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Informational Free Rides in Uniform Price Auctions: Exception or Norm?*

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Abstract
Multi-unit common value uniform price auctions with demand function bids are in widespread use. I analyze this auction when there is an informed bidder and other uninformed bidders. In such auctions it is easy to construct equilibria in which uninformed bidders earn a positive payoff by free riding on the informed bidder's information. Here I ask whether such free riding arises only in special cases, and should therefore be considered a pathological exception, or whether it is the norm in equilibrium. To answer this, I derive the necessary and sufficient condition for uninformed bidders to earn a zero payoff in all equilibria. The condition requires that there should be enough demand by uninformed bidders at least at low prices so that no single uninformed bidder is "pivotal" in deciding whether total uninformed demand equals or exceeds supply, and places a lower bound on the highest price submitted by the informed bidder (i.e. the highest price at which at least one unit is demanded by the informed bidder). Equilibria not satisfying the condition exist. In these, uninformed bidders appropriate some of the information rent. Further, the condition is quite strong in certain cases, casting doubt on existence of equilibria with zero uninformed payoff. If there is no such equilibrium, informational free riding characterizes all equilibria in uniform price auctions. I discuss application of the results to Treasury auctions as well as repo auctions.

KEYWORDS: *Uniform price auction, demand function bids, market clearing price, informational free riding, Treasury auctions, repo auctions*

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1. Introduction

Multi-unit common value auctions with a uniform pricing rule and demand function bids are in widespread use. Examples include Treasury bill auctions in the US and other countries, index-linked bond sales in the US, corporate bond auctions and IPO auctions conducted over the Internet (such as Hambrecht's OpenBook auction which sells corporate bonds online), and money market repo auctions. Under the uniform pricing rule, all winners pay the market clearing price, defined as the highest bid at which aggregate demand equals or exceeds supply (i.e. the lowest price at which some positive quantity is won by some bidder) [1].

In such auctions it is often important to know how uninformed bidders fare in the presence of informed bidders. For example, in treasury auctions, the designers might want to choose a format to encourage participation by uninformed bidders. For single unit first price auctions, Milgrom (1979)[2] provides general results to the effect that in the presence of a bidder with better information, less informed bidders cannot earn a strictly positive payoff. With multiple units, first price auctions correspond to discriminatory auctions. In a pure common value setting, and demand function bids, the entire supply is like one unit, and the results for single unit first price auctions extend easily to discriminatory auctions. In such auctions it is impossible for an uninformed bidder to free ride on the information of the informed bidder since winning bidders pay their own bids. However, under a uniform pricing rule, there is scope for demanding some units at a high price to ensure winning those units but still pay a lower price. Indeed, it is very easy to construct equilibria in uniform price auctions in which some uninformed bidders

(1) An alternative definition of market clearing price is the highest losing bid. Under this definition, the auction would become a second-price auction as the number of units is reduced to 1. Here, in contrast, the auction reduces to a first-price auction as the number of units is reduced to 1. The definition used here is the definition used in all practical applications including Treasury auctions, index-linked bonds sales, IPO auctions, corporate bond auctions, and repo auctions. The results derived in any analysis of such auctions do depend on this definition in an important way.

(2) See also Milgrom and Weber (1982) and Engelbrecht-Wiggans, Milgrom, and Weber (1983).

earn a positive payoff. The objective of this paper is to ask whether this arises only in special cases or generally.

To see how equilibria with free riding can be constructed, consider the following example. Suppose 10 units are being auctioned. There are three bidders. Each unit is valued at V by all bidders, which is a random variable with a uniform distribution on $[0, 1]$. Bidder 0 knows the realization v of V, and the other two bidders (1 and 2) know only the distribution of V. Suppose the reservation price is 0.

Suppose uninformed bidder 1 demands 9 units at price 1 and demands the last unit at price 0 (i.e. 1 submits a flat demand function at a price 1 on 9 units). Bidders 0 and 2 then compete for a single unit, and whoever posts a higher price over this unit wins the unit and pays their own bid since this bid is the market clearing price. Solving for the best response of bidders 0 and 2 is exactly like solving a single unit first price common value auction with one informed and one uninformed bidder. This latter auction has been analyzed by Engelbrecht-Wiggans, Milgrom, and Weber (1983). Using their results, the best responses of the informed bidder as well as bidder 2 can be calculated. Further, it turns out that given these strategies, the original demand function submitted by bidder 1 is a best response. Thus in equilibrium, bidder 1 wins 9 units at the market clearing price, which is the highest of the prices bid by bidders 0 and 2 on the first unit. I construct an example along these lines later in the paper and, for the sake of completeness, provide a detailed proof.

It is then important to ask whether such free riding occurs only in special cases (and therefore should be viewed as pathological), or whether it is the usual outcome. If there are natural conditions that guarantee a zero profit for any uninformed bidder in the presence of an informed bidder, we should conclude that free riding arises only in special cases, and is therefore not a very important phenomenon. To address the issue, I derive the necessary and sufficient condition for uninformed bidders to earn a zero payoff in any equilibrium. Clearly, any equilibrium that does not satisfy the condition is characterized by a positive payoff for some uninformed bidders. Roughly, the condition requires that there should be enough demand by uninformed bidders at least at low prices so that no single uninformed bidder can be "pivotal" in deciding whether total uninformed demand equals or exceeds supply, and places a lower bound on the highest price (i.e. the highest price at which at least one unit is demanded) submitted by the informed bidder.

The rough intuition for the result is as follows. If some uninformed bidder is pivotal in the above sense, he can reduce demand and force the market clearing price to coincide with some bid from the informed bidder. This helps in free riding. Further, if the highest price at which the informed bidder demands the first unit is low, this reduces the implicit "winner's curse" faced by uninformed bidders whenever they win units by outbidding the informed bidder. This, in turn, makes it easier for uninformed bidders to free ride.

While the question of existence of an equilibrium under the condition is moot, non-existence would suggest that all equilibria are characterized by positive profits for the uninformed. The issue is discussed further in section 7, which discusses applications to Treasury auctions as well as repo auctions.

Ausubel and Cramton (1998) have pointed out a property of uniform-price auctions in the presence of private value elements. They show that if there exists any equilibrium in such an auction, it is characterized by "demand reduction" by bidders (shading of bids to obtain a better price), which results in inefficiency. Here I focus on the complementary case of pure common values so that neither demand reduction, nor inefficiency is an issue[(3)].

The methods developed here to address the above issue rely on using properties of the marginal price distributions of bidders induced through their demand function bids. The technique employed here is new, and might help in characterizing equilibria in more complex multi-unit auctions.

The rest of the paper is organized as follows. The next section presents the model. Section 3 presents the main result. Sections 4 and 6 prove sufficiency and necessity, respectively. Section 5 provides an example of equilibrium with

[(3)] See Klemperer (1999) for a discussion of the recent literature on multi-unit auctions. The concerns here are somewhat tangential to this literature, so I do not discuss it in any detail here.

positive uniformed payoff. Finally, section 7 discusses whether the condition is very strong in some cases, and applies the results to Treasury auctions and repo auctions.

2. The Set-Up

$S > 1$ units are offered for sale. Each unit has a value V. This is a random variable with a continuous distribution function $F(\cdot)$ and density function $f(\cdot)$ over a support $[0, \overline{V}]$. The distribution of V is public information.

There is one informed bidder and 2 uninformed bidders. The bidders are indexed by $\{0, 1, 2\}$. Bidder 0 is informed and the others are uninformed. All bidders are risk neutral.

The informed bidder knows the realization of V prior to the auction. The uninformed bidders have access to only public information[(4)].

Bids and Strategies. A bid is any decreasing function $q(p)$ mapping the set of prices $[0, \overline{V}]$ to the set of quantities $\{0, 1, \ldots, S\}$ Since there are S discrete units, a bid function is a step function:

$$q(p) = \begin{cases} 0 & \text{for } p_1 < p \leq \overline{V}, \\ 1 & \text{for } p_2 < p \leq p_1, \\ \vdots & \vdots \\ S & \text{for } 0 \leq p \leq p_S, \end{cases} \tag{2.1}$$

The following representation of a bid function is very useful. Note that the inverse of a bid function can be derived as follows:

$$p(\tilde{q}) = \begin{cases} \max_p \{p | q(p) \geq \tilde{q}\} & \text{if this exists,} \\ 0 & \text{otherwise.} \end{cases}$$

[(4)]This simplifies the analysis. However, the same analysis would go through so long as the informed bidder has “better” information compared to the public information -i.e. he observes an informative signal which the others do not observe.

The resulting function $p(\cdot)$ is the inverse demand function, and thus a bid can be written as a vector $(p_1, \ldots, p_S)$, such that

$$p(q) = \begin{cases} p_1 & \text{over 1 unit,} \\ p_2 & \text{over 2 units,} \\ \vdots & \vdots \\ p_S & \text{over } S \text{ units,} \end{cases}$$

where

$$\overline{V} \geq p_1 \geq p_2 \geq \ldots \geq p_S \geq 0. \tag{2.2}$$

Let Ω be the set of vectors $(p_1, \ldots, p_S)$ that satisfy (2.2). Then $\Omega \subset \Re_+^S$ is the set of bid functions. Note that this is convex and compact.

Figure 1 shows the set of bids Ω for $S = 2$.

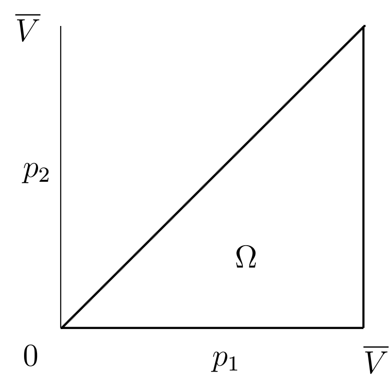

FIGURE 1. The set of bids for $S = 2$ is the triangle given by $0 \leq p_2 \leq p_1 \leq \overline{V}$.

Without loss of generality I restrict attention to pure strategies for the informed bidder. A pure strategy for the informed bidder is a mapping $\sigma_0 : [0, \overline{V}] \to \Omega$. The interpretation is that learning the realization of V, the informed bidder submits a bid $q_0(p)$. A strategy of the informed bidder is written either as $q_0(p)(V)$, or inverted as a price vector $(p_{01}(V), \ldots, p_{0S}(V))$, where $p_{0k}(V)$ is the price bid by bidder 0 on the k-th unit.

A pure strategy for uninformed bidder i, $i \in \{1, 2\}$ is simply a bid $q_i(p)$, which is inverted as a price vector $(p_{i1}, \ldots, p_{iS})$, where p_{ik} is the price bid by i on the

k-th unit. A mixed strategy for uninformed bidder i is given by a probability distribution μ_i over Ω.

Market Clearing Price, Allocation Rule. For any $K \leq S$, the market clearing price $m(K)$ is given by:[5]

$$m(K) = \sup_p \left(p | q_0(p)(\cdot) + \sum_{i=1}^{2} q_i(p) \geq K \right).$$

Finally, the quantity won by a particular bid needs to be specified. Suppose bidder ℓ submits a demand function $q_\ell(p)$ specifying positive prices for k units, $k \leqslant S$. Let $\hat{Q}$ be total demand at prices strictly above $m(S)$. The winning function $q_\ell^w(p)$ is specified below.

$$q_\ell^w(p) = \begin{cases} k & \text{if } p_k > m(S), \\ k' + \alpha_\ell(S - \hat{Q}) & \text{if } p_k = m(S), \\ 0 & \text{otherwise.} \end{cases}$$

where k' is the highest integer below k such that $p_{k'} > p_k$, and the fraction α_ℓ is given by

$$\alpha_\ell = \frac{k}{q_0(m(S))(\cdot) + \sum_{i=1}^{2} q_i(m(S))}$$

Thus if the price bid is above the market clearing price, the bidder wins the associated quantity bid, and if the price bid is equal to the market clearing price, a bidder wins a fraction of the remaining supply of $S - \hat{Q}$ proportional to the ratio of his own demand to total demand at the market clearing price[6].

Each winning bidder pays the market clearing price for each unit won.

[5] See also footnote (1).

[6] If $\alpha_\ell(S - \hat{Q})$ is not an integer, the bidder is allocated the greatest integer less than this. The remaining unit is then allocated randomly according to proportional probabilities.

3. Zero Uninformed Payoff: the Necessary and Sufficient Condition

The following theorem derives the necessary and sufficient condition for uninformed bidders to earn a zero payoff in any equilibrium. It is useful to define the following function.

$$\beta(t) \equiv E(V|V \leq t) \quad \text{for } t \in [0, \overline{V}]. \tag{3.1}$$

For example, $\beta(V) = V/2$ if V has a uniform distribution on $[0, \overline{V}]$. Note that the range of the function is the interval $[0, EV]$. Note also that $\beta(\cdot)$ is strictly increasing.

Let $Q(p)$ denote the aggregate demand by uninformed bidders. Let $Q_{-i}(p)$ denote the total demand by uninformed bidders other than i.

Finally, recall that $p_{01}(V)$ denotes the price at which the informed bidder demands 1 unit, which is at least as high as the price at which any subsequent unit is demanded. The following concept is useful in deriving results.

Definition 1. (Direct competition) *Two bidders are said to "compete directly over a unit" if the higher of the two prices bid on that unit wins that unit and is also the market clearing bid.*

Theorem 1. *The following conditions are jointly necessary and sufficient for uninformed bidders to earn a zero payoff in all equilibria:*

$$E(V - x|x \geq p_{01}(V)) < 0 \quad \textit{for any } x \textit{ that competes directly with } p_{01}(V), \tag{3.2}$$

and there is some interval $[0, \underline{p}]$ *where* $\underline{p} > 0$*, such that for any price* p *in this interval,*

$$Q_{-i}(p) \geq S - 1 \tag{3.3}$$

for any uninformed bidder $i \in \{1, 2\}$.

Condition (3.2) says that if an uninformed bidder directly competes with an informed bidder, the payoff of the former must be strictly negative. This basically places a lower bound on the highest price (i.e. the price at which the first unit is demanded) submitted by the informed bidder. In particular, note that

$$\begin{aligned} E(V - x|\beta(V) \leq x) &= E(V|V \leq \beta^{-1}(x)) - x \\ &= \beta(\beta^{-1}(x)) - x \\ &= 0. \end{aligned} \tag{3.4}$$

Thus condition (3.2) says that $p_{01}(V)$ cannot be lower than $\beta(V)$ for all values of V.

Next, condition (3.3) says that for at least some low prices, the total demand from uninformed bidders is high so that no single bidder can be "pivotal" in deciding whether total uninformed demand equals or exceeds supply. Under the condition, total uninformed demand always equals or exceeds supply as $2(S-1) \geq S$ for any $S \geq 2$. But if for some uninformed bidder (say 1), $Q_{-1} < S - 1$, 1 can bid on one or more units and still ensure total uninformed demand is lower than supply. Condition (3.3) rules out this possibility.

4. Theorem 1: Proof of Sufficiency

The proof proceeds through the following lemmas. I discuss intuition for the results. The formal proofs are in the appendix.

Lemma 1. *Under condition* (3.2)*, an uninformed bidder bids on at most* $S - 1$ *units. Thus for all* $i \in \{1, 2\}$*,* $Prob(p_{iS} > 0) = 0$*.*

The intuition is simple. First, in a single-unit auction, if condition (3.2) holds, any uninformed bid b clearly earns a negative payoff. Thus in a single unit case, direct competition against the informed bid is a dominated strategy.

Now compare two demand functions of uninformed bidder 1 that are identical with respect to the prices at which units $1, \ldots, (S-1)$ are demanded, but differs with respect to the price at which unit S is demanded. Demand function (1) demands S units at a positive price b and demand function (2) demands S units only at price 0.

The payoff from the two demand functions differ only if the first one actually wins all S units. Now, note that the competition over the S-th unit is exactly like the single unit case described above - to win the S-th unit in addition to $(S-1)$ units, b must exceed both $p_{01}(V)$ and p_{21}. Thus to win the S-th unit, b must compete directly with $p_{01}(V)$ and p_{21}. From (3.2), winning the S-th unit by directly competing with $p_{01}(V)$ reduces payoff. Thus demand function (2) yields a greater payoff than demand function (1), and demanding the S-th unit at a positive price is suboptimal.

Next, note that a mixed strategy of uninformed bidder i, $i \in \{1, 2\}$, induces a marginal price distribution for any $k \in \{1, \ldots, S\}$. Let H_{ik} denote the marginal price distribution over k units induced by the mixed strategy of bidder i. This is formally defined in section **A**.1 in the appendix.

For any $i \in \{1, 2\}$, and any $k \leq S$, define $\underline{p}_i(k)$ as follows:

$$\underline{p}_i(k) \equiv \inf_p (p | H_{ik}(p) > H_{ik}(0)).$$

Thus $\underline{p}_i(k)$ denotes the infimum of the set of prices strictly above zero at which demand of uninformed bidder i equals or exceeds k.

Lemma 2. *Under conditions* (3.2) *and* (3.3), $\underline{p}_i(k) \not> 0$ *for any* $i \in \{1, 2\}$, *and any* $k \in \{1, 2, \ldots, S-1\}$.

From lemma 1, we know that uninformed bidders never demand all S units at any positive price - at such prices they demand at most $(S-1)$ units. The result above now says that the marginal bid distribution on all other units cannot have a highest lower bound strictly exceeding 0. The intuition is similar to that of the previous result. To see this, suppose $S = 100$, and $\underline{p}_1(40) > 0$. Now, from condition (3.3), there is an interval of prices $[0, \underline{p}]$ such that $Q_{-1}(p) = q_2(p) \geq 99$ for p in that interval. This implies that 2 must demand at least 99 units at positive prices - and thus it is not possible to have $p_{2,60} = 0$ with probability 1. Thus $\underline{p}_2(60) \geq 0$, and $p_{2,60} > 0$ with positive probability.

Suppose $\underline{p}_2(60) \leq \underline{p}_1(40)$.

Consider a demand function of bidder 2 such that $p_{2,60} = b \in [\underline{p}_2(60), \underline{p}_1(40)]$. With such a bid, of course, 2 can win at most 60 units - as the prices at which he demands any further units are all below $\underline{p}_1(40)$. Now, 2 wins all 60 units only if b exceeds both $p_{01}(V)$ and $p_{1,41}$. But as in the last result, winning by beating $p_{01}(V)$ makes a negative profit given (from condition (3.2)) $p_{01}(V) > \beta(V)$. It is then easy to show that $\underline{p}_2(60)$ must exceed $\underline{p}_1(40)$, contradicting the original supposition. Further, while this applies to the case in which there is no atom at $\underline{p}_1(40)$, the formal proof in the appendix shows that the argument easily extends to the case when there is such an atom.

Therefore the only possibility in equilibrium is that $\underline{p}_1(k) = 0$ for all $k \in \{1, \ldots, (S-1)\}$. Thus a single straightforward intuition - competing directly against $p_{01}(V)$ is a dominated strategy - runs through the two results above.

Proof of sufficiency (completed)

The two lemmas above prove sufficiency as follows. Lemma 1 shows that an uninformed bidder demands at most $(S-1)$ units. Lemma 2 then shows that the marginal price distributions induced by the mixed strategy on units 1 to $(S-1)$ cannot have lower bounds strictly exceeding 0.

Thus, apart from $\underline{p}_i(k) = 0$, the only other possibility is that the uninformed bidders do not place any bids. They get a zero payoff in this case. The best response by the informed bidder is to bid so that the market clearing price is 0 for all values of V. These strategies, obviously, cannot be an equilibrium as any uninformed bidder can deviate and bid any price-quantity pair $(\epsilon, 1)$ ($\epsilon > 0$ and close to zero), and earn a positive payoff.

Thus any equilibrium characterized by the conditions in theorem 1 must satisfy $\underline{p}_i(k) = 0$, for $k \in 1, \ldots, S-1$. Further, $p_{i,S} = 0$. Now, any bid $(p_1, \ldots, p_S)$ where $p_1 = \ldots = p_S = 0$ earns a zero payoff. Thus if there exists any equilibrium that satisfies the condition in theorem 1, all uninformed bidders earn a zero equilibrium payoff.$\|$

Proof of Necessity: Before proceeding to prove necessity in section 6, I construct, in the next section, an equilibrium in which uninformed bidders earn a positive payoff. This is useful in providing intuition for parts of the proof of necessity. This also proves existence of equilibria with positive uninformed payoff.

5. Positive Uninformed Payoff

As discussed in the introduction, it is straightforward to construct equilibria with a positive payoff for uninformed bidders[(7)]. Below, I construct an equilibrium along the lines of the example in the introduction in which the informed bidder wins at most 1 unit, and the rest is won by uninformed bidders. The total payoff of uninformed bidders is strictly positive.

Recall from (3.1) that $\beta(t) \equiv E(V|V \leq t)$ for $t \in [0, \overline{V}]$.

Theorem 2. (**Example of Equilibrium**) *The following profile of strategies of bidders is an equilibrium.*

1. *The informed bidder's strategy is a flat demand of S units at a price $\beta(v)$ for all $v \in [0, \overline{V}]$.*
2. *Uninformed bidder 2 uses the following pure strategy: demand $S-1$ for prices $p \leq \overline{V}$, and demand S at price 0.*
3. *Uninformed bidder 1 uses the following mixed strategy: a flat demand of S units at a random price $\tilde{p}$, where the distribution of $\tilde{p}$ is given by $Prob(\tilde{p} \leq p) = Prob(\beta(V) \leq p)$, for any $p \in [0, EV]$.*

It is easy to see that the above strategies form an equilibrium. The strategy of uninformed bidder 2 is to submit "high" bids over all but 1 unit, in the sense that p^*_{S-1} exceeds even the highest market clearing price. Such a demand function always wins $S-1$ units, while the market clears at the maximum of the bids of the other two bidders. From the sufficiency proof of Engelbrecht-Wiggans, Milgrom, and Weber (1983), it follows directly that the stated price-bid strategies of 0 and 1 are indeed part of a Nash equilibrium on the one unit that they compete on - and thus a best response in the current game. Further, given the strategies of bidders 0 and 1, there is no gain for 2 from changing his strategy. He is already winning $S-1$ units, and if he wants to win even the remaining unit, he has to compete

[(7)] This example is from Daripa (1997). A result on the advantage of uninformed bidders in uniform-price auctions was independently derived by Hernando-Veciana in his PhD dissertation, University College London, 2000 (subsequently Hernando-Veciana (2004)). However, the model restricts bidders to single-unit demands (rather than demand functions) and defines market clearing price as the highest losing bid (see footnote (1)) - making for a very different analysis.

directly with the informed bidder on this unit. As noted at the beginning of the last section, this is never beneficial for the uninformed bidder - and only serves to raise the price paid on all infra-marginal units. Thus all bidders are playing a best response to the specified strategy profile.

For the sake of completeness, I provide the detailed proof in the appendix. The next result calculates the ex-ante expected payoffs of bidders in the above equilibrium..

Corollary 1. *The ex-ante expected payoff of the informed bidder and the expected payoffs of the uninformed bidders in the above equilibrium are given by*

$$\pi_0^* = E(F(V)(V - \beta(V))),\ \pi_1^* = 0,\ \textit{and}\ \pi_2^* = (S-1)\,\pi_0^*.$$

Clearly, one of the uninformed bidders earn a strictly positive payoff.

As an aside, note that it is not possible for the informed bidder to demand many units at very high prices and win most of the units. The intuition is straightforward - in any equilibrium, the market sometimes clears at a price submitted by an uninformed bidder. Suppose there is an equilibrium in which the informed bidder bids very high prices over some units. Now suppose he receives information that the true value is close to zero. In this case, he would have a profitable deviation to a low bid - otherwise he would win the units he demands at a high price and could pay a price posted by an uninformed bidder that is much higher than the true value. Therefore the informed bidder can never appropriate information rent by demanding units at high prices in the same way as uninformed bidders.

I now proceed to the proof of necessity of the condition in theorem 1.

6. Theorem 1: Proof of Necessity

Step 1: Necessity of (3.2) Suppose condition (3.2) does not hold. Then suppose uninformed bidder 1 deviates and submits an additional bid which is a price quantity pair $(b, 1)$ (demand 1 unit at price b), where b is in the support of the distribution of the market clearing price $m(S)$. The original payoff of uninformed bidder 1 in the candidate equilibrium is zero. Therefore, the deviation payoff (denoted by π_1) is simply the payoff from this new bid. This is given by the following. The steps are explained below.

$$\begin{aligned}\pi_1 &= \mathit{Prob}(m(S-1) \geq b \geq m(S))E(V-b|m(S-1) \geq b \geq m(S)) \\ &\quad + \mathit{Prob}(b > m(S-1))E(V-m(S-1)|b > m(S-1)) \\ &\geq \mathit{Prob}(b \geq m(S))E(V-b|b \geq m(S)) \qquad (6.1) \\ &\geq \mathit{Prob}(b \geq p_{11})\mathit{Prob}(b \geq p_{21})\mathit{Prob}(b \geq p_{01}(V))E(V-b|b \geq p_{01}(V)) \\ &\geq 0. \qquad (6.2)\end{aligned}$$

The inequality in the second line follows from the fact that $E(V - m(S-1)|b > m(S-1)) > E(V - b|b > m(S-1))$ and that fact that $\mathit{Prob}(b > m(S-1)) \geq 0$. The inequality in the third line follows from the fact that the toughest competition faced by the bid b is if it had to beat the highest prices posted by each bidder (including 1 himself) which would be the case, for example, if the supply were just 1 unit. The highest price posted by the informed bidder is the price at which he demands the first unit, given by $p_{01}(V)$. Similarly, p_{i1} denotes the price at which uninformed bidder i demands the first unit, $i \in \{1, 2\}$. Note that whenever b exceeds p_{11} and p_{21}, it directly competes with $p_{01}(V)$. Since we have started by supposing that condition (3.2) does not hold, the final inequality follows.

The deviation must not be successful. This implies we cannot have $E(V - b|b \geq p_{01}(V)) > 0$ (otherwise the inequality in (6.2) would be strict, making the deviation successful). Similarly, if $E(V-b|b \geq p_{01}(V)) = 0$, but $\mathit{Prob}(b > m(S-1)) > 0$, the inequality in (6.1) is strict, and the deviation is again successful. Therefore this case is ruled out as well.

We are then left with $E(V - b|b \geq p_{01}(V)) = 0$ and $\mathit{Prob}(b > m(S-1)) = 0$ for any b in the support of the market clearing price.

Now, since $E(V - b|b \geq p_{01}(V)) = 0$, from (3.4), we can also write (for a more compact form, which makes the algebra easier) $p_{01}(V) = \beta(V)$.

Thus to prove necessity, we need to prove that the uninformed bidders cannot earn a zero payoff in any equilibrium satisfying $p_{01}(V) = \beta(V)$, and $Prob(b > m(S-1)) = 0$ for any b in the support of the market clearing price. The following result proves this, which completes the proof of necessity of condition (3.2).

Lemma 3. *If $p_{01}(V) = \beta(V)$, and $Prob(b > m(S-1)) = 0$ for any b in the support of the market clearing price, all uninformed bidders cannot earn a zero payoff in any equilibrium.*

While the formal proof is in the appendix, the intuition derives from the last section. Since $Prob(b > m(S-1)) = 0$ for any b in the support of the market clearing price, there must be "high" prices (higher than the highest market clearing price) posted over $(S-1)$ infra-marginal units. As argued at the end of the last section, such high price bids cannot come from the informed bidder. In fact, as in the example provided by theorem 2, the high prices must be posted by at least one uninformed bidder, who wins all infra-marginal units. Since he pays the market clearing price, which never exceeds EV and is determined in part by the informed bidder's bids, he earns a strictly positive payoff. The formal proof needs to go through many cases to verify that this intuition is correct, and is relegated to the appendix.

This completes the proof of the necessity of condition (3.2).

Step 2: Necessity of (3.3) Next, suppose condition (3.3) is not satisfied in equilibrium. Then suppose $Q_{-1}(p) = q_2(p) \leq (S-k)$ for some $k \geq 2$ at all prices $p \in [0, \overline{V}]$.

First, suppose $q_1(p) < k$ at all prices. Then total uninformed demand is $Q(p) \leq (S-1)$ at all prices.

Let $(P_1, \ldots, P_K, P_{(K+1)}, \ldots, P_S)$ denote the aggregate demand of uninformed bidders where $Prob(P_K > 0) > 0$, $K \leq (S-1)$, and $P_{(K+1)} = \ldots = P_S = 0$ with probability 1.

Next, note that if market clears with positive probability at any $p_{0k}(V)$, it must be that $p_{0k}(V) < V$ in equilibrium. To see this, suppose not. Then for some k, and some realization v of V, $p_{0k}(V) = b \geq v$. Consider a deviation to $b' < v$. If market clears above b or below b', payoff of the informed bidder does not change. If market clears at some price $p \in [b', b]$, the deviation might reduce the number of units won, but lowers the price paid per unit won. Since at the original market clearing price b the payoff is non-positive, the deviation increases payoff. Contradiction.

Now, either the informed bidder's bid is such that $p_{0S}(V) > P_1$ and the uninformed bidders do not win any units, or the market clears at $\min(P_k, p_{0(S-k)}(V))$ for some $k \leq K$. Here $p_{0(S-k)}(V)$ denotes the price bid by the informed bidder on the $(S-k)$-th unit. Given that $Prob(P_k > 0) > 0$ for $k \leq K$, the latter case must happen with strictly positive probability. But from the argument above, $\min(P_k, p_{0(S-k)}(V)) \leq p_{0(S-k)}(V) < V$. Thus in equilibrium it cannot be that all uninformed bidders earn a zero payoff.

Next, suppose that $q_1(p) \geq k$ at all prices, and an equilibrium exists in which both uninformed bidders earn a zero payoff. Consider the following deviation by 1: he now bids prices above 0 only on at most $k-1$ units. Then total uninformed demand is $Q(p) \leq (S-1)$ at all prices - and by the argument for the preceding case, the deviation payoff must be strictly positive. Contradiction.

7. Discussion

In multi-unit common value uniform price auctions with demand function bids it is easy to construct equilibria in which uninformed bidders earn a positive payoff at the expense of the informed bidder. The paper asks under whether such equilibria should be seen as pathological special cases or are pervasively present. In single unit first price auctions, there are general results showing the impossibility of positive payoff for less informed bidders. These extend easily to multi-unit common value auctions with demand function bids. Here the focus is on uniform price auctions, and the question is whether there are natural conditions under which the result on zero rent for the uninformed bidders extends to such auctions. To answer this, the paper derives the necessary and sufficient condition for all equilibria to be characterized by zero payoff for uninformed bidders. The condition requires enough uninformed demand at least at low prices so that no single bidder can be "pivotal" in deciding whether total uninformed demand equals or exceeds supply, and places a lower bound on the highest price (i.e. the price at which the first unit is demanded) submitted by the informed bidder. The intuition, roughly, is as follows. If an uninformed bidder can reduce demand and force the market clearing price to coincide with some bid from the informed bidder, this helps him free ride on the informed bidder's information. Further, if the highest price at which the informed bidder demands the first unit is low, this reduces the implicit "winner's curse" faced by uninformed bidders whenever they win units by outbidding the informed bidder. This, in turn, makes it easier for uninformed bidders to free ride.

If there are a few uninformed bidders who demand small quantities, this condition is unlikely to be fulfilled. This is also true if uninformed bidders could collude. In such cases uninformed bidders would earn a positive payoff in all equilibria, making free riding a pervasive phenomenon. Examples show that such equilibria indeed exist.

US Treasury auctions provide an example of small uninformed bidders competing with a few large more-or-less equally well informed bidders. This scenario fits our model. There are primary dealers who participate regularly. They receive orders from customers (hence get a signal of aggregate demand) and bid on their behalf. There are also uninformed institutional bidders bidding on their own account occasionally - mostly for exogenous liquidity-demand-driven reasons.

In 1998, the US Treasury switched from using discriminatory auctions for most categories of government securities to uniform price auctions for selling all securities. This is puzzling to the extent that there is no theory establishing the superiority - in terms of either efficiency or revenue - of uniform-price auctions over discriminatory auctions (in fact, as mentioned in the introduction, Ausubel and Cramton (1998) show that in settings with private value elements, the former auction format is inefficient). Further, empirical studies have failed to find any significant difference in revenue generated by the two formats.

The results here suggest that since the uninformed bidders demand only a fraction of the supply in any auction, they must earn a strictly positive payoff in a uniform price auction. In a discriminatory auction, an uninformed bidder gets no such advantage. Thus uniform price auctions are likely to attract smaller, less informed bidders, perhaps reducing the chance of successful collusion by the informed bidders as well the chance of a single bidder being able to corner the entire market. This sheds some light on the rationale behind the switch to the uniform price format in US Treasury auctions[(8)]. Indeed, Friedman (1960) had suggested (informally) that a uniform-price auction makes it difficult to sustain collusion, as it encourages participation by relatively uninformed bidders[(9)]. The theory presented here provides a justification for the suggestion.

An opposite picture emerges in the case of repo auctions. Repo auctions constitute the principal instrument of money-market management by the Eurosystem. The European Central Bank (ECB) uses these auctions as the primary instrument for conducting monetary policy. In the longer-term repo auctions, private information about the future term structure of interest rates plays an important role. In March 1999, the ECB switched the auction format for longer-term repos from uniform-price to discriminatory. In a press conference, the governor said:[(10)] "The single rate method was chosen in order to encourage less experienced counterparties to participate in the tender. The Governing Council takes the view that all interested counterparties should by now be sufficiently accustomed to the longer-term refinancing operation also to be in a position to participate in this type of operation under the more market-oriented multiple rate (American)[(11)]

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(8) On the other hand, in countries like the UK and Germany, only the members of specific groups of investment banks ("Gilt-Edged Market Makers" in the UK, members of the "Bund Issues Auction Group" in Germany) are allowed to bid in Treasury auctions. In these cases, the question of attracting small bidders does not arise, and these countries have always used discriminatory auctions.

(9) Friedman (1991), an op-ed piece in the Wall Street Journal, reiterates this argument.

(10) Source: The President's introductory statement, Press conference, March 4, 1999, www.ecb.int/key/st990304.htm.

(11) Discriminatory auctions are often called American auctions in the banking literature. It is ironic, since the eponymous Treasury department has abandoned this format altogether in favor of uniform price auctions.

method of allotment." The concern clearly is that uniform price auctions give an undue advantage to less well informed bidders. In such auctions, it is very likely that the no-pivotal-uninformed-bidder condition does not hold. If the condition does not hold, our results clarify the impact of a uniform price format on the division of surplus between informed and uninformed bidders. From the governor's statement it seems the objective is to encourage all bidders to be informed participants, and given the nature of equilibria in uniform price auctions in this case, discriminatory auctions are clearly better suited for that purpose.

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8. Appendix

A.1. Marginal Price Distribution: A Formal Definition. A mixed strategy μ_i of uninformed bidder i, $i \in \{1, 2\}$, is a probability distribution on Ω. μ_i induces a marginal price distribution for any $k \in \{1, \ldots, S\}$. Let H_{ik} denote the marginal price distribution over unit k induced by the mixed strategy of bidder i. This is defined as follows:

$$H_{ik}(\tilde{p}) = \int_0^{\tilde{p}} H_{ik}(p_k)dp_k,$$

where $H_{ik}(p_k)$ is given by

$$H_{ik}(p_k) = \int_0^{p_k} \ldots \int_0^{p_{(S-1)}} \int_{p_k}^{\overline{V}} \ldots \int_{p_2}^{\overline{V}} \mu_i dp_1 \ldots dp_{(k-1)} dp_S \ldots dp_{(k+1)},$$

where the first integral refers to the last integrand and so on. Note that we are integrating over p_j for all $j \neq k$. Note also that for $\ell < k$, $p_\ell \in [p_{\ell+1}, \overline{V}]$, and for $\ell > k$, $p_\ell \in [0, p_{\ell-1}]$.

A.2. Proof of Lemma 1. Without loss of generality, I prove the result with reference to uninformed bidder 1.

Suppose, on the contrary, that $p_{1S} > 0$ with strictly positive probability. Consider the following demand functions of bidder 1:

$$q^{(1)}(p) \equiv \left(p_{11}^{(1)}, p_{12}^{(1)}, \ldots, p_{1S}^{(1)}\right), \tag{A.1}$$

$$q^{(2)}(p) \equiv \left(p_{11}^{(2)}, p_{12}^{(2)}, \ldots, p_{1S}^{(2)}\right), \tag{A.2}$$

where $p_{1k}^{(1)} = p_{1k}^{(2)}$ for $k \in \{1, \ldots, (S-1)\}$, but $p_{1S}^{(1)} = b > 0 = p_{1S}^{(2)}$.

The payoffs from the two demand functions above differ only when the market clears at b (i.e. 1 wins all S units). In this case, both $p_{01}(V) \leq b$ and $p_{21} \leq b$. Let $\lambda > 0$ be the probability that b wins.

Thus $\lambda = Prob(p_{01}(V) \leq b) Prob(p_{21} \leq b)$. Let D_1 denote the amount by which the payoff of bidder 1 from $q^{(1)}(p)$ exceeds that from $q^{(2)}(p)$.

$$\begin{aligned} D_1 &= \lambda \left[SE(V - b | p_{01}(V) \leq b) - (S-1)E(V - \max\left(p_{01}(V), p_{21}\right) | p_{01}(V) \leq b)\right] \\ &< \lambda E(V - b | p_{01}(V) \leq b) \\ &< 0, \end{aligned} \tag{A.3}$$

where the last step follows from condition (3.2). Thus the payoff of bidder 1 from $q^{(1)}(p)$ is strictly lower than that from $q^{(2)}(p)$. Therefore $p_{1S} = 0$ is strictly better than $p_{1S} > 0$. Contradiction. This proves that $Prob(p_{1S} > 0) = 0.\|$

A.3. **Proof of Lemma 2.** Without loss of generality, I prove the lemma for $i = 1$.

Suppose $\underline{p}_1(k) > 0$ for some $k \in \{1, \ldots, S-1\}$.

Recall that H_{ik} (defined in section **A**.1) denotes the marginal distribution on the k-th units induced by the mixed strategy of bidder i, $i \in \{1, 2\}$. Similarly, let $H_{i(S-k)}$ denote the marginal distribution on $(S-k)$-th unit.

Recall also that p_{ik} denotes the price bid by uninformed bidder i, $i \in \{1, 2\}$, on the k-th unit. Similarly, let $p_{i(S-k)}$ denote the price bid on the $(S-k)$-th unit.

Case 1: H_{1k} has no atom at $\underline{p}_1(k)$.

From condition (3.3), there is an interval of prices $[0, \underline{p}]$ such that $Q_{-1}(p) = q_2(p) \geq (S-1)$ for p in that interval. This implies that 2 must bid prices above 0 on at least $(S-1)$ units - and thus it is not possible to have $p_{2(S-k)} = 0$ with probability 1. Thus it must be that $\underline{p}_2(S-k) \geq 0$.

Suppose $\underline{p}_2(S-k) \leq \underline{p}_1(k)$.

Any $p_{2(S-k)} \in [\underline{p}_2(S-k), \underline{p}_1(k)]$ wins only if both $p_{1(k+1)} \leq p_{2(S-k)}$ and $p_{01}(V) \leq p_{2(S-k)}$[12]. Compare the following demand functions of bidder 2:

$$q_2^{(1)}(p) \equiv \left(p_{21}^{(1)}, p_{22}^{(1)}, \ldots, p_{2(S-1)}^{(1)}, p_{2S}^{(1)}\right), \quad \text{(A.4)}$$

$$q_2^{(2)}(p) \equiv \left(p_{21}^{(2)}, p_{22}^{(2)}, \ldots, p_{2(S-1)}^{(2)}, p_{2S}^{(2)}\right), \quad \text{(A.5)}$$

where $p_{2k}^{(1)} = p_{2k}^{(2)}$ for $k \in \{1, \ldots, (S-k-1)\}$, but $p_{2(S-k)}^{(1)} = b > 0 = p_{2(S-k)}^{(2)}$. Finally, $b \leq \underline{p}_1(k)$.

(12) Note that if H_{1k} has an atom at $\underline{p}_1(k)$, $p_{2(S-k)} = \underline{p}_{1k}$ could win even if these two conditions are not satisfied.

Let D_2 denote the amount by which the payoff of bidder 2 from $q_2^{(1)}(p)$ exceeds that from $q_2^{(2)}(p)$.

$$\begin{aligned} D_2 &= Prob(p_{01}(V) \leq b) Prob(p_{1(k+1)} \leq b) \Big[(S-k) E(V - b | p_{01}(V) \leq b) \\ &\quad - (S-k-1) E\left(V - \max\left(p_{01}(V), p_{1(k+1)}\right) | p_{01}(V) \leq b\right) \Big] \\ &< Prob(p_{01}(V) \leq b) Prob(p_{1(k+1)} \leq b) E(V - b | p_{01}(V) \leq b) \\ &< 0, \end{aligned} \tag{A.6}$$

where the last step follows from condition (3.2). Thus $q_2^{(2)}(\cdot)$ yields a strictly greater payoff. Since this is true for any $b \leq \underline{p}_1(k)$, $p_{2(S-k)} = \underline{p}_1(k)$ is strictly worse than $p_{2(S-k)} = 0$. Thus there exists $\xi > 0$ such that in equilibrium

$$p_{2(S-k)} \geq \underline{p}_1(k) + \xi. \tag{A.7}$$

This implies that $\underline{p}_2(S-k) > \underline{p}_1(k)$, which contradicts the original supposition.

Now suppose $\underline{p}_2(S-k) > \underline{p}_1(k)$. In the above analysis, switch the labels "$1k$" and "$2(S-k)$." This then leads to the conclusion that $\underline{p}_2(S-k) < \underline{p}_1(k)$. Contradiction.

Thus there is no equilibrium with $\underline{p}_i(k) > 0$ for any $i \in \{1, 2\}$ and any $k \in \{1, \ldots, (S-1)\}$.

Case 2: H_{1k} has an atom at $\underline{p}_1(k)$.

If $\underline{p}_2(S-k) < \underline{p}_1(k)$, the same argument as in case 1 applies. The only remaining possibility is that $\underline{p}_2(S-k) = \underline{p}_1(k)$.

Suppose $p_{2(S-k)} = b = \underline{p}_1(k)$ is preferred to $p_{2(S-k)} = 0$ (otherwise the proof is already done). This can happen only because b wins with strictly positive probability even when $p_{01}(V) > b$ (if b wins only when $p_{01}(V) \leq b$, then, as shown in case 1, it would be better to deviate to $p_{2(S-k)} = 0$).

Now, b can win with strictly positive probability even when $p_{01}(V) > b$ because $p_{1k} = \underline{p}_1(k)$ with strictly positive probability mass. Whenever b wins and also $p_{01}(V) > b$, the total demand by 1 and 2 at b is S units. Since the informed bidder demands at least 1 unit at some price above b, the total units available at price b is at most $(S-1)$.

It follows that whenever b wins and also $p_{01}(V) > b$, bidder 2 does not win all $(S-k)$ units - but wins at most $(S-k)-\eta$ (where $0 < \eta < 1$).

But since $p_{2(S-k)} = b$ is preferred to $p_{2(S-k)} = 0$, it must be that by raising $p_{2(S-k)}$ by a very small amount 2 can improve his payoff as price paid increases continuously while quantity won changes discontinuously from $(S-k)-\eta$ to $(S-k)$.

Thus if in equilibrium $\underline{p}_2(S-k) = \underline{p}_1(k)$, $H_{2(S-k)}$ has no atom at $\underline{p}_2(S-k)$. But then, by the argument in case 1, the bid $\underline{p}_1(k)$ of 1 (this is made with strictly positive probability) makes a strictly negative profit. Thus 1 cannot have an atom at $\underline{p}_1(k)$. Contradiction.

This completes the proof.||

A.4. **Proof of Theorem 2.** Given bidder 2's strategy, bidders 0 and 1 face a single-unit first-price auction. Thus for any demand function submitted by 0 or 1, only the price at which the first unit is demanded matters.

Now, given the mixed strategy of bidder 1, suppose $V = v$, and bidder 0 submits $p_{01}(V) = x$. Since bidder 2's bid does not exceed EV, $x > EV$ cannot be a best response. Thus $x \in [0, EV]$. Thus there is some $t \in [0, \overline{V}]$ such that $\beta(t) = x$. The payoff of bidder 0 from this is $\pi_0(t) = Prob(\tilde{p} < \beta(t))(v - \beta(t))$. Given the equilibrium strategy of bidder 1, $Prob(\tilde{p} < \beta(t)) = Prob(\beta(V) < \beta(t)) = Prob(V \leq t) = F(t)$. Thus

$$\pi_0(t) \;=\; F(t)(v-\beta(t)) \;=\; F(t)(v - E(V|V \leq t)) \;=\; F(t)v - \int_0^t Vf(V)dV.$$

Maximizing with respect to t, the first order condition is given by $f(t)v - tf(t) = 0$ which implies $t = v$.

Bidder 1's payoff from any bid with $p_1 > EV$ is negative. The payoff from any bid with $p_1 \in [0, EV]$ is

$$\pi_1(p_1) = Prob(\beta(V) \leq p_1)E(V - p_1|\beta(V) \leq p_1) \;=\; 0$$

where the second step follows from (3.4). Thus given 2's strategy, the strategies of bidders 0 and 1 are best responses to the strategies of other players.

It remains to be shown that given the strategies of 0 and 1, there is no profitable deviation for 2. If 2 deviates and lowers the prices currently bid on some units - he would either not win (if the bid is below the maximum of the bids of the other two bidders) or win at the same price as before (if the bid is above the maximum of the bids of the other two bidders).

There is clearly no scope for 2 to increase the prices currently bid on any of the $(S-1)$ units (even if scope existed, such a move would not affect 2's payoff). Next, suppose 2 decides to bid a positive price on the S-th unit. If $p_{2S} > EV$, the market clears at p_{2S} and the payoff is strictly negative. If $p_{2S} \leq EV$, 2 wins the S-th unit with positive probability (say $\lambda > 0$) and the extra payoff earned on the S-th unit is $\lambda E(V - p_{2S} | p_{2S} \geq \beta(V)) = \lambda(\beta(\beta^{-1}(p_{2S})) - p_{2S}) = 0$. But at the same time, this raises the distribution of the market clearing price, because whenever the market previously cleared *below* p_{2S}, it now clears *at* p_{2S}. Thus no component of 2's payoff increases by this deviation, but the expected price paid on all infra-marginal units strictly increases. Thus the deviation is not profitable.$\|$

A.5. Proof of Corollary 1. It has already been shown that $\pi_1^* = 0$. Next,

$$\begin{aligned} \pi_0^* &= \int_0^{\overline{V}} Prob(p_1 < \beta(v))(v - \beta(v)) f(v) dv \\ &= \int_0^{\overline{V}} F(v)(v - \beta(v)) f(v) dv \end{aligned}$$

Finally, for any given $p_1 \in [0, EV]$,

$$\begin{aligned} \frac{\pi_2(p_1)}{S-1} &= Prob(V \leq \beta^{-1}(p_1)) E(V - p_1 | V \leq \beta^{-1}(p_1)) \\ &\quad + Prob(V > \beta^{-1}(p_1)) E(V - \beta(V) | V > \beta^{-1}(p_1)) \end{aligned}$$

$E(V - p_1 | V \leq \beta^{-1}(p_1)) = \beta(\beta^{-1}(p_1)) - p_1 = 0$. Thus the first term on the right hand side is 0. As for the second term, note that for any given $p_1 \in [0, EV]$, $\pi_0(p_1) = Prob(V > \beta^{-1}(p_1)) E(V - \beta(V) | V > \beta^{-1}(p_1))$, and thus π_0^* can also be written as $\int_0^{\overline{V}} \pi_0(p_1) dG(p_1)$, where $G(\cdot)$ is the distribution of the price bid by uninformed bidder 1 given in theorem 2. From this, we have $\pi_2(p_1)/(S-1) = \pi_0(p_1)$. Now, $\pi_2^* = \int_0^{\bar{V}} \pi_2(p_1)dG(p_1)$. It follows directly that $\pi_2^* = (S-1)\pi_0^*$.‖

A.6. Proof of Lemma 3.

Step 1. Let $\overline{m}(S)$ denote the lowest upper bound of the distribution of market clearing prices. If $Prob(b > m(S-1)) = 0$ for any b in the support of the market clearing price, this implies that the price at which $S-1$ units are demanded is always higher than the market clearing bid, i.e.

$$Prob(m(S-1) \geq \overline{m}(S)) = 1. \qquad (\star)$$

Step 2. Since $p_{01}(V) = \beta(V)$ (and therefore $p_{0k}(V) \leq \beta(V)$ for all $k \in \{1, \ldots, S\}$), it must be that for any b in the support of the market clearing bid, there is a value of V low enough so that $\beta(V) < b$, i.e. the informed bidder demands units only at prices below b. Thus to satisfy condition $(\star)$, it must be that one or more uninformed bidders demand $(S-1)$ units at some price p_H above $\overline{m}(S)$.

Specifically, let $K_i \geq 0$, $i \in \{1, 2\}$, denote the demand of uninformed bidder i at price p_H, where (K_1, K_2) is such that $K_1 + K_2 = S - 1$.

Step 3. Now, if $\overline{m}(S) < EV$, from $(\star)$, it must be that $m(S) < EV$ with probability 1, which implies that both uninformed bidders cannot earn a zero payoff, and the proof is done.

Step 4. Next, suppose $\overline{m}(S) \geq EV$.

Step 4.1. Note that the support of $\beta(V)$ is $[0, EV]$. Thus $\overline{m}(S) > EV$ can happen only if at least 1 uninformed bidder i demands a further unit (further than K_i) at a price greater than EV. But since in this case the market clears at a price greater than EV, and uninformed bidders win the entire supply at this price, at least one uninformed bidder earns a strictly negative payoff. Any such bidder can profitably deviate to demanding the $(K_i + 1)$-th unit at a lower price or to a lower quantity demanded at positive prices (even demanding nothing is a profitable deviation).

This rules out $\overline{m}(S) > EV$.

Step 4.2. The only case left is $\overline{m}(S) = EV$. Suppose $m(S) = EV$ with probability 1. Then the informed bidder does not win any units with probability 1, and gets a zero payoff. But recall that at prices above EV, the demand is no more than $(S-1)$ units. Then for any value v of V such that $v > EV$, the informed bidder can deviate and demand a unit at some price $EV < p < v$, win that unit and pay at most p, thus earning a positive payoff.

Thus it must be that $m(S) < EV$ with strictly positive probability.

Step 4.3. Now, suppose uninformed bidder i bids a price $p_i \geq 0$ on the (K_i+1)-th unit. The above arguments establish that $p_i \in [0, EV]$. p_i directly competes with p_j, $i \neq j$, and $p_{01}(V)$ for the S-th unit. For any price p_i at which the (K_i+1)-th unit is demanded by uninformed bidder i, the payoff on this unit is $\pi_i(p_i) = Prob(p_i > p_{01}(V))E(V - p_i | p_i > p_{01}(V)) = 0$.

Now, the market clears at $\max(p_1, p_2, p_{01}(V))$, and this is the price which uninformed bidder i pays on K_i units. Further, using the argument in step (4.2) above, the market clears at $p_{01}(V)$ with strictly positive probability. Thus the payoff on K_i units is strictly greater than K_i times $\pi_i(p_i)$. Since $\pi_i(p_i) = 0$, the payoff of uninformed bidder i on K_i units is strictly positive.

Thus at least 1 uninformed bidder earns a strictly positive payoff on $K_i \leq (S-1)$ units[13]. This completes the proof.‖

[13] In fact, it is easy to show (but not needed here, so I do not show this) that if $Prob(b > m(S-1)) = 0$ for any b in the support of the market clearing price, the only possibility in equilibrium is for one uninformed bidder to bid "high" on $(S-1)$ units and the other uninformed bidder and the informed bidder is to compete directly over the S-th unit - exactly as in the example in theorem 2 in section 5. The uninformed bidder bidding high wins $(S-1)$ units and earns a strictly positive payoff given by corollary 1.