Birkbeck Working Papers in Economics & Finance

ISSN 1745-8587

Birkbeck UNIVERSITY OF LONDON

School of Economics, Mathematics and Statistics

BWPEF 0613

Belief Heterogeneity and Survival in Incomplete Markets

Tarek Coury
University of Oxford

Emanuela Sciubba
Birkbeck, University of London

November 2006

▪ Birkbeck, University of London ▪ Malet Street ▪ London ▪ WC1E 7HX ▪

Belief Heterogeneity and Survival in Incomplete Markets*

Tarek Coury
Dept of Economics, U of Oxford
Department of Economics
Manor Road Building
Manor Road, Oxford, OX1 3UQ, UK
Email: tarek.coury@economics.oxford.ac.uk

Emanuela Sciubba
Birkbeck College, U of London
School of Economics Mathematics and Statistics
Malet Street, London, WC1E 7HX, UK
Email: esciubba@econ.bbk.ac.uk

November 2006

*We thank seminar participants in Cambridge, Cornell, Leeds, and Oxford for helpful comments, along with conference participants at GET 2005, SAET 2005, ESWC 2005 and EEA 2005. In particular, we thank Lawrence E. Blume, David A. Easley, Sujoy Mukerji and John Quah for substantive comments on earlier versions of the paper. All errors remain our own.

Abstract

In complete markets economies (Sandroni [15]), or in economies with Pareto optimal outcomes (Blume and Easley [9]), the market selection hypothesis holds, as long as traders have identical discount factors. Traders who survive must have beliefs that merge with the truth. We show that in incomplete markets, regardless of traders' discount factors, the market selects for a range of beliefs, at least some of which do not merge with the truth. We also show that impatient traders with incorrect beliefs can survive and that these incorrect beliefs impact prices. These beliefs may be chosen so that they are far from the truth.

Keywords: Incomplete markets, market selection hypothesis, belief selection.

JEL Codes: D51, D52, D80, G10.

1 Introduction

Do markets select for correct expectations? The market selection hypothesis (Alchian [1], Friedman [12]) is one of the longest standing conjectures in economics. Traders who form more accurate predictions about future returns make more money at the expense of those who don't. In the long run all traders with inaccurate beliefs are driven out of the market and the only surviving ones have correct expectations. This hypothesis has a strong intuitive appeal and, if true, provides a robust justification to the assumption of rational expectations in both microeconomic and macroeconomic models. Given that long run market outcomes only reflect correct expectations, economists interested in the long run may as well assume rational expectations from the outset.

To test the validity of this conjecture, suppose that two traders disagree on the probability with which a particular state of nature occurs. If this disagreement does not have an impact on asymptotic wealth accumulation and survival, then Friedman's conjecture does not hold. Hence the market selection hypothesis requires that the trader with correct expectations is able to accumulate wealth at the other trader's expense by betting against him on the future realisation of that particular state of nature. It is only when there is a market that allows the two traders to make these bets that the trader with correct beliefs can actually accumulate more wealth than the other trader and drive him out of the market. When a state of nature can only be partially insured against by the existing market structure, the link between accuracy of beliefs and survival becomes weaker.

We know that when markets are complete [15], or when the allocation is Pareto optimal [9] correct beliefs are selected for by market forces[1]. In particular, heterogeneity of beliefs does not persist, and all surviving traders have either correct beliefs or beliefs which merge with the true probability distribution. Blume and Easley [9] argue by providing counterexamples that the same need not hold when markets are incomplete. In this paper we show that in incomplete markets economies, regardless of traders' discount factors, the set of beliefs which are consistent with traders' survival contains beliefs that are not equivalent to the true probability distribution. So the market selection hypothesis does not hold in incomplete markets. We also show in a class of economies that there exist surviving traders with beliefs that do not merge with the truth and these beliefs matter: were they to adopt correct beliefs, equilibrium prices would change and they may no longer survive. These surviving traders may be more impatient than other traders with correct beliefs. This stands in stark contrast to Blume and Easley's result that surviving traders must have either beliefs closer to the truth than other traders or be sufficiently patient to compensate for their incorrect beliefs.

We consider an economy with an open ended future and a finite number

[1] This assumes that traders discount future consumption at the same rate so that their degree of impatience does not affect their survival.

of traders. Every period, traders trade securities to hedge their stochastic endowment risk. Preferences are of the expected utility form and utility from future consumption is discounted at a rate that is allowed to differ across traders. There are many consumption goods each period, but the securities pay off only in terms of a numeraire good. Also, securities are short-lived. These last two assumptions do not affect the intuition of the result but considerably simplify the analysis and guarantee existence of an equilibrium (see Magill and Quinzii ([14])). Otherwise, the asset structure is rather general in that the payoff matrix may change from period to period. The infinite horizon economy that we model satisfies conditions for existence of an equilibrium with a transversality condition. This requires traders not to borrow and roll over their debt *ad infinitum*.

Our first result is that traders who survive admit beliefs that are not equivalent to the true probability distribution. To prove our result, we introduce the notion of *effectively identical beliefs* as the set of probability distributions for some trader that are consistent with the same overall equilibrium. Given an initial economy and its corresponding equilibrium, if some trader were to adopt beliefs that are effectively identical to his original beliefs, then the new equilibrium outcome would remain unchanged. We then show that the set of effectively identical beliefs is a singleton under complete markets. By contrast, this set is not a singleton in incomplete markets. Moreover, there exists a probability distribution that belongs to this set that is not equivalent to the truth. This has straightforward and important consequences for belief selection in incomplete markets. Suppose that a trader survives, our first result shows that there are probability distributions that are not equivalent to the truth which are consistent with his survival. Hence incomplete markets fail to select for traders with correct expectations.

While our first result shows that incomplete markets select for a wide range of beliefs, our second result shows that surviving traders whose beliefs are incorrect affect asset prices.We consider a two-trader economy and the corresponding no-trade outcome. Assuming that the first trader has correct beliefs, we can assign a discount factor and beliefs to the other trader such that she is more impatient than the first trader, has incorrect beliefs and survives. These beliefs matter because the equilibrium price sequence of assets would change were she to adopt beliefs that are correct. This is because the truth does not lie within her set of effectively identical beliefs. Hence traders with incorrect beliefs who survive need not behave as though they know the truth. Note that these results do not hold in comparable complete markets economies.

The structure of the paper is as follows. In section 2 we summarise the existing literature. In section 3, we present the model: we start by providing the intuition for our main results in a simple two-period model (subsection 3.1), then we go on to describe the infinite horizon economy which always admits an equilibrium with a transversality condition. Section 4 contains our first result. Section 5 contains our second result. Section 6 concludes the paper. For ease of exposition, all proofs are in the appendix.

2 Related Literature

The first attempts to validate the market selection hypothesis date back to the early 90s and address the related issue of whether markets select for rationality, with particular focus on the survival of noise traders. Shefrin and Statman [17] ask whether noise traders survive in financial markets by developing a model where rational and informed Bayesian traders interact with traders that make systematic cognitive errors. They show that, provided that noise traders are patient enough and that they do not commit errors that are “too serious”, they will not be driven to extinction by informed traders. De Long et al. [10] and [11] prove that noise traders can eventually come to dominate the market, if they unwillingly happen to make “good” cognitive mistakes. Biais and Shadur [5] consider a market where non-overlapping generations of buyers and sellers trade to share risk. They show that irrational traders, who misperceive the risk but enjoy a higher bargaining power, might outperform rational traders who correctly assess the distribution of the risk.

While this literature assumes asset prices to be exogenous, the paper by Blume and Easley [8] addresses the same problem in a market model, where asset prices are determined endogenously and reflect the dynamics of the wealth shares of the different types of traders, each represented by a portfolio rule. They find that, as long as traders save at the same rate, markets do not select for rationality, but rather for a specific attitude towards risk. In particular logarithmic utility maximisers with accurate beliefs accumulate wealth at a faster rate than any other trader. As a result, they determine asset prices asymptotically and drive to extinction any other trader. Hence within this framework markets do not select necessarily for rationality, but rather for a specific portfolio rule. Irrational traders, or traders with inaccurate expectations, may well survive if their mistakes or irrationality imply that their portfolio rules are closer to the portfolio rule of a log maximiser. On the other hand, rational traders with correct expectations may well vanish, if they happen to have the wrong attitude towards risk.

The results from this early literature are important in that they formalise through wealth dynamics what one might mean by market selection. They are also quite provocative because they make it very clear that expected utility maximisation and survival are distinct objectives. Hence rational behaviour is not necessarily selected for by market forces and the market selection hypothesis need not hold within this setting.

Sandroni [15] adopts the same notion of market selection and survival as in Blume and Easley [8], but differs from the earlier contributions in that he considers not only portfolio decisions but also savings decisions to be endogenous. In a Lucas trees complete markets economy where traders are expected utility maximisers and discount the future at the same rate, he finds that under mild conditions on traders' utility functions, only traders with correct beliefs survive. Hence, among rational traders, complete markets select for correct beliefs.

Blume and Easley [9] generalise Sandroni's result to any Pareto optimal allocation. For any optimal allocation, survival of traders is determined entirely by beliefs and discount factors; in contrast with [8], risk attitudes do not matter for survival. Among traders who discount future consumption at the same rate, it is those with most accurate beliefs that survive, irrespective of their utility function. In particular, if there are traders whose beliefs merge with the truth, they will be the only survivors. Blume and Easley [9] provide two interesting counterexamples that show that the same results need not carry through under market incompleteness, where in general allocations are not Pareto optimal.

In this paper we prove for a large class of incomplete markets economies that surviving traders need not have beliefs that merge with the true probability distribution. The fact that, under incomplete markets, opportunities to trade are restricted implies that traders with incorrect beliefs are not wiped out by market forces. As a result surviving traders' beliefs do not necessarily merge either with the truth or with other traders' beliefs, and so beliefs' heterogeneity is persistent and may matter (see section 5).

3 The Model

3.1 A Two-Period Example

Consider a two period economy with a unique consumption good and where there are S possible states of the world tomorrow. Time is indexed by $t = 0, 1$. Traders can trade $J \leq S$ securities whose period 1 payoff is the full rank

$S \times J$ matrix A. They trade these securities to hedge against their period 1 stochastic endowment $\omega \in \mathbb{R}^S_{++}$. Consumption takes place in period 1 only. In an equilibrium, period 0 asset prices $q \in \mathbb{R}^J_{++}$ have to satisfy the no arbitrage equation:

$$q = \frac{\pi_1^i}{\pi_0^i} A \tag{1}$$

where $\pi_1^i \in \mathbb{R}^S_{++}$ is trader i's utility gradient and where $\pi_0^i \in \mathbb{R}_{++}$ is a multiplier. The resulting ratio $\frac{\pi_1^i}{\pi_0^i}$ is trader i's normalized utility gradient or his state price vector. In the complete markets case, we get the usual condition that this ratio is equated across traders.[2] Note that $\pi_1^i(s) = \rho^i(s) v^{i\prime}(x^i(s))$ when traders have preferences of the expected utility form and their beliefs are represented by the probability distribution ρ^i. In the complete markets case, given an equilibrium outcome (x^*, q^*), there exists only one set of beliefs (an $S-$dimensional normalized vector ρ^i) such that $q^* \pi_0^i = \pi_1^i A$ where $\pi_1^i(s) = \rho^i(s) v^i(x^{*i}(s))$. This is because there are S equations in S unknowns. The only solution is the original normalized vector $\rho^i \in \mathbb{R}^S$.

In the incomplete markets case, $J < S$ and this system of equations may have multiple solutions. To guarantee multiple solutions to the no-arbitrage equation, one needs to assume that $J < S - 1$. The additional degree of freedom is used to ensure that the resulting solution is a probability distribution. So, given an economy and a resulting equilibrium outcome,

[2] Equation (1) can be given the familiar form $q = \psi \mathbb{E}(V)$ where the expectation is taken with respect to some probability distribution. In the complete markets case, this probability distribution is unique. The scalar ψ represents the price of a bond that pays off one unit of consumption in each state (if that bond exists).

and for any trader $i \in \mathbb{I}$, there exist many probability distributions that are consistent with the original equilibrium: The latter is also an equilibrium of any economy where all traders' preferences remain unchanged, except for trader i. His beliefs can be any $\lambda^i \neq \rho^i$ such that $q = \frac{\pi_1^{i\prime}}{\pi_0^i} A$ where $\pi_1^{i\prime}$ now represents trader i's utility gradient under the new beliefs λ^i. When $J < S - 1$, these beliefs exist. The intuition of this analysis is essentially the same in infinite horizon economies and ultimately drives our main result.

Turning to the case of infinite horizon economies, suppose that traders trade the same set of short-lived securities whose payoff next period is the matrix A. The no arbitrage equation takes the form:

$$q(s_t) = \frac{\pi_{t+1}^i(s_t)}{\pi^i(s_t)} A \tag{2}$$

where $\pi_{t+1}^i(s_t) \in \mathbb{R}_{++}^S$ is trader i's utility gradient for period $t+1$ when the current state of the world is s_t and where $\pi^i(s_t) \in \mathbb{R}_{++}$ is the marginal utility of consumption in node s_t of the date-event tree. Again, we consider the case of expected utility maximizers. Consider a particular economy and the resulting equilibrium outcome. Suppose that trader $i \in \mathbb{I}$ has beliefs represented by a probability distribution ρ^i. We wish to construct a probability distribution $\lambda^i \neq \rho^i$ such that the original equilibrium is still an equilibrium when trader i ad opts beliefs λ^i. We do this by rewriting the no arbitrage equation (2):

$$q_t = \rho^i(t+1|s_t)M^i(s_t) \tag{3}$$

Where $\rho^i(t+1|s_t) \in \mathbb{R}^S_{++}$ is the conditional probability distribution of period $t+1$ events, conditioning on the current state of the world s_t and where $M^i(s_t)$ is an $S \times J$ matrix determined in equilibrium. We show that there exists a unique probability distribution ρ^i which satisfies equation (3) in the complete markets case. When $J < S-1$, one can choose conditional probabilities $\lambda^i(t+1|s_t) \neq \rho^i(t+1|s_t)$ for each node in the date-event tree. Then one can construct a probability distribution λ^i over infinite events by using Kolmogorov's existence theorem. This implies that in the incomplete markets case, one can choose a probability distribution λ^i that is *effectively identical* to ρ^i but such that $\lambda^i \neq \rho^i$.

One can also choose λ^i such that λ^i and ρ^i are not equivalent. This requires that the marginals $\rho^i(t+1|s_t)$ are uniformly bounded away from the edges of the unit simplex. We can then choose λ^i uniformly bounded away from ρ^i. The theorem of Blackwell and Dubins [7] then implies that these distributions cannot be equivalent.

It follows that in an incomplete markets economy, observing a trader survive does not imply that his beliefs are equivalent to the truth. The above procedure can be used to construct beliefs for this trader that are not equivalent to the truth but that guarantee his survival in a way that is identical to the original economy. This is in contrast to the complete markets or Pareto efficient economy. In these economies and controlling for discount factors, traders who survive must have the truth be absolutely continuous with respect to their beliefs.

The second result, presented in section 5, shows that one can change both a trader's discount factor (by making her more impatient, for example) and her beliefs. If her beliefs are chosen such that the marginals are bounded away from the unit simplex, then one can find beliefs effectively identical to these new beliefs that are far away from the truth. The outcome is that this trader will survive and her wrong beliefs will influence equilibrium prices.

3.2 The Infinite Horizon Economy

The economy we model is a special case of the economy analyzed by Magill and Quinzii [14]. Our notation combines elements of [14], Araujo and Sandroni [3] and Sandroni [15]. Let $\mathbb{T} = \{0, 1, ..\}$ denote the set of time periods. Every period, the set of possible states is $T = \{1, .., S\}$, $S \in \mathbb{N}$. T^t is the $t-$Cartesian product of T. Let $\mathbb{S} = \{s^0\} \times T^\infty$ be the set of all possible infinite sequences of T where $s^0 \in T$ acts as the root element. Throughout, we use the notation $s_t = (s^0, s^1, .., s^t)$ for an element $s_t \in T^t$. All elements are taken to have $\{s^0\}$ as root so $s_t \in T^t$ necessarily means $s_t = \{s^0\} \times h_{t-1}$ where $h_{t-1} \in T^{t-1}$. We can represent the information revelation process in this economy through a sequence of finite partitions of the state space $\mathbb{S}$. In particular, define the cylinder with base on $s_t \in T^t$, $t \in \mathbb{T}$ as $C(s_t) = \{s \in T^\infty | s = (s_t, ..)\}$. Let $\mathbb{F}_t = \{C(s_t) : s_t \in T^t\}$ be a partition of the set $\mathbb{S}$. Clearly, $\mathbb{F} = (\mathbb{F}_0, .., \mathbb{F}_t, ..)$ denotes a sequence of finite partitions of $\mathbb{S}$ such that $\mathbb{F}_0 = \{\mathbb{S}\}$ and $\mathbb{F}_t$ is finer[3] than $\mathbb{F}_{t-1}$. We assume that all traders have identical information and that the information revelation process is represented by the sequence $\mathbb{F}$. Let $\mathbb{D} = \cup_{t\in\mathbb{T},\sigma_t\in\mathbb{F}_t} (t, \sigma_t)$ denote the date-event tree and $\mathbb{D}^+ = \mathbb{D} - \{(0, \sigma_0)\} = \mathbb{D} - \{s^0\}$. We use the short-hand notation $s_t \in \mathbb{D}$, meaning $(t, \sigma_t) \in \mathbb{D}$ where $\sigma_t = C(s_t)$. $\mathbb{D}_T(s_t)$ denotes the subset of successor nodes of s_t at date T, i.e. all elements $s^T \in T^T$ such that $s^T = (s_t, ..)$. Let $\mathcal{F}_t$ be the set consisting of all finite unions of cylinders with base on T^t. It is easily shown that $\mathcal{F}_t$ is a $\sigma-$field. Note that $\mathcal{F}_t = \sigma(\mathbb{F}_t)$. Define $\mathcal{F}_0$ as the trivial $\sigma-$field. Let $\mathcal{F} = \sigma(\cup_{t\in\mathbb{N}}\mathcal{F}_t)$. It can be shown that $\{\mathcal{F}_t\}_{t\in\mathbb{N}}$ is a filtration. Let ρ^i be trader i's beliefs on $\mathbb{S}$ represented by a probability measure on $(T^\infty, \mathcal{F})$. Let $\mathbb{E}^{\rho^i}$ be the expectation operator associated with ρ^i. Let $\mathbb{E}^{\rho^i}(.|\mathcal{F}_t)(s) = \mathbb{E}_t^{\rho^i}(.)(s)$ be the expectation operator associated with $\rho^i_{s_t}$ when $s = (s_t, ..)$ and where:

$$\rho^i_{s_t}(K) = \frac{\rho^i((T^t \times K) \cap C(s_t))}{\rho^i(C(s_t))} \text{ for any } K \in \mathbb{S} \text{ such that } T^t \times K \in \mathcal{F}$$

There are I$= \{1, .., I\}$ infinitely lived traders, $\mathbb{L} = \{1, .., L\}$ goods at each node. So $\mathbb{D} \times \mathbb{L}$ is the set of all goods over all nodes. Let $\mathbb{R}^{\mathbb{D}\times\mathbb{L}}$ denote the vector space of all maps $x : \mathbb{D} \times \mathbb{L} \to \mathbb{R}$. Let $l_\infty(\mathbb{D} \times \mathbb{L})$ denote sequences $x \in \mathbb{R}^{\mathbb{D}\times\mathbb{L}}$ such that $\sup_{(s_t,l)\in\mathbb{D}\times\mathbb{L}} |x_l(s_t)| < \infty$, the subspace of bounded maps. Let $\|x\|_\infty = \sup_{(s_t,l)\in\mathbb{D}\times\mathbb{L}} |x_l(s_t)|$ denote the sup-norm of $l_\infty(\mathbb{D} \times \mathbb{L})$. Also, let

[3] $\sigma_t \in \mathbb{F}_t, \sigma_{t-1} \in \mathbb{F}_{t-1}$ implies that either $\sigma_t \subset \sigma_{t-1}$ or $\sigma_t \cap \sigma_{t-1} = \varnothing$.

$l_1(\mathbb{D} \times \mathbb{L})$ denote sequences such that $\sum_{(s_t,l)\in\mathbb{D}\times\mathbb{L}} |x_l(s_t)| < \infty$. Agent i has endowment $\omega \in l_\infty^+(\mathbb{D} \times \mathbb{L}) = \{x \in l_\infty(\mathbb{D} \times \mathbb{L}) : x_l(s_t) \geq 0$ for all $\xi, l\}$.[4] Let $X^i = l_\infty^+(\mathbb{D} \times \mathbb{L})$ denote trader i's consumption set. Let $p \in \mathbb{R}^{\mathbb{D}\times\mathbb{L}}$ be the spot price process and set $p(s_t, 1) = 1$ for all $s_t \in \mathbb{D}$ so 1 is the numeraire good.[5] Further, we consider only short-lived numeraire securities. Let $J(s_t)$ be the set of securities issued at node $s_t \in T^t$. $j(s_t) = \#J(s_t) < \infty$ is the number of securities. $A_j(s_t, s)$ is the payoff of security $j \in J(s_t)$ in the immediate successor node $(s_t, s) \in T^{t+1}$. $A(s_t, s) = \left[A_1(s_t, s), .., A_{j(s_t)}(s_t, s)\right]$ is the $1 \times j(s_t)$ vector of security payoffs in immediate successor node $(s_t, s) \in T^{t+1}$. Finally, let $A_{t+1}(s_t)$ denote the $S \times j(s_t)$ matrix of payoffs in period $t+1$. Also, $A = (A(s_t, s) : (s_t, s) \in \mathbb{D}^+, t \in \mathbb{T}) \in \Pi_{s_t\in\mathbb{D}}\mathbb{R}^{S\times j(s_t)}$ is the process of security payoffs. We assume that all securities pay off in terms of the numeraire good. Let $q(s_t) = (q_j(s_t) : j \in J(s_t))$ be the $1 \times j(s_t)$ vector of node s_t security prices. $q = (q(s_t) : s_t \in \mathbb{D}) \in \Pi_{s_t\in\mathbb{D}}\mathbb{R}^{J(s_t)} = Q$ be the security price process, an element of the security price space. $z^i = (z^i(s_t) : s_t \in \mathbb{D}) \in \Pi_{s_t\in\mathbb{D}}\mathbb{R}^{J(s_t)} = Z$ be the portfolio process for trader i, an element of the portfolio space, where $z^i(s_t) = \left(z^i_j(s_t) : j \in J(s_t)\right)$ is the $j(s_t) \times 1$ portfolio vector of trader i at node s_t.

Let $\succeq_i$ represent trader i's preference ordering over X^i. Preferences $\succeq_i$

[4] Bewley [4] and subsequently Magill and Quinzii [14] impose the condition of Mackey contituity on traders' preferences. The Mackey topology on $l_\infty(\mathbb{D} \times \mathbb{L})$ is described in [4].

[5] We can do this because securities in this economy pay only in terms of the numeraire good.

are represented by an additively separable utility function:

$$u^i(x^i) = \sum_{s_t \in T^t, t \in \mathbb{T}} \rho^i(C(s_t))\delta_i^{t(s)} v^i(x^i(s_t)) = \mathbb{E}^{\rho^i}\left[\sum_{t \in \mathbb{T}} \delta_i^t v^i(x_t^i)\right]$$

Where $\rho^i(C(s_t))$ is the probability of $s_t \in T^t$, $\delta_i \in (0,1)$ is an intertemporal discount factor and $v^i : \mathbb{R}_+^L \to \mathbb{R}$ is a continuous, increasing and concave function with $v^i(0) = 0$. These assumptions on the utility function satisfy Mackey continuity (as shown in [4]).[6] Let $\succeq = (\succeq_1, .., \succeq_I)$, $\omega = (\omega^1, .., \omega^I)$. Finally, let $\mathcal{E}_\infty(\mathbb{D}, \succeq, \omega, A)$ denote the economy. When all traders' preferences are of the expected utility form, let $\rho = (\rho^1, .., \rho^I)$, $\delta = (\delta_1, .., \delta_I)$ and $v = (v^1, .., v^I)$ then $\mathcal{E}_\infty(\mathbb{D}, \rho, \delta, v, \omega, A)$ denote the economy in question.

Assumption A Endowments are uniformly bounded away from zero and aggregate endowments are uniformly bounded. Formally, there is an $m > 0$ such that $\omega_l^i(s_t) > m$ for all i, s_t, l; moreover there is an $m' > m > 0$ such that $\sum_i \omega_l^i(s_t) < m'$ for all s_t, l.

Assumption B There exists a riskless bond at every node $s_t \in D$. Formally, there is a $j \in J(s_t)$ so that $A_j(s_t, s) = 1$ for all $s \in T$.

Assumption B can be replaced with the condition that for each node $s_t \in D$, there exists a portfolio of securities $z \in R^{J(s_t)}$ such that $\sum_j A_j(s_t, s) z_j > 0$ for all $s \in T$.

[6] [2] shows that Mackey continuity is needed to prove existence of an equilibrum in economies with infinitely many commodities.

In this economy, assumptions A and B satisfy all conditions needed (see section 3 of [14]) for the existence of an equilibrium in open-ended incomplete markets economies. They are assumed to hold throughout this paper.

3.3 Equilibrium with a Transversality Condition

With the assumption that $z^i(s_{-1}) = 0$, and that preferences are strictly monotone, the trader's budget constraint at node $s_t \in \mathbb{D}$ is:

$$p(s_t)\left(x^i(s_t) - \omega^i(s_t)\right) = A(s_t)z^i(s_{t-1}) - q(s_t)z^i(s_t) \text{ for all } s_t \in \mathbb{D} \quad (4)$$

In infinite horizon economies, a trader can borrow and roll over his debt *ad infinitum.* So we need a transversality condition to ensure that there is a bound on the rate at which the trader accumulates debt.

$$\lim_{T\to\infty} \sum_{s_T \in \mathbb{D}_T(s_t)} \pi^i(s_T)q(s_T)z^i(s_T) = 0 \text{ for all } s_t \in \mathbb{D} \quad (5)$$

So the budget set for trader i is:

$$\mathcal{B}_\infty^{TC}(p, q, \pi^i, \omega^i, A) = \left\{x^i \in l_\infty^+(\mathbb{D} \times \mathbb{L}) : \exists z^i \in Z \text{ satisfying } (4) \text{ and } (5)\right\}$$

Definition 1 *An equilibrium of the economy* $\mathcal{E}_\infty(\mathbb{D}, \succeq, \omega, A)$ *is a pair* $(x, z), (p, q, (\pi^i)_{i\in\mathbb{I}}) \in l_\infty^+(\mathbb{D} \times \mathbb{L} \times I) \times Z^I \times \mathbb{R}^{\mathbb{D}\times\mathbb{L}} \times Q \times l_1^+(\mathbb{D} \times \mathbb{I})$ *such that:*

1. (x^i, z^i) *is* $\succeq_i$*maximal in* $\mathcal{B}_\infty^{TC}(p, q, \pi^i, \omega^i, A)$

2. *for each* $i \in \mathbb{I}$*:*

   (a) $\pi^i(s_t) > 0$*,for all* $s_t \in \mathbb{D}$ *and* $P^i \in l_1^+(\mathbb{D} \times \mathbb{L})$ *where* $P^i = (P^i(s_t), s_t \in \mathbb{D}) = (\pi^i(s_t)p(s_t), s_t \in \mathbb{D})$

   (b) x^i *is* $\succeq_i$*maximal in* $B_\infty(P^i, \omega^i) = \{x^i \in l_\infty^+(\mathbb{D} \times \mathbb{L}) : P^i(x^i - \omega^i) \leq 0\}$

   (c)

$$\pi^i(s_t)q_j(s_t) = \sum_{s_{t+1}=(s_t,s)} \pi^i(s_{t+1})A_j(s_{t+1}) \text{ for all } j \in j(s_t), s_t \in \mathbb{D} \tag{6}$$

3. $\sum_{i\in\mathbb{I}} (x^i - \omega^i) = 0$

4. $\sum_{i\in\mathbb{I}} z^i = 0$

Theorem 2 *Each economy* $\mathcal{E}_\infty(\mathbb{D}, \succeq, \omega, A)$ *satisfying the above assumptions has an equilibrium.*

Proof. Theorem 5.1 of [14]. ■

The assumption that assets must be short-lived and must pay off in terms of a numeraire good ensures that an equilibrium exists. Is it however only a simplifying assumption as the results in this paper rest on analyzing the no arbitrage equation which must hold in equilibrium regardless of the particular asset structure.

4 Belief Selection

4.1 Effectively Identical Beliefs

The set of beliefs that a trader adopts that yield the same equilibrium outcome is the set of *effectively identical beliefs* for this trader, defined below.

Definition 3 *Suppose that* $(x, z), (p, q, (\pi^i)_{i\in\mathbb{I}})$ *is an equilibrium of an economy* $\mathcal{E}_\infty(\mathbb{D}, \rho, v, \omega, A)$*. We say that trader i's beliefs* ρ^i *are effectively identical to* λ^i *(a probability measure on* $(\mathbb{S}, \mathcal{F})$*) if there exists an equilibrium* $(x, z), (p, q, (\psi^i)_{i\in\mathbb{I}})$ *of the economy* $\mathcal{E}_\infty(\mathbb{D}, \rho', v, \omega, A)$ *where* $\rho' = \left(\rho^1, .., \rho^{i-1}, \lambda^i, \rho^{i+1}, .., \rho^I\right)$*. We write* $\rho^i \in \left[\lambda^i\right]^i$ *.*

A sufficient condition for a probability distribution ρ^i to be effectively identical to the beliefs of some trader is that the no-arbitrage equation (6) is satisfied where $\pi^i(s_t) = \rho^i(C(s_t))\delta_i^t v_1^i(x^i(s_t))$. As beliefs change, so does the way traders value the future. Hence, the definition imposes that equilibrium allocations and prices are identical for different (but effectively identical) beliefs. The resulting state price process for trader i is different precisely because the probability distributions ρ^i and λ^i are different.

Equilibrium security prices can reveal some information about a trader's beliefs. The price of a security in node s_t represents trader i's marginal utility of consuming the stream of this security's payoff across successor nodes. Along with a trader's actual consumption over these nodes, one can extract some information about this trader's beliefs over successor nodes. In a complete markets economy, security prices reveal these beliefs perfectly. Equilibrium security prices and consumption for a given node s_t can be summarized in the no-arbitrage equation:

$$q_t = \rho^i(t+1|s_t)M^i(s_t) \tag{7}$$

$M^i(s_t)$ is a matrix determined by the equilibrium consumption of trader i in successor nodes of s_t. This is the traditional no-arbitrage equation (6) rewritten to make trader i's conditional beliefs more apparent. Given an equilibrium, this trader's conditional beliefs can then be extracted from this equation. These conditional beliefs, over all nodes, can be then put together to construct beliefs over the whole $\sigma-$field. We say that an economy $\mathcal{E}_\infty(\mathbb{D}, \succeq, \omega, A)$ has complete markets if $j(s_t) = b(s_t)$ for all $s_t \in \mathbb{D}$ and the $S \times j(s_t)$ matrix $A_{t+1}(s_t)$ has full rank for all $s_t \in \mathbb{D}$. The complete markets result is summarized in the proposition below.

Proposition 4 *Suppose that* $(x, z), (p, q, (\pi^i)_{i\in\mathbb{I}})$ *is an equilibrium of a complete markets economy* $\mathcal{E}_\infty(\mathbb{D}, \rho, \delta, v, \omega, A)$ *then the set of effectively identical beliefs for each trader is a singleton.*

In contrast, equation (7) doesn't determine trader i's conditional beliefs uniquely when markets are incomplete, because there are fewer security prices. This is shown in the next proposition, which makes use of the following

Assumption 1 (Markets are Incomplete at Some Node) There exists a finite path $\tilde{s}_{\tilde{t}} \in T^{\tilde{t}}$ such that $\text{Rank}[A_{\tilde{t}+1}(\tilde{s}_{\tilde{t}})] < S - 1$.

This assumption is stronger than the usual one for market incompleteness. The additional degree of freedom is used in the proof of the next proposition to ensure that candidate solutions to equation (7) are probability distributions.

Proposition 5 *Under assumption 1, suppose that* $(x, z), (p, q, (\pi^i)_{i \in \mathbb{I}})$ *is an equilibrium of an incomplete markets economy* $\mathcal{E}_\infty(\mathbb{D}, \rho, \delta, v, \omega, A)$ *then the set of effectively identical beliefs for each trader is not a singleton.*

The above proposition has some straightforward implications in terms of belief selection in incomplete markets. Let ρ be the true probability distribution on $(\mathbb{S}, \mathcal{F})$. We say that trader i has rational expectations (or correct beliefs) if $\rho^i = \rho$. Blume and Easley [9] define survival of trader i on a path $s \in \mathbb{S}$ if $\limsup_t x^i(s_t) > 0$. An implication of the above propositions in the incomplete markets case is that each trader with rational expectations has effectively identical beliefs which are not correct. Also, each trader that survives $\rho-$almost surely has effectively identical beliefs which are not correct.

Suppose we can observe all aspects of the economy except traders' beliefs. Then, given an equilibrium of that economy, we could not conclude that a trader who survives has correct beliefs. This definition of belief correctness is however very strong. A trader whose conditional beliefs are identical to the truth in all nodes except one node, has incorrect beliefs. In the Pareto optimal economy discussed in Blume and Easley [9], this trader may survive (if we control for other factors).

4.2 Homogeneity of Beliefs

Blume and Easley [9] show that a necessary condition for survival is that the truth is absolutely continuous with the beliefs of traders who survive. This formalizes the market selection hypothesis, that traders with incorrect beliefs are driven out of the market. Here, belief correctness refers to the concept of equivalence of a trader's beliefs with the truth. In this section, we show that survival in incomplete markets is consistent with beliefs not equivalent to the truth. To construct these beliefs, we require that all traders' conditional probabilities should be uniformly bounded away from the edges of the unit simplex by some $\varepsilon_0 > 0$.This ensures that effectively identical beliefs can be chosen sufficiently far away from original beliefs, thus allowing "sufficient room for disagreement" from a trader's original beliefs.

Assumption 2 There must exist an $\varepsilon_0 > 0$ such that the ε_0-ball[7] $B_{\varepsilon_0}(\rho(.|s_t)) \subset \mathbb{R}^S_{++}$ for all $s_t \in \mathbb{D}^+$.

The first step is to construct effectively identical beliefs that are not equivalent to a trader's original beliefs. We do this by constructing conditional beliefs uniformly bounded away from original beliefs, we then use Blackwell

[7] We use the sup norm ($\|x\|_S = \sup_{i\in S} |x_i|$).

and Dubin's theorem to show that these new beliefs cannot be equivalent to original beliefs. We recall the following definition and result.

Definition 6 *Agent $i \in \mathbb{I}$ and $j \in \mathbb{I}$'s beliefs eventually become homogeneous if there is a set $A \in \mathcal{F}$ such that : $P^k(A) = 1$ for $k = i, j$ and for all $s \in A$, $\sup_{B \in \mathcal{F}} |P^i_{s_t}(B) - P^j_{s_t}(B)| \to 0$ as $t \to \infty$.*

Proposition 7 *If two probability measures are equivalent (meaning: $\rho^i(B) = 0 \Leftrightarrow \rho^j(B) = 0$ for all $B \in \mathcal{F}$) then the posterior probabilities eventually become homogeneous.*

Proof. Blackwell and Dubins (1962). ■

Evidently, we must strengthen our notion of market incompleteness to ensure that we can choose effectively identical conditional beliefs sufficiently far away from original beliefs, infinitely often.

Assumption 3 (Markets are Sufficiently Incomplete Infinitely Often) For each $i \in \mathbb{I}$, there exists a set $A_i \in \mathcal{F}$ of positive measure ρ^i such that $\text{Rank}[A_{t+1}(s_t)] < S - 1$ i.o. on each path $s \in A_i$.

A sufficient condition for assumption 2 is that markets are incomplete at every node in the tree with $\text{Rank}(A_j(s_t, t+1)) < S - 1$.

Proposition 8 *Under assumptions 2 and 3, suppose that $(x, z), (p, q, (\pi^i)_{i \in \mathbb{I}})$ is an equilibrium of an economy $\mathcal{E}_\infty(\mathbb{D}, \rho, \delta, v, \omega, A)$ then the set of effectively identical beliefs for trader $i \in \mathbb{I}$ contains beliefs not equivalent to ρ^i.*

The main result of this paper is an implication of the following corollary.

Corollary 9 *Under assumptions 2 and 3, suppose that* $(x, z), (p, q, (\pi^i)_{i \in \mathbb{I}})$ *is an equilibrium of an economy* $\mathcal{E}_\infty(\mathbb{D}, \rho, \delta, v, \omega, A)$ *then the set of effectively identical beliefs for trader* $i \in \mathbb{I}$ *contains beliefs not equivalent to the true probability distribution* ρ*.*

Proof. If trader i's beliefs are not equivalent to ρ, then we're done. If they are, use the previous proposition. ■

If we can observe all aspects of the economy except for traders' beliefs, then given an equilibrium, a trader who survives $\rho-$a.s. has beliefs consistent with this survival that are not equivalent to ρ. This is in contrast to the Pareto optimal result of Blume and Easley [9]. Note that our result doesn't rely on assumptions about discount factors, or even the precise definition of survival. This is because it is the no-arbitrage equation along with the asset structure that determines a trader's set of effectively identical beliefs, in particular a surviving trader's beliefs.

We also obtain the result that two traders who survive may strongly disagree about the truth. This is a direct implication of the following

Corollary 10 *Under assumptions 2 and 3, suppose that* $(x, z), (p, q, (\pi^i)_{i \in \mathbb{I}})$ *is an equilibrium of an economy* $\mathcal{E}_\infty(\mathbb{D}, \rho, \delta, v, \omega, A)$ *then each trader has effectively identical beliefs that are not equivalent to another trader's beliefs.*

Finally, note that in incomplete markets economies with Pareto efficient outcomes, all traders beliefs must converge with the truth (see for example

Sandroni [15]). Because of the asset structure, we may construct effectively identical beliefs for the surviving traders that do not merge with the truth. In this case, the original outcome is still an equilibrium but it is no longer Pareto efficient: so any incomplete markets equilibrium outcome where traders with incorrect beliefs survive almost surely with respect to the truth must be Pareto efficient. This result is expected since outcomes are generically Pareto inefficient in incomplete markets economies, but it shows that the above results are not in contradiction with previous work on belief selection in Pareto efficient economies.

5 Survival in a Two Trader Economy

We have shown that incomplete markets select for a wide range of beliefs, including beliefs that do not merge with the truth. However when all surviving traders have beliefs that are effectively identical to the truth, incorrect expectations may not affect the asset price process. As a result, incomplete markets may select for beliefs that are incorrect in ways that are irrelevant for survival. In a simple two-trader economy, we show that traders with incorrect beliefs may both survive and affect asset prices.

We consider an economy with two identical traders i and j and the corresponding no-trade outcome. Initially, we assume that both traders know the truth. Because there is no trade, both traders survive according to any probability distribution. We modify the economy by assigning trader i a different discount factor and different beliefs such that the no-trade outcome is still an equilibrium of the new economy. Trader i's discount factor is chosen in a deleted neighborhood of trader j's discount factor. Trader i's beliefs are chosen such that the truth does not lie in her set of effectively identical beliefs. Trader i survives according to the truth. This occurs in the presence of trader j, who knows the truth (and who survives). For simplicity, we assume that markets are incomplete at all nodes.

Assumption 3' (Markets are Incomplete at all Nodes) There are S states of the world each period, and $\text{Rank}[A_{t+1}(s_t)] = J < S$ for all $s_t \in \mathbb{D}$.

Assumption 4 Traders have identical Bernoulli utilities v and identical endowment processes $\omega \in l_\infty(\mathbb{D} \times \mathbb{L})$ uniformly bounded away from zero and infinity. Given these processes, $[A_{t+1}(s_t)]$ lies outside a closed set of measure zero of endowments $A^*(s_t) \subset \mathbb{R}^{SJ}$ for all $s_t \in \mathbb{D}$.

The sets $A^*(s_t)$ are constructed in the proof of the following proposition.

Proposition 11 *Suppose that assumptions 2, 3' and 4 hold. Consider an economy* $\mathcal{E}_\infty(\mathbb{D}, \rho, \delta, v, \omega, A)$ *with two identical traders* i *and* j *and consider the corresponding no trade outcome. There exists a deleted neighborhood* N *of* δ^j *such that for all discount factors* $(\delta')^i$ *for trader* i *within that neighborhood* N*, there exist beliefs* $(\rho')^i$ *such that the no trade outcome is also an equilibrium for the economy* $\mathcal{E}'_\infty(\mathbb{D}, \rho', \delta', v, \omega, A)$ *where* $\rho' = \left((\rho')^i, \rho^j\right)$ *and* $\delta' = \left((\delta')^i, \delta^j\right)$. *We have the following properties in the new economy* $\mathcal{E}'_\infty$.

1. *Trader i survives $\lambda - a.s.$ for any $\lambda \in [\rho^j]^j$;*

2. $\lambda \notin \left[(\rho')^i\right]^i$;

3. $(\rho')^i$ *can be chosen such that λ is not equivalent to $(\rho')^i$.*

Trader i's incorrect beliefs may be far from the truth (property #3) and these beliefs matter (property #2): if trader i were to adopt correct beliefs λ, equilibrium prices will change and she may no longer survive $\lambda - a.s.$ Note that trader i may be chosen so that she is more impatient than the other trader.

For example, if endowments and payoffs are stationary and beliefs and the truth are iid, Blume and Easley [9] show that a trader who survives almost surely with respect to the truth must have the highest survival index[8]. Here, trader i may be chosen to be more impatient and have incorrect beliefs so her survival index is smaller than trader j's, yet she survives $\lambda - a.s.$

6 Conclusion

In this paper we model an infinite horizon economy, with a view to testing the market selection hypothesis under market incompleteness. We know from the literature (Sandroni [15], Blume and Easley [9]) that markets with a Pareto optimal outcome or, more narrowly, complete markets select for correct beliefs. All surviving traders have correct beliefs (i.e. beliefs that can

[8] $\log \delta^k - I_\rho(\rho^k)$ where δ^k is trader k's discount factor and $I_\rho(\rho^k)$ is the relative entropy of trader k's beliefs with respect to the truth.

be represented by probability distributions that merge with the truth). Both wealth and consumption of traders whose beliefs are incorrect converge to zero with true probability one. Hence in the long run heterogeneity of beliefs is not persistent and market outcomes reflect the true probability distribution over returns.

The motivation for our study lies in two counterexamples provided by Blume and Easley [9] that point to the fact that the same need not hold under market incompleteness. In this paper we show that incomplete markets do not select for correct beliefs. In particular we prove that when markets are incomplete the set of beliefs that is consistent with a trader's survival admits beliefs which are not equivalent to the truth, and these incorrect beliefs may matter.

We build our first result on the characterisation of the set of *effectively identical beliefs*. Given an economy and its corresponding equilibrium, this is the set of beliefs for a trader that are consistent with the same equilibrium allocation and prices. If a trader had to adopt different beliefs belonging to this set, the equilibrium outcome would remain unchanged. We show that, while in complete market economies the set of effectively identical beliefs admits only one element, under market incompleteness this set is not a singleton. Moreover, it always admits probability distributions that are not equivalent to the truth. This result holds for all traders and in particular for surviving traders. Hence one can always find beliefs that differ significantly from the true probability distribution and that still allow a trader to survive and have an impact on market outcomes in the long run.

An immediate corollary of our result is that heterogeneity of beliefs is persistent: surviving traders need not share the same beliefs in the long run. Under incomplete markets asset prices reflect a range of underlying probability distributions that generate them. These distributions offer conflicting evidence on the probability of some events and influence asset prices.

7 Appendix

7.1 Preliminary

The following proposition is used in the proof of proposition (5).

Proposition 12 *Suppose that* $(x,z),(p,q,(\pi^i)_{i\in\mathbb{I}})$ *is an equilibrium of an economy* $\mathcal{E}_\infty(\mathbb{D},\rho,\delta,v,\omega,A)$. *Let* $\left(\lambda^i\right)_{i\in\mathbb{I}}$ *be probability distributions on* $(\mathbb{S},\mathcal{F})$ *such that:*

$$q_j(s_t) = \sum_{s_{t+1}\in\{(s_t,s):s\in T\}} \frac{\lambda^i(C(s_{t+1}))\delta_i v_1^i(x^i(s_{t+1}))}{\lambda^i(C(s_t))v_1^i(x^i(s_t))} A_j(s_{t+1}) \textit{ for all } s_t\in T^t, j\in J, t\in\mathbb{T}$$

Then $\left(\lambda^i\right)_{i\in\mathbb{I}}$ *are effectively identical to* $(\rho^i)_{i\in\mathbb{I}}$.

Proof. Set:

$$\psi^i(s_t) = \delta_i^t v_1^i(x^i(s_t))\lambda^i(C(s_t)) \text{ for all } s_t\in T^t, t\in\mathbb{T}$$

So the no-arbitrage condition is satisfied:

$$\psi^i(s_t)q_j(s_t) = \sum_{s_{t+1}\in\{(s_t,s):s\in T\}} \psi^i(s_{t+1})A_j(s_{t+1}) \text{ for all } s_t\in T^t, j\in J, t\in\mathbb{T}$$

Note that the other FOCs of trader i's optimization problem are satisfied. Indeed, we know that:

$$\rho^i(C(s_t))\delta_i^t v_l^i(x^i(s_t)) = \pi^i(s_t)p_l(s_t) \text{ for all } s_t\in T^t, l\in\mathbb{L}, t\in\mathbb{T}$$

So that:

$$\frac{\rho^i(C(s_t))\delta_i^t}{\pi^i(s_t)} v_l^i(x^i(s_t)) = p_l(s_t) \text{ for all } s_t \in T^t, l \in \mathbb{L}, t \in \mathbb{T}$$

So that (with $p_1(s_t) = 1$):

$$\frac{p_l(s_t)}{p_1(s_t)} = p_l(s_t) = \frac{v_l^i(x^i(s_t))}{v_1^i(x^i(s_t))} \text{ for all } s_t \in T^t, l \in \mathbb{L}, t \in \mathbb{T}$$

So, given that:

$$\delta_i^t v_1^i(x^i(s_t))\lambda^i(C(s_t)) = \psi^i(s_t) \text{ for all } s_t \in T^t, t \in \mathbb{T} \tag{8}$$

It follows that:

$$\delta_i^t \frac{v_l^i(x^i(s_t))}{p_l(s_t)} \lambda^i(C(s_t)) = \psi^i(s_t) \text{ for all } s_t \in T^t, l \in \mathbb{L}, t \in \mathbb{T}$$

Or:

$$\lambda^i(C(s_t))\delta_i^t v_l^i(x^i(s_t)) = \psi^i(s_t)p_l(s_t) \text{ for all } s_t \in T^t, l \in \mathbb{L}, t \in \mathbb{T}$$

So all FOCs are satisfied. Since $(x, z), (p, q, (\pi^i)_{i\in\mathbb{I}})$ is an equilibrium with transversality condition for the economy $\mathcal{E}_\infty(\mathbb{D}, \rho, v, \omega, A)$, it follows from theorem 5.2 of [14] that $((x, z), (p, q))$ is an equilibrium with implicit debt constraint for the economy $\mathcal{E}_\infty(\mathbb{D}, \rho, v, \omega, A)$. So $(qz^i) \in l_\infty(\mathbb{D})$ for all $i \in \mathbb{I}$. So $((x, z), (p, q))$ is an equilibrium with implicit debt constraint for the economy $\mathcal{E}_\infty(\mathbb{D}, \succeq', \omega, A)$. Since preferences in the economy $\mathcal{E}_\infty(\mathbb{D}, \succeq', \omega, A)$ satisfy assumptions $A1 - A6$ in [14], theorem 5.2 of [14] implies the existence of present value vectors ν^i, $i \in \mathbb{I}$ so that $(x, z), (p, q, (\nu^i)_{i\in\mathbb{I}})$ is an equilibrium with transversality condition for the economy $\mathcal{E}_\infty(\mathbb{D}, \succeq', \omega, A)$. Incidentally, it follows that $\nu^i = \psi^i$ for all $i \in \mathbb{I}$, since $(\nu^i)_{i\in\mathbb{I}}$ satisfies equation (8). ■

7.2 Proof of Proposition (4)

Proof. Suppose not. Then there exists an equilibrium $(x, z), (p, q, (\psi^i)_{i\in\mathbb{I}})$ where trader i's preferences are represented by the expected utility $\mathbb{E}^{\lambda^i}\left[\sum_{t\in\mathbb{T}} \delta_i^t v^i(x_t^i)\right]$. Note that $(\psi^i)_{i\in\mathbb{I}}$ must satisfy:

$$q_j(s_t) = \sum_{s_{t+1}\in\{(s_t,s):s\in T\}} \frac{\psi^i(s_{t+1})}{\psi^i(s_t)} A_j(s_{t+1}) \text{ for all } s_t \in T^t, j \in J, t \in \mathbb{T}$$

Set $\psi_{t+1}^i(s_t) = \left(\psi^i(s_t, 1), .., \psi^i(s_t, S)\right)$. So, the above equation in matrix form is:

$$q(s_t) = \frac{\psi_{t+1}^i(s_t)}{\psi^i(s_t)} A_{t+1}(s_t) \text{ for all } s_t \in T^t, t \in \mathbb{T}$$

Where $A_{t+1}(s_t)$ is an $S \times j(s_t)$ matrix and $q(s_t)$ is a $1 \times j(s_t)$ vector. Since markets are complete, A is square and has full rank. So the above equation has a unique solution, which we know is $\frac{\pi_{t+1}^i(s_t)}{\pi^i(s_t)}$. Hence $\frac{\psi_{t+1}^i(s_t)}{\psi^i(s_t)} = \frac{\pi_{t+1}^i(s_t)}{\pi^i(s_t)}$ for all $s_t \in T^t, t \in \mathbb{T}$. Finally, in period 0, $\psi(s_0) = \pi(s_0)$ by construction. So $\psi^i = \pi^i$. So equation (8) implies that $\lambda^i(C(s_t)) = \rho^i(C(s_t))$ for $s = (s_t, ..) \in \mathbb{S}$. So λ^i and ρ^i agree on sets in $\cup_{t\in\mathbb{N}}\mathcal{F}_t$. This set is closed under finite intersections and hence is a $\pi-$system. The $\pi - \lambda$ theorem and it's implication (theorem 3.3 in [6]) in turn implies that $\lambda^i = \rho^i$, a contradiction.

■

7.3 Proof of Proposition (5)

Proof. Choose a process $\lambda^i(C(s_t)) \in [0,1]$ for all $s_t \in \mathbb{D}$ so that $\lambda^i(C(s_0)) = 1$ and:

$$\begin{aligned}
\sum_{s_{t+1} \in \{(s_t,s):s\in T\}} \left[\delta_i v_1^i(x^i(s_{t+1}))A_j(s_{t+1})\right] \lambda^i(C(s_{t+1})) &= v_1^i(x^i(s_t))q_j(s_t)\lambda^i(C(s_t)) \\
\sum_{s_{t+1} \in \{(s_t,s):s\in T\}} \lambda^i(C(s_{t+1})) &= \lambda^i(C(s_t))
\end{aligned}$$

Then, by Kolmogorov's Existence Theorem [see theorem 36.1 in Billingsley [6]], λ^i is a probability distribution on $(T^\infty, \mathcal{F})$, proposition (12) applies and $\left(\lambda^i\right)_{i\in\mathbb{I}}$ are effectively identical to $(\rho^i)_{i\in\mathbb{I}}$. We simplify this system by rewriting it.

$$\begin{aligned}
\sum_{s_{t+1} \in \{(s_t,s):s\in T\}} \left[\delta_i v_1^i(x^i(s_{t+1}))A_j(s_{t+1})\right] \lambda^i(s_{t+1}|s_t) &= v_1^i(x^i(s_t))q_j(s_t) \quad (9) \\
\sum_{s_{t+1} \in \{(s_t,s):s\in T\}} \lambda^i(s_{t+1}|s_t) &= 1
\end{aligned}$$

Given a process $\lambda^i(s_{t+1}|s_t)$, one can reconstruct a probability distribution on $(T^\infty, \mathcal{F})$ by setting, recursively:

$$\begin{aligned}
\lambda^i(C(s_1)) &= \lambda^i(s_1|s_0)\lambda^i(C(s_0)) = \lambda^i(s_1|s_0) \text{ for all } s_1 = (s_0, s) \\
\lambda^i(C(s_{t+1})) &= \lambda^i(s_{t+1}|s_t)\lambda^i(C(s_t)) \text{ for all } s_{t+1} = (s_t, s) \text{ for } t \in \mathbb{T} - \{0\}
\end{aligned}$$

Set $\lambda^i(.|s_t) = \rho^i(.|s_t)$ for all $s_t \neq \tilde{s}_{\tilde{t}}$. $\lambda^i(.|\tilde{s}_{\tilde{t}})$ is chosen such that $\lambda^i(.|\tilde{s}_{\tilde{t}}) \neq \rho^i(.|\tilde{s}_{\tilde{t}})$ and such that system of equations (9) is satisfied (this is possible because markets are incomplete, see below). Then the resulting probability distribution λ^i is different from ρ^i but effectively identical to ρ^i, by proposition (12) in section (7.1).

How to choose an appropriate $\lambda^i(.|\tilde{s}_{\tilde{t}}) \neq \rho^i(.|\tilde{s}_{\tilde{t}})$: Note that the set of equations in (9) can be rewritten as:

$$M^i(s_t)\lambda^i(.|s_t) = q(s_t)$$

Where:

$$M^i(s_t) = \frac{\delta_i}{v_1^i(x^i(s_t))}\begin{bmatrix} v_1^i(x^i(s_{t+1}^1))A_1(s_{t+1}^1) & \ldots & v_1^i(x^i(s_{t+1}^S))A_1(s_{t+1}^S) \\ \vdots & \ddots & \vdots \\ v_1^i(x^i(s_{t+1}^1))A_J(s_{t+1}^1) & \ldots & v_1^i(x^i(s_{t+1}^S))A_J(s_{t+1}^S) \end{bmatrix}$$

Note that $M^i(\tilde{s}_{\tilde{t}})$ has full rank equal to the rank of $A_{\tilde{t}+1}(\tilde{s}_{\tilde{t}},) < S - 1$. Since we know that $\rho^i(.|\tilde{s}_{\tilde{t}})$ solves the system of equations in (9), we know the solution set $\Lambda(\tilde{s}_{\tilde{t}})$ is linear and of dimension at least 1. We know that

$\rho^i(.|\tilde{s}_{\tilde{t}}) \in \mathbb{R}^S_{++}$ and is interior to the unit simplex, by construction. Using the sup norm ($\|x\|_S = \sup_{i\in S}|x_i|$), choose an $\varepsilon > 0$ sufficiently small such that $B_\varepsilon(\rho^i(.|\tilde{s}_{\tilde{t}})) \subset \mathbb{R}^S_{++}$, and choose an element $\bar{\lambda}^i(.|\tilde{s}_{\tilde{t}}) \in B_\varepsilon(\rho^i(.|\tilde{s}_{\tilde{t}})) \cap \Lambda(\tilde{s}_{\tilde{t}})$ such that $\bar{\lambda}^i(.|\tilde{s}_{\tilde{t}}) \neq \rho^i(.|\tilde{s}_{\tilde{t}})$. ■

7.4 Proof of Proposition (8)

Proof. We use the construction in the proof of proposition (5) by choosing $\varepsilon = \varepsilon_0$ at the end of the proof. On each path $s \in A_i$, build a probability distribution λ^i by choosing $\lambda^i(.|s_t) \in [B_{\varepsilon_0}(\rho^i(.|s_t)) \cap \Lambda(s_t)] - B_{\varepsilon_0/2}(\rho^i(.|s_t))$ for all $s_t, t \in \mathbb{T}$ such that $\text{Rank}(A_j(s_t, t+1)) < S-1$ and such that $s = (s_t, ..)$. If the rank condition is not satisfied on these paths, choose $\lambda^i(.|s_t) = \rho^i(.|s_t)$. For paths $s \notin A_i$, choose $\lambda^i(.|s_t) = \rho^i(.|s_t)$.

For each path $s \in A_i$, we show that:

$$\lim_{t\to+\infty} \sup_{B\in\mathcal{G}} |\lambda^i_{s_t}(B) - \rho^i_{s_t}(B)| \geq \frac{\varepsilon_0}{2} \tag{10}$$

Where $\mathcal{G} = \{C(s_t) : s = (s_t, ..) \text{ for all } t \in \mathbb{T}\}$. Then we show that:

$$\lim_{t\to+\infty} \sup_{B\in\mathcal{G}} |\lambda^i_{s_t}(B) - \rho^i_{s_t}(B)| \leq \lim_{t\to+\infty} \sup_{B\in\mathcal{F}} |\lambda^i_{s_t}(B) - \rho^i_{s_t}(B)| \text{ when } \mathcal{G} \subset \mathcal{F} \tag{11}$$

This in turn implies that $\lim_{t\to+\infty}\sup_{B\in\mathcal{F}} |\lambda^i_{s_t}(B) - \rho^i_{s_t}(B)| > 0$ on a set of paths that trader i assigns positive measure. Blackwell and Dubins' result implies in turn that $\lambda^i$and ρ^i are not equivalent.

We now show inequality (10). On a path $s \in A_i$, let $a_t = \sup_{B \in \mathcal{G}} |\lambda^i_{s_t}(B) - \rho^i_{s_t}(B)|$ and $a = \lim_{t \to +\infty} a_t$. Suppose that $a < \frac{\varepsilon_0}{2}$. Choose $\delta > 0$ such that $B_\delta(a) \cap \{\frac{\varepsilon_0}{2}\} = \varnothing$. There is a $T_\delta \in \mathbb{T}$ such that $t \geq T_\delta \Rightarrow |a_t - a| < \delta$. Since $a_t < \frac{\varepsilon_0}{2}$ for $t \geq T_\delta$, it follows that $|\lambda^i_{s_t}(B) - \rho^i_{s_t}(B)| < \frac{\varepsilon_0}{2}$ for $t \geq T_\delta$. But this contradicts the existence of a $B \in \mathcal{G}$ such that $|\lambda^i_{s_t}(B) - \rho^i_{s_t}(B)| \geq \frac{\varepsilon_0}{2}$ i.o. on path $s \in A_i$. Take $B = C(s_{t+1})$ where $s_{t+1} = (s_t, s)$ and where s is chosen such that $\left|\lambda^i(s|s_t) - \rho^i(s|s_t)\right| \geq \frac{\varepsilon_0}{2}$. This s must exist by construction of $\lambda^i(.|s_t)$.

Inequality (11) is obvious: let $a_t = \sup_{B \in \mathcal{G}} |\lambda^i_{s_t}(B) - \rho^i_{s_t}(B)|$ and $a = \lim_{t \to +\infty} a_t$ and $b_t = \sup_{B \in \mathcal{F}} |\lambda^i_{s_t}(B) - \rho^i_{s_t}(B)|$ and $b = \lim_{t \to +\infty} b_t$. Suppose that $a > b$. Let $\eta = a - b > 0$. Choose $\varepsilon = \frac{\eta}{4}$. There exists a $T_\varepsilon \in \mathbb{T}$ such that $t \geq T_\varepsilon \Rightarrow |a_t - a| < \varepsilon$ and $|b_t - b| < \varepsilon$. So if $t \geq T_\varepsilon$, $a_t > b_t$ so $a_t > \frac{a_t + b_t}{2} \geq \sup_{B \in \mathcal{G}} |\lambda^i_{s_t}(B) - \rho^i_{s_t}(B)|$ so a_t is not the sup, a contradiction. ■

7.5 Proof of Proposition (11)

Proof. Consider an economy with 2 identical traders where ρ, δ, v, ω represent common beliefs, discount factors, Bernoulli utility and endowment processes. Given the common endowment process, consider the matrices:

$$B(s_t) = \begin{bmatrix} v_1(\omega\left(s^1_{t+1}\right))A_1(s^1_{t+1}) & ... & v_1(\omega\left(s^S_{t+1}\right))A_1(s^S_{t+1}) \\ \vdots & \ddots & \vdots \\ v_1(\omega\left(s^1_{t+1}\right))A_J(s^1_{t+1}) & ... & v_1(\omega\left(s^S_{t+1}\right))A_J(s^S_{t+1}) \end{bmatrix}$$

The set $A^*(s_t) \subset \mathbb{R}^{SJ}$ is the set of all $A_{t+1}(s_t) \in \mathbb{R}^{SJ}_{++}$ such that the $1 \times S$ vector $I = (1, .., 1) \in \text{Span}B(s_t)$. So $A^*(s_t)$ is the set of all payoffs such that the matrix $\begin{bmatrix} B(s_t) \\ I \end{bmatrix}$ is not of full rank, a closed set of measure zero (this is a direct application of the pre-image theorem. See pages 21 and 27 of Guillemin and Pollack [13]).

Let $(x, z), (p, q, (\pi^k)_{k \in \{i,j\}})$ be the corresponding equilibrium outcome. This is the no-trade equilibrium. The asset price process is therefore $q(s_t) = M(s_t)\rho(.|s_t)$ where $M(s_t) = \frac{\delta B(s_{t+1})}{v_1(\omega(s_t))}$. This price process $q \in l_\infty(\mathbb{D} \times J)$. This is because the endowment process is uniformly bounded away from zero and from infinity. For each node s_t, choose $\mu(s_t) > 0$ and $\rho^i(.|s_t) >> 0$ such that:

$$\frac{q(s_t)}{(1 + \mu(s_t))} = M(s_t)\rho^i(.|s_t) \tag{12}$$

And such that:

$$\sum_{s_{t+1} \in \{(s_t, s): s \in T\}} \rho^i(s_{t+1}|s_t) = 1 \tag{13}$$

Rewriting equations (12) and (13), the system of equations $B(s_{t+1})\rho^i(.|s_t) = \frac{q(s_t)v_1(\omega(s_t))}{\delta(1+\mu(s_t))}$ and $\sum_{s_{t+1} \in \{(s_t,s):s \in T\}} \rho^i(s_{t+1}|s_t) = 1$ has a non-empty solution set because by construction, $\begin{bmatrix} B(s_t) \\ I \end{bmatrix}$ is of full rank less than or equal to S. Denote this set by $\Lambda(s_t)$. Choose $\mu(s_t) > 0$ sufficiently close to zero so that $\Lambda(s_t) \cap B_{\frac{\varepsilon_0}{2}}(\rho(.|s_t)) \neq \varnothing$. This guarantees a solution $\rho^i(.|s_t) \in \mathbb{R}^S_{++}$ which is $\frac{\varepsilon_0}{2}-$bounded away from the unit simplex. Choose such $(\mu(s_t), \rho^i(.|s_t))$ for all $s_t, t \in \mathbb{T}$. Note that a unique $\mu = \mu(s_t)$ can be chosen such that $\mu > 0$. This is because the price sequence $q(s_t)$ is uniformly bounded. Using Kolmogorov's existence theorem, construct a ρ^i, a probability distribution on $(\mathbb{S}, \mathcal{F})$ that represents trader i's beliefs whose marginals equal $\rho^i(.|s_t)$, marginals that are $\frac{\varepsilon_0}{2}-$bounded away from the unit simplex. Also, choosing μ such that $\delta(1+\mu) < 1$, let $\delta^i = \delta(1+\mu)$ represent trader i's discount factor. Because $\frac{q(s_t)}{1+\mu} \neq q(s_t)$ for all $s_t, t \in \mathbb{T}$, $\rho \notin [\rho^i]^i$. This shows property #2. Property #1 holds because it's a no-trade equilibrium and endowments are assumed uniformly bounded away from zero.

The same construction can be made by choosing μ such that $\mu < 0 < \delta(1+\mu) < 1$ hence we can construct a small deleted neighborhood around δ. Property #3 follows from the fact that ρ^i's conditional probabilities are uniformly bounded away from the unit simplex and proposition (8) applies.

■

References

[1] A. Alchian, Uncertainty, Evolution and Economic Theory, J. Polit. Economy 58 (1950), 211-221.

[2] A. Araujo, Lack of Pareto Optimal Allocations in Economies with Infinitely Many Commodities: The Need for Impatience, Econometrica 53 (1985), 455-461.

[3] A. Araujo and A. Sandroni, On the Convergence to Homogeneous Expectations when Markets are Complete, Econometrica 67 (1999), 663-672.

[4] T. Bewley, Existence of Equilibria in Economies with Infinitely Many Commodities, J. of Econ. Theory 4 (1972), 514-540.

[5] B. Biais and R. Shadur, Darwinian selection does not eliminate irrational traders, Europ. Econ. Rev. 44 (2000), 469-490.

[6] P. Billingsley, Probability and Measure, John Wiley & Sons, Inc., 1995.

[7] D. Blackwell and L. Dubins, Merging of Opinions with Increasing Information, Annals of Mathematical Statistics 38 (1962), 882-886.

[8] L. Blume and D. Easley, Evolution and Market Behavior, J. Econ. Theory 58 (1992), 9-40.

[9] L. Blume and D. Easley, If You're So Smart, Why Aren't You Rich? Belief Selection in Complete and Incomplete Markets, Econometrica, Forthcoming.

[10] J.B. De Long, A. Shleifer, L. Summers and R. Waldman, Noise Traders Risk in Financial Markets, J. Polit. Economy 98 (1990), 703-738.

[11] J.B. De Long, A. Shleifer, L. Summers and R. Waldman, The Survival of Noise Traders, J. Bus. 64 (1991), 1-19.

[12] M. Friedman, Essays in Positive Economics, University of Chicago Press, Chicago, 1953.

[13] V. Guillemin and A. Pollack, Differential Topology, Prentice-Hall, Englewood Cliffs, New Jersey,1974.

[14] M.F. Magill and M. Quinzii, Infinite Horizon Incomplete Markets, Econometrica 62 (1994), 853-880.

[15] A. Sandroni, Do Markets Favor Agents Able to Make Accurate Predictions?, Econometrica 68 (2000), 1303-1341.

[16] A. Sandroni, Markets Selection when Markets are Incomplete, J. Math. Econ. , Forthcoming.

[17] H. Shefrin and M. Statman, Behavioral capital asst pricing theory, J. Finan. Quant. Anal. 29 (1994), 323-349.