ISSN 1745-8587

School of Economics, Mathematics and Statistics

BWPEF 0706

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March 2007

Birkbeck Working Papers in Economics & Finance

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Multiple Applications Matching Function: An Alternative

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March 5, 2007

Abstract

The multiple applications matching function derived in Albrecht, Gautier, Tan and Vroman (2004) involves terms which are very “convoluted”. This letter proposes an alternative more concise function, derived using applicant types as the source of randomness rather than the resulting application outcomes.

JEL Classification: J41, J64

Keywords: search, matching function

1 Introduction

Original matching functions were derived in an urn-ball set-up, with workers making single applications and firms offering single vacancies. A typical example is Blanchard and Diamond (1990). More recent works have introduced multiplicity to this framework. For example Burdett, Shi and Wright (2001) investigate matching outcomes

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when firms offer varying numbers of vacancies. Albrecht, Gautier, Tan and Vroman (2004) on the other hand allow multiple applications by workers. Then with v vacancies and u applicants, each making $a \geq 1$ applications, the number of matches is given by,

$$M(u, v; a) = u(1 - \Psi) \tag{1}$$

where Ψ is the probability that an applicant receives no job offer from any of the a firms that he applies to,

$$\Psi = \sum_{i=1}^{u} \sum_{j=1}^{u} ... \sum_{l=1}^{u} \left(1 - \frac{1}{i}\right) \left(1 - \frac{1}{j}\right) ... \left(1 - \frac{1}{l}\right) \Delta_1(i)\Delta_2(i, j)...\Delta_a(i, j, ..., l) \tag{2}$$

$\Delta_m(i, j, ..., k)$ is the conditional probability that there are k applicants at the m^{th} vacancy to which the worker applies to, given $i, j, ...$ applicants at vacancies $1, ..., m-1$.[1] The expressions for these conditional probabilities, given in Tan (2003), rapidly become very "convoluted" (Tan, 2003) for later vacancies. For example for $m = 2$,

$$\begin{aligned} \Delta_2(i, j) \quad = \quad & \sum_z \binom{i-1}{z} \binom{u-i}{j-1-z} \\ & \times \left(\frac{a-1}{v-1}\right)^z \left(1 - \frac{a-1}{v-1}\right)^{i-1-z} \left(\frac{a}{v-1}\right)^{j-1-z} \left(1 - \frac{a}{v-1}\right)^{u-i-(j-1-z)} \end{aligned}$$

In this letter I propose an alternative derivation for Ψ which yields a much more concise expression of the multiple-application matching function. This is done in

[1] Albrecht et al. (2003a) originally derived the multiple-applications matching function assuming independence between the probabilities of the number of applicants at each vacancy, a mistake which was pointed out by Tan (2003) and was later corrected by Albrecht et al. (2003b).

Section 2 by modelling applicant types as the source of randomness rather than the resulting application outcomes at vacancies. Section 3 checks the limiting case as $u, v \to \theta$ and $\frac{v}{u} \to \theta < \infty$. The letter concludes with a remark in Section 4.

2 The Matching Function

First I introduce a type matrix T, the column vectors of which indicate the vacancies applied by different types of applicants. For example if $v = 3$ and $a = 2$,

$$\mathbf{T} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \tag{3}$$

For example here type 1 workers apply to vacancies 1 and 2. The number of types is given by $\tau = \binom{v}{a}$, which in this case is $\binom{3}{2} = 3$. $\mathbf{T}$ is thus a $v \times \tau$ matrix. Applicants are otherwise assumed homogeneous.

Next denote by vector $\mathbf{n} = (n_1, n_2, ..., n_\tau)'$ the realized number of each type. The restriction is that $\sum_{t=1}^{\tau} n_t = u$. The number of applications at each vacancy $\boldsymbol{\alpha}(\mathbf{n}) = (\alpha_1(\mathbf{n}), \alpha_2(\mathbf{n}), ..., \alpha_v(\mathbf{n}))'$ is then, where $\sum_{j=1}^{v} \alpha_j(\mathbf{n}) = au$,

$$\boldsymbol{\alpha}(\mathbf{n}) = \mathbf{T}\mathbf{n} \tag{4}$$

Given realizations of $\mathbf{n}$, a worker can then calculate the number of his competitors at each of the vacancies that he applies to. Assume without loss of generality that the representative applicant is of type 1. Then the constraint for the realization $\mathbf{n}$ is that $n_1 \geq 1$. Denote the set of all possible such realizations by Ω, i.e. $\Omega \equiv \{\mathbf{n}\,|n_1 \geq 1; \sum_{t=1}^{\tau} n_t = u\}$. There are $\binom{\tau+u-2}{u-1}$ elements in Ω.[2] Given that workers choose their type randomly, the probability that $\mathbf{n}$ takes particular values $(n_1, n_2, ..., n_\tau)' \in \Omega$ is the ratio of the number of distinct ways that this set of values can be attained, and the total number of distinct ways that $u-1$ workers can be assigned to τ types,

$$p(\mathbf{n}) = \frac{(u-1)!}{(n_1-1)!\prod_{t=2}^{\tau} n_t!} \frac{1}{\tau^{u-1}} \tag{5}$$

Now assuming that firms randomly select one candidate when they receive more than one applications, the probability of an applicant receiving zero job offer ex-post of the realizations of $\mathbf{n}$ is the product of the probabilities $1 - \frac{1}{\alpha_j(\mathbf{n})}$ that he receives no job offer from each of the vacancies he applies to, which for the type 1 applicant is the first a vacancies. Taking the expectation of this over all possible values of $\mathbf{n}$ therefore yields the ex-ante probability of zero job offer from any of the a jobs,

$$\Psi = \sum_{\mathbf{n}\in\Omega} p(\mathbf{n}) \prod_{j=1}^{a} \left(1 - \frac{1}{\alpha_j(\mathbf{n})}\right) \tag{6}$$

The number of matches m is then given by (1).

As an example consider the case $u = 3$, $v = 3$ and $a = 2$. The type matrix was given in (3). Assuming again that the representative applicant is of type 1, there are $\binom{3+3-2}{2} = 6$ possible realizations of the types of the 3 applicants,

[2]This can be proved by induction.

$\Omega \equiv \{(3,0,0)', (2,1,0)', (2,0,1)', (1,2,0)', (1,0,2)', (1,1,1)'\}$. For each of these realizations the probability and the application outcome are, using (5) and (4), $p(\mathbf{n}) \in \{\frac{1}{9}, \frac{2}{9}, \frac{2}{9}, \frac{1}{9}, \frac{1}{9}, \frac{2}{9}\}$ and $\boldsymbol{\alpha}(\mathbf{n}) \in \{(3,3,0)', (3,2,1)', (2,3,1)', (3,1,2)', (1,3,2)', (2,2,2)'\}$ respectively. Then (6) gives the probability of no job offer as

$$\Psi = \frac{1}{9}.\frac{4}{9} + \frac{2}{9}.\frac{1}{3} + \frac{2}{9}.\frac{1}{3} + \frac{1}{9}.0 + \frac{1}{9}.0 + \frac{2}{9}.\frac{1}{4} = \frac{41}{162}$$

which is the same result as attained by Tan (2003).

3 The Limiting Case

Investigate now the limiting case of (6) as $u, v \to \infty$ and $\frac{v}{u} \to \theta < \infty$. It is intuitive that then the covariances of the numbers of applicants at each vacancy go to zero. To show it, rewrite the type matrix $\mathbf{T}$ as, for $a \geq 2$,

$$T = \left(\begin{array}{ccc|ccc|ccc|ccc} 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & \cdots & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 1 & 0 & \cdots & 1 & 0 & \cdots & 1 \end{array}\right) \tag{7}$$

$$\underbrace{\qquad\qquad}_{\tau_{11}}\underbrace{\qquad\qquad}_{\tau_{12}}\underbrace{\qquad\qquad}_{\tau_{21}}\underbrace{\qquad\qquad}_{\tau_{22}}$$
$$\underbrace{\qquad\qquad\qquad\qquad}_{\tau_{1}}\underbrace{\qquad\qquad\qquad\qquad}_{\tau_{2}}$$

where the number of columns of each block are,

$$\begin{aligned} \tau_1 &= \binom{v-1}{a-1}, \quad \tau_2 = \binom{v-1}{a} \\ \tau_{11} &= \binom{v-2}{a-2}, \quad \tau_{12} = \tau_{21} = \binom{v-2}{a-1}, \quad \tau_{22} = \binom{v-2}{a} \end{aligned} \tag{8}$$

Then for example $\alpha_1(\mathbf{n})$ and $\alpha_2(\mathbf{n})$ are, using (4),

$$\begin{aligned}
\alpha_1(\mathbf{n}) &= \sum_{t=1}^{\tau_{11}} n_t + \sum_{t=\tau_{11}+1}^{\tau_{11}+\tau_{12}} n_t \\
\alpha_2(\mathbf{n}) &= \sum_{t=1}^{\tau_{11}} n_t + \sum_{t=\tau_1+1}^{\tau_1+\tau_{21}} n_t
\end{aligned}$$

Two things happen in the limit: as $u \to \infty$, n_t's become independent, and as $v \to \infty$, the second terms dominate, as $\frac{\tau_{12}}{\tau_{11}} = \frac{v-a}{a-1} \to \infty$. Hence $cov\,[\alpha_1(\mathbf{n}), \alpha_2(\mathbf{n})]$ goes to zero. By symmetry this is true for all $\{\alpha_i(\mathbf{n}), \alpha_j(\mathbf{n})\}$, $i \neq j$. Therefore

$$\lim_{u,v\to\infty} \sum_{\mathbf{n}\in\Omega} p(\mathbf{n}) \prod_{j=1}^{a} \left(1 - \frac{1}{\alpha_j(\mathbf{n})}\right) = \left\{ \sum_{\mathbf{n}\in\Omega} p(\mathbf{n}) \left(1 - \frac{1}{\alpha_1(\mathbf{n})}\right) \right\}^a \tag{9}$$

Next evaluate the following term, where $\Omega_{\alpha_1} \equiv \{\mathbf{n}\,|n_1 \geq 1; \sum_{t=1}^{\tau_1} n_t = \alpha_1\}$ for given α_1,

$$\begin{aligned}
\sum_{\mathbf{n}\in\Omega} p(\mathbf{n}) \frac{1}{\alpha_1(\mathbf{n})} &= \sum_{\alpha_1=1}^{u} \left(\sum_{\mathbf{n}\in\Omega_{\alpha_1}} p(\mathbf{n}) \right) \frac{1}{\alpha_1} \\
&= \sum_{\alpha_1=1}^{u} \left\{ \frac{\binom{u-1}{\alpha_1-1} \tau_1^{\alpha_1-1} \tau_2^{u-\alpha_1}}{\tau^{u-1}} \right\} \frac{1}{\alpha_1} \\
&= \sum_{\alpha_1=1}^{u} \frac{a^{\alpha_1-1}}{\alpha_1!} \left(\prod_{j=1}^{\alpha_1-1} \frac{u-j}{v-a} \right) \left(\frac{v-a}{v} \right)^{u-1}
\end{aligned}$$

In the limit then,

$$\lim_{\substack{u,v\to\infty \\ v/u\to\theta}} \sum_{\mathbf{n}\in\Omega} p(\mathbf{n}) \frac{1}{\alpha_1(\mathbf{n})} = \sum_{\alpha_1=1}^{\infty} \frac{1}{\alpha_1!} \left(\frac{a}{\theta}\right)^{\alpha_1-1} e^{-\frac{a}{\theta}} = \frac{\theta}{a} \left(1 - e^{-\frac{a}{\theta}}\right)$$

Applying this to (9) yields the same limiting result as that derived by Albrecht, Gautier and Vroman (2003a),

$$\lim_{\substack{u,v\to\infty \\ v/u\to\theta}} \Psi = \left\{1 - \frac{\theta}{a}\left(1 - e^{-\frac{a}{\theta}}\right)\right\}^{a} \tag{10}$$

4 Remark

One major advantage of working with the applicant types, rather than the resulting application outcomes, is that the framework can easily incorporate workers' optimizing behaviors. This leads naturally to a model of directed search, where workers choose their application patterns by taking into consideration factors such as wage levels and skills match. A potential extension would then be a model of wage dispersion, with the equilibrium wage levels determined endogenously.[3]

[3] Many assume workers' non-observance of wage levels prior to application to explain wage dispersion (e.g. Burdett and Mortensen, 1998). Moen (1997) introduces submarkets within which matching takes place to explain wage dispersion even when applicants observe wage levels. However in his paper the wage levels are assigned exogenously. Galenianos and Kircher (2005) analyze the direct search process as a portfolio choice problem, and show that workers' willingness to send applications to separate wage levels incentivize firms to post different wages.

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