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# **Asymptotic Skew under Stochastic Volatility**

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# Asymptotic skew under stochastic volatility <sup>\*</sup>

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## Abstract

The purpose of this paper is to improve and discuss the asymptotic formula of the implied volatility (when maturity goes to infinity) given in [3]. Indeed, we are here able to provide more accurate at-the-money asymptotics. Such analytic formulas are useful for calibration.

## 1 Introduction : Brief review of Lewis' work

This section has two purposes : first, to introduce the notations we will use hereafter, then to briefly review the methodology as well as the main results concerning the asymptotic implied volatility smile developed in [3]. In the following, we will consider an European Call option  $C$  written on an asset  $S$ , with maturity  $T$  and strike  $K$ . We note  $\tau$ , the time to maturity. The underlying stock price follows  $\frac{dS_t}{S_t} = rdt + \sigma_t dW_t$  and the variance process,  $V_t = \sigma_t^2$  follows  $dV_t = b(V_t)dt + a(V_t)dB_t$ , the two Brownian Motions being correlated with correlation  $\rho$ . The functions  $a(\cdot)$  and  $b(\cdot)$  are at least twice differentiable. These two dynamics are considered under the risk-adjusted measure. With these notations, the PDE satisfied by the European call option is therefore

$$\frac{\partial C}{\partial t} + rS \frac{\partial C}{\partial S} + \frac{1}{2}VS^2 \frac{\partial^2 C}{\partial S^2} + b(V) \frac{\partial C}{\partial V} + \frac{1}{2}a^2(V) \frac{\partial^2 C}{\partial V^2} + \rho a(V) \sqrt{V}S \frac{\partial^2 C}{\partial S \partial V} - rC = 0$$

with terminal condition  $C(S, V, t = T) = (S_T - K)_+$ . As proved in [3], the above PDE can be transformed into a one-space variable PDE of the following form :

$$\frac{\partial \hat{h}}{\partial \tau} = \frac{1}{2}a^2(V) \frac{\partial^2 \hat{h}}{\partial V^2} + \left[ b(V) - ik\rho a(V) \sqrt{V} \right] \frac{\partial \hat{h}}{\partial V} - \frac{k^2 - ik}{2}V\hat{h} \quad (1)$$

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<sup>\*</sup>Similar corrections can be found in [4], by Alan Lewis, whom the author thanks very much for his comments. We also would like to thank Claude Martini and Francois-Xavier Vialard from Zeliade Systems, as well as Raymond Brummelhuys from Birkbeck College.

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With initial condition  $\hat{h}(k, V, \tau = 0) = -\frac{K^{ik+1}}{k^2 - ik}$  and  $k \in \mathbb{C}$  such that  $\Im(k) > 1$ . Here  $\tau = T - t$  is the time to maturity. The passage from this function  $\hat{h}$  to the call price is fast and explained in [3]. We sketch it briefly : first, let  $x = \ln(S)$ , then  $C(S, V, t) = f(x, V, t)$ . Applying the chain rule, we have  $\frac{\partial f}{\partial x} = S \frac{\partial C}{\partial S}$  and  $S^2 \frac{\partial^2 C}{\partial S^2} = \frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial x}$  and  $f$  solves

$$-\frac{\partial f}{\partial t} = -rf + \left(r - \frac{V}{2}\right) \frac{\partial f}{\partial x} + \frac{V}{2} \frac{\partial^2 f}{\partial x^2} + b(V) \frac{\partial f}{\partial V} + \frac{1}{2} a^2 \frac{\partial^2 f}{\partial V^2} + \rho a(V) \sqrt{V} \frac{\partial^2 f}{\partial x \partial V}$$

We now consider the Fourier transform of  $f$  :

$$\hat{f}(k, V, t) = \int_{-\infty}^{+\infty} e^{ikx} f(x, V, t) dx$$

We refer to [3] for a discussion on the existence of this integral. We thus have

$$\frac{\partial \hat{f}}{\partial t}(k, V, t) = \int_{-\infty}^{+\infty} e^{ikx} \frac{\partial f}{\partial t}(x, V, t) dx$$

Plugging the above PDE into the last equation, we get

$$\frac{\partial \hat{f}}{\partial t} = - \int_{-\infty}^{+\infty} e^{ikx} \left[ -rf + \left(r - \frac{V}{2}\right) \frac{\partial f}{\partial x} + \frac{V}{2} \frac{\partial^2 f}{\partial x^2} + b(V) \frac{\partial f}{\partial V} + \frac{1}{2} a^2 \frac{\partial^2 f}{\partial V^2} + \rho a(V) \sqrt{V} \frac{\partial^2 f}{\partial x \partial V} \right] dx$$

Integrating by parts and neglecting the boundary terms (see [3] for an explanation), we eventually get

$$\frac{\partial \hat{f}}{\partial t} = -r(1 + ik) \hat{f} + \frac{V}{2} (k^2 - ik) \hat{f} + \left(b(V) - ika(V) \sqrt{V}\right) \frac{\partial \hat{f}}{\partial V} + \frac{1}{2} a^2(V) \frac{\partial^2 \hat{f}}{\partial V^2}$$

Eventually, letting  $\tau = T - t$  and  $\hat{h}(k, V, \tau) = \hat{f}(k, V, \tau) e^{(1+ik)r\tau}$ , we obtain (1), where the initial condition  $\hat{h}(k, V, \tau = 0) = -\frac{K^{ik+1}}{k^2 - ik}$  is the Fourier Transform of the payoff at maturity.

**Lemma 1** *The Call option price as defined above is given by :*

$$C(S, V, t) = S_t - \frac{K e^{-r\tau}}{2\pi} \int_{ik_i - \infty}^{ik_i + \infty} e^{-ikx} \frac{\hat{H}(k, V, \tau)}{k^2 - ik} dk$$

Where  $k_i = \Im(k) > 1$ ,  $x = \ln\left(\frac{S}{K e^{-r\tau}}\right)$ , and  $\hat{H}$  is the Fundamental Transform of the reduced PDE of the call option under stochastic volatility (i.e. it satisfies (1)).

**Proof :** See [3]. ■

**Comment :** The Fundamental Transform  $\hat{H}$  is any solution to (1) with the initial condition  $\hat{H}(k, V, \tau = 0) = 1$ . This formula for the call option price is obtained by inverting the above steps from the PDE for  $C$  to the PDE for  $\hat{h}$ . We also changed the definition of  $x$  to make it dimensionless. The restriction  $\Im(k) > 1$  is for the existence of the (generalized) Fourier Transform. For other payoffs, the restriction would be different.

**Lemma 2** When  $\tau \rightarrow \infty$ , the Fundamental Transform  $\hat{H}$  has the following form :

$$\hat{H}(k, V, \tau) = e^{-\lambda(k)\tau} u(k, V)$$

Where  $\lambda$  and  $u$  solves the following Eigenvalue equation :

$$\mathcal{L}_k u = \lambda(k) u$$

With

$$\mathcal{L}_k u = -\frac{1}{2} a^2(V) \frac{d^2 u}{dV^2} - \left[ b(V) - ik\rho a(V) \sqrt{V} \right] \frac{du}{dV} + c(k) V u$$

and  $c(k) = \frac{k^2 - ik}{2}$

**Proof :** See [3]. ■

**Comment :** The differential operator here corresponds to (1). In the following, as we will only consider  $\tau \rightarrow \infty$ , the dominant term of the above equation is the smallest eigenvalue, so we do not need to develop the solution over the whole spectrum of the eigenvalues. In fact, we do not specify here the nature of the spectrum (discrete or continuous). This analysis, and hence the full eigenfunction expansion of the fundamental transform is left for further research. Indeed, depending on the nature of the spectrum, the term ‘first eigenvalue’ may refer to the minimum of the (positive) spectrum. We refer the interested reader to sections 7 and 8, chapter 6 of [3] for the way of finding the so-called first eigenvalue as well as the saddlepoint that we mention below.

**Summary :** When plugging **Lemma 2** into **Lemma 1**, the Call option formula becomes, when  $\tau \rightarrow \infty$  :

$$\frac{C(S, V, \tau)}{K e^{-r\tau}} = e^x - \frac{1}{2\pi} \int_{ik_i - \infty}^{ik_i + \infty} \frac{u(k, V)}{k^2 - ik} e^{-\lambda(k)\tau} dk$$

## 2 From the Fundamental Transform to the asymptotic smile

This section relies on the above results and provides general formulas for the asymptotic implied volatility smile. Similar corrections can be found in [4].

### 2.1 The asymptotic Call price formula

**Lemma 3** Let us consider an European Call option  $C(S, V, \tau)$ , written on a stock  $S$ , the risk-neutral dynamics of which reads  $\frac{dS_t}{S_t} = rdt + \sigma dW_t$ . Let  $K$  be the strike of the option and  $\tau = T - t$  the time to maturity. Then

$$\frac{C(S_t, \sigma, \tau)}{K e^{-r\tau}} \approx_{\tau \rightarrow \infty} e^x - \frac{\sqrt{8}}{\sigma \sqrt{\pi \tau}} e^{-\frac{1}{2} d^2}$$

Where  $x = \ln\left(\frac{S_t}{Ke^{-r\tau}}\right)$ , and  $d_- = \frac{x}{\sigma\sqrt{\tau}} - \frac{1}{2}\sigma\sqrt{\tau}$ .

**Proof :** We recall the Black-Scholes formula, in terms of log-moneyness  $x = \ln\left(\frac{S_t}{Ke^{-r\tau}}\right)$ ,  $\frac{C(x,\sigma,\tau)}{Ke^{-r\tau}} = e^x N(d_+) - N(d_-)$ , where  $d_{\pm} = \frac{x}{\sigma\sqrt{\tau}} \pm \frac{1}{2}\sigma\sqrt{\tau}$ . First, we note that,  $N(d_+) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_+} e^{-\frac{x^2}{2}} dx = 1 - \frac{1}{\sqrt{2\pi}} \int_{d_+}^{+\infty} e^{-\frac{x^2}{2}} dx$ . Let us note  $I$  the integral. Integrating by part, we have  $I = -\int_{d_+}^{+\infty} \frac{1}{x} d\left(e^{-\frac{x^2}{2}}\right) dx = \left[-\frac{1}{x} e^{-\frac{x^2}{2}}\right]_{d_+}^{+\infty} - \int_{d_+}^{+\infty} \frac{1}{x^2} e^{-\frac{x^2}{2}} dx = \frac{1}{d_+} e^{-\frac{d_+^2}{2}} - \int_{d_+}^{+\infty} \frac{1}{x^2} e^{-\frac{x^2}{2}} dx$ . The last integral is considered null, as we want to keep terms up to order  $(\sigma\sqrt{\tau})^{-1}$ . We therefore have :  $e^x N(d_+) \approx_{\tau \rightarrow \infty} e^x \left[1 - \frac{1}{\sqrt{2\pi}d_+} e^{-\frac{1}{2}d_+^2}\right]$ . But,  $e^x e^{-\frac{1}{2}d_+^2} = e^{x - \frac{1}{2}\left(\frac{x^2}{\sigma^2\tau} + \sigma^2\tau + x\right)} = e^{-\frac{1}{2}d_-^2}$ . Therefore,  $e^x N(d_+) \approx_{\tau \rightarrow \infty} e^x - \frac{1}{\sqrt{2\pi}d_+} e^{-\frac{1}{2}d_-^2}$ . Also,  $d_+ \approx_{\tau \rightarrow \infty} \frac{1}{2}\sigma\sqrt{\tau}$ .

Now, we use the fact that  $\frac{n(x)}{x} \left(1 - \frac{1}{x^2}\right) \leq N(-x) \leq \frac{n(x)}{x}$ , where  $n$  is the derivative of  $N$ . When  $\tau \rightarrow \infty$ ,  $d_- \approx -\frac{1}{2}\sigma\sqrt{\tau}$ , and  $\frac{1}{d_-^2} \rightarrow 0$ . Hence,  $N(d_-) \approx -\frac{n(-d_-)}{d_-} \approx \frac{\sqrt{2}}{\sigma\sqrt{\pi\tau}} e^{-\frac{1}{2}d_-^2}$ . Eventually, we find that

$$\frac{C(S_t, \sigma, \tau)}{Ke^{-r\tau}} \approx_{\tau \rightarrow \infty} e^x - \left[ \frac{\sqrt{2}}{\sigma\sqrt{\pi\tau}} + \frac{\sqrt{2}}{\sigma\sqrt{\pi\tau}} \right] e^{-\frac{1}{2}d_-^2}$$

■

Let us now consider the Call option, written on a stock with stochastic volatility dynamics, expressed in terms of the Fundamental Transform  $\hat{H}$  then we have the following result

**Lemma 4** *With the same notations as in Section 1, the asymptotic call price formula can be written as*

$$\frac{C(S, V, \tau)}{Ke^{-r\tau}} \approx_{\tau \rightarrow \infty} e^x - \frac{1}{\sqrt{2\pi\lambda''(k_0)}\tau} \frac{u(k_0, V)}{k_0^2 - ik_0} e^{-\lambda(k_0)\tau - ik_0x - \frac{x^2}{2\lambda'(k_0)\tau}}$$

Where we assume the existence of a saddlepoint  $k_0$  for the eigenvalue function  $\lambda(k)$ .

**Proof :** From **Summary**, we have the asymptotic value of the call option as an integral of a function depending on  $u$  and  $\lambda$ . As explained in [3], the method is here to expand the eigenvalue  $\lambda$  up to order 2 around the saddlepoint  $k_0$ . Indeed, as the Fundamental Transform is an analytic characteristic function, it satisfies the so-called *Ridge property*, that is any saddlepoint must lie along the purely imaginary axis. Thus, using Cauchy theorem, we can move the integration contour to the imaginary part of this very saddle point (we keep the assumption that the imaginary part of the saddle point lies within the strip of regularity of the option formula). Therefore, around the saddle point  $k_0$ , we have ( $k_r$  stands for the real part of  $k$ , and  $k_0$  is purely imaginary) :

$\lambda(k) = \lambda(k_r + k_0) \approx \lambda(k_0) + \frac{1}{2}k_r^2\lambda''(k_0)$ . We do not have a first derivative here as a property of the saddle point. Plugging this into the call option formula, we get

$$\begin{aligned}
\frac{C(S, V, \tau)}{Ke^{-r\tau}} &\approx e^x - \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i(k_0+k)x} e^{-[\lambda(k_0) + \frac{1}{2}k^2\lambda''(k_0)]\tau} \frac{u(k_0, V)}{k_0^2 - ik_0} dk \\
&\approx e^x - \frac{e^{-ik_0x} e^{-\lambda(k_0)\tau}}{2\pi} \frac{u(k_0, V)}{k_0^2 - ik_0} \int_{-\infty}^{+\infty} e^{-ikx - \frac{1}{2}k^2\lambda''(k_0)\tau} dk \\
&\approx e^x - \frac{e^{-ik_0x} e^{-\lambda(k_0)\tau}}{2\pi} \frac{u(k_0, V)}{k_0^2 - ik_0} \int_{-\infty}^{+\infty} e^{-\frac{\lambda''(k_0)\tau}{2} \left[ \left(k + \frac{ix}{\lambda''(k_0)\tau}\right)^2 + \left(\frac{x}{\lambda''(k_0)\tau}\right)^2 \right]} dk \\
&\approx e^x - \frac{e^{-ik_0x} e^{-\lambda(k_0)\tau}}{2\pi} \frac{u(k_0, V)}{k_0^2 - ik_0} e^{-\frac{x^2}{2\lambda''(k_0)\tau}} \int_{-\infty}^{+\infty} e^{-\frac{\lambda''(k_0)\tau}{2} \left(k + \frac{ix}{\lambda''(k_0)\tau}\right)^2} dk \\
&\approx e^x - \frac{e^{-ik_0x} e^{-\lambda(k_0)\tau}}{\sqrt{2\pi\lambda''(k_0)\tau}} \frac{u(k_0, V)}{k_0^2 - ik_0} e^{-\frac{x^2}{2\lambda''(k_0)\tau}} \\
&\approx e^x - \frac{1}{\sqrt{2\pi\lambda''(k_0)\tau}} \frac{u(k_0, V)}{k_0^2 - ik_0} e^{-\lambda(k_0)\tau - ik_0x - \frac{x^2}{2\lambda''(k_0)\tau}}
\end{aligned}$$

■

## 2.2 Deriving an asymptotic polynomial form

This subsection contains the main result of our paper, namely the implied squared volatility expressed in terms of the log-moneyness, when maturity goes to infinity. From **Lemma 3** and **Lemma 4**, let us note

$$A = \frac{u(k_0, V)}{\sqrt{2\pi\lambda''(k_0)\tau}} \frac{1}{k_0^2 - ik_0} e^{-\lambda(k_0)\tau - ik_0x - \frac{x^2}{2\lambda''(k_0)\tau}}$$

and

$$B = \frac{\sqrt{8}}{\sigma\sqrt{\pi\tau}} e^{-\frac{1}{2}\left(\frac{x}{\sigma\sqrt{\tau}} - \frac{1}{2}\sigma\sqrt{\tau}\right)^2}$$

Where  $V$  stands for the initial variance in the first formula, and  $\sigma$  is the implied volatility, in the second formula. When comparing the two call prices we obtained, first using the  $\hat{H}$  function, then deriving Black-Scholes asymptotics, we want to have

$$\forall x, \lim_{\tau \rightarrow \infty} \frac{A}{B} = 1$$

First, let us consider the at-the-money case, that is  $x = 0$ , and let  $W_\tau = \sigma\sqrt{\tau}$ . The above ratio can be written as  $\Phi_\tau W_\tau e^{-\frac{1}{8}W_\tau^2 + \lambda(k_0)\tau}$ , where the function  $\Phi_\tau$  does not depend on  $W_\tau$ . (more precisely,  $\Phi_\tau = \frac{u(k_0, V)}{4(k_0^2 - ik_0)\sqrt{\lambda''(k_0)\tau}}$ ) Hence, if the process  $W_\tau$  is unbounded, then - as it is positive - the ratio is also unbounded, which contradicts its convergence to 1. Hence  $W_\tau$  is bounded. Let  $y$  be a limit

point of  $(W_\tau)$ . We henceforth have  $-\frac{1}{8}y + \lambda(k_0)\tau = 0$ , that is  $y = 8\lambda(k_0)\tau$ , which is unique.

We now consider the case  $x \neq 0$ . Regrouping terms in  $x$  and  $x^2$  in the exponential, we can write

$$\frac{A}{B} = \Phi_\tau W_\tau e^{\psi_\tau - (ik_0 + \frac{1}{2})x + \frac{x^2}{2} \left( \frac{1}{W_\tau^2} - \frac{1}{\lambda''(k_0)\tau} \right)}$$

Where  $\psi_\tau = \frac{1}{8}W_\tau^2 - \lambda(k_0)\tau$ . The term in  $x^2$  converges to 0 as maturity becomes infinite. Replacing  $W_\tau$  by the limit point we just found, we obtain

$\lim_{\tau \rightarrow +\infty} \frac{A}{B} = \Phi_\tau \sqrt{8\lambda(k_0)\tau} e^{\psi - (ik_0 + \frac{1}{2})x}$ . As the ratio must tend to 1, this means

$\psi_\tau = (ik_0 + \frac{1}{2})x - \ln(\Phi_\tau \sqrt{8\lambda(k_0)\tau})$ . Now, returning to the implied variance

$V_\tau = \frac{W_\tau^2}{\tau}$ , and noting that the term in the log is actually independent of  $\tau$  (simplifying by  $\tau$ , we set,  $\Phi = \tau\Phi_\tau$ ), we eventually obtain

$$V_{\tau \rightarrow \infty}(x) \approx 8\lambda(k_0) - \frac{8}{\tau} \ln(\Phi \sqrt{8\lambda(k_0)}) + (8ik_0 + 4) \frac{x}{\tau} + \mathcal{O}\left(\frac{1}{\tau^2}\right)$$

And plugging the value of  $\Phi$  :

$$V_\tau(x) \approx_{\tau \rightarrow \infty} 8\lambda(k_0) - \frac{8}{\tau} \ln\left(\frac{u(k_0, V) \sqrt{\lambda(k_0)}}{(k_0^2 - ik_0) \sqrt{2\lambda''(k_0)}}\right) + (8ik_0 + 4) \frac{x}{\tau} + \mathcal{O}\left(\frac{1}{\tau^2}\right)$$

We were here able to derive an additional term in  $\frac{1}{\tau}$  for the at-the-money implied variance smile. Concerning the curvature term, we can not in fact rigorously obtain it. We claim that the curvature term in [3] is wrong, or at least terms are missing. Indeed, this very term is  $-\frac{1}{2\lambda(k_0)\tau^2}$ ; but,  $\lambda(k_0)$  is actually strictly positive, at least in the Heston model. It means that the smile becomes concave, which entails arbitrage opportunities. Hence, this is not possible.

### 3 Application to the Heston model

Let us now consider the Heston model, namely

$$dV_t = \kappa(\theta - V_t) dt + \xi \sqrt{V_t} dZ_t$$

Where  $Z_t$  has a correlation  $\rho$  with the Brownian process underlying the dynamics of the stock price. The functions  $u$  and  $\lambda$  are given in [3] for the square-root model, namely :

$$\begin{cases} \lambda(k) = \frac{\kappa\theta}{\xi^2} \left\{ \sqrt{(\kappa + ik\rho\xi)^2 + (k^2 - ik)\xi^2} - (\kappa + ik\rho\xi) \right\} \\ u(k, V) = e^{-\frac{V}{\kappa\theta} \lambda(k)} \end{cases}$$



The saddle point  $k_0$  as well as the eigenfunction  $\lambda(k_0)$  are also given :

$$\begin{cases} k_0 = \frac{i}{1-\rho^2} \left[ \frac{1}{2} - \frac{\rho}{\xi} \left( \kappa - \frac{1}{2} \sqrt{4\kappa^2 + \xi^2 - 4\rho\kappa\xi} \right) \right] \\ \lambda(k_0) = \frac{\kappa\theta}{2(1-\rho^2)\xi^2} \left[ \sqrt{(2\kappa - \rho\xi)^2 + (1-\rho^2)\xi^2} - (2\kappa - \rho\xi) \right] \end{cases}$$

We also need the second derivative of  $\lambda$  at the point  $k_0$ . This latter is the following

$$\lambda''(k_0) = \frac{\kappa\theta}{2\xi^2} \left\{ \Psi''(k_0) \Psi^{-\frac{1}{2}}(k_0) - \frac{1}{2} \Psi'(k_0)^2 \Psi^{-\frac{3}{2}}(k_0) \right\}$$

with

$$\begin{cases} \Psi(k_0) = [\kappa - \rho\xi\mathfrak{S}(k_0)]^2 + [-\mathfrak{S}(k_0)^2 + \mathfrak{S}(k_0)] \xi^2 \\ \Psi'(k_0) = -2\rho\xi[\kappa - \rho\xi\mathfrak{S}(k_0)] + [1 - 2\mathfrak{S}(k_0)] \xi^2 \\ \Psi''(k_0) = -2\xi^2(1-\rho^2) \end{cases}$$

Henceforth, when plugging this into the asymptotics we just found above, we obtain :

$$V_{\tau \rightarrow \infty}(0) \approx \frac{4\kappa\theta}{(1-\rho^2)\xi^2} \left\{ \sqrt{(2\kappa - \rho\xi)^2 + (1-\rho^2)\xi^2} - (2\kappa - \rho\xi) \right\} - \frac{8}{\tau} \ln \left( \Phi \sqrt{8\lambda(k_0)} \right)$$

The first term in the right hand side is exactly the term given by Gatheral in [1]. If we now explicit the form of the second term, we obtain

$$\frac{8}{\tau} \ln \left( \Phi \sqrt{8\lambda(k_0)} \right) = \frac{8}{\tau} \ln \left( \frac{u(k_0, V)}{(k_0^2 - ik_0)} \sqrt{\frac{\lambda(k_0)}{2\lambda''(k_0)}} \right)$$

Replacing  $u$  by its semi-explicit form (in terms of  $\lambda$  and rewriting the asymptotic formula above, we eventually obtain

$$\boxed{V_{\tau \rightarrow \infty}(0) \approx \frac{4\kappa\theta}{(1-\rho^2)\xi^2} \left\{ \sqrt{(2\kappa - \rho\xi)^2 + (1-\rho^2)\xi^2} - (2\kappa - \rho\xi) \right\} - \frac{8}{\tau} \ln \left[ \frac{u(k_0, V)}{(k_0^2 - ik_0)} \sqrt{\frac{\lambda(k_0)}{2\lambda''(k_0)}} \right]}$$

And the skew term is

$$\boxed{\left. \frac{\partial V}{\partial x} \right|_{x=0} \approx_{\tau \rightarrow \infty} \frac{1}{\tau} \left\{ 4 - \frac{8}{1-\rho^2} \left[ \frac{1}{2} - \frac{\rho}{\xi} \left( \kappa - \frac{1}{2} \sqrt{4\kappa^2 + \xi^2 - 4\rho\kappa\xi} \right) \right] \right\}}$$

## 4 Conclusion

In this paper, we were able to give an analytic formula for the implied volatility as a function of the log-moneyness, when maturity gets very large. Compared to [3], [1] or [2], we provide here an improved and more precise analytical skew for the asymptotic implied volatility smile. However, we were not able to determine the analytical form of the curvature term of the smile, which is therefore left for further research. Similar results and supplements may also be found in [4].

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