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Wage-Directed Job Match with Multiple Applications and Multiple Vacancies: The Optimal Job Application Strategy and Wage Dispersion

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Wage-Directed Job Match with Multiple Applications and Multiple Vacancies: The Optimal Job Application Strategy and Wage Dispersion

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Abstract

This paper develops a model of directed-search where workers' preference for a higher wage is explicitly modelled into their application strategy. In a general setting where jobs offer non-uniform wages and different probabilities of a job offer, the optimal strategy for selecting the set of applied jobs is established. In applying this to a homogeneous-workers job-matching market, the equilibrium outcome is then shown to entail wage dispersion when firms have non-uniform labour demand. Finally a matching function is derived that captures both urn-ball and multiple-applications frictions, that nests many of the existing functions.

JEL Classification: J31, J64

Keywords: directed-search, matching function, wage dispersion

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1 Introduction

This paper investigates a search model combining two of the branches of search theory that have received much attention recently, namely those of multiple-applications matching and directed-search, to first investigate a matching process that incorporates workers' preference for a higher wage, and second to use the result to suggest a new explanation for wage dispersion. Traditional matching functions are derived in an urn-ball set-up where a matching market contains firms consisting of a single vacancy, and workers each making a single application (e.g. Pissarides, 1979; Blanchard and Diamond, 1994). However Albrecht, Gautier and Vroman (2006) point out that there are two coordination frictions that operate simultaneously in a matching market: (1) urn-ball friction, where some vacancies receive no application while others receive more than one, and (2) multiple-applications friction, where some workers receive multiple offers while others receive none. The traditional models only capture the first. On the other hand Julien, Kennes and King (2000), in which firms make a job offer to one worker after observing all applicants' reservation wages, capture the second but not the first. More recent models allow multiple-applications (e.g. Galenianos and Kircher, 2005; Albrecht et al., 2006). Amongst them, Albrecht, Gautier, Tan and Vroman (2004) and Hori (2007) both derive matching functions with workers making multiple applications to firms consisting of a single vacancy. The model here builds on the latter by allowing firms to advertise more than one vacancies, an extension which leads to a result that uniform-wage cannot be an equilibrium outcome even when workers are homogeneous.[1] This is the first feature of the model in this paper.

The second feature is that it is a model of directed-search.[2] Shi (2007) points out three problems with undirected-search models: (1) in reality searching workers do often have information about posted wages, (2) it has been a long tradition in

[1]Burdett, Shi and Wright (2001) consider multiple-vacancies firms, but with single-application workers. They conclude that with heterogeneous firms, frictions are more problematic when there are more firms with limited number of vacancies. Hori (2005) also considers heterogeneous multiple-vacancies firms, and derives an aggregate matching function that captures both the frictions caused by co-ordination failure and heterogeneity.

[2]Examples of these include Montgomery (1991), Acemoglu and Shimer (1999), Burdett, Shi and Wright (2001), and more recently Galenianos and Kircher (2005) and Albrecht et al. (2006).

economics to treat prices as a useful mechanism to direct the allocation of resources ex-ante, and (3) with undirected-search wage dispersion disappears if searching workers can view the posted wages.[3] In general in macroeconomic matching models the equilibrium wage is determined as a result of bargaining ex-post of the match (e.g. Pissarides, 2000). This paper follows those models that reinstate the traditional role of wage as the ex-ante resource allocation tool and establish wage dispersion as the equilibrium outcome. An earlier attempt at this was Moen (1997), in which the labour market is divided into submarkets, each of which with an assigned exogenously determined wage. Observing this workers choose a submarket to join, within which matchings occur. Wage dispersion is then the result of a trade-off between the wage level and the expected duration of unemployment period in the submarket. Unfortunately with this set-up the matching mechanism itself, and therefore also the matching function, are independent of the wage level within the submarkets. In this paper the preference of the applicants for higher paying jobs is explicitly modelled into the matching mechanism. In equilibrium when homogeneous workers all play the same equilibrium strategy, the distribution of the wage levels affect the resulting application pattern of the workers. This in turn affects two probabilities at a hiring firm: for workers, the probability of a job offer from the firm, and for the firm, the probability of job acceptance when an offer is made. A higher wage lowers the former and raises the latter, which are the two trade-offs that allow non-uniform wages. The result attained here is that when firms' labour demands are non-uniform, then the wage offers must also be non-uniform. This is because the different number of job vacancies advertised at firms affect both of the probabilities above, which must then be offset by different wage levels for the trade-offs. Hence with a job-matching market with multiple-vacancies firms, uniform-wage cannot be an equilibrium wage offer even when workers are homogeneous.

More specifically, the paper begins with a formal modelling of a wage-directed job search, in a general set-up where each job is defined by its wage and its probability

[3] Some models allow firms to post wages but applicants do not know who posted which. For examples of these see Burdett and Mortensen (1998) or Burdett and Coles (2003).

of a job offer. The wage is uniform to all workers, but the probability depends on factors such as worker-job skills match, and hence is unique to an applicant. Once job offers are made, an applicant selects the job that pays the highest wage. The question posed is then when multiple-applications is allowed, to which jobs, and to how many of them, should a worker apply to. The problem is complicated by the fact that an additional job application reduces the benefit of those jobs already applied to that pay less wages, due to the reduced probability of accepting them (the 'survival rate') if job offers are received. By analyzing this survival rate, here the application strategy that outlines the optimal selection process of the set of jobs to apply to is derived. The rule is to always choose the Next Best Choice, defined as the job that yields the highest marginal benefit from its addition to the set of already applied jobs, or the one offering the highest wage if there were more than one such. The optimal number of job applications is then determined by the condition that the marginal benefit be larger than the cost of application.

Once this wage-directed search mechanism for jobs is established, this is then applied to the homogeneous-workers job-matching market to investigate the equilibrium outcome. By defining applicant types by the jobs to which workers apply to, the workers' problem becomes that of selecting a mixed-strategy of choosing the types. In equilibrium all workers play the same mixed-strategy, which determines the probability distribution of the applications patterns outcome (i.e. how many applications are received at each job). Given these probabilities, in equilibrium workers are indifferent between the jobs, as a result of a trade-off between the wage level and the probability of a job offer, and the firms are indifferent between different wage levels that they can offer, a result of a trade-off between the wage level and the job acceptance probability. The matching outcomes of both the uniform-wage and distinct-wages cases are analyzed, and it is shown that for heterogeneous job demands of the firms, wages must also vary to establish the trade-offs as already stated. Thus wage dispersion exists in equilibrium despite workers being homogeneous. The matching function is then derived for this multiple-applications, multiple-vacancies job matching market, that captures both the urn-ball and the multiple-applications frictions. Most matching functions in the literature are nested in this derived function.

Similar results are obtained but for different reasons by Galenianos and Kircher (2005, 2007) who also consider wage-directed search. In their first model multiple-applications workers view their applications as a portfolio choice problem, and are thus willing to apply to jobs offering different wage levels (i.e. 'risk-diversify'). This incentivise firms to post different wages. In equilibrium every worker applies once to each distinct wage. In their second model wage dispersion is driven by fundamentals where more productive firms post higher wages. However both of these assume single-vacancy firms; here the multiple-vacancy element drives the wage dispersion result.

Some evidence of wage dispersion are surveyed in Mortensen (2003), where it is estimated that "observable worker characteristics that are supposed to account for productivity differences typically explain no more than 30 percent of the variation in compensation across workers" (p.1). Hornstein, Krusell and Violante (2006) also state that factors related to human capital theory, the theory of compensating differentials and the models of discrimination (see Hornstein et al. for explanations of these) explain at most one third of the total wage variation. Different explanations are offered in the literature for the remaining wage variation (sometimes termed as the 'frictional wage dispersion' (Hornstein et al., 2006)); some rely on productivity heterogeneity of the firms (Montgomery, 1991; Acemoglu and Shimer, 2000; Galenianos and Kircher, 2007), while others on the reservation wage heterogeneity of the workers (Albrecht and Axell, 1984). Those assuming homogeneous firms and workers rely on asymmetric information, with workers having only partial information regarding wages offered by firms (Burdett and Judd, 1983; Mortensen, 2003). This paper adds to this list an alternative explanation.

This paper is structured as follows. Section 2 establishes the optimal job application strategy. Section 3 then investigates the equilibrium wage offers in a homogeneous-workers job-matching market. Section 4 derives the matching function and compares it with existing functions in the literature. Section 5 then .

2 Job Application Strategy

2.1 The Labour Market

Consider a job-matching market where firms offer one or more vacancies for each advertised job, and workers apply to one or more vacancies. In this paper I distinguish between a job and a vacancy: a job is defined by its characteristics, and consists of one or more of its vacancies. The job characteristics are factors such as its job specification, the wage level and the number of vacancies. In line with reality, it is assumed that workers can only apply once to a particular job, irrespective of the labour demand at that job.

The job-matching market operates in the following manner:

1. Firms $j = 1, ..., J$ determine the characteristics of the jobs, including the wage level w_j and the number of vacancies L_j, and advertise. Each firm's wage offer is the same for all applicants.

2. Workers $i = 1, ..., I$ view the advertised jobs and their characteristics, and each worker estimates his probability p_{ij} of receiving a job offer from job j, which depends on the degree of his match to the job's specification, as well as the popularity of the job by other candidates. The workers then each select an optimal set of jobs to apply to, and apply.

3. Firms with more than one applications per vacancy select one candidate for each vacancy and make a job offer.

4. Applicants with one or more job offers accept one job of their highest preference.

5. Applicants with no job offer remain unemployed. Vacancies with no application, or with a selected candidate who rejects its job offer, remain unfilled.

I assume no post-match wage renegotiation. Also in this section no assumptions are made about the probabilities of a job offer p_{ij}. Hence for example a possibility of a better paying job offering a higher probability of successful application is not ruled out. For now I take these probabilities to be given. The labour demand at each job $\mathbf{L} = (L_1, ..., L_J)$ is also assumed given exogenously; the situation envisaged is one where firms are replacing a random number of lost workers. $\mathbf{L}$, as well as the wage vector $\mathbf{w} = (w_1, ..., w_J)$, are known by all parties. Applications are on the other hand private in that the application pattern of a worker is only known by the applicant himself, and the number of received applications at each firm is only known by the firm.

Now let the set of available jobs be denoted by $\Omega = \{1, 2, ..., J\}$. The set of applied jobs for worker i is a subset $A_i \subset \Omega$. The investigation in this section is to formally model the mapping from Ω to A_i, which is the optimal job application strategy. Without loss of generality then, let jobs be numbered in the descending order of their wage levels $w_1 \geq w_2 \geq ... \geq w_J$. Jobs offering the same wage level are ordered, for a particular worker, in the descending order of their associated job offer probabilities. Identical jobs (i.e. those with the same wage level and the same probability) are ordered randomly between them. The ordering is therefore unique to a worker. In this analysis I focus on a representative applicant i, and for notational brevity the subscript i is suppressed for the rest of this section. Now given the set Ω of jobs ordered as described above, again without loss of generality the following assumptions are made regarding a worker's preference between the jobs. First in choosing jobs to apply to, when an applicant has a choice between two or more jobs with identical preferences, then he will always choose the one furthest to the left (i.e. the smallest numbered). Second once jobs are offered, the worker will always prefer the one with the highest wage. If he receives more than one job offers with the same wage, then again the applicant chooses the one assigned with the smallest number.

For simplification it is assumed that workers' utility from a matched job is solely monetary, although this can easily be modified. Therefore once an applicant is matched with job j, his benefit before cost is the wage income w_j. In a more general framework this may be the discounted present value of the expected life-time benefit. As the purpose here is to establish the optimal job application strategy, the static one-period model of jobs is adopted. The cost of application is assumed to be uniform at c. The expected net benefit from a worker's first application, to say job j, is then

$p_j w_j - c$. For his second application, this time to job k, the total expected net benefit from the two applications is either $p_j w_j + (1-p_j)p_k w_k - 2c$ or $p_k w_k + (1-p_k)p_j w_j - 2c$, depending on whether job k is to the right or to the left of j in the numbering system. The applicant's objective is then to choose the optimal set A^* of a^* applied jobs, with its corresponding set of wages W^*, that maximizes the following total expected net benefit,

$$
\begin{aligned}
&\max_{A \subset \Omega} Ew(A) - ac \\
\text{where } Ew(A) &= \sum_{j \in A} s(A^{<j}) p_j w_j \qquad (1) \\
\text{and } s(A^{<j}) &= \prod_{k \in A, k<j} (1 - p_k)
\end{aligned}
$$

Here $A^{<j}$ denotes the subset $\{ k \in A | \, k < j \}$ of the set A of chosen a jobs, and $s(A^{<j})$ is the survival rate for job j, i.e. the probability that, if offered, the worker will accept job j. $Ew(A)$ is then the expected wage income from applying to set A of chosen jobs. The worker would only apply to another job, say l, if the marginal increase in the expected wage income $Ew_l(A')$ of forming a new set A' is strictly greater than c, where $Ew_l(A')$ is given by,

$$
\begin{aligned}
Ew_l(A') &= s(A'^{<l}) \left[p_l w_l - p_l \sum_{j \in A', j>l} s(A'^{(l,j)}) p_j w_j \right] \\
&= s(A'^{<l}) \left[Ew(\{l\}) - p_l Ew(A'^{>l}) \right] \qquad (2)
\end{aligned}
$$

where $A'^{(l,j)}$ denotes the set $\{ k \in A' | \, l < k < j \}$. Adding a new job to a set of already applied jobs therefore has two opposing effects: a positive effect of increasing the expected income by its own contribution $s(A'^{<l}) Ew(\{l\})$, and a negative effect of reducing the contribution of those jobs on the right of it by a factor p_l. The condition $Ew_l(A') > c$ then determines the number of applications a. Note however that different sequences of chosen jobs may lead to different stopping points; the challenge is to find the optimal set of jobs A^* with its corresponding optimal a^*.

To demonstrate this problem, consider the following example of a set of available jobs, with their corresponding wage levels and the probabilities of a job offer:

j	1	2	3	4	5	6	7	8	9	10
w_j	100	90	80	70	60	50	40	30	20	10
p_j	0.13	0.12	0.25	0.24	0.20	0.25	0.49	0.50	0.95	0.90
$p_j w_j$	13.0	10.8	20.0	16.8	12.0	12.5	19.6	15.0	19.0	9.0

$<$ **Table 1: Example of Wages and Job Offer Probabilities** $>$

The last row shows the expected wage income $Ew(\{j\}) = p_j w_j$ from applying to the job on its own. As shown this is the highest for job 3. So consider the case that the worker first applies to job 3. Contemplate further the case that the applicant uses a strategy whereby he selects the next job to apply to that yields the largest marginal increase in the expected wage income using (2). This in this case turns out to be job 7, for which $Ew_7(\{3,7\}) = (1-p_3) \times Ew(\{7\}) = 14.7$. Now instead consider the case where the worker first applies to job 4. Then the job with the largest $Ew_j(\{4,j\})$ is this time job 3, for which $Ew_3(\{3,4\}) = 15.8$. Therefore if the cost of application c was 15.0, then in the first case the applicant would stop applying after the first job, while in the second case he would continue applying to the second. However the worker in this example is in fact better off in the first case, as the expected net benefits are $20.0 - 15.0 = 5.0$ and $16.8 + 15.8 - 15.0 \times 2 = 2.6$ respectively. It turns out that this strategy of always adding the next job with the highest marginal contribution, as long as it is above the cost of application, is the optimal job application strategy, with its optimal number of applied jobs a^*. For example then for $c = 9$, the optimal set of applied jobs is $A^* = \{3,4,7\}$, with $a^* = 3$.[4]

2.2 Optimal Job Application Strategy

To prove the optimal job application strategy it is first useful to note the following mathematical property of $Ew(.)$,

[4] It is therefore not optimal for example to apply to the job with the highest individual expected wage income $Ew(\{j\})$. In this case job 9 is not in A^*, despite $Ew(\{9\})$ being larger than $Ew(\{4\})$.

Property 1 *Given $w_1 \geq w_2 \geq ... \geq w_n$ for jobs $j = 1, ..., n$ in the set of n applied jobs $A_n \subset \Omega$,*

$$Ew(A_n) \leq w_1$$

This property is intuitive: the best an applicant can do is to be offered the highest paying job for sure.[5] Using this I first show the following,

Proposition 1 *Given jobs offering the same wage level, the applicant will always choose to apply to the left-most job.*

This is true by assumption for the case that the jobs have the identical probability of a job offer. When they do not, then the job with the highest probability, say job k, would be on the furthest left. It is not immediately obvious that k would be the best choice of job to apply to, due to its larger negative effect on the jobs $j \in A^{>k}$ to the right of job k in the set of applied jobs A (i.e. the second term in (2)). This is checked in the following proof,

Proof. Consider two jobs k and $k+1$, with $w_k = w_{k+1}$ but $p_k > p_{k+1}$. The marginal increases in the expected wage income of adding each of these jobs to the set A of already applied jobs are, using (2), $Ew_k(A \cup \{k\}) = s(A^{<k}) \left[Ew(\{k\}) - p_k Ew(A^{>k+1})\right]$ and $Ew_{k+1}(A \cup \{k+1\}) = s(A^{<k}) \left[Ew(\{k+1\}) - p_{k+1} Ew(A^{>k+1})\right]$ respectively. Substituting $Ew(\{k\}) = p_k w_k$ and $Ew(\{k+1\}) = p_{k+1} w_k$ the difference is,

$$Ew_k(A \cup \{k\}) - Ew_{k+1}(A \cup \{k+1\}) = s(A^{<k})(p_k - p_{k+1}) \left\{w_k - Ew(A^{>k+1})\right\}$$

which is non-negative using Property 1. Hence k is weakly preferred to $k+1$, and the applicant will choose the left-most job k. ■

[5] Mathematically, for example for $n = 3$,

$$\begin{aligned} Ew(A_3) &= p_1 w_1 + (1 - p_1) p_2 w_2 + (1 - p_1)(1 - p_2) p_3 w_3 \\ &\leq \{p_1 + (1 - p_1) p_2 + (1 - p_1)(1 - p_2) p_3\} w_1 \\ &\leq [p_1 + (1 - p_1)\{p_2 + (1 - p_2)\}] w_1 \\ &= w_1 \end{aligned}$$

This is true for all $n \geq 1$.

I can now investigate the optimal set of applied jobs A^*. First define the following,

Definition 1 (Best Choice) *Given a set of advertised jobs Ω, with the corresponding set of wages W, the Best Choice job for an applicant is $b_1 \in \Omega$ such that,*

1. $p_{b_1} w_{b_1} \geq p_j w_j \; \forall j \in \Omega$, *and*
2. $b_1 < j \; \forall j$ *such that* $p_{b_1} w_{b_1} = p_j w_j$.

Then,

Proposition 2 $b_1 \in A^*$.

Proof. Suppose not. Then from Proposition 1 the set A^* cannot contain any jobs offering the same wage as b_1. Consider then the following two cases:

1. $b_1 < \max\{A^*\}$, i.e. $\exists \; r \in A^*$ with corresponding wage w_r that is the next largest in W^*. Thus $w_{b_1} > w_r$. Now compare two sets A^* and A', the latter of which is formed by replacing r in A^* with b_1. The expected wage income of each set can be expanded as,

$$\begin{aligned} Ew(A^*) &= Ew(A^{*<r}) + s(A^{*<r})Ew(\{r\}) + s(A^{*<r})(1-p_r)Ew(A^{*>r}) \\ Ew(A') &= Ew(A'^{<b_1}) + s(A'^{<b_1})Ew(\{b_1\}) + s(A'^{<b_1})(1-p_{b_1})Ew(A'^{>b_1}) \end{aligned}$$

Then, as subsets $A'^{<b_1} = A^{*<r}$ and $A'^{>b_1} = A^{*>r}$,

$$Ew(A') - Ew(A^*) = s(A'^{<b_1}) \left\{ Ew(\{b_1\}) - Ew(\{r\}) - (p_{b_1} - p_r)Ew(A'^{>b_1}) \right\} \quad (3)$$

By the definition of b_1, $Ew(\{b_1\}) \geq Ew(\{r\})$. For the case that $p_{b_1} < p_r$ then, this is clearly strictly positive. For the case $p_{b_1} > p_r$,

$$Ew(\{b_1\}) - Ew(\{r\}) = p_{b_1}w_{b_1} - p_r w_r > (p_{b_1} - p_r)w_r \geq (p_{b_1} - p_r)Ew(A'^{>b_1})$$

The last inequality uses Property 1. Hence (3) is again strictly positive. Finally for $p_{b_1} = p_r$, $w_{b_1} > w_r$ implies that $Ew(\{b_1\}) > Ew(\{r\})$ strictly, and hence once again (3) is strictly positive. Thus $Ew(A') > Ew(A^*)$, and A^* cannot be the optimal set.

2. $b_1 > \max\{A^*\}$, i.e. $w_{b_1} < w_j$ (again strictly) $\forall w_j \in W^*$. Let then $\max\{A\} = r$. This time, where A' is again formed by replacing r in A^* with b_1,

$$\begin{aligned} Ew(A^*) &= Ew(A^{*<r}) + s(A^{*<r})Ew(\{r\}) \\ Ew(A') &= Ew(A'^{<b_1}) + s(A'^{<b_1})Ew(\{b_1\}) \end{aligned}$$

As again $A'^{<b_1} = A^{*<r}$ the only term that is affected is $Ew(\{b_1\})$. This is strictly greater than $Ew(\{r\})$, as by the definition of b_1 no jobs with equal $Ew(\{.\})$ can be on the left of b_1. Hence again $Ew(A') > Ew(A^*)$ and A^* cannot be the optimal set.

■

Thus we now know that in Table 1, $b_1 = 3$ must be in the final set of applied jobs A^*. The next question is then how to select the rest of the jobs in A^*. To investigate this I now define the following,

Definition 2 (Next Best Choice) *Given a set of chosen jobs $\{j, k, l, ...\}$, the next best choice (**NBC**) $\beta(j, k, l, ...)$ of jobs $\{j, k, l, ...\}$ is defined by following properties,*

1. *$Ew(\{\beta(j, ...), j, k, l, ...\}) \geq Ew(\{m, j, k, l, ...\})$ $\forall m \in \Omega | \{j, k, l, ...\}$, and*

2. *$\beta(j, k, l, ...) < m$ $\forall m$ such that $Ew(\{\beta(j, ...), j, k, l, ...\}) = Ew(\{m, j, k, l, ...\})$.*

I denote the set of the series of N NBCs $B_N = \{b_1, b_2, ..., b_N\}$, where $b_2 = \beta(b_1)$, $b_3 = \beta(b_1, b_2)$, ... , $b_N = \beta(b_1, b_2, ..., b_{N-1})$. Then,

Proposition 3 (Optimal Job Application Strategy) *The optimal set of jobs to apply to is found by adding each time the NBC b_{j+1} to the already chosen set B_j. Then $A^* = B_{a^*}$, where $Ew_{b_{a^*}}(B_{a^*}) > c$ and $Ew_{b_{a^*+1}}(B_{a^*+1}) \leq c$.*

With the example in Table 1, it can be checked that $b_2 = \beta(3) = 7$ and $b_3 = \beta(3,7) = 4$, and hence the proposition predicts that when $c = 9$, the optimal set A^* is $B_3 = \{3,4,7\}$. The proof that follows is in two stages. First it is shown that for any given $N \geq 1$, the NBC set B_N is the expected wage maximising (hereafter Ew-max) set of all sets of N applied jobs $\{A_N\}$. In the second stage it is shown that B_a with its optimal stopping point a is the optimal set A^* of all possible sets of applied jobs $\{A_n\}$, $1 \leq n \leq J$.

Proof. First I prove by contradiction that, given all possible sets of N applied jobs $\{A_N\}$, the set of NBCs $B_N \subset \{A_N\}$ yields the highest expected wage income $Ew(A_N)$. So suppose not, and assume that there exists a set A_N^* not equal to B_N, which is the Ew-max set of $\{A_N\}$. Now without loss of generality, assume that A_N^* contains the n first NBCs $B_n = \{b_1, ..., b_n\}$ where $n < N$. Appendix A shows, in a proof analogous to that for Proposition 2 above, that if an element $r \in A_N^* | B_n$ is selected and replaced with b_{n+1} using the rule outlined in the appendix, then the new set A'_N yields higher expected wage income $Ew(A'_N)$ than $Ew(A_N^*)$. Therefore A_N^* cannot be the Ew-max set. This is true for all $n = 1, ..., N-1$, and hence the set of N NBCs B_N is the Ew-max set of N applied jobs A_N^*. So it remains to show that B_a such that $Ew_{b_a}(B_a) > c$ but $Ew_{b_{a+1}}(B_{a+1}) \leq c$ is the optimal set A^* of all possible sets $\{A_n\}$, $1 \leq n \leq J$, i.e. $a = a^*$. Consider then any other set of applied jobs $A_{a'}$ with its optimal stopping point a'. As $Ew(B_{a'}) > Ew(A_{a'})$ and a is the optimal stopping point of $\{B_n\}$, $n \geq 1$, it follows that $Ew(B_{a^*}) - a^*c \geq Ew(B_{a'}) - a'c > Ew(A_{a'}) - a'c$, $\forall a'$. Hence $A_{a'}$ cannot be the optimal set of applied jobs, and thus $A^* = B_a$ with the optimal stopping point $a^* = a$. ■

I have therefore established the optimal application strategy for an individual worker for the general case. This can now be applied to investigate the labour market equilibrium outcome. In the next section I do this for the case of homogeneous workers. This allows me to derive the multiple-applications matching function that is comparable to those in the literature, which follows in Section 4.

3 Homogeneous Workers Labour Market

First I investigate the equilibrium application strategy of the applicants.

3.1 Equilibrium Applications

Up till now the probability p_{ij} of a job offer at job j was unique to worker i, depending partly on the suitability of the worker to the job specification. With homogeneous workers the probability p_j is the same for all i, and it now depends purely on the number of applications received by j, denoted α_j. The optimal number of applications for each applicant is also now the same, and is from here on treated given as a. Wages are for now assumed to be given and distinct; an argument for non-degenerate wage distribution is given later in this section.

Given uniform a then, the problem of individual worker choosing his set of applied jobs A becomes simplified to that of choosing an applicant type $t \in \{1, ..., \tau\}$, where each t is represented by the distinct permutation of selecting a jobs out of J. The number of possible types is $\tau = \binom{J}{a}$. The applicant types can be represented by a $J \times \tau$ type matrix $\mathbf{T}$, where $T_{jt} = 1$ if type t applies to job j, and is 0 otherwise. For example for $J = 4$ and $a = 2$, there are $\binom{4}{2} = 6$ possible applicant types,

$$\mathbf{T} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \tag{4}$$

In this case a type 1 applicant applies to the first two jobs. If now the number of each type chosen by the workers is given by a $\tau \times 1$ vector $\mathbf{n} = (n_1, ..., n_\tau)'$, then the resulting number of applicants at each job is given by a $J \times 1$ vector $\boldsymbol{\alpha}(\mathbf{n}) = (\alpha_1(\mathbf{n}), ..., \alpha_J(\mathbf{n}))'$ calculated by

$$\mathbf{T}\mathbf{n} = \boldsymbol{\alpha}(\mathbf{n}) \tag{5}$$

By assumption firms do not know the types chosen by the workers, and hence neither the resulting applications pattern $\mathbf{n}$. Given the number of workers I and the number of types τ, the number of possible outcomes of $\mathbf{n}$ is given by $\binom{I+\tau-1}{I}$.[6] Denote the set of all such possible realizations $\mathbf{n}$ by Λ. Given a realized applications outcome $\boldsymbol{\alpha}(\mathbf{n})$, $\mathbf{n} \in \Lambda$, firms with more applications than its number of vacancies choose their candidates randomly. The workers' probability of a job offer from firm j when the realized application pattern is $\mathbf{n}$ is therefore,

$$p_j(\mathbf{n}) = \begin{cases} \min\left[\frac{L_j}{\alpha_j(\mathbf{n})}, 1\right] & , \text{ if } \alpha_j(\mathbf{n}) > 0 \\ 0 & , \text{ if } \alpha_j(\mathbf{n}) = 0 \end{cases} \tag{6}$$

Using the result from Section 2, it is now possible to state the following regarding the equilibrium strategy of the applicants,

Equilibrium Applications *Assuming that $a < J$ and the cost of advertising a vacancy is non-zero, the equilibrium strategy of the applicants is given by a mixed-strategy $\boldsymbol{\theta} = (\theta_1, ..., \theta_\tau)'$ of selecting type t, such that $\boldsymbol{\theta}$ solves, for an equilibrium constant λ,*

$$\frac{1}{\Phi_j(\boldsymbol{\theta})} E\left[p_j(\mathbf{n}) w_j\right] = \lambda \; \forall j \in \Omega \text{ such that } \boldsymbol{\theta} \geq \mathbf{0} \text{ and } \sum_{t=1}^{\tau} \theta_t = 1 \tag{7}$$

where the expectation operator $E[X(\mathbf{n})] = \sum_{\mathbf{n}\in\Lambda} \phi_{\mathbf{n}}(\boldsymbol{\theta}) X(\mathbf{n})$ takes the average over all possible realizations of $\mathbf{n} \in \Lambda$ with the probability density function $\phi_{\mathbf{n}}(\boldsymbol{\theta})$, which for given $\boldsymbol{\theta}$ is given by,

$$\phi_{\mathbf{n}}(\boldsymbol{\theta}) = \frac{I!}{\prod_{t=1}^{\tau} n_t!} \prod_{t=1}^{\tau} \theta_t^{n_t} \tag{8}$$

and $\Phi_j(\boldsymbol{\theta})$ is the probability that job j receives at least one application,

$$\Phi_j(\boldsymbol{\theta}) = \sum_{\{\mathbf{n}'\in\Lambda \mid \alpha_j(\mathbf{n}')>0\}} \phi_{\mathbf{n}'}(\boldsymbol{\theta}) \tag{9}$$

Thus in equilibrium the expected wage incomes from all jobs, averaged over all outcomes of $\mathbf{n}$ for which the jobs receive at least one application, equate. Intuitively

[6] This can be proved by induction.

(7) formulates the trade-off between higher wage and lower probability of a job offer. In equilibrium then applicants are indifferent between all jobs. The level of λ is determined by the relative bargaining power between the firms and the workers.

Proof. Consider first the case where there is an upward deviation $\frac{1}{\Phi_k(\boldsymbol{\theta})} E\left[p_k w_k\right] > \lambda$ for some $k \in \Lambda$. Then job k would be the Best Choice job as defined in Definition 1, and hence by Proposition 2 it will be in all applicants' set of applied jobs. This would drive $\frac{1}{\Phi_k(\boldsymbol{\theta})} E\left[p_k w_k\right]$ back down to λ, which is ensured by the fact that it is not in the interest of the firm to offer a wage level any higher than the level at which the equality occurs. On the other hand if there is a downward deviation $\frac{1}{\Phi_k(\boldsymbol{\theta})} E\left[p_k w_k\right] < \lambda$, then this time all jobs apart from k are the Best Choice for all applicants, which means that for $a < J$ job k would receive no application. With a non-zero cost of vacancies firms will avoid this outcome by raising w_k such that $\frac{1}{\Phi_k(\boldsymbol{\theta})} E\left[p_k w_k\right] = \lambda$. Thus in equilibrium (7) holds. $\phi_{\mathbf{n}}(\boldsymbol{\theta})$ given in (8) is then the probability that the applications pattern is $\mathbf{n}$, given the application strategy $\boldsymbol{\theta}$. ■

3.2 Wages

Consider now the firms. Firm j's expected profit per vacancy is given by,[7]

$$E\pi_j = \frac{m_j}{L_j}\left(y - w_j\right) \tag{10}$$

where y is the uniform productivity of the workers, and m_j is the expected number of filled vacancies, which is a function of the wages and the labour demand at all firms $(\mathbf{w}, \mathbf{L})$, as well as the number of applicants I and the number of applications per applicant, a. This number of matches at each firm is now investigated separately for the two cases where firms either offer uniform or distinct wages. First take the case of the uniform wages $w_j = w \ \forall j \in \Omega$,

[7] The cost of maintaining vacancies is now ignored.

Uniform-Wages Match *Under uniform wages, the number of matches at firm j is,*

$$m_j^U(I,\mathbf{L},w,a) = E\left[\sum_{t=1}^{\tau} q_{jt}(\mathbf{n}) T_{jt} n_t p_j(\mathbf{n})\right] \tag{11}$$

where given realization $\mathbf{n}$, $q_{jt}(\mathbf{n})$ is the probability that type t accepts the job offer from j given that t applies to j,

$$q_{jt}(\mathbf{n}) = \sum_{i=0}^{a-1} \frac{(-1)^i}{(i+1)!} \underbrace{\sum_{k=1,k\neq j}^{J} \dots \sum_{r=1,r\neq j,k,l,\dots}^{J}}_{i \text{ summations}} T_{kt} p_k(\mathbf{n}) \dots T_{rt} p_r(\mathbf{n}) \tag{12}$$

and $p_j(\mathbf{n})$ is the probability of a job offer given by (6).

Proof. By symmetry type t worker who applies to j would accepts j's offer with probability $\frac{1}{i+1}$ if the worker has i other offers. Then for a realized applications pattern $\mathbf{n}$ the probability of job acceptance by type t is given by,

$$q_{jt}(\mathbf{n}) = \sum_{i=0}^{a-1} \frac{1}{(i+1)!} \underbrace{\sum_{k=1,k\neq j}^{J} \dots \sum_{r=1,r\neq j,k,l,\dots}^{J} T_{kt} p_k(\mathbf{n}) \dots T_{rt} p_r(\mathbf{n})}_{i \text{ summations}} \underbrace{\prod_{s=1,s\neq j,\dots,r}^{J} (1 - T_{st} p_s(\mathbf{n}))}_{J-1-i \text{ product sums}} \tag{13}$$

For example when $i = 1$, the worker receives 1 other job offer apart from j, and the probability of job acceptance is $\frac{1}{2!}\sum_{k=1,k\neq j}^{J} T_{kt} p_k(\mathbf{n}) \prod_{s=1,s\neq j,k}^{J} (1 - T_{st} p_s(\mathbf{n}))$. The term $\frac{1}{(i+1)!}$ reflects the $i!$ symmetries in the i-summations as well as the $\frac{1}{i+1}$ probability of job acceptance. This is summed over all possible numbers of job offers i, the maximum number of which is $a-1$, to yield (13). Expanding the product-sum terms in (13) and collecting the terms for each i-summation yields (12). The expected number of matches at firm j for a given realized applications pattern $\mathbf{n}$ is then,

$$\begin{aligned} m_{\mathbf{n}j}^U &= \begin{cases} \sum_{t=1}^{\tau} q_{jt}(\mathbf{n}) \frac{T_{jt} n_t}{\alpha_j(\mathbf{n})} \min(L_j, \alpha_j(\mathbf{n})) &, \text{ if } \alpha_j(\mathbf{n}) > 0 \\ 0 &, \text{ if } \alpha_j(\mathbf{n}) = 0 \end{cases} \\ &= \sum_{t=1}^{\tau} q_{jt}(\mathbf{n}) T_{jt} n_t p_j(\mathbf{n}) \end{aligned} \tag{14}$$

where $\frac{T_{jt}n_t}{\alpha_j(\mathbf{n})}$ is the probability of each type being chosen (note $\sum_{t=1}^{\tau} T_{jt}n_t = \alpha_j(\mathbf{n})$) and $\min(L_j, \alpha_j(\mathbf{n}))$ is the number of job offers made, taking into account that if the firm receives less applications than its number of vacancies then it can only offer $\alpha_j(\mathbf{n})$ jobs. The second line then uses (6). The overall number of matches is then the expectation of this evaluated over all possible realizations of $\mathbf{n}$. ■

Next consider the case when firms offer non-uniform wages. For simplicity I assume that the offered wage levels are distinct. As before firms are numbered in the decreasing order of their wage levels, and given two or more job offers workers accept the highest paying (i.e. the lowest numbered) job. Then this time,

Distinct-Wages Match *With distinct wages the number of matches at firm j is,*

$$m_j^D(I, \mathbf{L}, \mathbf{w}, a) = E\left[\sum_{t=1}^{\tau} s_{jt}(\mathbf{n}) T_{jt} n_t p_j(\mathbf{n})\right] \tag{15}$$

where for realization $\mathbf{n}$, s_{jt} is the survival rate that a candidate of type t would not receive any job offers from firms offering higher wages, given that t applies to j,

$$s_{jt}(\mathbf{n}) = \prod_{k=1}^{j-1} (1 - T_{kt} p_k(\mathbf{n})) \tag{16}$$

For example in the example (4), for firm 3 represented by row 3, the chosen candidate may be of the types 2, 4 or 6. For types 2 and 4 the candidate's acceptance depends on whether he has also been offered a job by firms 1 or 2, while for type 6 applicant $s_{36} = 1$. If the worker survives all $j-1$ firms (i.e. does not receive any job offers from these firms), then he will always accept the offer from j with probability 1. The number of matches is then given by replacing the job acceptance probability $q_{jt}(\mathbf{n})$ for the uniform-wages case in (11), with this survival rate (16).

Note that the number of matches in both (11) and (15) reflect the probability of job acceptance, as well as the possible number of applications received. This contrasts with the matching models in literature that generally only consider the latter. The matching model here therefore captures both the urn-ball friction and the multiple-applications friction, as defined by Albrecht et al. (2006).

3.3 Wage Dispersion Equilibrium

Now the equilibrium wage offer can be investigated. First I define the equilibrium,

Equilibrium Wage Offer *Given* $\mathbf{L}$ *the equilibrium wage offer* $\mathbf{w}$ *is one that satisfies,*

$$E\pi_j = \frac{m_j}{L_j}(y - w_j) = \pi \;\forall j = 1, ..., J \tag{17}$$

for an equilibrium constant π*, where* m_j *is the number of matches at firm* j *for the wage offer* $\mathbf{w}$*, given that the workers play the equilibrium application strategy (7).*

This simply states that the expected profit from each vacancy is the same for all firms, if the applicants all play the equilibrium strategy. This is the second of the two trade-offs, namely that of between higher wage and lower probability of job acceptance. Again the level of π is determined by the relative bargaining power between the firms and the workers. Then,

Proposition 4 *For non-uniform* $\mathbf{L}$*, uniform wages* $\mathbf{w}$ *cannot be an equilibrium outcome.*

Proof. For uniform wages $w_j = w$, (7) implies that workers choose their equilibrium application strategy $\boldsymbol{\theta}$ such that the resulting probability distribution $\phi_{\mathbf{n}}(\boldsymbol{\theta})$ for $\mathbf{n} \in \Lambda$ satisfies $\frac{1}{\Phi_j(\boldsymbol{\theta})} E[p_j(\mathbf{n})] = \frac{\lambda}{w} \;\forall j = 1, ..., J$, for the equilibrium constant λ. However for non-uniform $\mathbf{L}$, for this probability distribution the resulting $\frac{m_j^U}{L_j}$ evaluated using (11) are non-uniform. Therefore (17) cannot hold for $w_j = w$. ■

In fact the relative sizes of L_j has opposing effects on $\frac{m_j^U}{L_j}$ and consequently on the sizes of $E\pi_j$:

1. While a larger L_j for firm j does not affect the firm's own job acceptance probabilities $q_{jt}(\mathbf{n})$ given realizations $\mathbf{n}$, it does reduce $q_{kt}(\mathbf{n})$ at other firms $k \neq j$ for which $T_{kt} = 1$, i.e. $\frac{\partial q_{kt}(\mathbf{n})}{\partial L_j} < 0$ in (12). This is a benefit of being a bigger player in the labour market: for $L_j > L_k$, on average there are proportionally more applicants at k with firm j's offer than those with k's offer at firm j.

2. For (7) to hold, changes in L_j also affect the probabilities of the realizations of $\mathbf{n}$. More specifically, an increase in L_j leads to an adjustment in the workers' equilibrium application strategy $\boldsymbol{\theta}$ such that $\phi_{\mathbf{n}}(\boldsymbol{\theta})$ is higher for the realizations $\mathbf{n}$ for which $\alpha_j(\mathbf{n})$ is higher. This has two opposing effects on $E\pi_j$. Firstly it increases the per vacancy profit by reducing the probabilities of outcomes for which $\alpha_j(\mathbf{n}) = 0$. Secondly given that firm j receives one or more applications, an increase in L_j reduces $E\pi_j$ by increasing the probabilities of $\mathbf{n}$ for which $\alpha_k(\mathbf{n})$, $k \neq j$, is lower, i.e. $q_{jt}(\mathbf{n})$ is lower. Intuitively this is an effect where, given the fixed number of applications a, more applications at j means less applications at elsewhere, which implies a higher probability of the applicants at j having job offers from rival firms.

These effects do not in general offset each other, leading to non-uniform $E\boldsymbol{\pi}$, and hence uniform wages cannot be an equilibrium outcome when $\mathbf{L}$ is non-uniform. The point is that when the number of vacancies at the firms are non-uniform, the effects of this on the probability $p_j(\mathbf{n})$ of a job offer and the probability $\frac{m_j}{L_j}$ of a job acceptance must be offset by non-uniform wages for the two trade-offs in (7) and (17) to be re-established. Note however that when firms are totally homogeneous, in that the labour demands L_j are the same as well as the wage offers w_j, then by symmetry the equilibrium application strategy for homogeneous workers is to choose the type randomly,

$$\boldsymbol{\theta}^H = \left(\frac{1}{\tau}, \frac{1}{\tau}, ..., \frac{1}{\tau}\right)' \tag{18}$$

In this case on average all firms receive an equal number of applications,

$$E\boldsymbol{\alpha}^H = \left(\frac{aI}{J}, \frac{aI}{J}, ..., \frac{aI}{J}\right)' \tag{19}$$

which leads to uniform $E\boldsymbol{\pi}$. Hence the uniform wage offers $\mathbf{w}^H$ can be an equilibrium when firms have uniform labour demand.

It still remains to show that there is an unambiguous trade-off between higher wages and higher matching ratio $\frac{m_j^D}{L_j}$, for the trade-off in (17). The effect of w_j on m_j^D is again a combination of opposing factors,

1. Given realizations $\mathbf{n}$, the job acceptance probabilities $s_{jt}(\mathbf{n})$ are higher when the firm moves up the rank by increasing w_j,

$$s_{j-1t}(\mathbf{n}) = \frac{s_{jt}(\mathbf{n})}{1 - T_{j-1t}p_{j-1}(\mathbf{n})} > s_{jt}(\mathbf{n})$$

This is a discontinuous effect that occurs whenever the ranking is altered.

2. As with the uniform wage case, the probabilities $\phi_{\mathbf{n}}(\boldsymbol{\theta})$ are affected in a way that it both reduces the probability of no applications $\alpha_j(\mathbf{n}) = 0$ (i.e. a positive effect on $\frac{m_j^D}{L_j}$) and increases the probabilities of the realizations $\mathbf{n}$ for which $s_{jt}(\mathbf{n})$ is lower (i.e. a negative effect on $\frac{m_j^D}{L_j}$).

The step function nature of the first effect however ensures that there is an unambiguous trade-off. When a firm increases its wage offer by a small amount, both effects in the second point above are small. However if the wage increase results in the firm moving up the rank then the positive effect on m_j^D, and hence on $E\pi_j$, will be non-insignificant. Hence at this point the sign of $\frac{\partial m_j^D}{\partial w_j}$ is unambiguously positive, allowing a trade-off between higher wage payout and higher match.

3.4 Illustration: Case $(J, I, a) = (3, 3, 2)$

Consider the case $(J, I, a) = (3, 3, 2)$. The type matrix is

$$\mathbf{T} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Let the labour demand at the three firms be $\mathbf{L} = (1, 1, 2)'$. Consider first the case of uniform-wage. The equilibrium application strategy $\boldsymbol{\theta}$ that satisfy (7) is calculated to be $\boldsymbol{\theta}^U$=$(0.02, 0.49, 0.49)'$, reflecting the fact that the applicants would choose types 2 and 3, who apply to the firm offering 2 vacancies, with higher probabilities than type 1. These randomization probabilities determine the probability of each of the $\binom{3+3-1}{3} = 10$ possible realizations of $\mathbf{n} \in \Lambda$, given by (8). The realization probabilities are, in turn, used to average over the number of matches $m^U_{\mathbf{n}j}$ given in (14), yielding the expected matches (11) of $\mathbf{m}^U = (0.59, 0.59, 1.43)'$ with the corresponding expected profits per vacancy of $E\boldsymbol{\pi} = (0.26, 0.26, 0.32)'$. These are non-uniform as predicted, and therefore the uniform wages are not the equilibrium wage offer.

Next consider the non-uniform wage offer $\mathbf{w} = (1.00, 0.95, 0.75)'$, with as before firm 3 offering the 2 vacancies. The equilibrium application strategy is this time $\boldsymbol{\theta}^D = (0.18, 0.43, 0.39)'$, indicating a shift in the application intension of the workers towards jobs offering higher wages compared with $\boldsymbol{\theta}^U$. The resulting expected matches (15), calculated by evaluating the survival rates (16) for each t and each $\mathbf{n}$, is $\mathbf{m}^D = (0.94, 0.79, 0.94)'$, or the matching rates $\frac{m^D_j}{L_j}$ are $(0.94, 0.79, 0.47)'$. Then if the worker productivity was $y = 1.25$, the expected per vacancy profits are uniform at $E\pi_j = 0.24$ $\forall j$, implying that this is an equilibrium wage offer for the given values of $\mathbf{L}$ and y.

4 Application: Matching Function

One application of the results obtained in Section 3 is to derive the aggregate matching function for the homogeneous workers labour market that is comparable to those in the literature. A matching function is an aggregate macroeconomic function that gives the number of matches m given the number of the unemployed U and the number of vacancies V. The function $m = M(U, V)$ and its properties are generally assumed in the macroeconomic literature. In the above model of wage-directed match with multiple applications and multiple vacancies, where $U = I$ and $V = \sum_{j=1}^{J} L_j$, the expected number of matches when wage offers are either uniform or distinct are, for given I, $\mathbf{L}$, $\mathbf{w}$ and a,

$$m^X(I, \mathbf{L}, \mathbf{w}; a) = \sum_{j=1}^{J} m^X_j(I, \mathbf{L}, \mathbf{w}, a)\ ,\quad X \in \{U, D\} \tag{20}$$

where m^X_j are given by (11) and (15) for the cases $X \in \{U, D\}$. There is however an alternative derivation for the matching function, which uses the fact that as firms can only make one offer per vacancy, an applicant with at least one job offer is guaranteed a job. Then,

Aggregate Matching Function *The number of matches when all workers adopt the equilibrium application strategy* $\boldsymbol{\theta}$ *is,*

$$m(I, \mathbf{L}, \mathbf{w}; a) = I\left(1 - \Psi(I, \mathbf{L}, \mathbf{w}; a)\right) \tag{21}$$

where $\Psi(I, L, w; a)$ *is the probability that a worker receives zero job offer from all his job applications,*

$$\Psi(I, \mathbf{L}, \mathbf{w}; a) = E\left[\sum_{t=1}^{\tau} \frac{n_t}{I} \prod_{j=1}^{J} \left(1 - T_{jt} p_j(\mathbf{n})\right)\right] \tag{22}$$

Here $\prod_{j=1}^{J}\left(1 - T_{jt}p_j(\mathbf{n})\right)$ is the probability of type t not getting a single job offer for the given realization $\mathbf{n}$. Averaging this over t and over $\mathbf{n} \in \Lambda$ gives the expected probability of no job offer for an applicant. Now,

Proposition 5 *(20) and (21) are equivalent for both uniform and distinct wage offers.*

Proof. First consider the uniform wage job acceptance probabilities q_{jt} in (12),

$$\begin{aligned}
\sum_{j=1}^{J} q_{jt} T_{jt} p_j(\mathbf{n}) &= \sum_{i=1}^{a} \frac{(-1)^{i-1}}{i!} \underbrace{\sum_{j=1}^{J} \sum_{k=1, k\neq j}^{J} \cdots \sum_{r=1, r\neq j,k,l,\ldots}^{J} T_{jt}p_j(\mathbf{n}) \ldots T_{rt}p_r(\mathbf{n})}_{i \text{ summations}} \\
&= 1 - \prod_{j=1}^{J}\left(1 - T_{jt}p_j(\mathbf{n})\right)
\end{aligned}$$

noting that only a of T_{jt}'s equal 1. Substituting this into (20) using (11), as $\sum_{t=1}^{\tau} n_t = I$,

$$\begin{aligned}
m^U(I, \mathbf{L}, \mathbf{w}; a) &= E\left[\sum_{t=1}^{\tau} n_t \left\{1 - \prod_{j=1}^{J}\left(1 - T_{jt}p_j(\mathbf{n})\right)\right\}\right] \\
&= I\left(1 - \Psi(I, \mathbf{L}, \mathbf{w}; a)\right)
\end{aligned}$$

Next consider distinct wages. Note that for any series X_k, using $X_j = 1 - (1 - X_j)$,

$$\begin{aligned}
\sum_{j=1}^{J}\prod_{k=1}^{j-1}(1-X_k)\,X_j &= \sum_{j=1}^{J}\prod_{k=1}^{j-1}(1-X_k) - \sum_{j=1}^{J}\prod_{k=1}^{j}(1-X_k) \\
&= \sum_{j=1}^{J}\prod_{k=1}^{j-1}(1-X_k) - \sum_{j=2}^{J+1}\prod_{k=1}^{j-1}(1-X_k) \\
&= 1 - \prod_{k=1}^{J}(1-X_k)
\end{aligned}$$

Then by substituting in (15) and (16) into (20) and using the above,

$$\begin{aligned}
m^D(I,\mathbf{L},\mathbf{w};a) &= \sum_{j=1}^{J} E\left[\sum_{t=1}^{\tau}\prod_{k=1}^{j-1}(1-T_{kt}p_k(\mathbf{n}))\,T_{jt}\sum_{j=1}^{J}p_j(\mathbf{n})\right] \\
&= E\left[\sum_{t=1}^{\tau} n_t\left\{1-\prod_{k=1}^{J}(1-T_{kt}p_k(\mathbf{n}))\right\}\right]
\end{aligned}$$

which again equals (21). ■

Therefore the aggregate matching function is the same irrespective of the wage policies of the firms. What differ are the actual values of $m(I,\mathbf{L},\mathbf{w};a)$, due to the different equilibrium application strategies $\boldsymbol{\theta}$ chosen by the applicants to satisfy (7), and the distribution of the matches at the firms given by (11) and (15).

Now of the family of matching functions, (21) is a most general one in that it allows multiple-application workers and multiple-vacancies firms. As such many matching functions in the literature are nested in this. For example both Albrecht et al. (2004) and Hori (2007) consider workers with multiple applications but single-vacancy firms. I have already stated in (18) that with uniform $\mathbf{L}$ the workers choose all types with identical probabilities $\frac{1}{\tau}$, and thus applying this to (8) and substituting into (22),

$$\begin{aligned}
\Psi(I,J;a) &= \sum_{\mathbf{n}\in\Lambda}\frac{I!}{\prod_{t=1}^{\tau}n_t!}\frac{1}{\tau^I}\sum_{t=1}^{\tau}\frac{n_t}{I}\prod_{j=1}^{J}(1-T_{jt}p_j(\mathbf{n})) \\
&= \sum_{\mathbf{n}\in\Lambda}\frac{1}{\tau^{I-1}}\frac{(I-1)!}{(n_1-1)!\prod_{t=2}^{\tau}n_t!}\prod_{j=1}^{a}(1-p_j(\mathbf{n})) \qquad (23)
\end{aligned}$$

where $p_j(\mathbf{n}) = \frac{1}{\alpha_j(\mathbf{n})}$ if $\alpha_j(\mathbf{n}) > 0$, and 0 otherwise. The second line is derived by using symmetry and selecting type 1 that applies to the first a jobs as the representative applicant. This applied to (21) is the matching function derived in Hori (2007). Hori further demonstrates that in taking the limit $I, J \to \infty$ and $\frac{J}{I} \to \mu < \infty$, (23) yields the same limiting result as that derived by Albrecht, Gautier and Vroman (2003),

$$\Psi(I, J; a) = \left\{1 - \frac{\mu}{a}\left(1 - e^{-\frac{a}{\mu}}\right)\right\}^a \tag{24}$$

More traditional forms of matching functions are derived in an urn-ball set-up, where workers apply to one firm only, and firms offer a single vacancy and randomly select one candidate (see for example in Petrongolo and Pissarides (2001)). This is the case where $a = 1$ and $L_j = 1\ \forall j$, which when applied to (23) yields,

$$\begin{aligned}\Psi(I, J; 1) &= \sum_{n_1=1}^{I} \binom{I-1}{n_1-1} \left(\frac{1}{J}\right)^{n_1-1} \left(1 - \frac{1}{J}\right)^{I-n_1} \left(1 - \frac{1}{n_1}\right) \\ &= 1 - \frac{J}{I}\left\{1 - \left(1 - \frac{1}{J}\right)^I\right\}\end{aligned}$$

and therefore,

$$m(I, J; 1) = J\left\{1 - \left(1 - \frac{1}{J}\right)^I\right\} \tag{25}$$

This is equivalent to the matching function derived in Pissarides (1979). Petrongolo and Pissarides further state that for large J this becomes

$$m(I, J; 1) = J\left(1 - e^{-\frac{I}{J}}\right) \tag{26}$$

which is simply (21) using (24) when $a = 1$.[8] As noted by Albrecht et al. (2006), these functions capture the urn-ball friction of the job-matching market, i.e. the coordination failure that results in some vacancies receiving no application, while others receive more than one. Julien et al. (2000) on the other hand envisage a case

[8] Blanchard and Diamond (1994) also uses this limiting form of matching function, with an additional exogenous parameter in the exponent representing the acceptable application probability.

where firms choose one applicant to make one job offer, and workers with multiple offers auction their employment. In the derived model this is the case $L_j = 1\ \forall j$ and $a = J$, i.e. workers in effect apply to all firms. Then as there is only one type of applicant with $n_1 = I$, (23) becomes,

$$\Psi(I, \mathbf{1}, \mathbf{w}; J) = \left(1 - \frac{1}{I}\right)^J$$

and hence

$$m(I, \mathbf{1}, \mathbf{w}; J) = I\left\{1 - \left(1 - \frac{1}{I}\right)^J\right\}$$

which is the matching function derived by Julian et al. In the Albrecht et al.'s classification this captures the multiple-applications friction, i.e. the coordination failure that results in some workers receiving multiple offers, while others receive none. The matching function derived here encapsulates both of these job-matching market frictions.

5 Conclusions

Three main results are attained in this paper. First in contrast to the undirected-search models, the application process of workers with preferences for a higher wage is formally modelled, and the equilibrium application strategy for the wage-directed search is derived. Second in applying this to the homogenous-workers labour market, it is shown that the equilibrium outcome cannot be uniform-wage when firms have non-uniform labour demands. Finally a matching function is derived that capture both urn-ball and multiple-applications frictions, which nests many of the derived matching functions in the literature.

There are many possible future extensions for the model. First, it is argued here that by eliminating the possibility of a uniform-wage equilibrium, the equilibrium outcome must exhibit wage dispersion. A formal analysis of the existence or the uniqueness of the equilibria may be one possible future analysis (see for example Galenianos and Kircher (2007) for the existence of their wage dispersion equilibrium).

Secondly I note that for uniform labour demand, the equilibrium wage policy can be either uniform or non-uniform. For example with the example considered in Section 3.4, if $L_j = 1 \ \forall j$ then the uniform wage $w_j = 0.92$ and the distinct wages $\mathbf{w} = (1.0, 0.91, 0.79)'$ both lead to uniform expected profits $E\pi_j = 0.246$ for all firms when $y = 1.25$. The results so far do not predict which equilibrium should be chosen. This raises a further question of whether, other things being equal, it is also in the interest of the firms to vary their labour demands. Indeed as well as frictional wage dispersion, there may be an effect of the labour market friction on firms' determination of labour demands. Third possible extension is to introduce heterogeneity in jobs and workers. The optimal job application strategy derived in Section 2 does not make any restrictions on the probabilities of job offers p_{ij}, and thus this can be applied to cases where workers' skills match to the offered jobs affect the probabilities. An extension of this may then lead to a matching model that captures both the job-matching market frictions caused by coordination failure and heterogeneity, along the line of Hori (2005). Finally the paper does not consider efficiency issues. Albrecht et al. (2006) state that when both urn-ball and multiple-applications frictions are present, the equilibrium outcome is inefficient as the market cannot correct both of the frictions at the same time. If it is the case that firms have two tools - the wage level and the labour demand as suggested above, then this may lead to an improvement in efficiency.

A Supplement to Proof of Proposition 3

The supplementary proof requires the following rule,

Definition 3 (Replacement Rule) *Consider the case where $B_n \subset A_N$ but $b_{n+1} \notin A_N$, where $n < N$. Then define the following rule that replaces one element of $A_N|B_n$ with b_{n+1},*

1. *If $\exists r \in A_N$ such that $r \notin B_n$ and $r > b_{n+1}$, then replace the smallest such (i.e. the largest w_r) element.*

2. *Else replace the largest element (i.e. one with the smallest w_r) such that $r \notin B_n$ and $r < b_{n+1}$.*

For example, given $b_{n+1} \notin A_N$, if the smallest element in A_N such that $r > b_{n+1}$ was one of $\{b_1, b_2, ..., b_n\}$, then the rule will continue to look for the next smallest non-NBC element on the right-hand side. If no such element is found (i.e. all elements on the right of b_{n+1} in A_N is a member of B_n), then the rule will look for the largest non-NBC on the left-hand side of b_{n+1}. Using this,

Supplementary Proof to Proposition 3 . I prove that given a set A_N^* containing the n first NBCs B_n, $n < N$, but not b_{n+1}, replacing $r \in A_N^* | B_n$ with b_{n+1} to form A'_N using the above Replacement Rule results in a higher expected wage income, and hence A_N^* cannot be the Ew-max set of $\{A_N\}$. I consider the two cases where b_{n+1} is smaller or greater than r:

1. $b_{n+1} < r$. Note that there may or may not be one or more NBCs $b_j \in B_n$ such that $b_{n+1} < b_j < r$. Denote the range of such b_j's, if they exist, by $[\underline{b}, \overline{b}]$. Then as b_{n+1} was not chosen for A_N^*, we know from Proposition 1 that $w_{b_{n+1}} > w_{\underline{b}} \geq w_r$. Now the expected wage income of A_N^* and A'_N can be expanded as,

$$
\begin{aligned}
Ew(A_N^*) &= Ew(A_N^{*<\underline{b}}) + s(A_N^{*<\underline{b}})Ew(A_N^{*[\underline{b},r]}) + s(A_N^{*\leq r})Ew(A_N^{*>r}) \\
Ew(A'_N) &= Ew(A_N^{\prime<b_{n+1}}) + s(A_N^{\prime<b_{n+1}})Ew(A_N^{\prime[b_{n+1},\overline{b}]}) + s(A_N^{\prime\leq\overline{b}})Ew(A_N^{\prime>\overline{b}})
\end{aligned}
$$

As $A_N^{*<\underline{b}} = A_N'^{<b_{n+1}}$ and $A_N^{*>r} = A_N'^{>\bar{b}}$, and noting that $A_N^{*[\underline{b},r]} = B_n^{[\underline{b},\bar{b}]} \cup \{r\}$ and $A_N'^{[b_{n+1},\bar{b}]} = B_{n+1}^{[b_{n+1},\bar{b}]}$, then,

$$\begin{aligned} Ew(A'_N) - Ew(A^*_N) &= s(A_N'^{<b_{n+1}}) \left[Ew(B_{n+1}^{[b_{n+1},\bar{b}]}) - Ew(B_n^{[\underline{b},\bar{b}]} \cup \{r\}) \right] \\ &\quad -(p_{b_{n+1}} - p_r) s(A_N^{*\leq\bar{b}}) Ew(A_N'^{>\bar{b}}) \end{aligned} \tag{27}$$

If $[\underline{b}, \bar{b}]$ is an empty set, then $\underline{b}$ and $\bar{b}$ in these equations are replaced by r and b_{n+1} respectively. In analogy to the proof for Proposition 2, the sign of (27) now needs to be checked for each of the three cases $p_{b_{n+1}} <,=,> p_r$. First consider the case $p_{b_{n+1}} < p_r$. Whilst the sign of $Ew(B_{n+1}^{[b_{n+1},\bar{b}]}) - Ew(B_n^{[\underline{b},\bar{b}]} \cup \{r\})$ is uncertain, we do know that $Ew(B_{n+1}) > Ew(B_n \cup \{r\})$ by the definition of NBC and the fact that an applicant will always choose the left-most job given two or more equally preferred choices. Expanding these,

$$\begin{aligned} Ew(B_n \cup \{r\}) &= Ew(B_n^{<\underline{b}}) + s(B_n^{<\underline{b}}) Ew(B_n^{[\underline{b},\bar{b}]} \cup \{r\}) + s(B_n^{\leq\bar{b}})(1 - p_r) Ew(B_n^{>\bar{b}}) \\ Ew(B_{n+1}) &= Ew(B_{n+1}^{<b_{n+1}}) + s(B_{n+1}^{<b_{n+1}}) Ew(B_{n+1}^{[b_{n+1},\bar{b}]}) + s(B_{n+1}^{\leq\bar{b}}) Ew(B_{n+1}^{>\bar{b}}) \end{aligned}$$

Noting that $B_n^{<\underline{b}} = B_{n+1}^{<b_{n+1}}$ and $B_n^{>\bar{b}} = B_{n+1}^{>\bar{b}}$ then, using these (27) becomes,

$$\begin{aligned} Ew(A'_N) - Ew(A^*_N) &= \frac{s(A_N'^{<b_{n+1}})}{s(B_{n+1}^{<b_{n+1}})} \left[Ew(B_{n+1}) - Ew(B_n \cup \{r\}) \right] \\ &\quad -(p_{b_{n+1}} - p_r) s(A_N^{*\leq\bar{b}}) \left[Ew(A_N'^{>\bar{b}}) - Ew(B_{n+1}^{>\bar{b}}) \right] \end{aligned} \tag{28}$$

Further $B_{n+1}^{>\bar{b}} \subseteq A_N'^{>\bar{b}}$ implies that $Ew(A_N'^{>\bar{b}}) \geq Ew(B_{n+1}^{>\bar{b}})$. Thus for $p_{b_{n+1}} < p_r$, (28) is strictly positive. This is true also for the case $p_{b_{n+1}} = p_r$. For $p_{b_{n+1}} > p_r$, I use the equality $\sum_{k \in [\underline{b},\bar{b}]} s(B_n^{[\underline{b},k)}) p_k = 1 - s(B_n^{[\underline{b},\bar{b}]})$, i.e. the probability of getting at least one job offer within $[\underline{b}, \bar{b}]$ is 1 minus the probability of receiving no job offer. Then expanding the [ ] term on the right-hand side of (27),

$$\begin{aligned}
& Ew(B_{n+1}^{[b_{n+1},\overline{b}]}) - Ew(B_n^{[\underline{b},\overline{b}]} \cup \{r\}) \\
= \;& p_{b_{n+1}} w_{b_{n+1}} + (1-p_{b_{n+1}}) Ew(B_n^{[\underline{b},\overline{b}]}) - Ew(B_n^{[\underline{b},\overline{b}]}) - s(B_n^{[\underline{b},\overline{b}]}) p_r w_r \\
= \;& p_{b_{n+1}} \left\{ w_{b_{n+1}} - \sum_{k \in [\underline{b},\overline{b}]} s(B_n^{[\underline{b},k)}) p_k w_k \right\} - s(B_n^{[\underline{b},\overline{b}]}) p_r w_r \\
> \;& p_{b_{n+1}} \left\{ 1 - \sum_{k \in [\underline{b},\overline{b}]} s(B_n^{[\underline{b},k)}) p_k \right\} w_{b_{n+1}} - s(B_n^{[\underline{b},\overline{b}]}) p_r w_{b_{n+1}} \\
= \;& s(B_n^{[\underline{b},\overline{b}]}) \left(p_{b_{n+1}} - p_r \right) w_{b_{n+1}}
\end{aligned}$$

The inequality uses $w_{b_{n+1}} > w_{\underline{b}} \geq w_r$. Substituting this back into (27) yields,

$$Ew(A'_N) - Ew(A^*_N) > (p_{b_{n+1}} - p_r) s(A_N^{*\leq \overline{b}}) \left\{ w_{b_{n+1}} - Ew(A_N'^{>\overline{b}}) \right\}$$

As we know from Property 1 that $Ew(A_N'^{>\overline{b}}) < w_{b_{n+1}}$, this implies that $Ew(A'_N) > Ew(A^*_N)$ also for $p_{b_{n+1}} > p_r$. Therefore in all cases $Ew(A'_N) > Ew(A^*_N)$.

2. $b_{n+1} > r$. Again there may or may not be one or more NBCs $b_j \in [\underline{b},\overline{b}] \subset B_n$ between b_{n+1} and r, such that $r < [\underline{b},\overline{b}] < b_{n+1}$. Now the fact that there are no non-NBCs on the right-hand side of b_{n+1} means that $A_N^{*>r} = B_n^{\geq \underline{b}}$ and $A_N'^{\geq \underline{b}} = B_{n+1}^{\geq \underline{b}}$. Then this time,

$$\begin{aligned}
Ew(A^*_N) &= Ew(A_N^{*<r}) + s(A_N^{*<r}) Ew(\{r\} \cup B_n^{\geq \underline{b}}) \\
Ew(A'_N) &= Ew(A_N'^{<\underline{b}}) + s(A_N'^{<\underline{b}}) Ew(B_{n+1}^{\geq \underline{b}}) \\
Ew(A'_N) - Ew(A^*_N) &= s(A_N'^{<\underline{b}}) \left[Ew(B_{n+1}^{\geq \underline{b}}) - Ew(\{r\} \cup B_n^{\geq \underline{b}}) \right] \qquad (29)
\end{aligned}$$

using $A_N^{*<r} = A_N'^{<\underline{b}}$. Now again we know that $Ew(B_{n+1}) > Ew(B_n \cup \{r\})$, which is strictly true this time from the definition of b_{n+1} for any r on the left-hand side of b_{n+1}. When expanded these are,

$$\begin{aligned} Ew(B_n \cup \{r\}) &= Ew(B_n^{\underline{<b}}) + s(B_n^{\underline{<b}})Ew(\{r\} \cup B_n^{\underline{\geq b}}) \\ Ew(B_{n+1}) &= Ew(B_{n+1}^{\underline{<b}}) + s(B_{n+1}^{\underline{<b}})Ew(B_{n+1}^{\underline{\geq b}}) \end{aligned}$$

Since $B_n^{\underline{<b}} = B_{n+1}^{\underline{<b}}$ then, by substitution (29) becomes,

$$Ew(A'_N) - Ew(A^*_N) = \frac{s(A_N'^{\underline{<b}})}{s(B_{n+1}^{\underline{<b}})} \left[Ew(B_{n+1}) - Ew(B_n \cup \{r\})\right] > 0 \qquad (30)$$

Hence again $Ew\left(A^*_N\right)$ cannot be the Ew-max set of $\{A_N\}$.

■

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