simply explained by the Volga of the dispersion trade, hence, by the vol of vol terms.

Theta/Gamma flat Strategy
Suppose we want to get rid of the Gamma P&L of the dispersion. Recalling its value :
$$P\&L^{\Gamma}_{\text{Disp}} = \frac{1}{2} \sum_{i=1}^n \alpha_i \Gamma_i S_i^2 \left[\left(\frac{dS_i}{S_i} \right)^2 - \sigma_i^2 dt \right] - \frac{1}{2} \Gamma_I I_t^2 \left[\left(\frac{dI_t}{I_t} \right)^2 - \sigma_I^2 dt \right]$$

we thus need to set

$$\alpha_i = \frac{\Gamma_I I_t^2 \left[\left(\frac{dI_t}{I_t} \right)^2 - \sigma_I^2 dt \right]}{\sum_{i=1}^n \Gamma_i S_i^2 \left[\left(\frac{dS_i}{S_i} \right)^2 - \sigma_i^2 dt \right]}$$

replacing the Γ by their values, we get

$$\alpha_i = \frac{\left[\left(\frac{dI_t}{I_t} \right)^2 - \sigma_I^2 dt \right]}{\sum_{i=1}^n \left[\left(\frac{dS_i}{S_i} \right)^2 - \sigma_i^2 dt \right]}$$

On a very short period, we almost have $\left(\frac{dS}{S}\right)^2 = 0$, and hence, this is also a Theta flat strategy (with no interest rate. Actually the difference between the Gamma and the Theta flat strategies is the difference between the risk-free rate and the return of the stocks over the period we consider). Furthermore, the dispersion trade is fully exposed to moves in volatility, namely through the Vegas and the Vannas of the variance swaps.

6 Conclusion

We here dealt with dispersion trading, and we showed the P&L of such a strategy, considering both Variance Swaps and Gamma Swaps. The first one are particularly appealing because of their Greeks, which enable us to have a clear vision of our exposure. The main result of our paper is that we proved that the observed spread between implied correlation through Variance Swaps dispersion trades and fair values of correlation swaps is totallyy explained by a vol of vol parameter. We also developped results for Gamma Swaps Dispersion trades and different weighting schemes, one of them - the Vega Flat weighting strategy - being an arbitrage bound. This also gives us a way of estimating the vol of vol parameter, based on the observed prices of variance and correlation swaps.
This work could be analyzed deeper when considering third-generation exotic product such as Corridor Variance Swaps, Up Variance Swaps,. . . . They indeed allow investor to bet on future realized variance at a lower cost. Similar results should be found, but with less elegant formulas, as the stock price - just like for Gamma Swaps - will have to be taken into account.

References

[Avellaneda] Avellaneda, M., (2002), Empirical aspects of dispersion trading in US Equity markets, *Petit Déjeûner de la Finance, Nov 2002*

[Blanc1] Blanc, N., (2004), Index Variance Arbitrage : Arbitraging Component Correlation, *BNP Paribas Technical Studies*

[Bossu1] Bossu, S., (2006), A new approach for modelling and pricing correlation swaps in equity derivatives, *Global Derivatives Trading & Risk Management, May 2006*

[Bossu3] Bossu, S., (2005), Arbitrage pricing of equity correlation swaps, *JP Morgan Equity Derivatives*

[Bossu4] Bossu, S., Gu, Y., (2004), Fundamental relationship between an index's volatility and the correlation and average volatility of its components, *JP Morgan Equity Derivatives*

[Branger] Branger, N., Schlag, C., (2003), Why is the index smile so steep ?, *EFMA 2003 Helsinki Meetings*

[Brenner] Brenner, M., Ou, E.Y., Zhang, J.E. (2006), Hedging volatility risk, *Journal of Banking & Finance* 30, 811-821

[CM] Carr, P., Madan, D., (2005), FAQ in option pricing theory, *Journal of Derivatives, forthcoming*

[CaMad] Carr, P., Madan, D., (1998), Towards a theory of volatility trading, *Volatility : New estimation techniques for pricing derivatives*, 417-427

[Dup] Dupire, B., (2004), Understanding and implementing volatility and correlation arbitrages, *Global Derivatives Conference*

[Mougeot1] Mougeot, N., (2005), Volatility Investing Handbook, *BNP Paribas Technical Studies*

[Mougeot2] Mougeot, N., (2006), Smile trading, *BNP Paribas Technical Studies*

[Parilla1] Parilla, R., (2006), Trade pure volatility via Variance Swaps, *SGCIB Hedge Fund Group*

[Parilla2] Parilla, R., (2006), Play Dispersion Trades via Variance Swaps, *SGCIB Hedge Fund Group*

[Parilla3] Parilla, R., (2006), Correlation Swap, the only instrument to trade "pure" realized correlation, *SGCIB Hedge Fund Group*

A Vega of a Gamma Swap

As we developed it above, the value of a Gamma Swap at inception is worth

$$\mathbb{E}_0\left(\frac{1}{T}\int_0^T \sigma_t^2 \frac{S_t}{S_0} dt\right) = \frac{2e^{2rT}}{TS_0}\left[\int_0^{F_0} \frac{1}{K} P\left(S_0, K\right) dK + \int_{F_0}^{+\infty} \frac{1}{K} C\left(S_0, K\right) dK\right]$$

Its Vega at inception is then (we here use the fact that the Vega of a Call option is equal to the Vega of a Put option)

$$\begin{aligned}
\Upsilon_\sigma^\Gamma &= \frac{2e^{2rT}}{TS_0}\int_0^{+\infty} \frac{1}{K}\frac{\partial O_K}{\partial \sigma} dK \\
&= \frac{2e^{2rT}}{TS_0}\int_0^{+\infty} \frac{1}{K} S_0 \sqrt{T}\phi\left(d_1\right) dK \\
&= \frac{2e^{2rT}\sqrt{T}}{TS_0} S_t \int_0^{+\infty} \frac{1}{K}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2 T}\left[ln(S_0/K)+\left(r+\sigma^2/2\right)T\right]^2} dK \\
&= \frac{2e^{2rT}\sqrt{\tau}}{TS_0} S_0 \int_0^{+\infty} \frac{1}{K}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2 T}\left[ln(K)-\left(ln(S_0)+\left(r+\sigma^2/2\right)T\right)\right]^2} dK
\end{aligned}$$

Let us do the following change of variable : $x = \frac{1}{\sigma\sqrt{T}}\left[ln\left(K\right) - \left(ln\left(S_0\right) + \left(r + \frac{\sigma^2}{2}\right)T\right)\right]$. We then have :

$$\begin{aligned}
\Upsilon_\sigma^\Gamma &= \frac{2e^{2rT}\tau\sigma}{S_0 T} S_0 \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= 2\sigma e^{2rT}
\end{aligned}$$

Moreover, at time $t = T - \tau$, we have

$$\mathbb{E}_t\left(\frac{1}{T}\int_0^T \sigma_u^2 \frac{S_u}{0} du\right) = \frac{1}{T}\int_0^t \sigma_u^2 \frac{S_u}{0} du + \frac{\tau}{T}\frac{S_t}{S_0} K_\Gamma^{t,T}$$

Hence, the Vega of the Gama Swap at time t is worth

$$\Upsilon_\sigma^\Gamma\left(t\right) = 2\sigma e^{2r\tau}\frac{\tau}{T}\frac{S_t}{S_0}$$

B Gamma of a Gamma Swap

As for the Vega of the Gamma Swap, we have :

$$\Gamma^{\Gamma} = \frac{2e^{2rT}}{TS_0} \int_0^{+\infty} \frac{1}{K} \frac{\partial^2 O_K}{\partial S^2} dK$$

$$= \frac{2e^{2rT}}{TS_0} \int_0^{+\infty} \frac{1}{K} \frac{\phi(d_1)}{S_0 \sigma \sqrt{T}} dK$$

$$= 2 \frac{e^{2rT}}{TS_0^2}$$

We here used the same change of variable as for the Vega of the Gamma Swap. We can also calculate the Gamma of the Gamma Swap at time $t = T - \tau$

$$\Gamma^{\Gamma}(t) = \frac{2}{T} \frac{e^{2r\tau}}{S_0 S_t}$$

C P&L of a Gamma Swap

We here consider the P&L of a Gamma Swap. The calculations below are almost identical to those of Part 5.3.

$$\sum_{i=1}^{n} \alpha_i P\&L_i^{\Gamma} - P\&L_I^{\Gamma} = \sum_{i=1}^{n} \frac{1}{2} S_i^2 \left[\left(\frac{dS_i}{S_i} \right) - \sigma_i^2 dt \right] \left(\alpha_i \Gamma_i - p_i^2 \Gamma_I \right) + \frac{1}{2} \Gamma_I I_t^2 \sum_{i \neq j} w_i w_j \sigma_i \sigma_j (\rho - \hat{\rho})\, dt$$

Since the Gamma of a Gamma swap for a maturity T, at time t is $\Gamma = \frac{2}{T} \frac{e^{2r\tau}}{S_0 S_t}$, we have

$$P\&L_{\text{Disp}}^{\Gamma} = \frac{1}{2} \sum_{i=1}^{n} \left[\left(\frac{dS_i}{S_i} \right) - \sigma_i^2 dt \right] \left(\alpha_i \Gamma_i S_i^2 - w_i^2 \Gamma_I I_t^2 \right) + \frac{1}{2} \Gamma_I I_t^2 \sum_{i \neq j} w_i w_j \sigma_i \sigma_j (\rho - \hat{\rho})\, dt$$

$$= \frac{2}{T} e^{2r\tau} \sum_{i=1}^{n} \left[\left(\frac{dS_i}{S_i} \right) - \sigma_i^2 dt \right] \left(\alpha_i \frac{S_i}{S_0} - w_i^2 \frac{I_t}{I_0} \right) + \frac{1}{T} e^{2r\tau} \frac{I_t}{I_0} \sum_{i \neq j} w_i w_j \sigma_i \sigma_j (\rho - \hat{\rho})\, dt$$

The sum that multiplies the correlation spread does not depend on the correlation, but only on the components of the index. Hence, we can note $\beta^{\Gamma} = \frac{1}{T} e^{2r\tau} \frac{I_t}{I_0} \sum_{i \neq j} w_i w_j \sigma_i \sigma_j$ and eventually write

$$P\&L_{\text{Disp}}^{\Gamma} = \frac{2}{T} e^{2r\tau} \sum_{i=1}^{n} \left[\left(\frac{dS_i}{S_i} \right) - \sigma_i^2 dt \right] \left(\alpha_i \frac{S_i}{S_0} - w_i^2 \frac{I_t}{I_0} \right) + \beta^{\Gamma} (\rho - \hat{\rho})\, dt$$

We can now write the total P&L of the Gamma Swap, as well as the one for the dispersion trade via Gamma Swaps :

$$P\&L_I = P\&L_I^\Gamma + \text{Vega}_I d\sigma_I + \frac{1}{2}\text{Volga}_I \xi_I^2 \sigma_I^2 dt + \text{Vanna}_I \sigma_I I \rho_I \xi_I dt$$

$$= P\&L_I = P\&L_I^\Gamma + 2\sigma e^{2r\tau} \frac{I_t}{I_0} \left[d\sigma_I + \frac{1}{2}\xi_I^2 \sigma_I dt + \rho_I \xi_I dt \right]$$

Hence

$$P\&L_{\text{Disp}} = P\&L_{\text{Disp}}^\Gamma + 2e^{2r\tau} \left[\sum_{i=1}^{n} \alpha_i \sigma_i \frac{S_i}{S_0} \left(d\sigma_i + \frac{1}{2}\xi_i^2 \sigma_i dt + \rho_i \xi_i dt \right) - \sigma_I \frac{I_t}{I_0} \left(d\sigma_I + \frac{1}{2}\xi_I^2 \sigma_I dt + \rho_I \xi_I dt \right) \right]$$

D Arbitrage opportunity condition and Vega weighted flat strategy for VarSwap Dispersion

We here analyse the Vega weighted flat strategy in terms of arbitrage opportunities. We only consider a Dispersion Trade via Variance Swaps. $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an arbitrage opportunity if and only if for any $\hat{\sigma} = (\hat{\sigma}_I, \hat{\sigma}_1, \ldots, \hat{\sigma}_n)$, we have

$$\sigma_I^2 - \hat{\sigma}_I^2 + \sum_{i=1}^{n} \left(\hat{\sigma}_i^2 - \sigma_i^2 \right) \geq 0$$

Rearranging the terms, we have :

$$\forall \hat{\sigma} : \left(\sum_{i=1}^{n} \hat{\sigma}_i^2 - \hat{\sigma}_I^2 \right) + \left(\sigma_I^2 - \sum_{i=1}^{n} \sigma_i^2 \right) \geq 0$$

In particular, if $\hat{\sigma} = 0$, then $\sigma_I^2 - \sum_{i=1}^{n} \alpha_i \sigma_i^2 \geq 0$, i.e :

$$\frac{\sum_{i=1}^{n} \alpha_i \sigma_i^2}{\sigma_I^2} \leq 1$$

If we consider the Vega weighted flat strategy, we have

$$\alpha_i = \frac{w_i \sigma_I^2}{\sigma_i \sum_{j=1}^{n} \alpha_j \sigma_j}$$

With this weighting schemes, we see that

$$\frac{\sum_{i=1}^{n} \alpha_i \sigma_i^2}{\sigma_I^2} = 1$$

Hence, the Vega weighted flat strategy represents the boundary condition for arbitrage opportunity.