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Optimal Sale: Auctions with a Buy-Now Option

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Abstract

We characterize the optimal selling mechanism in a scenario where similar goods are sold to "high end" buyers through a posted price and to "lower end" buyers through an auction. We show that the optimal mechanism involves an auction which is a standard optimal auction (Myerson (1981)) up to a critical type. Types above the critical type are pooled. Further, the allocation probability jumps up at the critical type and is the maximal possible for the pooled types. Therefore other than pooling at the top, the optimal mechanism allocates the object as efficiently as in a standard optimal auction. We show that posted price selling followed by auctions with a "temporary" buy-now option implements the optimal mechanism. Auctions with such an option are in widespread use on eBay.

KEYWORDS: Optimal Auction, eBay Auctions, Buy-Now Option, Posted Price, Price Discrimination

JEL CLASSIFICATION: D44

1 Introduction

Low transactions cost of trading on the Internet has led to a tremendous growth in the scope of auctions. Moreover, the advent of listing sites such as eBay and Yahoo! have made it very easy to sell an item through an auction and provide access to a large pool of buyers. Initially, most of the goods sold through online auctions were collectables(1). Increasingly, however, goods traditionally sold at posted prices (through bricks-and-mortar stores and web sites) are being offered also through online auctions. A large part of the standard goods sold through auctions are overstock items from wholesalers and standard retailers. Overstock.com, which describes itself as "an Internet leader for name-brands at clearance prices," sells items through both posted prices and auctions. IBM has launched its own eBay auctions to sell discontinued products(2). Other companies such as Dell, HP, Gateway, and Sears, who have well established posted price sales channels, have opened up eBay stores to sell overstock items and refurbished systems through auctions(3).

These auctions allow sellers to sell to customers with values not high enough to buy at the prices posted at stores. In this paper we focus on the role of online auctions for sellers who use such auctions in *addition* to the traditional sales channels, allowing them to expand their market base to customers with lower values(4). Our objective is to characterize the optimal sales mechanism in such environments. However, the theory we develop applies more generally to the study of optimal price discrimination across different types of buyers using different venues.

Specifically, we study a scenario in which goods are sold to "high end" buyers through a posted price and to "lower end" buyers through an auction. An important insight emerging from our analysis is that the optimal design results in second degree price discrimination *across* individual mechanisms, and therefore individual components of the overall optimal mechanism may seem suboptimal if considered in isolation. As

(1)See Lucking-Reiley (2000).

(2)`www.channeladvisor.com/downloads/IBM_public.pdf`

(3)Also see Miller (2005) for a variety of examples.

(4)A useful interpretation might be that traditionally a seller could only reach the "high end" of the market; subsequent innovation in marketing technology (e.g. the possibility of selling over the Internet) makes it possible for the seller to reach the lower segment of the market as well.

we discuss later, our analysis also provides a novel theoretical justification for auctions with buy now options by showing that they form part of the optimal selling mechanism[5].

To capture the environment discussed above, we model the demand for an item as follows: with probability μ a buyer has valuation v_h for the item, an event that can be interpreted as the "high end" of the market. With probability $(1-\mu)$ the buyer can have a range of possible valuations all of which are less than v_h. More concretely, with probability $(1-\mu)$ the buyer's valuation is drawn from the interval $[\underline{v}, \overline{v}]$ and we suppose that the high value $v_h > \overline{v}$. We make the standard assumption that a buyer's valuation is private information. Given this environment, we ask what the optimal selling strategy for the seller is and use techniques from the mechanism design literature to answer this question.

The presence of an atom at v_h implies, unsurprisingly, that the optimal mechanism involves a posted price. Indeed we need this feature since we want our model to match up with the observed fact that the goods sold through online auctions are also sold (in other venues) through a posted price method. Assuming an atom at v_h is the simplest way to ensure that the optimal mechanism involves a posted price, and allows us to study the design of the optimal auction in the presence of such a price[6]. The level of the posted price is, of course, determined endogenously.

Let $F(v)$ denote the cumulative distribution function of buyer valuations on the domain $[\underline{v}, \overline{v}]$. We show that the optimal mechanism has non-standard features even under standard assumptions about the distribution $F(v)$. Specifically, types above a cutoff must be pooled. Such pooling "at the top" contrasts with an optimal auction in the standard setting, which can admit pooling anywhere *except* at the top. The intuition is that the high-end type v_h can choose to participate in the auction rather than buy at the posted price, leaving the seller to solve a second-degree price discrimination problem.

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(5) The literature typically describes environments where such auctions are superior to *standard auctions*. As far as we know, we are the first to show that these auctions can arise as part of the *optimal* mechanism in certain environments.

(6) An alternative model would be to assume that there is also a high end distribution over some interval $[v_h, v_h + M]$, with $M > 0$, and then restrict the seller to choosing a posted price to serve this end of the market (perhaps because running auctions in shopping malls is very costly). This would simply add to algebra without adding to the results.

Satisfying the incentive constraint of the high type requires "downgrading" the ability of such a type to secure the object in an auction. The auction outcome is therefore distorted relative to standard auctions and our analysis shows that the "best" way for the seller to introduce this distortion is to have pooling at the top[(7)]. (Without the atom at v_h, the standard optimal auction is inefficient only if there is a non-trivial reserve price.)

As explained above, the pooling at the top is a device for price discrimination. Of course, if v_h or μ is very high, the optimal strategy for the seller is simply to sell only to the high end buyers at a price v_h and exclude the lower types. Whenever parameter values are such that price discrimination is profitable, the price that can be extracted from the high end buyers decreases in the probability of winning for a type in the pooling region. Despite this, we show that for the pooled types, the optimal allocation function takes the highest possible value. In other words, once the pooling cutoff is determined optimally, any further transfer of the probability of allocation from pooling types to types below is suboptimal. Therefore, beyond the pooling at the top, there is no added inefficiency.

This "pooling at maximal value" feature of the optimal mechanism is very important. As we explain below, it is precisely this feature that allows the mechanism to be linked directly with observed selling procedures.

A puzzling non-standard feature of many online auctions is the availability of a "buy-now" option. This is a posted price at which a bidder can buy the object immediately, superseding the auction[(8)]. In some cases (e.g. Yahoo!, Amazon, Overstock) the buy price is available throughout the duration of the auction. However, the predominant site eBay uses a format where the buy now option is temporary: it vanishes as soon as the auction becomes active – i.e. a bid higher than the reserve price (but lower than the buy price) is placed.

[(7)]Even though the intuition is clear when one considers the overall actions of the seller, we need to emphasize that in the game that results from the optimal mechanism, *in equilibrium*, the types that show up for the auctions lie in the interval $[0, 1]$. Hence to an observer studying the auctions *only*, the pooling would appear puzzling and a direct refutation of well-known auction theory.

[(8)]This is called "Buy-It-Now" on eBay auctions, "Buy" price on Yahoo! auctions, "Take It" price on Amazon auctions, "UBuy it" on uBid auctions and "Make it Mine" price on Overstock.com auctions. Many smaller auctions also have such a buy-now feature.

We show that a posted price followed by an auction with a buy now option implements the optimal mechanism. In an online auction if at least one bidder exercises the buy now option, the object is sold for sure at the buy price. The "pooling at maximal value" feature of our optimal auction corresponds exactly to this feature. Moreover, in our implementation scheme the buy price phase precedes the auction. Thus the buy now option is *temporary*. Moreover, we show that in general it is not possible to implement the optimal mechanism using a permanent buy now option. This contrasts with the literature – discussed below – which focuses primarily on a permanent buy now option and note (implicitly or explicitly) the suboptimality of a temporary buy now option in their settings.

Of course, online auctions have a rich set of institutional details, and we do not claim to capture all of them. Our principal objective is to point out that many online auctions act as an addition to pre-existing sales channels, and gaining a proper understanding of these auctions requires considering the overall sales strategy. Indeed, a phenomenon such as an auction with a temporary maximum price might seem very puzzling viewed in isolation, but such an auction might well form part of an optimal overall sales mechanism.

Relating to the literature

The (relatively small) literature on auctions with a buy now option usually takes the auction format as given (this is often taken to be the English auction) and focuses on cases under which revenue might increase if a buy now option is added. There is no suggestion that an auction with such an option is an optimal arrangement in such cases.

Our approach, in contrast, is to model the online pricing environment, and ask what the optimal selling mechanism is. We derive the optimal mechanism and show that implementing it involves selling through a posted price followed by an auction with a "temporary" buy now option. Such auctions are in widespread use on eBay. As far as we are aware, ours is the first attempt to explain an auction with a buy now option as part of an optimal response to the online pricing environment. Our approach also generates different testable implications compared to the literature. We discuss this further in the conclusion.

An early contribution to the literature is by Budish and Takeyama (2001). They provide a simple example with discrete values to show that adding a maximum price to an English auction increases revenue if the bidders are risk averse. Others have subsequently investigated this idea in more general settings. Reynolds and Wooders (2004) and Hidvégi, Wang, and Whinston (2005) show that this result holds in a setting in which bidders draw values from a continuous distribution. The latter paper as well as Mathews and Katzman (2004) show that the result holds even with risk neutral buyers if the seller is risk averse. However, auctions with a buy now option are not optimal in the environments they model. With continuous distributions, and risk averse buyers, Maskin and Riley (1984) characterize the optimal auction(9). They also show that even if we limit attention to standard auctions, so long as either the bidders are risk averse or the seller is risk averse, first price auctions are preferred to English auctions by the seller(10). However, none of the papers that study buy-price English auctions under a continuous distribution and risk averse bidders/seller compare such auctions with the optimal auction in their setting, or even the first price auction(11). Therefore even with risk averse bidders/seller, the reason for choosing a buy-price auction remains unclear.

Some authors have looked at auctions with a buy now option in the context of sequential auctions. Kirkegaard and Overgaard (2005) model a sequence of single-unit auctions. Buyers have multi-unit demand and there are competing sellers. They show that if only one seller is allowed to introduce a buy price, he would do so in earlier auctions and increase own revenue compared to second price auctions without such an option. They also show that if there is only one seller, he benefits by introducing a buy price in later auctions. Hendricks, Onur, and Wiseman (2004) conduct a theoretical and empirical study of two auctions held in sequence. In a model with two possible values for bidders, they derive conditions under which the first seller can increase revenue by using a buy now option. In our paper the optimal mechanism can be implemented by an indirect mechanism which is sequential: the object is offered through

(9)See also Matthews (1983). The optimal auction is in fact quite complex, involving payments by some losing bidders, and is very different from standard auctions with a buy-now option.

(10)Holt (1980) previously derived this result for the case of risk averse bidders and a risk neutral seller.

(11)Budish and Takeyama (2001) show that with a discrete value distribution (two possible values), an English auction with a buy-now option can raise more revenue than a first price auction.

a posted price first, and then through an auction with a buy now option. However, the auction itself is static rather than sequential.

Milgrom (2004) considers a model with sequential entry of bidders who face a cost of learning their own type. A bidder who wants to buy at a (permanent) buy price wins only if no other bidder in a higher queue position exercises this right. Bidders do not know own queue positions, and consider all positions equally likely. In this setting types above a certain cutoff have an incentive to buy at the buy price, and the revenue maximizing auction involves a buy price. Our model, in contrast, is much closer to the standard private values model so that entry is simultaneous, and there is no cost of learning own value. The only departure from the standard model is the presence of an atom at v_h.

A further difference between the literature and our approach is the treatment of a temporary buy now option. Such an option is seemingly even more puzzling (compared to a permanent buy now option) in the sense that even if one could find a reason for introducing a buy now option, taking it away as the auction starts begs a further question. Indeed, the literature has noted that in the standard independent private values setting (with risk-neutral seller and bidders), while an auction involving a suitably chosen permanent buy price is revenue equivalent to an English auction (and therefore an optimal auction), an auction with a temporary buy price seems not to be optimal. Both Reynolds and Wooders (2004) and Hidvégi, Wang, and Whinston (2005) conclude that auctions with a temporary buy now option are revenue inferior to those with a permanent buy now option. The same conclusion applies to the sequential costly entry model of Milgrom (2004). In contrast, our analysis shows that posted price selling followed by a standard auction with a temporary buy now option – used widely by eBay – implements the optimal mechanism.

Our results also relate to the literature on optimal auctions. In a standard private value setting the bidders draw values from a continuous distribution over some interval $[\underline{v}, \overline{v}]$. As Myerson (1981) and Riley and Samuelson (1981) show, so long as the virtual value is increasing, the revenue maximizing auction allocates the object efficiently. If the virtual value is not increasing over some interval, types over that interval are pooled. However, for values close to $\overline{v}$ the virtual value is always increasing so that there is never any pooling "at the top." Our model departs from the standard setting since there is an atom at a value v_h higher than $\overline{v}$. In all other respects it coincides with the standard model. We assume that a modified version of the virtual value is increasing, so that the usual reason for pooling is eliminated. However, the addition of an atom at a high value generates pooling for types close to $\overline{v}$. The intuition for this kind of pooling - impossible in the standard setting - is as follows. If the auction that serves types $[\underline{v}, \overline{v}]$ is exactly like a standard optimal auction, the high type v_h pays no more than type $\overline{v}$. But pooling at the top makes it possible to extract a higher payment from this type. We show that the optimal mechanism involves a posted price (the price charged to type v_h) and an auction with pooling at the top at the maximal possible value, and then show that this type of pooling implies precisely that the optimal mechanism involves an auction with a buy price.

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 formulates and solves for the optimal direct mechanism. Section 4 shows that the optimal mechanism can be implemented by a combination of a posted price and an auction with a temporary buy now option, and also that in general it is not possible to implement the optimal mechanism using a permanent buy now option. Section 5 concludes.

2 The Model

An object is offered for sale by a risk neutral seller with reservation utility zero. There are two potential risk neutral buyers. The buyers are ex ante symmetric and draw values independently from the following distribution. With probability μ a buyer has valuation v_h. With probability $(1-\mu)$ a buyer has valuation drawn from the distribution $F(v)$ with support $[0,1]$. Assume $v_h > 1$.

We assume $F(v)$ has continuous density $f(v)$, where $f(v) > 0$ for all $v \in [0,1]$, and that $F(v)$ satisfies the monotone hazard rate. In other words, other than the atom at v_h, the rest of the model is essentially the standard independent private values model[(12)].

(12)We could, for the sake of generality, write the type space as $[\underline{v}, \overline{v}] \cup \{v_h\}$ with the stipulation that $v_h > \overline{v}$. However, since we allow the seller to have a non-trivial reserve price in the mechanism in any case, there is no gain in allowing $\underline{v}$ to be less than 0 (the seller's valuation). Further, as will be clear later, it is not important for the theoretical analysis to explore the so-called "gap" and the "no gap" case.

3 The Optimal Mechanism

3.1 Formulating the seller's optimization problem

The seller is the mechanism designer. Without loss of generality, we restrict attention to direct mechanisms. Let (r_1, r_2) denote a profile of reported values, where $r_i \in [0,1] \cup \{v_h\}$ for all $i \in \{1,2\}$.

For any (r_i, r_j), the direct mechanism specifies a vector $(t_i(r_i, r_j), x_i(r_i, r_j))$ for all $i, j \in \{1,2\}$, $i \neq j$, where $t_i(\cdot,\cdot)$ is the transfer from buyer i to the seller, and $x_i(\cdot,\cdot)$ is the probability that i wins the object. Later, we focus on symmetric mechanisms, and the convention followed throughout is that the first entry of $x(r_i, r_j)$ and $t(r_i, r_j)$ refer to one's own report.

Let $X_i(r_i)$ denote the expected probability with which i wins the item conditional on reporting r_i, and $T_i(r_i)$, the expected transfer conditional on reporting r_i. The expectation is with respect to the other buyer's valuation distribution; in other words, as is standard, buyer i assumes that buyer j is reporting truthfully and the mechanism is incentive compatible when buyer i wants to report truthfully as well.

Since the seller is risk neutral and the buyers are ex ante symmetric, we can, without any loss of generality, restrict the search for the optimal mechanism by considering symmetric mechanisms only. In what follows, we drop the subscripts in the functions $T_i(\cdot)$ and $X_i(\cdot)$.

The seller's task consists of finding the optimal $X(\cdot)$ and $T(\cdot)$ subject to the incentive compatibility and the individual rationality constraints. As is well known this can be transformed into a problem of finding the optimal $X(\cdot)$. We now proceed to derive this problem.

A bidder of type v chooses r to maximize $U(v,r) = vX(r) - T(r)$. Satisfying the incentive compatibility constraint implies that the maximum occurs at $r = v$. Following standard arguments it is easy to verify that incentive compatibility requires $X(\cdot)$ to be non-decreasing[(13)].

Therefore, in order to not carry extra notation, we use the unit interval instead of $[\underline{v}, \overline{v}]$.

[(13)] Consider two possible values v and v' of a bidder. Incentive compatibility requires $vX(v) -$

It is convenient to consider types $v \in [0,1]$ and type v_h separately. From the envelope theorem[(14)] we have,

$$U(v) = U(0) + \int_0^v X(t)\ dt \tag{3.1}$$

Putting $U(0) = 0$, we get,

$$T(v) = vX(v) - \int_0^v X(t)\ dt \tag{3.2}$$

Next, consider type v_h. The highest payoff this type can get by reporting falsely is $\max_{v\in[0,1]} (v_h X(v) - T(v))$.

For any $v', v'' \in [0,1]$ with $v' > v''$. We have,

$$\begin{aligned}
& \left[v_h X(v') - T(v')\right] - \left[v_h X(v'') - T(v'')\right] \\
= \ & v_h \left[X(v') - X(v'')\right] - v'X(v') + v''X(v'') + \int_{v''}^{v'} X(t)dt \\
\geqslant \ & v_h \left[X(v') - X(v'')\right] - v'X(v') + v''X(v'') + \left(v' - v''\right) X(v'') \\
= \ & \left[v_h - v'\right] \left[X(v') - X(v'')\right] \\
\geqslant \ & 0
\end{aligned}$$

Where the first inequality follows from the fact that $X(v)$ is non-decreasing in v and the last inequality follows since in addition $v_h > v'$. Thus type v_h gets a higher utility by reporting a higher type. Therefore, without loss of generality, we can write,

$$\max_{v\in[0,1]} v_h X(v) - T(v) = v_h X(1) - T(1). \tag{3.3}$$

It is clear that the IC constraint of type v_h must bind at the optimal mechanism. Hence we must have

$$v_h X(v_h) - T(v_h) = \max_v v_h X(v) - T(v) \tag{3.4}$$

$T(v) \geqslant vX(v') - T(v')$ and $v'X(v') - T(v') \geqslant v'X(v) - T(v)$ Combining the inequalities we get $(v - v')\,(X(v) - X(v')) \geqslant 0$. Therefore $X(v) \geqslant X(v')$ whenever $v \geqslant v'$.

(14)See Milgrom and Segal (2002). Even though our model does not fit exactly their framework since our entire type space, $[0,1] \cup v_h$ is not a connected interval, it is clear that since v_h is the highest type, the optimal mechanism involves $x(v_h, v) = 1$ and $t(v, v_h) = 0$ for all $v \in [0,1]$. Hence without loss of generality we can consider $X(v) = (1-\mu)\int_0^1 x(v,v')dF(v')$ and $T(v) = (1-\mu)\int_0^1 t(v,v')dF(v')$ for all $v \in [0,1]$. We show later that $X(v)$ is in fact differentiable almost everywhere for $v \in [0,1]$.

Using equations (3.2)-(3.3), $T(v_h)$ is given by

$$T(v_h) = v_h X(v_h) - v_h X(1) + X(1) - \int_0^1 X(t)\, dt \tag{3.5}$$

Denote the seller's expected revenue as ER. The seller's *per capita* expected revenue, $\frac{ER}{2}$ is given by

$$\frac{ER}{2} = \mu T(v_h) + (1-\mu) \int_0^1 T(v) f(v) dv$$

Following standard procedure (see Myerson (1981)), we can write:

$$\int_0^1 T(v) f(v) dv = \int_0^1 \left(v - \frac{1-F(v)}{f(v)} \right) X(v) f(v) dv$$

Hence, the per capita expected revenue is given by

$$\begin{aligned} \frac{ER}{2} &= \mu \left(v_h X(v_h) - (v_h - 1) X(1) - \int_0^1 X(v) dv \right) \\ &\quad + (1-\mu) \int_0^1 \left(v - \frac{1-F(v)}{f(v)} \right) X(v) f(v) dv \end{aligned}$$

This can be further rewritten as:

$$\frac{ER}{2} = \mu v_h X(v_h) - \mu (v_h - 1) X(1) + \int_0^1 \psi(v) X(v) f(v) dv \tag{3.6}$$

where

$$\psi(v) \equiv (1-\mu) \left(v - \frac{1-F(v)}{f(v)} \right) - \frac{\mu}{f(v)} \tag{3.7}$$

We call $\psi(v)$ the "modified virtual valuation" of type v.

As mentioned in the introduction, our main result shows that as long as the seller is selling to some types in $[0,1]$, the optimal mechanism must involve pooling (type-bunching) around 1. The most forceful way to show this is to make our environment as close as possible to that under which standard auctions are optimal. In other words, we want to rule out other reasons for pooling. We therefore assume the following.

Assumption 1 ***(Modified regularity)*** *$\psi(v)$ is increasing in v*[15].

[15] Monotone hazard rate of $F(v)$ ensures that the standard virtual valuation, i.e. $v - \frac{1-F(v)}{f(v)}$ is increasing in v. Here, this is no longer sufficient to ensure that the modified virtual valuation $\psi(v)$ is increasing in v. To get a sufficient condition that is independent of μ, we also need convexity of $F(v)$ (satisfied by distributions such as uniform and triangular). For other distributions this assumption holds if, along with monotone hazard rate, μ is not too high.

Next, if $\psi(v)$ is always negative for $v \in [0,1]$, the seller's problem is trivial: never allocate to these types. To avoid this uninteresting case, the weakest possible assumption is to ensure $\psi(1) > 0$ We assume

Assumption 2 ***(Non-trivial Optimal Mechanism)*** $\frac{\mu}{1-\mu} < f(1)$.

This ensures $\psi(1) > 0$ and hence by continuity that there is a range of values of v such that $\psi(v) > 0$ for all v in that range[(16)].

Since there is no type higher than v_h (i.e., there is no type who can obtain rent by reporting v_h), the seller should sell with probability 1 if at least one bidder announces v_h. Thus

$$X(v_h) = (1-\mu) + \frac{\mu}{2} = 1 - \frac{\mu}{2}. \tag{3.8}$$

Since this is a constant, we can ignore this term in maximizing the seller's revenue (given by (3.6)) exercise. This proves the following result.

Proposition 1 *The optimal mechanism is the solution of the following problem*

$$\max_{X(v)} \quad -\mu\,(v_h - 1)\,X(1) + \int_0^1 \psi(v)X(v)f(v)\,dv$$

when $X(v)$ is non-decreasing and where $\psi(v)$ is given by equation (3.7) *and the transfers $T(v)$ and $T(v_h)$ are given by equations (3.2) and (3.5) respectively.*

Intuitively, the first term is the loss from having an auction which allows types below v_h to participate. This generates a loss because a pure posted price mechanism can charge v_h, while the mechanism including an auction must charge a lower posted price to satisfy incentive compatibility. The second term is the gain from including an auction. Note that the form of the second term is similar to the corresponding expression in a standard setting. However, the solution for $X(\cdot)$ does not coincide with the standard solution because of the first term. This also clarifies the price discriminating role of

(16)However, this assumption, though necessary, is not sufficient for the seller to want to sell to types other than v_h. If μ is high, it is optimal for the seller to not price discriminate, and only sell to type v_h. The precise condition is derived in proposition 2 below.

the auction: $X(\cdot)$ must be set so that we can separate the high type from the rest and extract surplus optimally from both segments of the market.

Note that we have already imposed the condition that $X(v)$ must satisfy $U(0) = 0$. The condition is used following equation (3.1), which also shows that the solution to the above problem satisfies individual rationality. To satisfy incentive compatibility, the solution to the above problem must also satisfy monotonicity of $X(v)$. The following analysis takes this into account explicitly in deriving the optimal mechanism.

3.2 Solving the seller's optimization problem

We now derive a set of results that characterize the optimal mechanism.

Note that for v close to zero, $\psi(v) < 0$. From assumptions 1 and 2 it follows that there is a $v_* < 1$ such that $\psi(v_*) = 0$.

Since $\psi(v) < 0$ for $v < v_*$, a simple inspection of the seller's objective function shows that the optimal mechanism must be characterized by

$$X(v) = 0 \ \text{ for any } v \in [0, v_*). \tag{3.9}$$

In other words, similar to standard auctions, the seller does not sell to types whose (modified) virtual valuation is negative.

If μ is close to 1, it is clearly optimal to simply sell to type v_h at price v_h and not sell to lower types at all. In other words, in this case a large information rent must be ceded to type v_h if the seller wants to sell to lower types and maintain incentive compatibility. The next result clarifies the precise condition required for the optimal mechanism to involve a positive probability of allocating to lower types.

Proposition 2 *Let* $\lambda(v) \equiv \int_v^1 \psi(t) f(t) dt - \mu(v_h - 1)$. *The optimal mechanism involves non trivial price discrimination only if* $\lambda(v_*) > 0$. *In other words, if* $\lambda(v_*) \leqslant 0$ *the optimal mechanism is simply a posted price of* v_h.

Proof. Using equation (3.9), the seller's optimization problem can be rewritten as

$$\max_{X(v)} \quad - \mu \left(v_h - 1\right) X(1) + \int_{v_*}^1 \psi(v) X(v) f(v) \, dv.$$

Let us reduce $X(v)$ for all v by ϵ. The maximand increases by $-\epsilon\lambda(v_*)$. Further, $\lambda'(v) = -\psi(v)f(v) < 0$ for $v > v_*$ and therefore $\lambda(\cdot)$ is maximized at $v = v_*$. Therefore if $\lambda(v_*) \leqslant 0$, $\lambda(v) < 0$ for all $v > v_*$. Therefore reducing $X(v)$ increases the maximand by a positive amount, and it is then optimal to set $X(v) = 0$ for all v. ❋

If $\lambda(v_*) \leqslant 0$, the information rent type v_h gets if the seller tries to reach the "lower end" of the market as well is too high and it is better for the seller to sell to type v_h only. In what follows, we assume that the optimal design is non-trivial, i.e. $\lambda(v_*) > 0$.

The next result, which is one of our main results, shows that the optimal mechanism must involve pooling for some neighborhood of 1.

Proposition 3 *Let $X^*(v)$ denote the optimal mechanism. There exists a $\widehat{v} < 1$, such that $X^*(v)$ is constant for $v \in [\widehat{v}, 1]$.*

Proof. Let $\widetilde{X}(v)$ be incentive compatible, individually rational, and strictly increasing for any neighborhood of 1 no matter how small. The following argument shows that such a $\widetilde{X}(v)$ cannot be optimal.

Consider $\varepsilon > 0$, small, and define $K(\varepsilon)$ as

$$K(\varepsilon) = E\left[\widetilde{X}(v)|1-\varepsilon \leq v \leq 1\right] = \int_{1-\varepsilon}^{1} \widetilde{X}(v)\frac{f(v)}{1-F(1-\varepsilon)}dv$$

Note that because $\widetilde{X}(v)$ is strictly increasing, $\widetilde{X}(1-\varepsilon) < K(\varepsilon) < \widetilde{X}(1)$.

Consider the following $X(\cdot)$:

$$X(v) = \begin{cases} \widetilde{X}(v) & \text{for } v \in [0, 1-\varepsilon) \\ K(\varepsilon) & \text{for } v \in [1-\varepsilon, 1] \end{cases}$$

$X(v)$ is non-decreasing since $\widetilde{X}(v)$ is non-decreasing. In particular, $X(v)$ must satisfy the incentive compatibility and individual rationality constraints given that $\widetilde{X}(v)$ does.

Now, let

$$\widetilde{R} = -\mu(v_h - 1)\widetilde{X}(1) + \int_0^1 \widetilde{X}(v)\psi(v)f(v)dv$$

and

$$R = -\mu(v_h - 1)X(1) + \int_0^1 X(v)\psi(v)f(v)dv.$$

We have,

$$\begin{aligned} R - \widetilde{R} &= \left[\widetilde{X}(1) - K(\varepsilon)\right]\mu(v_h - 1) + K(\varepsilon)\int_{1-\varepsilon}^1 \psi(v)f(v)dv - \int_{1-\varepsilon}^1 \psi(v)\widetilde{X}(v)f(v)dv \\ &> \left[\widetilde{X}(1) - K(\varepsilon)\right]\mu(v_h - 1) + K(\varepsilon)\int_{1-\varepsilon}^1 \psi(v)f(v)dv - \widetilde{X}(1)\int_{1-\varepsilon}^1 \psi(v)f(v)dv \\ &= \left[\widetilde{X}(1) - K(\varepsilon)\right]\left[\mu(v_h - 1) - \int_{1-\varepsilon}^1 \psi(v)f(v)dv\right] \end{aligned}$$

where the inequality follows since $\widetilde{X}(v)$ is strictly increasing. Now, for ε sufficiently small but strictly positive both terms in the last line above is positive and hence $R > \widetilde{R}$ which shows that $\widetilde{X}(v)$ cannot be optimal. It follows that any optimal mechanism must satisfy the condition in the statement of the proposition. ❋

Let $\widehat{v}$ denote the cutoff type such that all types above $\widehat{v}$ are pooled. Let K denote the allocation probability of the pooled types. In other words, $K = X(v)$ for $v \in [\widehat{v}, 1]$. Using this, as well as (3.9), the seller's objective function can be rewritten as

$$\max_{X(v),\,K,\,\widehat{v}} \; K\left[-\mu\left(v_h - 1\right) + \int_{\widehat{v}}^1 \psi(v)f(v)dv\right] + \int_{v_*}^{\widehat{v}} X(v)\psi(v)f(v)dv \tag{3.10}$$

The remaining task is to characterize K, $\widehat{v}$, and $X(v)$ for $v \in [v_*, \widehat{v})$. Since the allocation function can be discontinuous at $\widehat{v}$, we use K to represent $X(\widehat{v})$ and we let $X(\widehat{v}_-)$ denote the left hand limit of $X(v)$ at $\widehat{v}$. That is $X(\widehat{v}_-) \equiv \lim_{v\uparrow\widehat{v}} X(v)$. We show below that the optimal $X(v)$ is discontinuous at $\widehat{v}$; furthermore the extent of the upward jump (ICC implies that the jump cannot be downwards) is the maximal possible.

In order to derive the optimal mechanism it is useful to think directly in terms of the actual allocation function $x(v, v')$ rather than the expected allocation function $X(v)$. A few observations are helpful. First, because of the pooling, the allocation of the object over the entire interval $[v_*, 1]$ is clearly inefficient. However, given the need to pool, there is no further gain for the seller in making the allocation rule any less efficient than it needs to be. Put differently, for a given $\widehat{v}$, the seller needs to determine how to allocate the probabilities across the types in $[v_*, \widehat{v})$ and in $[\widehat{v}, 1]$ but there is no gain in "wasting probabilities" (i.e., not selling with positive probability even if at least one bidder's type is greater than v_*), and there is no gain from having an inefficient allocation rule if both bidders' types lie in the interval $[v_*, \widehat{v})$. Hence, without loss of generality, the optimal mechanism can be written as follows.

From equation (3.9), $x(v, v') = 0$ for $v < v_*$. For $v \in [v_*, \widehat{v})$

$$x(v, v') = \begin{cases} 1 & \text{if } v' \leqslant v \\ 0 & \text{if } v' \in (v, \widehat{v}) \\ \beta(v) & \text{if } v' \in [\widehat{v}, 1] \end{cases}$$

For $v \in [\widehat{v}, 1]$

$$x(v, v') = \begin{cases} 1 & \text{if } v' < v_* \\ 1 - \beta(v') & \text{if } v' \in [v_*, \widehat{v}) \\ \frac{1}{2} & \text{if } v' \in [\widehat{v}, 1] \end{cases}$$

where the function $\beta(\cdot)$ is to be determined as part of the seller's optimization exercise. From the above we can derive the expected allocation function $X(\cdot)$. Recall that types in $[0, 1]$ win only if there is no other bidder of type v_h. Thus all terms in the expression for $X(v)$ contain a common factor $(1 - \mu)$. For economy of expression, we keep this common factor on the left hand side and derive the solution for $X(v)/(1 - \mu)$. The allocation function described above implies the following expected allocation function:

$$\frac{X(v)}{(1-\mu)} = \begin{cases} 0 & \text{if } v < v_* \\ F(v) + \beta(v)\,(1 - F(\widehat{v})) & \text{if } v \in [v_*, \widehat{v}) \\ F(v_*) + \int_{v_*}^{\widehat{v}} (1 - \beta(t))\, f(t)\, dt + \frac{1 - F(\widehat{v})}{2} & \text{if } v \in [\widehat{v}, 1] \end{cases} \tag{3.11}$$

The next (and crucial) observation is that the function $\beta(\cdot)$ should optimally be such that the allocation function for $v > v_*$ has only a single point of discontinuity, and is flat beyond this point at the maximal height. Since the types above the point of discontinuity are pooled, we refer to this as the "pooling cutoff."

In other words, there exists a value of v, let's call it v_ℓ, with $v_\ell \in [v_*, \widehat{v})$ such that it is optimal to set $\beta(v) = 0$ for $v < v_\ell$, and for $v \in [v_\ell, 1)$ set $\beta(v)$ such that $X(v)/(1 - \mu)$ is flat at the highest feasible value.

Now compare a mechanism with $\beta(v) = 0$ for all v with one in which $\beta(v) > 0$ for some v and set optimally as described above. In the first case, the allocation function $X(v)/(1-\mu)$ equals $F(v)$ up to $\widehat{v}$ and is flat beyond this point at the maximal height $(1 + F(\widehat{v}))/2$. In the second case, the function equals $F(v)$ up to v_ℓ and is flat beyond this point at the maximal height $(1 + F(v_\ell))/2$.

The only difference between the two is that the pooling cutoff is set at different points. Thus the result simplifies the search for the optimal mechanism – the problem reduces to simply finding the optimal pooling cutoff. Since we already introduced the notation $\widehat{v}$ to denote a pooling cutoff, instead of introducing new notation to denote the optimal cutoff, we simply denote it as $\widehat{v}^*$.

Before stating the result formally, let us explain the intuition behind it. In the following discussion, for economy of expression let $Y(v) \equiv \dfrac{X(v)}{(1-\mu)}$.

Simplifying from (3.11), $Y(\widehat{v})$ is given by $(1 + F(\widehat{v}))/2 - \int_{v_*}^{\widehat{v}} \beta(v) f(v)\, dv$. Suppose $\widehat{v}$ is fixed at some value, and $Y(\widehat{v})$ is fixed at K. For $\beta(v)$ to be not equal to zero for all $v \in [v_*, \widehat{v})$, we clearly need $K < (1 + F(\widehat{v}))/2$.

Given such a K, the seller must allocate $(1 + F(\widehat{v}))/2 - K$ to types in $[v_*, \widehat{v})$. What is the best way of doing this? Given that $\psi(v)$ is increasing, it pays to "pack the right side" of the interval $[v_*, \widehat{v})$ as much as possible. Specifically, so long as some probability is being transferred from the types above $\widehat{v}$ to types below, it is optimal for the seller to transfer probability to the highest possible types below $\widehat{v}$. But to preserve incentive compatibility it must be that $Y(v)$ is non-decreasing and $Y(v) \leqslant K$ for any $v < \widehat{v}$. This puts an upper bound on $\beta(v)$, which is the probability transferred to any type $v < \widehat{v}$.

Taking into account such constraints, the highest possible value of $\beta(v)$ must be such that $Y(v) = K$[17]. We claimed before that $Y(v)$ should be flat at this maximal height beyond the cutoff point. To see why this is true, consider a small interval $[\widehat{v} - \varepsilon, \widehat{v})$, and suppose $\beta(v)$ is such that $Y(v) = K$ over this interval but is strictly less than K everywhere else. Let $v_c \geqslant v_*$ be such that $\beta(v) = 0$ for $v \leqslant v_c$ and $\beta(v) > 0$ for $v > v_c$. Now suppose we reduce $\beta(v)$ to zero over a small interval $[v_c, v_c + \eta]$, and transfer this

[17] Specifically, as can be seen from (3.11), the highest possible value of $\beta(v)$ is given by $\beta(v)(1 - F(\widehat{v})) = K - F(v)$.

probability weight to an interval $[\widehat{v} - \varepsilon - \delta, \widehat{v} - \varepsilon]$ (this is possible since previously we had $Y(v) < K$ on this interval). This increases expected revenue. Indeed, this process should go on until $Y(v)$ can be raised to K over the latter interval. In this way we can keep "packing" the right end of the interval $[v_*, \widehat{v})$, raising $Y(v)$ to K for successively lower intervals until we run out of the "probability budget" we started with. This gives us a pooling cutoff v_ℓ such that $(1 + F(v_\ell))/2 = K$.

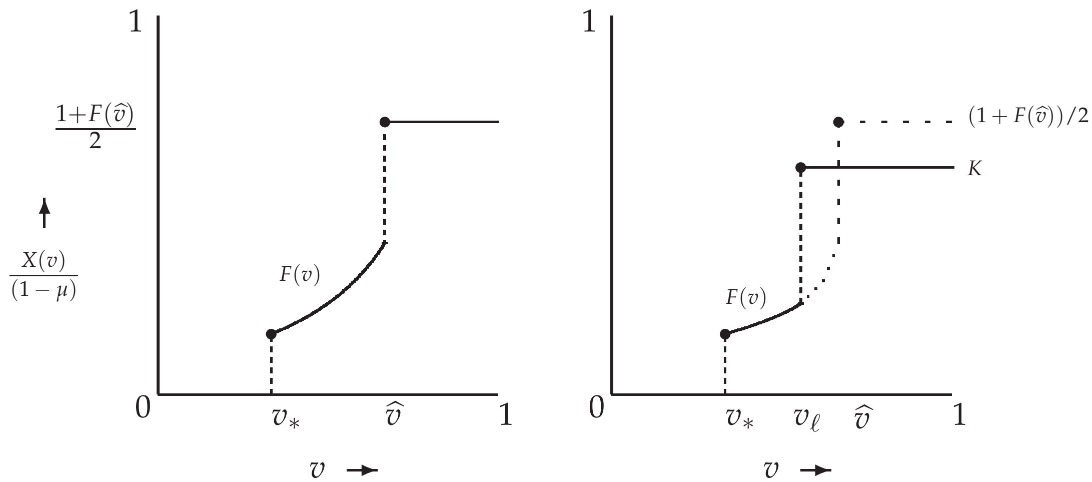

Figure 1: In the left hand picture $X(v)/(1-\mu) = (1 + F(\widehat{v}))/2$ for $v \geqslant \widehat{v}$, and therefore $\beta(\cdot) = 0$. In the picture on the right side, for $v \geqslant \widehat{v}$, $X(v)/(1-\mu) = K$ where $K < (1 + F(\widehat{v}))/2$. Thus $\beta(v) > 0$ for some v to the left of $\widehat{v}$. The optimal $\beta(v)$ function for $v \in [v_*, \widehat{v})$ is obtained by packing the right side until successively lower $X(v)$ is pushed up to K, as shown. This continues until v_ℓ such that $(1 + F(v_\ell))/2 = K$. Note that under the optimal $\beta(\cdot)$ function, the general form of the allocation function remains the same as that in the picture on the left. Thus we only need to optimize with respect to the pooling cutoff.

The upshot is that no matter what the value of K is, the form of the optimal allocation function remains the same: $Y(v)$ equals $F(v)$ up to a cutoff point, and is flat at the maximal height beyond the cutoff. Figure 1 shows the optimal $Y(v)$ function.

We state the result formally below. The formal proof requires an approximation argument and is somewhat lengthy. We have relegated this to the appendix.

Proposition 4 *The optimal mechanism involves solving the following maximization problem*

$$\max_{y} \quad \int_{v_*}^{y} F(v)\psi(v)f(v)\,dv + \frac{1+F(y)}{2}\left[-\mu(v_h-1)+\int_{y}^{1}\psi(v)f(v)\,dv\right]$$

where the argmax gives the optimal pooling cutoff.

As mentioned before, we denote the optimal pooling cutoff as $\widehat{v}^*$. From the result above it follows that $\widehat{v}^*$ is characterized implicitly by the equation:

$$\psi(\widehat{v}^*) = \frac{1}{1-F(\widehat{v}^*)}\left(-\mu(v_h-1)+\int_{\widehat{v}^*}^{1}\psi(v)f(v)\,dv\right) \tag{3.12}$$

The next result shows that $\widehat{v}^*$ exists and is unique.

Proposition 5 *The optimal pooling cutoff $\widehat{v}^*$, implicitly defined by equation* (3.12)*, exists and is unique.*

Proof. Define $G(\cdot)$ as follows.

$$\begin{aligned} G(v) &\equiv (1-F(v))\,\psi(v)+\mu(v_h-1)-\int_{v}^{1}\psi(t)f(t)dt \\ &= (1-F(v))\,\psi(v)-\lambda(v), \end{aligned}$$

where $\lambda(v)$ is defined in proposition 2. Note that $G(\cdot)$ is a continuous function. Following proposition 2, we assumed that $\lambda(v_*)>0$ (to ensure a non-trivial price discrimination problem). This, along with the fact that $\psi(v_*)=0$, implies $G(v_*)<0$. Further, $G(1)=\mu(v_h-1)>0$. Hence there must exist $\widehat{v}^*\in(v_*,1)$ such that $G(\widehat{v}^*)=0$. Finally, $\frac{dG}{dv}=(1-F(v))\,\psi'(v)>0$. Hence $\widehat{v}^*$ is unique. ❋

From proposition 4, the general form of the mechanism can be rewritten as follows. As before, the common term $(1-\mu)$ is placed on the left hand side.

$$\frac{X(v)}{(1-\mu)} = \begin{cases} 0 & \text{if } v<v_* \\ F(v) & \text{if } v\in[v_*,\widehat{v}^*) \\ \dfrac{1+F(\widehat{v}^*)}{2} & \text{if } v\in[\widehat{v}^*,1] \end{cases} \tag{3.13}$$

where $\widehat{v}^*$ is given implicitly by equation (3.12).

Finally, we collect together the results from above and describe the optimal mechanism. This is given by the allocation function $X^*(\cdot)$ and the expected payment function $T^*(\cdot)$ given by equations (3.14) to (3.17) below.

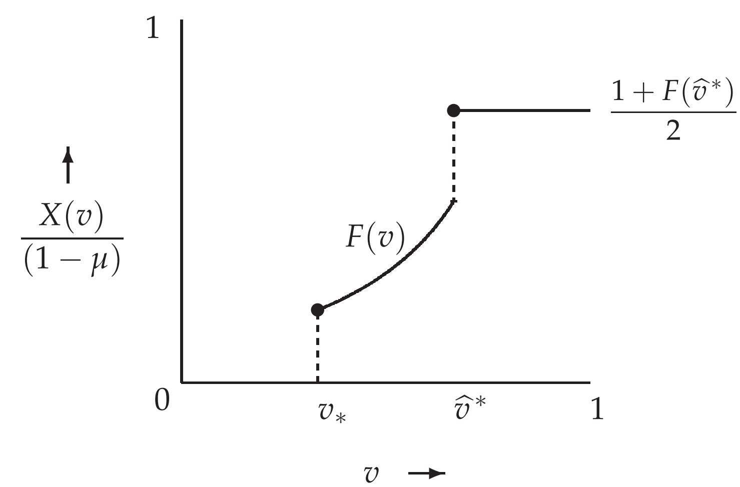

Figure 2: The figure shows the optimal allocation function $X^*(v)/(1-\mu)$. The function jumps up at the reserve cutoff v_* and at the pooling cutoff $\widehat{v}^*$. The extent of the jump at the pooling cutoff is maximal and given by $(1+F(\widehat{v}^*))/2 - F(\widehat{v}^*) = (1-F(\widehat{v}^*))/2$.

$$X^*(v_h) = 1 - \frac{\mu}{2}, \tag{3.14}$$

$$X^*(v) = \begin{cases} 0 & \text{if } v < v_* \\ (1-\mu)\, F(v) & \text{if } v \in [v_*, \widehat{v}^*) \\ (1-\mu)\, \dfrac{1+F(\widehat{v}^*)}{2} & \text{if } v \in [\widehat{v}^*, 1] \end{cases} \tag{3.15}$$

$$T^*(v_h) = v_h X^*(v_h) - (v_h - 1)\, X^*(1) - \int_0^1 X^*(t)\, dt, \tag{3.16}$$

$$T^*(v) = vX^*(v) - \int_0^v X^*(t)\, dt \qquad \text{for } v \in [0,1]. \tag{3.17}$$

where the "reserve type" v_* is given by $\psi(v_*) = 0$, and the "pooling cutoff" $\widehat{v}^*$ is given by equation (3.12).

$X^*(v_h)$ follows from equation (3.8), and $T^*(v_h)$ follows from equation (3.5). The optimal allocation $X^*(v)$ follows from equation (3.13). $T^*(v)$ then follows from equation (3.2).

Figure 2 shows the optimal mechanism. There are two jump points, v_* and $\widehat{v}^*$, and the rest of the mechanism is continuous (and differentiable).

Also note that other than the pooling of types in $[\widehat{v}^*, 1]$ and the presence of the reserve type v_*, the optimal mechanism is efficient. In particular, whenever at least one bidder draws a type above v_*, the object is sold with probability 1, and other than the pooling region, the optimal auction coincides with the standard optimal auction (Myerson (1981)).

The fact that the optimal mechanism is characterized by "pooling at the top at maximal feasible value" implies that the auction includes a posted-price-like feature. Indeed, we show in the next section this property translates into a temporary buy now option in an auction. To explain the precise nature of such an option, and clarify the nature of the optimal mechanism further, we explicitly construct an indirect selling mechanism that implements the mechanism. However, before we do that we first present a simple numerical example that compares the optimal mechanism both with a simple posted price selling (only) and a mechanism that uses a posted price followed by a standard auction. The latter comparison is particularly interesting since it gives an idea of the (numerical) importance of the pooling feature of the optimal mechanism.

3.4 An Example

Suppose the bidders draw values from a uniform distribution over the unit interval. Then $F(v) = v$, and $f(v) = 1$ for all $v \in [0, 1]$. From equation (3.7), the modified virtual value is given by

$$\psi(v) = 2v(1-\mu) - 1. \tag{3.18}$$

The optimal mechanism is trivial if it involves a pure posted price of v_h. In this case the seller simply sells to the high type and does not sell to any other lower types. For the optimal mechanism to involve selling to lower types as well, we need to satisfy assumption 2, which requires $\mu < \frac{1}{2}$. From proposition 2, we also need $\lambda(v_*) > 0$.

Using the value of $\psi(v)$ from above, this requires $4\mu(1-\mu)v_h < 1$. Solving for μ and using the restriction $\mu < 1/2$, we get the following constraint on μ which must be satisfied for a non-trivial mechanism.

$$\mu < \frac{1}{2}\left(1 - \sqrt{\frac{v_h - 1}{v_h}}\right) \equiv \overline{\mu}(v_h) \tag{3.19}$$

Figure 3 shows this upper limit.

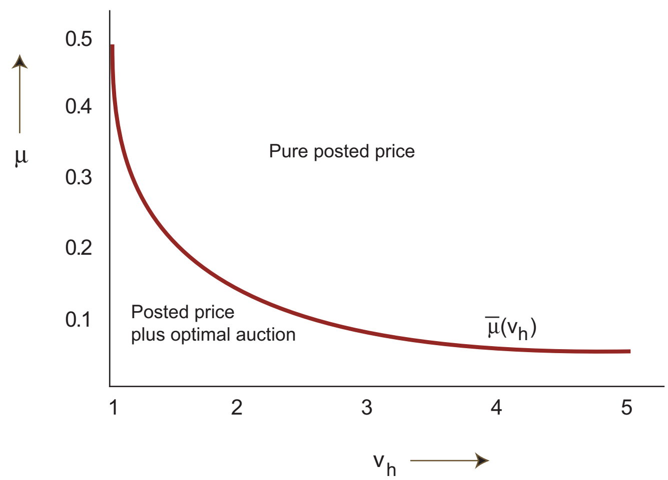

Figure 3: For any (μ, v_h) to the left of $\overline{\mu}(v_h)$, the optimal mechanism is to have a posted price plus an optimal auction, which itself includes a pooling region (which, as the next section shows, implies an auction with a buy-now price). In the region to the right of $\overline{\mu}(v_h)$, the optimal mechanism coincides with a pure posted price of v_h.

For any $\mu < \overline{\mu}(v_h)$, the optimal mechanism involves an auction. Under a uniform distribution, the reserve type v_* (given by $\psi(v_*) = 0$), and the "pooling cutoff" $\widehat{v}^*$ (given by equation (3.12)) are

$$\begin{aligned} v_* &= \frac{1}{2(1-\mu)} \\ \widehat{v}^* &= 1 - \sqrt{\frac{\mu(v_h - 1)}{1-\mu}} \end{aligned}$$

From equation (3.6), the expected revenue is given by

$$ER = 2\left[\mu v_h X^*(v_h) - \mu\,(v_h - 1)\,X^*(1) + \int_0^1 \psi(v) X^*(v) f(v) dv\right]$$

where $X^*(\cdot)$ is specified in the previous section. For $\mu \geqslant \overline{\mu}(v_h)$, the optimal mechanism is a posted price of v_h and the expected revenue is[18]

$$ER = 2\mu v_h X^*(v_h)$$

Finally consider a mechanism that combines a posted price with a standard auction. This is clearly suboptimal[19], but calculating the expected revenue from this mechanism clarifies the gain from the pooling at the top in the optimal auction. The expected revenue from the suboptimal mechanism is given by

$$ER = 2\left[\mu v_h X^*(v_h) - \mu\left(v_h - 1\right)X(1) + \int_0^1 \psi(v)X(v)f(v)dv\right]$$

where $X^*(v_h)$ is as before, but now $X(\cdot) = (1-\mu)F(v)$ for all $v \in [v_*, 1]$.

As an example, consider $v_h = 1.5$ and $\mu = .1$. Since $\overline{\mu}(1.5) = 0.21$, the optimal mechanism involves a posted price and an auction. The auction is characterized by $v_* = .56$ and $\widehat{v}^* = .76$, i.e. the lowest 56% types are not served and the top 24% types in the unit interval are pooled. The revenue from the optimal mechanism is 0.47. A simple posted price fetches a revenue of .29. Thus there is a gain from price discrimination. The following figure shows the optimal allocation.

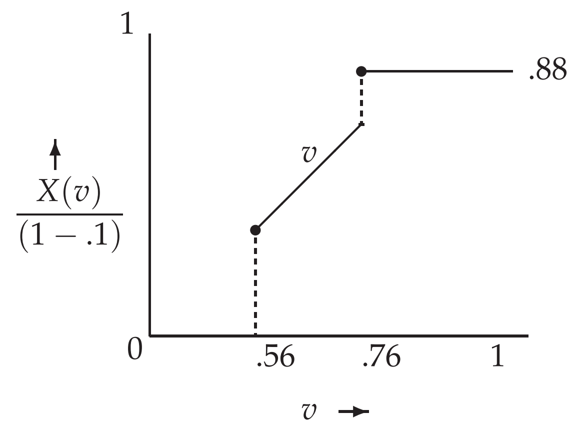

Figure 4: The optimal allocation function $X^*(v)/(1-\mu)$ for $\mu = .1$ and $v_h = 1.5$.

[18] Alternatively, this can be derived as v_h times the probability that at least one bidder is of type v_h, i.e. $ER = (1-(1-\mu)^2)v_h$.

[19] Such an auction (with the right reserve price) is optimal in the standard independent private values setting (i.e. without an atom at v_h), but suboptimal in our setting.

Figure 5 shows the expected revenue from the optimal mechanism for different values of v_h for $\mu = .1$. For $v_h < 2.78$, the optimal mechanism involves an auction, and the dotted line shows the revenue if instead of the optimal mechanism, a pure posted price of v_h is used. The figure clarifies the revenue gain from price discrimination.

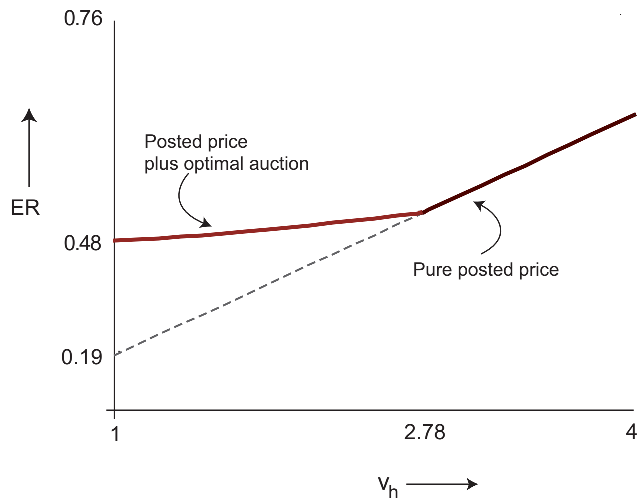

Figure 5: The expected revenue from the optimal mechanism for $\mu = .1$ under different values of v_h. For $v_h < 2.78$ the optimal mechanism involves a posted price as well as price discrimination through an auction, and generates more revenue compared to a simple posted price. For higher values of v_h the optimal mechanism coincides with a simple posted price of v_h.

The following table now compares the expected revenue from the optimal mechanism (ER Opt), posted price plus standard auction (ER Std), and a pure posted price mechanism (ER PP) across different values of v_h. The numbers reported are for $\mu = 0.1$. The table compares revenue for $v_h \in [1, 2.75]$. For $v_h \leqslant 2.75$, $\overline{\mu}(v_h) > 0.1$. Therefore over this range of v_h, the optimal mechanism involves non-trivial price discrimination, and generates more revenue than a pure posted price mechanism. The table also shows the gain from using the optimal mechanism compared to a mechanism that combines a posted price with a standard auction. Finally, the table compares the values of the optimal posted price (PP Opt) with the posted price if a standard auction is used (PP Std). For either mechanism, the posted price is given by $T(v_h)/X(v_h)$.

v_h	ER Opt	ER Std	ER PP
1	0.463	0.463	0.190
1.25	0.468	0.465	0.238
1.5	0.475	0.468	0.285
1.75	0.483	0.470	0.333
2	0.493	0.473	0.380
2.25	0.503	0.475	0.428
2.5	0.514	0.478	0.475
2.75	0.526	0.480	0.523

PP Opt	PP Std
0.673	0.673
0.705	0.686
0.755	0.699
0.815	0.712
0.883	0.725
0.959	0.738
1.042	0.751
1.130	0.765

The following figure compares the expected revenue from the optimal mechanism with that from a mechanism that combines a posted price with a standard auction for $\mu = 0.1$.

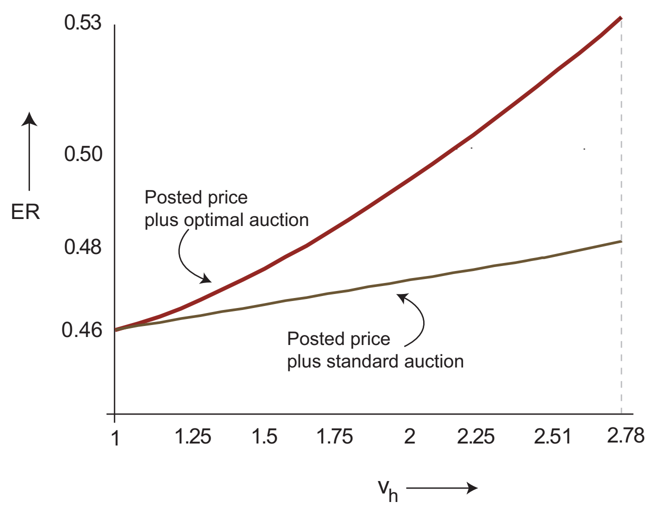

Figure 6: The expected revenue from the optimal mechanism and a mechanism that combines posted price with a standard auction for $\mu = 0.1$. For this μ, the optimal mechanism involves an auction for $v_h \leqslant 2.78$, beyond which the optimal mechanism is a pure posted price of v_h.

4 Implementing the Optimal Mechanism

In this section we discuss a selling mechanism (an "indirect" mechanism) that implements the optimal direct mechanism from the previous section. As we show below, part of the mechanism is a posted price selling; the rest is an auction *augmented* by some non-standard feature that we show to be a buy now option. In particular, we show that an auction with a temporary – as opposed to permanent – buy now option implements the optimal mechanism described in the previous section. We discuss this point further after describing the selling mechanism.

4.1 Description of the selling mechanism

The selling mechanism is implemented in two stages.

Stage 1. (Posted Price) In stage 1, the item is offered for sale at a posted price P. If any buyer wants to buy at that price, the item is sold and the game is over. If there is a tie, it is resolved by randomly allocating the item to one of the tied buyers. If the item is not sold in stage 1, we proceed to stage 2.

Stage 2. (Auction) Stage 2 is an auction augmented by a buy now option. Let B denote the buy price. Stage 2 has two sub-stages.

First sub-stage. (Buy now option) In the first sub-stage, the auction opens with a buy price B. If a single bidder bids B then the item is awarded to that bidder. If both bidders submit B then the item is allocated randomly with the winning bidder paying the price B. If no bidder bids the B price, the game proceeds to the second sub-stage.

Second sub-stage. (Vickrey auction) The second sub-stage is a standard Vickrey auction with a reserve price. In this stage the item is allocated to the highest bidder at a price which is the maximum of the reserve price and the second highest bid provided the highest bid is above the reserve price. If the highest bid is below the reserve price, the seller keeps the item and the game is over.

We now show that this selling mechanism implements the optimal direct mechanism from the previous section.

4.2 Implementation

To economize on notation, in what follows we refer to the optimal pooling cutoff, given by equation (3.12), as $\widehat{v}$ (rather than $\widehat{v}^*$).

Proposition 6 *The two stage selling mechanism described above implements the optimal mechanism when the reserve price in the Vickrey auction is chosen as v_*, and the posted price P and temporary buy price B are chosen as:*

$$P = v_h - \left(\frac{1-\mu}{1-\mu/2}\right)\left[\left(v_h - F(\widehat{v})\right)\left(\frac{1+F(\widehat{v})}{2}\right) + \int_{v_*}^{\widehat{v}} F(t)\, dt\right] \tag{4.1}$$

$$B = \widehat{v} - \left(\frac{2}{1+F(\widehat{v})}\right)\int_{v_*}^{\widehat{v}} F(t)\, dt \tag{4.2}$$

where v_* is given by $\psi(v_*) = 0$ and $\widehat{v}$ is given by equation (3.12).

The proof works in two steps. We first describe the equilibrium strategies and show that if the buyers follow the prescribed strategies then the expected probabilities of winning and the expected payments are exactly as in the optimal mechanism from the last section. We then show that the described strategies do indeed constitute an equilibrium of the selling mechanism.

Proof. STEP 1. Consider the following strategies: Type v_h buys the item in stage 1 by paying the posted price P. Types in the interval $[\widehat{v}, 1]$ submit the buy price B in the first sub-stage of stage 2. Types in $[v_*, \widehat{v})$ submit their true valuations as bids in the second sub-stage. (Types below v_* do not win the item so it is irrelevant whether they participate or not. Without loss of generality, assume they submit bids equal to their valuations also.)

Given the strategies, type v_h is the only type to buy at price P which means that if a buyer of type v_h follows the prescribed strategy, the expected probability of winning the item is $(1-\mu) + \frac{\mu}{2}$ which is exactly the same as $X^*(v_h)$ (given by equation (3.14)).

Using the values of $X^*(v_h)$ and $X^*(v)$ for $v \in [0,1]$, we get

$$\int_0^1 X^*(t)\,dt = (1-\mu)\left[\int_{v_*}^{\widehat{v}} F(t)\,dt + (1-F(\widehat{v}))\left(\frac{1+F(\widehat{v})}{2}\right)\right]$$

Substituting in equation (3.16),

$$T^*(v_h) = v_h\left(1-\frac{\mu}{2}\right) - (1-\mu)\left[\left(v_h - F(\widehat{v})\right)\left(\frac{1+F(\widehat{v})}{2}\right) + \int_{v_*}^{\widehat{v}} F(t)\,dt\right]$$

Hence, if the posted price P is given by equation (4.1), the expected payment of type v_h is the same in the indirect mechanism as it is in the optimal mechanism. For types in $[v_*, \widehat{v})$, the probability of winning and expected payments are exactly as in the optimal mechanism (given by equation (3.15)) since they are simply taking part in a Vickrey auction. The types in $[\widehat{v}, 1]$ also have the same expected probability of winning the item as in the optimal mechanism since if they all submit the buy price, they are all being "pooled" in the indirect mechanism in exactly the same way they are in the optimal direct mechanism and hence their expected probability of winning the item in the indirect mechanism is $(1-\mu)\left(\frac{1+F(\widehat{v})}{2}\right)$. Now, the expected payment of these pooled types in the optimal mechanism is given by $\widehat{v}\,X^*(\widehat{v}) - \int_0^{\widehat{v}} X^*(t)\,dt$. Since $\int_0^{\widehat{v}} X^*(t)\,dt = (1-\mu)\int_{v_*}^{\widehat{v}} F(t)\,dt$, the expected payment of the pooled types can be written as

$$(1-\mu)\left[\widehat{v}\left(\frac{1+F(\widehat{v})}{2}\right) - \int_{v_*}^{\widehat{v}} F(t)\,dt\right]$$

Conditional on reaching stage 2, if a type $v \in [\widehat{v}, 1]$ bids B, the expected payment is $(1+\frac{F(\widehat{v})}{2})\,B$, and hence this type's overall expected payment from participating in the mechanism is given by $(1-\mu)(\frac{1+F(\widehat{v})}{2})\,B$. Hence if the buy price B is given by equation (4.2), the expected payments are the same in the direct and the indirect mechanisms.

STEP 2. We now show the strategies described above do constitute an equilibrium. From the way the payoffs are constructed, type v_h is indifferent between buying at the price P at the first stage and waiting to bid B in the second stage and strictly prefers buying at price P to bidding any other price in the second sub-stage auction. Consider now a type $v \in [0,1]$. It is clear that no type $v \in [0,1]$ wants to mimic type v_h's strategy of buying the item in the first stage. To derive the optimal bid of a type $v \in [0,1]$ in the second stage, all we need to do is to compare the type's expected payoff from only two possible bids: bidding the buy price B in the first sub-stage, or bidding the true value of v in the Vickrey auction (in the second sub-stage).

Let $D(v)$ denote the difference in expected surplus of type v from the bids B and v. The proof now proceeds through the following lemma, which is proved in the appendix.

Lemma 1 *$D(v)$ is strictly increasing in v and $D(\widehat{v}) = 0$.*

This shows that, as required, type $\widehat{v}$ is indeed indifferent between bidding B and $\widehat{v}$. Further, since $D'(v) > 0$, types below $\widehat{v}$ strictly prefer to bid their true value rather than B, and types $v \in (\widehat{v}, 1]$ strictly prefer to bid B. ❋

4.3 Properties of the selling mechanism: A temporary buy now option

We now clarify that the indirect mechanism which implements the optimal direct mechanism includes a *temporary* buy now option. Later, in section 4.5 we point out the connection between the temporary buy now option we use and the one used by eBay.

In stage 2 of the selling mechanism described in the previous (sub)section, bidders are offered the chance to buy the item at price B, followed by a Vickrey auction. In other words, the mechanism involves an auction but with a buy now option that is only temporary - the option is withdrawn when the actual auction takes place. It is interesting to note that under such a buy now option, the auction price can actually exceed the buy price. To see this, note that types above $\widehat{v}$ plan to buy the item at buy price B and types $[v_*, \widehat{v})$ follow the standard strategy of a Vickrey auction. Hence, at the auction stage, the price range belongs to the interval $[v_*, \widehat{v}]$ and since $B < \widehat{v}$ (see equation (4.2)), once the buy now option vanishes, the subsequent auction price in the optimal mechanism could be higher than the buy price.

Next, we show that in general it is not possible to implement the optimal direct mechanism using a permanent buy now option.

4.4 Suboptimality of a permanent buy now option

As noted in the introduction, the literature has focused mainly on augmenting standard auctions with a permanent buy price feature and has remarked that this generates more revenue compared to a standard auction augmented a temporary by a temporary buy-price auction. We show, on the other hand, that in general it is not possible to implement our optimal mechanism with a permanent buy price.

Hidvégi, Wang, and Whinston (2005) (henceforth HWW) provide a detailed analysis of an English auction augmented by a permanent buy price b[20]. We make use of some of their results to show that in our context a permanent buy price is suboptimal. HWW consider an independent private values setting with bidders drawing values from a distribution on $[\underline{v}, \overline{v}]$.

Result (HWW): Consider an English auction with a reserve price r and permanent buy price b. In (the unique) equilibrium, there are cutoffs v_{c} and v_{uc} such that the type space can be partitioned into the following (possibly empty) intervals:

- types $v \in [r, b)$ use a "traditional strategy" : remain active till price rises to v,
- types $v \in [b, v_{\text{c}})$ use a "threshold strategy" : remain active till the auction price reaches a critical point $t(v, b)$ and then jumps to the buy price b.
- types $v \in [v_{\text{c}}, v_{\text{uc}})$ use a "conditional strategy" : bid b if (and only if) the auctions clocks moves from r (implying that there is at least one other bidder with value above r), and
- types $v \in [v_{\text{uc}}, \overline{v}]$ use an "unconditional strategy" : bid b right at the start of the auction.

The threshold $t(v, b)$ is decreasing in v, and if $\lim_{v \to \overline{v}} t(v, b) \geq r$, $v_{\text{c}} = v_{\text{uc}} = \overline{v}$, and all types $v \geqslant b$ play the threshold strategy. On the other hand if $\lim_{v \to \overline{v}} t(v, b) < r$, there are types who play the conditional and unconditional strategies.

Translating this to our context, given a permanent buy price B and the reserve type v_*, a posted price followed by an auction can implement the optimal mechanism if

[20] It should be noted that their basic environment is different from ours. They have a standard private-value auction setting, and the distribution of values does not have a counterpart of the atom at v_h that features in our model. However, this distinction is not important for what follows.

types $[\widehat{v}, 1]$ play the conditional strategy (i.e. bid B immediately) and types $[v_*, \widehat{v})$ play strategies such that their bids are revenue-equivalent to the bids in the temporary buy-price auction.

It is clear that in the temporary buy-price auction (as in the optimal direct mechanism), *only* types $v \in [\widehat{v}, 1]$ are pooled, and there is no pooling in the interval $[v_*, \widehat{v})$. In this interval the allocation probability and expected payment are strictly increasing. Therefore if any indirect mechanism involves pooling (same allocation probability, same expected payment) in the latter region, this would introduce additional inefficiency, and prevent the mechanism from implementing the optimal mechanism.

Consider an auction with a permanent buy price B. Types playing the conditional strategy or unconditional strategy are clearly pooled. Therefore to avoid pooling of types below $\widehat{v}$, it must be that that no sub-interval of types in $[v_*, \widehat{v})$ play either the conditional or unconditional strategies. But from the result above, we know that this happens if and only if $\lim_{v \to \widehat{v}} t(v, B) \geq v_*$. We state this below.

Corollary of the HWW result: A necessary condition for implementing the optimal direct mechanism using a posted price combined with an English auction with a permanent buy price B is given by $\lim\limits_{v \to \widehat{v}} t(v, B) \geqslant v_*$.

Can we ensure this condition holds? As the following result shows, the problem is that $\widehat{v}$, B as well as the function $t(\cdot, \cdot)$ are already determined by conditions of implementation of the optimal mechanism. Thus there is no parameter we could vary to ensure this inequality holds. It might hold for some distribution F by lucky coincidence, but in general it is not possible to ensure this. The proof of the following result shows that the inequality is not satisfied for a uniform distribution.

For the following result, we use the usual definition of implementation: for any distribution F (satisfying the basic assumptions) an indirect mechanism implements the optimal mechanism if there exists an equilibrium of the indirect mechanism whose outcome is the same as that of the optimal mechanism. The following result shows that the permanent buy now option fails this test.

Proposition 7 *It is not possible to implement the optimal direct mechanism using an English auction with a permanent buy price B.*

Proof. Since the indirect mechanism, by definition, is required to implement the optimal mechanism for all (allowable) possible parameter values and distributions, it suffices to show an example where it does not. In particular, we show that under a uniform distribution $\lim_{t\to\widehat{v}} t(v,B) < v_*$.

Note first that if a permanent buy-price auction implements the optimal mechanism, then, given that the temporary buy-price auction does implement the optimal mechanism as shown in the preceding subsection, all types should win with the same probability and get the same expected surplus in the two buy-price auctions.

Let us derive explicitly the threshold strategies of the types in $[B,\widehat{v})$.[21] In the temporary buy-price auction a type $v \in [B,\widehat{v})$ obtains a surplus

$$(v - v_*)\, F(v_*) + \int_{v_*}^{v} (v-y) f(y) dy$$

Under a permanent buy price, assuming, for the time being, that there is no pooling sub-interval (i.e. the condition in the corollary is satisfied), expected surplus from following the threshold strategy is:

$$(v - v_*)\, F(v_*) + \int_{v_*}^{t(v,B)} (v-y) f(y) dy + (v-B)\, (F(\widehat{v}) - F(t(v,B)))$$

Suppose now that $F(.)$ is the uniform distribution. Equating the two expressions above, we can solve for $t(v,B)$ and obtain $t(v,B) = 2B - v$. Using the value of B under the uniform distribution (from equation (4.2)), we have

$$\lim_{t\to\widehat{v}} t(v,B) - v_* \;=\; \widehat{v} - \frac{2}{1+\widehat{v}}\left(\widehat{v}^2 - v_*^2\right) - v_* \;=\; \frac{(\widehat{v}-v_*)\,(1-\widehat{v}-2v_*)}{1+\widehat{v}}$$

Under a uniform distribution, v_* (defined as $\psi(v_*) = 0$) is given by $v_* = \dfrac{1}{2(1-\mu)}$, which finally gives us

$$\lim_{t\to\widehat{v}} t(v,B) - v_* \;=\; \frac{(\widehat{v}-v_*)\left(1-\widehat{v}-\frac{1}{1-\mu}\right)}{1+\widehat{v}} \;<\; 0$$

where the inequality follows from the fact that $\mu \in (0,1)$. This violates the implementation condition in the corollary above. ❋

[21] Types $v \in [v_*, B)$ play the standard Vickrey auction strategy and English auction strategy in the two auctions so their probability of winning and expected surplus are the same in the two auctions.

4.5 The temporary buy now option used by eBay

In the final part of this section we discuss the relation between our temporary buy-price auction and the one used by eBay.

In our indirect mechanism the seller makes an initial offer to sell the item at a pre-specified price B but this buy price offer is temporary since it is withdrawn by the seller if there are no takers. Therefore the buy now option is absent during the subsequent auction. eBay uses a slightly different form of temporary buy now option. The chief distinction is that in the eBay auction the disappearance of the buy price is endogenous: the buy now option vanishes whenever a bidder places a bid above the reserve price. Even though the eBay auction differs from ours, we now argue that in the scenario modeled here, the outcome of the two should be the same. To see this, recall the four possible strategies in a buy-price auction from the HWW result stated in section 4.4 above. These are traditional, threshold, conditional and unconditional strategies. Now, the threshold strategy (under which a bidder waits for the price in the auction to rise to a certain level before exercising the buy now option), and the conditional strategy (under which a bidder waits and bids the buy price only if some other bidder bids above the reserve price) are not available under the rules of eBay auctions, since in both cases the buy now option would vanish. Therefore, in an eBay buy-price auction, only two – the standard and the unconditional strategies – are feasible. But this is just like our selling mechanism. Hence, under the setting of our model, the optimal eBay auction is to choose the reserve price and the buy price to be v_* and B respectively, and in this case the eBay auction should implement the optimal mechanism.

5 Conclusion

We study the optimal selling mechanism when similar goods are sold through auctions as well as posted prices. The starting point of our analysis is the observation that in addition to traditional posted price selling through "bricks-and-mortar" stores, sellers can now use the Internet to sell their items through auctions and posted prices. In particular, lower transactions cost of online trading enables sellers to reach types of buyers who are typically priced out in traditional markets. Since the sellers use different sales mechanisms to sell to different groups of buyers, the optimal design of the overall mechanism involves second degree price discrimination *across* selling methods[(22)].

We attempt to capture this aspect of online pricing in our model. We characterize the optimal selling mechanism in this environment, and show that it involves a posted price combined with an auction with "pooling at the top." Moreover, the allocation function assumes the maximal value over the pooled region. This feature of the optimal mechanism corresponds exactly with a buy now option. Thus, the phenomenon of a buy now option in an auction, something that might appear puzzling when seen in isolation (i.e. in the context of the auction alone) emerges as a *necessary* feature of the overall optimal selling mechanism. Further, we show that posted price selling followed by a standard auction with a *temporary* buy now option – used by eBay and seemingly even more of a puzzling phenomenon[(23)] – implements the optimal mechanism.

Of course, eBay as well as other online auctions are rich in institutional detail, all of which we do not claim to capture[(24)]. However, we do believe that for many sellers online auctions form *part* of an optimal selling strategy, and therefore it is important to understand the overall strategy in order to analyze its constituent parts. The main lesson from our results is that such considerations play a role in understanding how non-standard auction forms can emerge as an optimal response. While an advantage of our theory is that it is based on optimal design, further empirical tests taking into account overall market data are required to distinguish between competing theories.

[(22)]The ease of eliciting bids from customers has lead to many new and interesting opportunities for price discrimination on the Internet. Here we have focused on auctions. Another example is the "name your own price" strategy used by sites such as Priceline to sell airlines tickets, which downgrades the flexibility of flights (e.g. buyers can name their own price, but flight times are shown after purchase) in order to sell to lower end customers. See Hall (2001) for a detailed discussion.

[(23)]As noted in the introduction, this is more puzzling in the sense that even if one could find a reason for introducing a buy-now option, taking it away as the auction starts begs a further question. Indeed, as also noted in the introduction, the literature often concludes that auctions with a temporary buy-now option are revenue inferior to the permanent buy-now counterpart.

[(24)]For example, one aspect of online auctions is that bidders arrive at random points of time over a certain interval. Modeling random entry requires a substantially different setup. Here our purpose is to derive an optimal mechanism that implies a buy now price in a setting as close as possible to the standard private values model.

To this end, we briefly mention how the testable implications of our model differ from other theories.

Testable implications

The introduction mentions the risk-aversion and sequential auction theories analyzed in the literature. These theories look at auctions in isolation, and imply that there is no systematic relation between prices posted elsewhere by the seller for similar goods and the buy price in the auction. A further implication of the risk aversion theory is that if similar items are offered for sale through posted prices elsewhere, this might weaken the incentive to have a buy price, since the most risk averse buyers might prefer to buy at a posted price. Thus the theory implies that if similar items are sold elsewhere through posted prices by a seller, this makes it less likely that the seller would use a buy now option.

The theory presented here, on the other hand, shows that posted price sales make it more likely that the seller uses a buy price in the auction. Further, our theory predicts a negative relation between the posted price and the buy price. A higher posted price necessitates greater inefficiency in the auction to preserve incentives. Further empirical work is needed to test the relative importance of theories in explaining the emergence of a buy now option in standard auctions.

6 Appendix: Proofs

A.1 Proof of Proposition 4

From (3.10), the objective is to maximize

$$\int_{v_*}^{\widehat{v}} \psi(v)X(v)f(v)dv + \left[-\mu(v_h - 1) + \int_{\widehat{v}}^{1} \psi(v)f(v)\,dv\right] X(\widehat{v})$$

For any $v \in [v_*, \widehat{v})$, and $v' \geqslant \widehat{v}$, let $x(v, v') = \beta(v)$. Thus $\beta(v)$ is the transfer away from top types.

For $v \in [v_*, \widehat{v})$,

$$X(v) = F(v) + \beta(v)(1 - F(\widehat{v})). \tag{A.1}$$

For $v \in [\widehat{v}, 1]$,

$$\begin{aligned} X(v) &= F(v_*) + \int_{v_*}^{\widehat{v}} (1 - \beta(t))f(t)\,dt + \frac{1 - F(\widehat{v})}{2} \\ &= \frac{1 + F(\widehat{v})}{2} - \int_{v_*}^{\widehat{v}} \beta(t)f(t)\,dt \end{aligned}$$

For $v \in [\widehat{v}, 1]$, let $X(v) < \dfrac{1 + F(\widehat{v})}{2}$, so that the transfer of probability away from top types is positive. Specifically, let

$$X(v) = (1 - \theta)\frac{1 + F(\widehat{v})}{2} \qquad \text{for } v \in [\widehat{v}, 1] \tag{A.2}$$

where $\theta \in (0, 1)$ and fixed exogenously.

Further, incentive compatibility requires

$$X(v_-) \equiv \lim_{v \uparrow \widehat{v}} X(v) \leqslant (1 - \theta)\frac{1 + F(\widehat{v})}{2} \tag{A.3}$$

Finally, the "probability budget" must be balanced so that $\int_{v_*}^{\widehat{v}} \beta(t)f(t)\,dt$ must equal the total probability transferred away from each type above $\widehat{v}$, given by $\theta\dfrac{1 + F(\widehat{v})}{2}$.

Thus

$$\int_{v_*}^{\widehat{v}} \beta(t) f(t)\, dt = \theta \frac{1+F(\widehat{v})}{2} \tag{A.4}$$

For any given $(\widehat{v}, \theta)$, the optimization problem can be written (using equations (A.1) and (A.2)) as

$$\max_{\beta(v)} \int_{v_*}^{\widehat{v}} \left[F(v) + \beta(v)(1 - F(\widehat{v}))\right] \psi(v) f(v)\, dv + \left[-\mu(v_h - 1) + \int_{\widehat{v}}^{1} \psi(v) f(v)\, dv\right] (1-\theta) \frac{1+F(\widehat{v})}{2}$$

subject to (A.3) and (A.4).

Given $(\widehat{v}, \theta)$, the second term is a constant. Thus the optimization problem is simply given by:

$$\max_{\beta(v)} \int_{v_*}^{\widehat{v}} \psi(v)[F(v) + \beta(v)(1 - F(\widehat{v}))] f(v) dv$$

subject to (A.3) and (A.4).

Let us approximate the $\beta(v)$ function by a step function with n steps at $v_0 < v_1 < \ldots < v_n$. Let $v_0 = v_*$ and $v_n = \widehat{v}$.

Let $\widehat{R}$ denote the maximand. The optimization problem is

$$\max_{\beta_1, \ldots, \beta_n} \widehat{R}$$

where

$$\widehat{R} \equiv \int_{v_*}^{v_1} \left[F(v_1) + \beta_1(1 - F(\widehat{v}))\right] \psi(v) f(v) dv + \int_{v_1}^{v_2} \left[F(v_2) + \beta_2(1 - F(\widehat{v}))\right] \psi(v) f(v) dv + \ldots + \int_{v_{n-1}}^{\widehat{v}} \left[F(\widehat{v}) + \beta_n(1 - F(\widehat{v}))\right] \psi(v) f(v) dv$$

Subject to

$$\theta \frac{1+F(\widehat{v})}{2} = \beta_1(F(v_1) - F(v_*)) + \ldots + \beta_n \left(F(\widehat{v}) - F(v_{n-1})\right) \tag{A.5}$$

We ignore the incentive compatibility constraint (A.3) for the moment - we take this into account below.

Let us increase β_k and reduce β_j in a way such that the constraint is preserved.

From the constraint,

$$\frac{\partial \beta_j}{\partial \beta_k} = -\left(\frac{F(v_k) - F(v_{k-1})}{F(v_j) - F(v_{j-1})}\right)$$

Using this, $\dfrac{d\widehat{R}}{d\beta_k} = (1 - F(\widehat{v}))\,(F(v_k) - F(v_{k-1}))\;\; Z$, where

$$Z = \int_{v_{k-1}}^{v_k} \psi(v) \frac{f(v)}{(F(v_k) - F(v_{k-1}))}\, dv - \int_{v_{j-1}}^{v_j} \psi(v) \frac{f(v)}{(F(v_j) - F(v_{j-1}))}\, dv$$

Clearly, $Z \gtreqless 0$ as $k \gtreqless j$.

Therefore, if constraint (A.3) is slack to start with, it is optimal to increase β_n until it binds, so that

$$F(\widehat{v}) + \beta_n(1 - F(\widehat{v})) = (1-\theta)\frac{1 + F(\widehat{v})}{2}$$

Note that at this point the incentive compatibility constraint (A.3) binds, so that β_n cannot be increased any further. Thus the optimal value of β_n is given by

$$\beta_n = \frac{(1-\theta)(1 + F(\widehat{v})/2 - F(\widehat{v})}{(1 - F(\widehat{v}))}$$

Next, in the same way, increase β_{n-1} until $F(v_{n-1}) + \beta_{n-1}(1 - F(\widehat{v})) = (1-\theta)\dfrac{1 + F(\widehat{v})}{2}$.

Continuing in this manner, we get $\ell \in 1, \ldots, n-1$ such that

$$G(v_\ell) \geqslant 0 \quad \text{and} \quad G(v_{\ell-1}) < 0,$$

where

$$G(v_\ell) = \sum_{k=\ell+1}^{n} \int_{v_{k-1}}^{v_k} \frac{(1-\theta)(1 + F(\widehat{v}))/2 - F(v_k)}{1 - F(\widehat{v})} f(v)\, dv - \theta\, \frac{1 + F(\widehat{v})}{2}$$

Note that the expression on the right hand side simply reflects the constraint (A.5), with β_k replaced by its optimal value for $k \in \{\ell, \ldots, n\}$.

As we consider a finer and finer grid (i.e. as $(v_k - v_{k-1}) \to 0$ for all k), in the limit v_ℓ is such that

$$\int_{v_\ell}^{\widehat{v}} \frac{(1-\theta)(1 + F(\widehat{v}))/2 - F(v)}{1 - F(\widehat{v})} f(v)\, dv = \theta\, \frac{1 + F(\widehat{v})}{2}$$

Thus the general form of the optimization problem is given by

$$\max_{\widehat{v}} \int_{v_*}^{v_\ell} F(v)\psi(v)f(v)\,dv + (1-\theta)\frac{1+F(\widehat{v})}{2}\int_{v_\ell}^{\widehat{v}} \psi(v)f(v)\,dv$$
$$+ (1-\theta)\frac{1+F(\widehat{v})}{2}\left[-\mu(v_h-1) + \int_{\widehat{v}}^{1} \psi(v)f(v)\,dv\right] \quad \text{(A.6)}$$

$$\text{Subject to} \quad \int_{v_\ell}^{\widehat{v}} \frac{(1-\theta)(1+F(\widehat{v}))/2 - F(v)}{1-F(\widehat{v})} f(v)\,dv = \theta\,\frac{1+F(\widehat{v})}{2}$$

Let

$$M \equiv (1-\theta)\frac{1+F(\widehat{v})}{2}.$$

Then, from the constraint, we have

$$\int_{v_\ell}^{\widehat{v}} (M - F(v))f(v)dv = (1-F(\widehat{v}))\left(\frac{1+F(\widehat{v})}{2} - M\right)$$

Which implies

$$M(F(\widehat{v}) - F(v_\ell)) - \frac{F(\widehat{v})^2 - F(v_\ell)^2}{2} = \frac{1-F(\widehat{v})^2}{2} - (1-F(\widehat{v}))M$$

Solving, the constraint reduces to

$$M = \frac{1+F(v_\ell)}{2}$$

Using this in (A.6), we can rewrite the optimization problem as the following unconstrained maximization problem:

$$\max_{v_\ell} \int_{v_*}^{v_\ell} F(v)\psi(v)f(v)\,dv + \frac{1+F(v_\ell)}{2}\left[-\mu(v_h-1) + \int_{v_\ell}^{1} \psi(v)f(v)\,dv\right]$$

Thus the general form of the maximization problem is as stated. This completes the proof. ❋

A.2 Proof of Lemma 1

Case 1: $v \leqslant \widehat{v}$.

In stage 2, in equilibrium, the expected payment from bidding v is $v_*F(v_*) + \int_{v_*}^{v} tf(t)dt$.

Since $\int_{v_*}^{v} tf(t)dt = vF(v) - v_*F(v_*) - \int_{v_*}^{v} F(t)dt$, the expected payment from bidding v can be written as

$$vF(v) - \int_{v_*}^{v} F(t)dt$$

The expected value is $vF(v)$. Therefore the expected net payoff from bidding v is $\displaystyle\int_{v_*}^{v} F(t)dt$. The expected payoff from bidding B is $(v - B)\left(F(\widehat{v}) + \dfrac{1 - F(\widehat{v})}{2}\right)$.

$D(v)$ denotes the difference between the expected payoff from bidding B in the first stage or v in the second is. From the above, this is given by

$$D(v) = (v - B)\left(\frac{1 + F(\widehat{v})}{2}\right) - \int_{v_*}^{v} F(t)dt$$

Using the value of B from equation (4.2), this can be rewritten as

$$D(v) = (v - \widehat{v})\left(\frac{1 + F(\widehat{v})}{2}\right) + \int_{v}^{\widehat{v}} F(t)dt$$

Clearly, $D(\widehat{v}) = 0$, and $D'(v) = \frac{1+F(\widehat{v})}{2} - F(v) > 0$, where the inequality follows from the fact that $v \leq \widehat{v}$.

Case 2: $v > \widehat{v}$.

The expected payment from bidding v is $v_*F(v_*) + \displaystyle\int_{v_*}^{\widehat{v}} tf(t)dt$. This simplifies to $\widehat{v}F(\widehat{v}) - \displaystyle\int_{v_*}^{\widehat{v}} F(t)dt$. The expected value is $vF(\widehat{v})$. Therefore the expected net payoff is

$$(v - \widehat{v})\, F(\widehat{v}) + \int_{v_*}^{\widehat{v}} F(t)dt$$

Hence for types $v > \widehat{v}$, $D(v) = (v - B)\left(\dfrac{1 + F(\widehat{v})}{2}\right) - \left((v - \widehat{v})\, F(\widehat{v}) + \displaystyle\int_{v_*}^{\widehat{v}} F(t)dt\right)$. Using the value of B from equation (4.2), this can be rewritten as $D(v) = (v - \widehat{v})\left(\frac{1-F(\widehat{v})}{2}\right)$. Clearly, $D(\widehat{v}) = 0$, and $D'(v) = \dfrac{1 - F(\widehat{v})}{2} > 0$. ❋

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