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Abstract

This paper analyzes equilibria in sequential take-it-or-leave-it sales and sequential auctions when demand is stochastic. It is shown that equilibria in the former mechanism trade-off allocative efficiency and competing buyers' opportunities to acquire an item to be sold, permitting prices and expected revenue above those of one-shot offers. Hence Coase-type conjectures are invalid in this setting. Moreover, sequential take-it-or-leave-it sellers can achieve expected revenue in early periods in excess of those in sequential auctions, and, if sufficiently patient, higher expected revenue in the entire game than slightly less patient auction sellers. This provides one explanation why some goods are typically sold in take-it-or-leave-it deals, while others are sold in auctions. An asymptotic revenue equivalence result is shown to reconcile the two mechanisms as the time horizon of the dynamic game gets large.

KEYWORDS: Dynamic Monopoly, Stochastic Demand, Optimal Auctions, Coase Conjecture, Revenue Equivalence.

1 Introduction

This paper builds on three basic observations: (i) Many economic decisions on markets, such as sellers' pricing and buyers' purchase decisions, are dynamic; (ii) economic agents' preferences and valuations are private information, and hence market demand is stochastic; (iii) in many markets, the number of potential buyers is small. The implications of these conditions are analyzed for a situation in which a monopolistic seller of one item faces a group of n potential buyers with unit demands, whose valuations for the item are private information. There are two periods. If the item does not sell in period 1, it is common knowledge that it may be offered again in the final period 2. Several questions are examined. Does the dynamic nature of the transaction game and the intrinsic market uncertainty favor one side of the market? Under what circumstances? What will be the seller's preferred selling mechanism in this setting? What happens in the limit, as the time horizon becomes infinite?

Examples of markets that give rise to observations (i)-(iii) are markets for fine art, exclusive real estate, vintage cars, clothes and jewellery, rare wines, yearling horses, literary rights, and many others. Some of these goods are typically sold in auctions, while others are sold via private posted-price, take-it-or-leave-it sales, and some through both mechanisms, with a possibility of re-offering the item, should it not be sold in the first offering. The most prominent online trading platform, eBay, next to its traditional auction sales now offers sellers the option of posted-price sales, under the"Buy It Now" label, and meanwhile earns a quarter of its revenue from it.[1] Traditional auction houses also offer private sales, such as, for instance, Sotheby's "salon privé" for jewellery.

The analysis presented in this paper shows that the dynamic nature of potentially sequential offers induces strategic behavior in small markets which has three main consequences: first, in terms of expected revenue, it may allow the take-it-or-leave-it seller to benefit from a second period, contrary to Coase-type conjectures; second, it may imply expected revenue in early offers of sequential take-it-or-leave-it sales above and beyond those in early auctions of sequential auction sales with optimal reserve prices; and third, it may induce higher total expected revenue for sufficiently patient sequential take-it-or-leave-it sellers compared to slightly less patient sequential auction sellers.

The critical distinction between the two mechanisms is that auctions yield efficient allocations, i.e. the buyer with the highest valuation wins the item, while take-it-or-leave-it deals provide opportunities for buyers other than the highest valuation buyer to acquire the item, possibly leading to inefficient allocations[2]. Buyers are willing to pay for this opportunity, and the seller may be able to exploit this. In equilibrium under the take-it-or-leave-it mechanism, the increased risk of not obtaining the item in the second period, when the equilibrium price is lower, induces

[1]BusinessWeek online, 27 August 2004

[2]The seller, of course, has no a priori interest in allocative efficiency per se.

high valuation buyers to accept a higher price in the first period. Competition among buyers with stochastic demand, thus, alters buyers' strategic incentives compared to Coase-type analyses with deterministic demand. This creates a competitive externality which the seller can exploit in take-it-or-leave-it sales, but not in auctions. Depending on the degree to which second period pay-offs are discounted and how many potential buyers compete, this competitive externality allows the take-it-or-leave-it seller to achieve higher expected revenue in the dynamic (two-period) than the static (one-period) game, and to achieve higher expected revenue in early periods than a sequential auction seller, even with optimal reserve prices.

A limit revenue equivalence result reconciles these mechanisms. It shows that, under mild conditions, the sequential take-it-or-leave-it mechanism generates the same expected revenue as a single, optimal final auction when the number of periods becomes large and there is no discounting. The reason is that the sequential take-it-or-leave-it mechanism asymptotically assigns zero probability to inefficient allocations; i.e. asymptotically, the highest bidder obtains the item, while the lowest valuation bidder expects zero surplus. Therefore, the conditions of the classical revenue equivalence theorem (Vickrey (1961), Riley and Samuelson (1981), Myerson (1981), Milgrom and Roberts (1982); see also Milgrom (1987), and Klemperer (2002) for a comprehensive survey) are met. An intuitive way to think about this result is that, asymptotically, the sequential take-it-or-leave-it mechanism mimics a Dutch (declining-price) auction.

Related Literature

This paper contributes to the literatures on auctions, optimal mechanisms and the Coase conjecture. Its results complement work by Bulow and Klemperer (1996) on static auctions as compared to negotiations. Bulow and Klemperer show the value of an extra bidder in an auction, compared to negotiations and to setting an optimal reserve price. Hence they examine the value of additional competition, due to the additional buyer, in a class of mechanisms yielding an efficient allocation. One can interpret the results of this paper this paper as showing the value of competition due to the design of the mechanism, given the same number of potential buyers, when allocative efficiency is not the primary objective. Wang (1993, 1998) compares static auctions with posted-price selling.

The paper also adds to the literature on optimal auctions in general (Vickery (1961), Myerson (1981), Riley and Samuelson (1981), Harris and Raviv (1981)), and optimal sequential auction in particular (McAfee and Vincent (1997), Laffont and Robert (2002)). The asymptotic revenue equivalence result connects the literature on endogenous market regimes (e.g. Harris and Raviv (1981a)) to the literature on optimal auctions. McAfee and Vincent (1997) prove a result reminiscent of Theorem 3 in this paper, for sequential second-price sealed-bid auctions, where no bid shading occurs, and treat the case of sequential first-price sealed bid case implicitly by an appeal to the revenue equivalence theorem. This paper develops the theory of sequential first-price sealed bid auctions explicitly. A setup similar to the sequential take-it-or-leave-it mechanism is considered in Bulow and Klemperer (1994), but their analysis treats prices as exogenous.

The paper also sheds new light on Coase-type conjectures. In the context of deterministic demand, it has been shown that a durable-goods monopolist, somewhat paradoxically, competes with himself, charging the competitive rather than the monopoly price in equilibrium (Coase, 1972).[3] Equilibria of durable-goods monopolies have been analyzed extensively (Stokey (1981); Bagnoli (1989, 1995); Bulow (1982); Chen and Wang (1999); Gul, Sonnenschein and Wilson (1986); Ausubel and Deneckere (1989,1992); Majerus (1992); Riley and Zeckhauser (1983); Thépot (1998)). This paper contributes to this discussion of the Coase conjecture, showing that take-it-or-leave-it sellers may benefit from a dynamic setting with demand uncertainty: If future pay-offs are not discounted too heavily, first period prices and the expected revenue of the entire game are higher than in a one shot game.

The paper proceeds as follows. Section 2 introduces the main themes and insights in the context of two tractable examples of sequential take-it-or-leave-it sales and first-price sealed bid auctions. Section 3 lays out the key theorems that provide the underlying theoretical foundation of the motivating examples. Section 4 reconciles the two mechanisms by providing an asymptotic revenue equivalence theorem. All proofs are in the appendix.

2 Uniform Examples

Consider a monopolistic seller of a single unit of an item who faces n potential buyers. Buyers are assumed to have independent and identically uniformly distributed valuations of the item, $X_i \sim u[0,1]$, $i = 1, \ldots, n$. Each buyer's valuation is private information. The item has no value to the seller if it is not sold. It is common knowledge that if the item is not sold in period one, it is re-offered for sale in period two, following the same sales mechanism whose parameters can be adapted according to the first period outcome. Period two pay-offs are discounted with common discount factor $\delta \in [0,1]$. Two mechanisms for sale are considered: sequential take-it-or-leave-it sales and sequential second-price sealed bid auctions.[4]

[3] For the purpose of this and related analyses, durability can be broadly defined. Compare the discussion in Bulow (1982), p. 316. Buyers care about the service the asset provides, and the asset can be considered durable if no repeat purchases are necessary for the service provision. This interpretation includes perishable assets like airline seats, hotel rooms, or concert tickets.

[4] It is a well known revenue equivalence result that other auction formats, e.g. sequential second-price sealed bid auctions, yield the same expected revenue as the auction considered here, provided buyers are risk-neutral, the item is sold to the highest valuation buyer and unsuccessful bidders have zero gain; see, for example Myerson (1981) and Riley and Samuelson (1981)

2.1 Sequential Take-it-or-leave-it Sales

In this mechanism, the seller announces a period one price, $p_1 \in [0,1]$, and buyers decide whether they wish to submit a purchase order at this price. If no orders are submitted, the game proceeds to period two. In this second and final period, the seller announces a second period price, $p_2 \in [0,1]$, and buyers subsequently decide again whether to submit an order. If at any point in the game more than one purchase order is submitted, it is assumed that buyers that have submitted an order acquire the item with equal probability.[5]

Since buyers' valuations in this sequential game are private information, the game is a dynamic game with incomplete information. The equilibrium in this game, therefore, is a Perfect Bayesian Equilibrium (PBE).[6] In the two period case, a PBE is defined as first and second period prices and buyers' order submission strategies such that, given buyers' strategies, the prices maximize the seller's expected total revenue, and, given these prices, buyer's strategies maximize their expected surplus. The PBE can be found using a standard backward induction algorithm.

It is common knowledge in period one that the item will be re-offered in period two if it is not sold in period one. Therefore, some buyers may find it optimal to wait until period two, even though their valuation of the item exceeds p_1. This is so because, if the item does not sell in period one, its price in period two, p_2, will not be any higher than p_1. Denote the valuation threshold of buyers just indifferent between buying at price p_1 in period one and at price p_2 in period two by y. This threshold value can be interpreted as a measure of how strategic buyers behave in equilibrium. If $y^\star = 0$ in equilibrium, then buyers will submit purchase orders according to their true valuations, i.e. there is no waiting for lower second period prices. If, on the other hand, in equilibrium $y^\star = 1$, then all buyers would out-wait the seller, i.e. no buyer will submit a purchase order in period one and postpone purchase decisions to period two when the lower period two price may be expected to prevail. This PBE outcome, if it exists, is reminiscent of the so-called Coase conjecture.

Consider period two. To reach this stage, it must be that no buyer has $X_i > y$, $i = 1, \ldots, n$. Otherwise, by definition of y, this buyer would have purchased the item in period one. Hence, conditional on reaching period two, the seller maximizes period two expected revenue, i.e. the seller determines p_2 as

$$p_2(y,n) = \arg\max_{p_2} p_2 \left(1 - \left(\frac{p_2}{y}\right)^n\right) = \lambda_n y,$$

where the first expression is the product of the second period price and the probability of at least one buyer submitting an order, and where $\lambda_n = (n+1)^{-1/n} < 1$. Hence, $p_2(y,n) < y$ for any $y \in [0,1]$. Note that the more strategic buyers act in period one, i.e. the higher y in equilibrium,

[5] An alternative way of thinking about this mechanism is that, after a price is posted at the beginning of each period, buyers visit the sales outlet and make their purchase decision, and that each buyer is equally likely to get to the sales outlet first.

[6] It will be seen that such a PBE actually exists in this game and is unique.

the higher the price the seller can charge in period two.

Now consider period one. Given y, the seller will choose p_1 such that the marginal buyer with valuation y will be just indifferent between his or her expected gain from buying in either period. Given the assumed allocation mechanism which assigns equal probability of being fulfilled to any submitted purchase order, the marginal buyer's expected surplus in period one is

$$\begin{aligned} s_1(y, p_1) &= (y - p_1)\left(y^{n-1} + \frac{1}{2}(n-1)(1-y)y^{n-2} + \ldots + \frac{1}{n}(1-y)^{n-1}\right) \\ &= (y - p_1)\frac{1}{n}\frac{1}{1-y}\left(1 - y^n\right), \end{aligned}$$

while the expected discounted gain in period two, given $p_2(y, n)$, is

$$s_2(y, n, \delta) = \delta y(1 - \lambda_n)y^{n-1}\frac{1}{n}\frac{1}{1-\lambda_n}\left(1 - \lambda_n^n\right).$$

Therefore, given y, the seller's optimal period one price is

$$p_1(y, , n, \delta) = y - \delta y^n(1 - \lambda_n)\frac{\frac{1}{n}\frac{1}{1-\lambda_n}(1 - \lambda_n^n)}{\frac{1}{n}\frac{1}{1-y}(1 - y^n)} = y - \delta\frac{n}{n+1}(1-y)\frac{y^n}{1-y^n}.$$

Hence, the seller's problem can be cast as choosing the threshold y and subsequently evaluating the prices $p_1(y, n, \delta)$ and $p_2(y, n)$ so as to maximize expected revenue in the sequential game. The seller will obtain $p_1(y, n, \delta)$ in period one with a probability that equals the probability of at least one buyer having a valuation $X > y$, i.e. with probability $1 - y^n$; and the seller's discounted revenue in period two will be $\delta p_2(y, n, \delta)$ with a probability equal to the joint probability of the event that all buyers have valuations below y, so that the game gets to period two, and that at least one buyer has a valuation above $p_2(y, n, \delta)$, i.e. with probability $\left(1 - \left(\frac{p_2(y,n)}{y}\right)^n\right)y^n = \frac{n}{n+1}y^n$. Therefore, the seller's expected profit of the entire game is

$$\begin{aligned} \pi(y, n, \delta) &= \left(y - \delta\frac{n}{n+1}(1-y)\frac{y^n}{1-y^n}\right)(1 - y^n) + \delta\lambda_n\frac{n}{n+1}y^{n+1} \\ &= y(1 - y^n) - \delta\frac{n}{n+1}y^n + \delta\frac{n}{n+1}y^{n+1}(1 + \lambda_n). \end{aligned}$$

This function is a polynomial of degree $n + 1$ in y and can be maximized over y numerically. For the case of 2 buyers, Figure 1 shows expected profits as a function of δ as well the associated strategic threshold y. Details on the underlying computations are in an appendix.

This example provides the following observations which will be shown to hold more generally. The strategic threshold y is always interior, i.e. for any degree of discounting, it lies strictly between 0 and 1. Hence, it is always in the seller's interest not to annihilate the two-period structure of the game by charging a prohibitively high period one price. Moreover, for high enough discount factors, the seller benefits from having a second round available. This invalidates Coase-type conjectures according to which buyers would out-wait the monopolist in equilibrium. It can be interpreted in terms of a trade-off between efficiency and opportunities. The take-it-or-leave-it mechanism

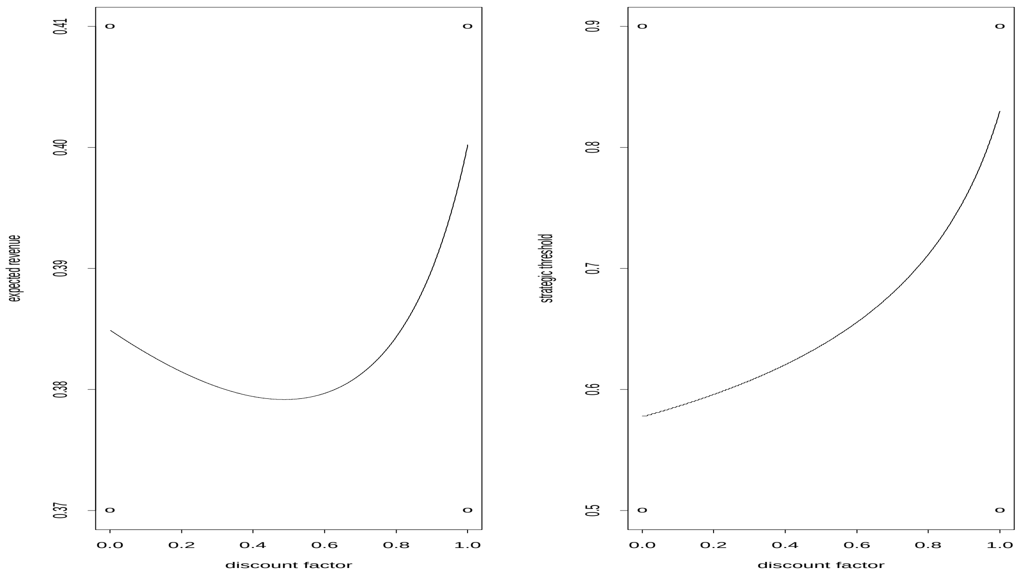

Figure 1: Figure 1: Sequential Sales, $n = 2$.

considered here does not guarantee an efficient allocation in which the highest valuation buyer obtains the item, in contrast to the auction mechanism considered in the next section. Instead, it allows eligible buyers, whose valuations are above the quoted price, to acquire the item, even if they are not the highest valuation buyer. Buyers are willing to pay for this opportunity, which the auction does not afford them. This, in turn, raises the seller's expected revenue. In fact, since competition is no softer in period two than in period one, buyers' incentives to strategically defer purchases are reduced, thus allowing the seller to credibly charge prices in the first period that exceed the optimal price in a one-time offering. This can be interpreted as a competitive externality in period two which has a positive, intertemporal external effect for the seller in period one. This competitive externality is diminished as the number of buyers increases, because with large numbers of buyers the marginal increase in risk of not obtaining the item in the second period in the presence of an extra buyer is small and therefore has little impact on buyers' strategic behavior. This is akin to the diminished winner's curse in common value auctions, where the marginal impact on the negative signal conveyed by winning the auction declines with the number of bidders.

2.2 Sequential Auctions

Now maintain the same setup as before, but consider a sequential auction mechanism in which buyers turn in sealed bids and the winning bidder pays his or her bid. This is referred to as a first-price sealed-bid auction. The item is sold to the highest bidder in the auction taking place in period one if the highest bid meets the seller's period one reserve price. If not, a second first-price sealed-bid auction takes place in period two. The first-price sealed-bid auction format is chosen for the exposition here because, unlike in the revenue equivalent second-price sealed-bid auction format where truthful bidding is an equilibrium strategy, bids are shaded, i.e. bidders submit bids below their valuations. This makes the strategic features of the auction analysis more comparable to the case of sequential take-it-or-leave-it offers considered above.

Again, the dynamic game can be solved by backward induction. As before, denote by y the valuation threshold of the marginal bidder who is indifferent between submitting an eligible[7] bid now or in the second period auction. Consider bidding in the second auction. Standard arguments reveal that, when the second period reserve price is R_2 and the bidder's valuation is X_i, his bidding function is

$$b(X_i, R_2, n) = X_i - \int_{R_2}^{X_i} u^{n-1} du / X_i^{n-1} = X_i \frac{n-1}{n} + \frac{1}{n} \frac{R_2^n}{X_i^{n-1}}.$$

Note that the period two bidding function does not depend directly on the strategic threshold y, as conditioning on the event $\{X_j < y, \forall j \neq i\}$ cancels out in the shading of bids. The reason is that the entire support of competing bidder's valuations does not enter bidder i's bidding function directly, since it effectively conditions on $\{X_j < X_i \; \forall j \neq i\}$, where X_i is less than y. Of course, it will depend indirectly on y through the dependence of the optimally chosen reserve price R_2 on y; this dependence will be derived below.

Conditional on the game getting to the second period, bidder i's expected surplus is

$$s_2(X_i, y, R_2, n) = \int_{R_2}^{X_i} u^{n-1} du / X_i^{n-1} \, (X_i/y)^{n-1} = \frac{1}{n} \frac{X_i^n - R_2^n}{y^{n-1}}.$$

The auction seller obtains an expected payment of

$$\begin{aligned} \pi_2(y, R_2, n) &= E\left[\max\{b(X_i, R_2), i = 1, \ldots, n\}; X_i \in [R_2, y]\right] \\ &= n \int_{R_2}^{y} \left[x \frac{n-1}{n} + \frac{1}{n} \frac{R_2^n}{x^{n-1}} \right] x^{n-1} dx \\ &= n \left[\frac{n-1}{n(n+1)} y^{n+1} + \frac{1}{n} R_2^n y - \frac{2}{n+1} R_2^{n+1} \right]. \end{aligned}$$

The auction seller then chooses R_2 so as to maximize expected revenue in this auction,

$$R_2(y) = \arg\max_{R_2} \pi_2(y, R_2, n) = \frac{y}{2}.$$

Note that, given y, the optimal reserve price does not depend on the number of bidders n. Anticipating future results, note also, however, that the equilibrium value of y generally is a function of n and δ. Consequently, in contrast to standard auction results[8], the optimal reserve price depends indirectly on the number of bidders through the strategic threshold y.

Now consider bidding in the first period. Bids will be shaded beyond the usual offset, to reflect the expected surplus in the second period auction, in case bids for the item do not meet

[7] Here, a bid is referred to as eligible if it is above the reserve price. It is a well-known theoretical result that it is immaterial whether the reserve price is secret or not; see, e.g. Ashenfelter (1989).

[8] See, e.g., Riley and Samuelson (1981)

the reserve price in period one.[9] Given $R_2(y)$, the marginal bidder expects $s_2(y,y,y/2,n)y^{n-1} = y^n \frac{1}{n}(1-2^{-n})$ in the second period auction, and hence bidder i's bidding function for the first auction is

$$\beta(X_i, y, n, \delta) = X_i - \frac{1}{n}\frac{X_i^n - y^n}{X_i^{n-1}} - \delta \frac{y^n}{X_i^{n-1}}\frac{1}{n}(1-2^{-n}).$$

Note that, as a consequence of bid shading, y must exceed the period one reserve price R_1, so that, unlike $b(X_i, y, R_2)$, the period one bidding function does not depend on the reserve price. In fact, the period one reserve price is given by the bid of the marginal bidder,

$$R_1(y, n, \delta) = \beta(y, y, n, \delta) = y - \delta y \frac{1}{n}(1-2^{-n}).$$

The auction seller obtains period one expected revenue

$$\begin{aligned}
\pi_1(y,n,\delta) &= E\left[\max\{\beta(X_i,y,n,\delta), i=1,\ldots,n\}; X_i \in [y,1]\right] \\
&= n\int_y^1 \left[x - \frac{x^n - y^n}{y^{n-1}} - \delta y \frac{1}{n}(1-2^{-n})\right] x^{n-1}dx \\
&= \frac{n}{n+1}(1-\frac{1}{n})(1-y^{n+1}) + y^n(1-\delta(1-2^{-n}))(1-y).
\end{aligned}$$

Total expected auction revenue then is

$$\begin{aligned}
\pi(y,n,\delta) &= \pi_1(y,n,\delta) + \delta\pi_2(y, y/2, n) \\
&= \frac{n-1}{n+1}(1-y^{n+1}) + y^n(1-\delta(1-2^{-n}))(1-y) + \delta y^{n+2}\frac{2^n(n-1)+1}{2^n(n+1)}.
\end{aligned}$$

This also is a polynomial in y which can be maximized numerically over y. As in the case of the take-it-or-leave-it seller, the auction seller's optimization problem can be cast as choosing a threshold y and then evaluating the optimal reserve prices $R_1(y,n,\delta)$ and $R_2(y)$. For the case of two bidders, figure 2 shows total expected revenue from the sequential auction procedure as well as the strategic threshold y, both as a function of δ. Details on the underlying computations are again in an appendix.

The sequential auction mechanism yields an outcome that differs from the one induced by the sequential take-it-or-leave-it sales. In contrast to the sequential take-it-or-leave-it seller, the auction seller does not benefit from the second period. In fact, a patient auction seller may find it optimal to credibly annihilate the first auction by choosing a prohibitively high period one reserve price and defer any auction revenue to the second period. This case arises when the equilibrium threshold y is equal (or close) to unity. The auction seller does not benefit from a second round because bidders, anticipating the possibility of a future auction, shade their bids in the first auction by more than they would in a single auction, thus reducing the auction seller's period one expected revenue. If an auction takes place in period two, then period one has revealed that all bidders have valuations below y. This is only bad news for the seller, who will choose a period two reserve price accordingly. Bidders will bid according to the bidding function $b(\cdot, y/2, n)$, with shading increasing in y. This depresses the auction seller's expected revenue in the second auction. Note that this

[9] See also Bernhardt and Scoones (1994)

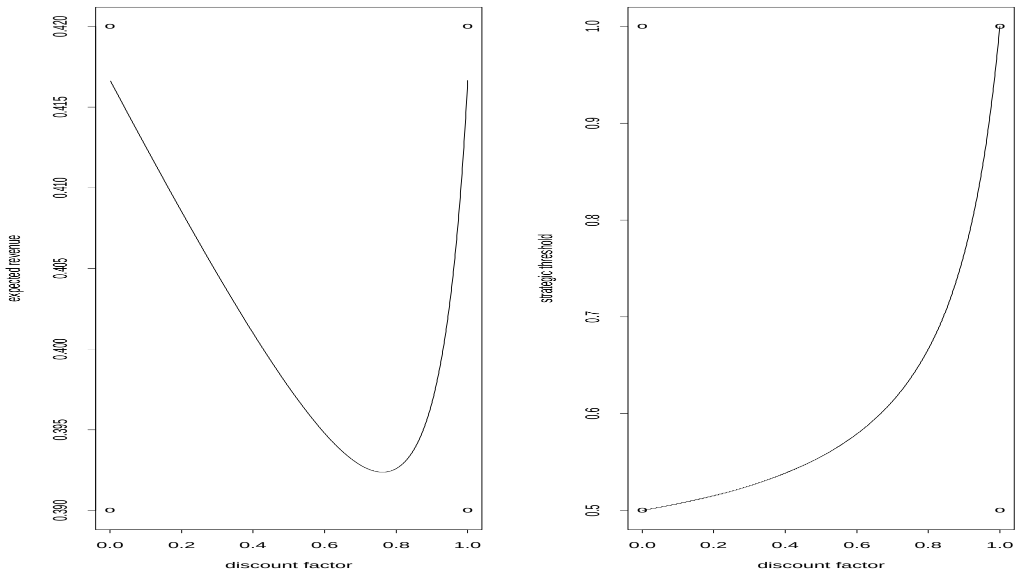

Figure 2: Figure 2: Sequential Auctions, $n = 2$.

auction mechanism always yields an efficient allocation, with the highest valuation bidder obtaining the item. This is so because bidding functions are monotonically increasing in valuations. Hence there is no intertemporal competitive externality as there is in the case of take-it-or-leave-it offers that would allow the auction seller to charge higher reserve prices.

A comparison of the two mechanisms reveals that there exist situations in which the sequential take-it-or-leave-it mechanism dominates the sequential auction mechanism in terms of period one expected venue. The economic reason behind this feature is that the take-it-or-leave-it seller, in light of the intertemporal competitive externality among competing buyers, can play off buyers against each other, effectively charging high valuation buyers a premium to reduce the risk of losing out in the more competitive second period. Figure 3 illustrates this for the uniform examples presented in these two subsections. A comparison of total expected revenues in Figures 1 and 2 reveals furthermore that this effect can make patient sequential take-it-or-leave-it sellers better off than slightly less patient sequential auction sellers.

The next section provides the general theoretical foundations of the strategic PBE features in dynamic monopoly games with incomplete information that were illustrated by the foregoing examples.

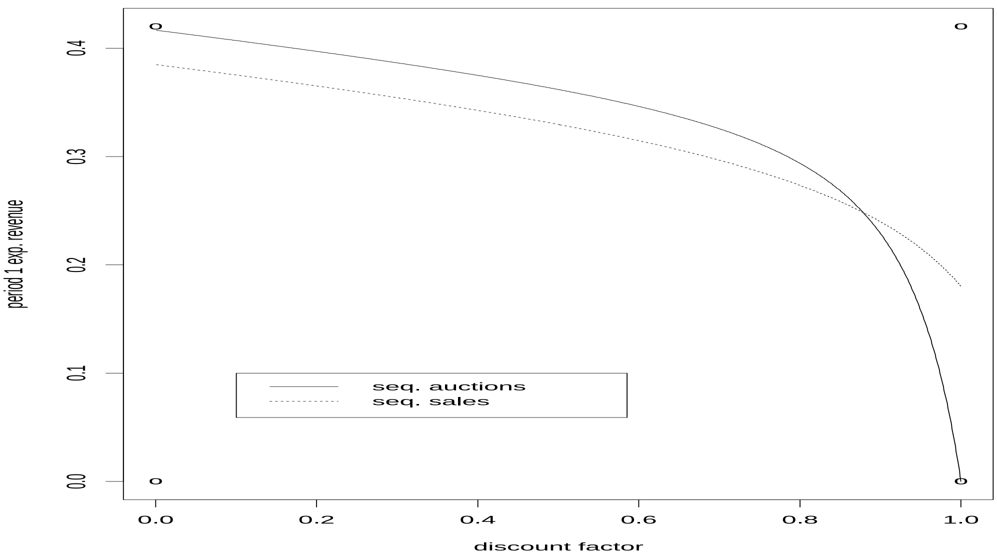

Figure 3: Figure 3: Comparison of First Period Expected Revenues, $n = 2$.

3 General Theory of Dynamic Monopolies with Stochastic Demand

The analysis maintains the following assumptions:

Assumption A1: Buyers' valuations X_i are independently and identically distributed with CDF $F(x)$.

Assumption A2: $F(x)$ is twice continuously differentiable, with pdf $f(x)$ which is positive on the support $\mathcal{X}$ of F.

Assumption A3: Marginal revenue $1 - F(x) - xf(x)$ is downward sloping.[10]

Assumption A4: The item has zero value for the seller if it is not sold; the seller maximizes expected revenue, and the buyers maximize expected surplus, and buyers and the seller are risk-neutral.

To develop some general results on Perfect Bayesian equilibria of sequential dynamic monopoly games, the following benchmark result will be useful.

Theorem 1: Consider a one period game in which the seller of an item can either make a final

[10] In auction theory, assumption A3 is sometimes referred to as regularity in the sense of Myerson (1981). See, e.g., Klemperer (2000)

take-it-or-leave-it offer with expected revenue π_S, or hold a final first-price sealed bid auction with expected revenue π_A. Under assumptions A1-A4, $\pi_S \leq \pi_A$.

The proof of this as well as subsequent results are in an appendix to the paper. Theorem 1 states the intuitively plausible proposition that in a one-shot situation a monopolist cannot do better than to hold an auction. The proof relies on techniques advanced in Bulow and Klemperer (1996) and borrows from monopoly theory. It shows that the expected marginal revenue of the winning bidder in an auction is always higher than the expected marginal revenue of the nth buyer in a take-it-or-leave-it offering. The result is relevant in this discussion, as it pins down the ranking between the take-it-or-leave-it sale and auction mechanisms when $\delta = 0$ or $\delta = 1$, i.e. when the item is either instantaneously perishable or very durable. The result suggests that in both cases the seller chooses the auction mechanism; Theorem 3 below will confirm this for $\delta = 1$. The further results below will, moreover, address the equally interesting intermediate cases in which δ is bounded away from zero or one, $\delta \in (0, 1)$.

Now consider the dynamic cases with two consecutive periods. The same definition of PBE as above applies. The following result establishes the key features of the uniform example as general properties of sequential take-it-or-leave-it offers with incomplete information.

Theorem 2: Consider a two period game with consecutive take-it-or-leave-it sales. Under assumptions A1-A4, there exists a unique PBE with first and second period prices $p_1(y^\star, n, \delta) \geq p_2(y^\star, n)$, which are monotonic in $y^\star$, and where $y^\star$ is in the interior of $\mathcal{X}$.

The result has an obvious and significant corollary.

Corollary 2.1: Under assumptions A1-A4, (i) there exists $\check{\delta} \in (0, 1)$ such that, for all $\delta \in [\check{\delta}, 1]$, a take-it-or-leave-it seller achieves higher expected revenue in a two period game than in a one period game, and (ii) the equilibrium first period price $p_1(y^\star, n) > p^\star = \arg\max_p p(1 - F(p)^n)$, where $p^\star$ is the optimal price in a one-shot offering.

In other words, with stochastic demand a take-it-or-leave-it seller always benefits from the sales opportunities provided by future periods. Enhanced competition among buyers if the item is not sold in period one induces a competitive externality in demand. This externality allows the seller to charge a first period price that would be higher than optimal in a simple one-shot take-it-or-leave-it offering.

This result can be contrasted with the following theorem which deals with the sequential auction counterpart. The PBE in the case of auctions comprises period one and two reserve prices and bidding strategies, such that, given the bidders' strategies, the reserve prices maximize the auction seller's expected revenue, and given the reserve prices, the bidding strategies maximize the bidders' expected surpluses.

Theorem 3: Consider a two period game with consecutive first-price sealed-bid auctions. Under assumptions A1-A4, there exists a unique PBE with first and second period reserve prices $R_1(y^\star, n, \delta)$ and $R_2(y^\star, n)$, which are monotonic in $y^\star$; furthermore, there exists $\bar{\delta} \in (0, 1) : \delta > \bar{\delta} \Rightarrow y^\star = \sup\{x : x \in \mathcal{X}\}$.

The key distinction between theorem 2 and 3 is that the equilibrium threshold $y^\star$ is interior in the former, but can be on the boundary in the latter. This implies that, in the case of sequential auctions, there exist Perfect Bayesian Equilibria in which the seller foregoes the first period auction by choosing a prohibitively high period one reserve price. Hence, Theorem 3 identifies circumstances in which the possibility of subsequent auctions in future periods are harmful to the seller.

Theorems 1 - 3 and Corollary 2.1, considered jointly, provide a few additional insights. The source for the result of Corollary 2.1 is the intertemporal competitive externality induced by competition among buyers. This externality becomes more pronounced as the discounted value of period 2 payoffs rises, i.e. as δ rises. By Theorem 1, period 1 expected payoffs for low δ are higher for sequential auctions than for sequential take-it-or-leave-it sales. In Coasian logic, they decline as δ rises, but do so less rapidly for take-it-or-leave-it sales, due to the countervailing competitive externality. Hence, situations may arise in which sequential take-it-or-leave-it sellers with sufficiently high δ earn higher period 1 expected revenue than sequential auction sellers. Ultimately, for $\delta > \check{\delta}$, this allows sequential take-it-or-leave-it sellers to benefit from a second period in terms of total expected revenue, as Corollary 2.1 concludes. Since sequential auction sellers never earn more than π_A, but earn strictly less for intermediate values of δ (implied by Theorem 3), situations may arise in which patient sequential take-it-or-leave-it sellers achieve higher total expected revenue than slightly less patient sequential auction sellers.

4 A Limit Revenue Equivalence Theorem

As the preceding sections demonstrated, the two mechanisms yield different strategic implications and expected revenues. The source of these differences is the trade-off between the efficiency of the final allocation, which the sequential auction mechanism guarantees, and the opportunities for buyers other than the highest valuation buyer to obtain the item, which the sequential take-it-or-leave-it mechanism grants. Since the latter introduces the possibility of an inefficient final allocation, this discrepancy in expected revenues is not a failure of the classical revenue equivalence theorem, as one might conjecture. This section shows, however, that, under a mild additional assumption, the sequential take-it-or-leave-it mechanism possesses properties in the limit, as the number of periods approaches infinity, which imply the conditions underlying the Revenue Equivalence Theorem. Hence, it will be seen that, in the limit, the expected revenue of this mechanism equals the epected revenue of a single, final optimal auction, regardless of its format.

The key step towards a revenue equivalence theorem is a result which eliminates the possibility of an inefficient final allocation as the number of periods, m, gets large. To establish this limiting property, one additional assumption will be maintained:

Assumption A5: (bounded support) $\sup\{x : x \in \mathcal{X}\} = \bar{x} < \infty$.

Under assumptions A1-A5, Theorem 2 can be generalized to the case of m periods.

Theorem 4: Consider an m-period game with sequential take-it-or-leave-it offers. Under assumptions $A1 - A5$, there exists a PBE with prices

$$p_{m+1}(\mathbf{y}^\star, n, \delta) < p_m(\mathbf{y}^\star, n, \delta) < \ldots < p_1(\mathbf{y}^\star, n, \delta) < p_0(\mathbf{y}^\star, n, \delta),$$

where $p_{m+1}(\mathbf{y}^\star, n, \delta) = 0$ and $p_0(\mathbf{y}^\star, n, \delta) = \bar{x}$, and strategic thresholds $\mathbf{y}^\star = (y_1^\star, \ldots, y_{m-1}^\star)'$, such that

$$y_1^\star > \ldots > y_{m-1}^\star.$$

The theorem is proven by induction on the results provided by Theorem 2, following in the induction step essentially parallel arguments to the ones given in the proof of that theorem. This result, for finite m, still permits a moderate probability of an inefficient final allocation. In the limit, one might expect the interval between equilibrium prices and strategic thresholds to shrink to zero, so that this residual probability vanishes and the final allocation is bound to be efficient. This is the essence of the following limiting result.

Theorem 5: (Asymptotic Revenue Equivalence) Suppose $A1 - A5$ hold in the model for m periods, with $\delta = 1$. Let π_S^m denote the PBE expected revenue of m sequential take-it-or-leave-it offers, and let π_A denote the expected revenue of a final optimal auction. Then, $\pi_S^m \to \pi_A$ as $\to \infty$.

This result establishes revenue equivalence in the limit. The reasoning behind the result is straightforward. Index PBE prices and strategic thresholds by superscript m. As $m \to \infty$, one might expect (and the proof demonstrates) that $\max\{y_k^{m\star} - y_{k+1}^{m\star}, k = 0, \ldots, m+1\} \to 0$, and $\max\{p_k(\mathbf{y}^\star, n, 1) - p_{k+1}(\mathbf{y}^\star, n, 1), k = 1, \ldots, m+1\} \to 0$ as $m \to 0$. Hence, under $A2$, the probability of any two buyers falling into the same price acceptance region vanishes. Since then the highest valuation buyer will submit a purchase order first and acquire the item, while other buyers expect zero surplus, arguments in Myerson (1981) and Riley and Samuelson (1981) imply revenue equivalence. Hence, for the purposes of the theorem, the format of the auction is irrelevant. The result has an immediate, intuitive interpretation. The limiting sequential take-it-or-leave-it mechanism produces a (continuously) declining equilibrium price sequence, and buyers submit a purchase order once the price has dropped to their respective valuation of the item. This is the same format as in the case of a Dutch auction, or descending price auction. In this auction format, the highest valuation bidder obtains the item, and losing bidders expect zero surplus. These are precisely the conditions of the revenue equivalence theorem to apply. Since sequential take-it-or-leave-it mechanism in the limit shares these characteristics, the theorem applies here as well and hence predicts equal expected revenues.

5 Conclusion

This paper emphasizes the strategic implications of market uncertainty for equilibria in dynamic markets. It highlights the trade-off between allocative efficiency and opportunities for buyers other than the highest valuation buyer to obtain an item to be sold; this trade-off induces a competitive externality in demand, due to competing buyers who are willing to pay for this opportunity. And it concludes that there exist dynamic situations in which a take-it-or-leave-it seller can intertemporally exploit this trade-off to raise prices and expected revenue above the one obtained in a one-shot take-it-or-leave-it offer. This demonstrates that market uncertainty invalidates Coase-type conjectures. The paper also demonstrates that there exist situations in which sequential take-it-or-leave-it sales dominate sequential auctions in terms of short term expected gains. This may provide an explanation for why a popular online auction trading platform like eBay may find it profitable to add posted-price selling options to its menu of concurrent services. Finally, the paper also shows that the two mechanisms can be reconciled asymptotically. As the number of periods gets large, the sequential take-it-or-leave-it mechanism mimics a Dutch auction which is revenue equivalent to a single, final auction of any format.

6 Appendix

6.1 Proof of Theorem 1

Since, by virtue of the Inverse Transform method for CDFs, using the inverse CDF $F^{-1}(\cdot)$ every problem satisfying A1 and A2 can be mapped into a problem in which $\mathcal{X} = [0,1]$, without loss of generality the case $\mathcal{X} = [0,1]$ is considered in this and all subsequent proofs.

The marginal revenue of a bidder in the auction is $1 - F(p) - pf(p) = 1 - F(p)(1+\epsilon(p))$, where $\epsilon(p) = f(p)p/F(p)$ is the elasticity of F at p. The marginal revenue of the nth potential buyer in the sale is $1 - F(p)^n(1 + n\epsilon(p))$. Therefore, using the approach as in Bulow and Klemperer (1996),

$$\begin{aligned}
\pi_A &= E\left[\max\{1 - F(X_i)(1+\epsilon(X_i)), i = 1, \ldots, n\}\right] \\
&= \int_0^1 (1 - F(x)(1+\epsilon(x)))n(1-F(x))^{n-1} f(x)dx \\
&= 1 - \int_0^1 F(x)(1+\epsilon(x))n(1-F(x))^{n-1} f(x)dx,
\end{aligned}$$

where integration is with respect to the density of the minimum of the X_i, in light of A3. Similarly,

$$\begin{aligned}\pi_S &= E\left[\max\{1 - F(x)^n(1 + n\epsilon(x)), 0\}\right] \\ &= \int_0^{p^\star} (1 - F(x)^n(1 + n\epsilon(x))f(x)dx,\end{aligned}$$

where $p^\star < 1$ is the optimal price quote which solves $1 - F(p^\star)^n(1 + n\epsilon(p^\star)) = 0$. Note that

$$\pi_S \leq 1 - \int_0^{p^\star} F(x)(1/n + \epsilon(x))nF(x)^{n-1}f(x)dx.$$

Now add $a = \int_{p^\star}^1 nF(x)^{n-1}f(x)dx = 1 - F(p^\star)^n < 1$ to the last expression, and subtract

$$\begin{aligned}b &= \frac{n-1}{n}\int_0^{p^\star} F(x)nF(x)^{n-1}f(x)dx + \int_{p^\star}^1 F(x)nF(x)^{n-1}f(x)dx \\ &= \frac{n}{n+1}\left(1 - \frac{1}{n}F(p^\star)^{n+1}\right).\end{aligned}$$

Note that

$$\pi_A = 1 - \int_0^{p^\star} F(x)(1/n + \epsilon(x))nF(x)^{n-1}f(x)dx + a - b.$$

Next, it is shown that $1 - F(p^\star)^n \geq \frac{n}{n+1}$ for all n. For the purpose of this argument, the notation recognizes that $p^\star_n$ depends on n. Suppose that the opposite inequality were true. Then, $p^\star_n(1 - F(p^\star)^n) < \frac{n}{n+1}$ for any n, and hence $\lim_{n\to\infty} p^\star_n(1 - F(p^\star_n)^n) < 1$, which contradicts that in the limit the seller earns 1 with probability 1.

Therefore, $a > \frac{n}{n+1}$, $b < \frac{n}{n+1}$, so $a - b > 0$ and $\pi_S < \pi_A$, completing the proof. $\square$.

6.2 Proof of Theorem 2 and Corollary 2.1

Proceed by backward induction. In period two, the seller's problem is

$$\max_{p_2} p_2\left(1 - \left(\frac{F(p_2)}{F(y)}\right)^n\right),$$

so that $p^\star_2 = p_2(y, n)$ solves

$$F(y)^n - F(p^\star_2)^n - p^\star_2 nF(p^\star_2)^{n-1}f(p^\star_2) = 0.$$

Regularity A3 implies that the solution is unique, given y. Also, notice that increasing y shifts up the left-hand side of the last expression, so that A4 implies that $p^\star_2$ must rise to return to equality. Hence, $p_2(y, n)$ is monotonically increasing in y.

Following the same reasoning as in the uniform example, given y the first period price $p_1(y, n, \delta)$ is such that the marginal buyer is indifferent between buying in either period, hence,

$$p_1(y, n, \delta) = y - \delta(y - p_2(y, n))\left[\frac{F(y)^n - F(p_2(y, n))^n}{F(y) - F(p_2(y, n))}\right]\left[\frac{1 - F(y)}{1 - F(y)^n}\right].$$

The seller will choose $y^\star$ such as to equate expected marginal revenue from selling in periods one and two. Expected revenue in either period is the product of period price and probability of sale. Since an increase in y implies that the probability of selling in period two rises while the probability of selling in period one falls, and since $p_2(y,n)$ is increasing in y, $p_1(y,n,\delta)$ must be monotonically increasing in y.

Since for any $p>0$ and $y>0$,

$$\Pr(\max\{X_i, i=1,\dots,n\} < p | X_i > y\ \forall i) < \Pr(\max\{X_i, i=1,\dots,n\} < p | X_i \leq y\ \forall i),$$

it follows that $p_1(y,n,\delta) > p_2(y,n)$ for any y,n,δ.

Finally, suppose $y^\star$ were not unique, i.e. $\exists\, y_1^\star < y_2^\star$, both equating expected marginal revenues and yielding the same expected revenue. Then, for any n and δ,

$$p_1(y_j^\star, n, \delta) \geq p_2(y_j^\star, n), \quad j=1,2$$

and

$$\begin{aligned} \Pr(\max\{X_i, i=1,\dots,n\} > y_1^\star) &> \Pr(\max\{X_i, i=1,\dots,n\} > y_2^\star) \\ \Pr(\max\{X_i, i=1,\dots,n\} \in [p_2(y_1^\star,n), y_1^\star]) &< \Pr(\max\{X_i, i=1,\dots,n\} \in [p_2(y_2^\star,n), y_2^\star]). \end{aligned}$$

Since by hypothesis expected revenue is equal for $y_1^\star$ and $y_2^\star$, it must be that

$$\begin{aligned} p_1(y_2^\star, n, \delta) &> p_1(y_1^\star, n, \delta) \\ p_2(y_2^\star, n) &< p_2(y_1^\star, n), \end{aligned}$$

a contradiction to the monotonicity of $p_2(y,n)$. Hence $y^\star$ is unique.

If $\delta = 0$, then y is irrelevant. Consider $\delta \in (0,1]$ and suppose $y^\star = 1$. Then, $p_1(1,n,\delta) = 1$ and $p_2(1,n) < 1$. Therefore, the expected surplus of the marginal buyer, whose valuation is $y^\star = 1$, is zero in period one, and positive in period two. Hence, $y = 1$ cannot be part of a PBE. Alternatively, suppose $y^\star = 0$. Then, $p_2(0,n) = 0$ and $p_1(0,n,\delta) > 0$. In this case, the expected surplus of the marginal bidder is zero in period two and negative in period one. Hence $y = 0$ cannot be part of a PBE. Therefore, $y^\star$ must be in the interior of $\mathcal{X}$. This completes the proof of the theorem. $\Box$

The corollary follows immediately. Part (i) is implied by $y^\star$ in the interior of $\mathcal{X}$: In any PBE, since $y^\star < \sup\{x : x \in \mathcal{X}\}$, the seller never finds it optimal to choose a prohibitively high first-period price which would induce all buyers to wait for the second period. Hence collapsing the dynamic game to a one-shot game, which is only possible by annihilating the first period subgame, is not optimal. Since Theorem 2 for $\delta = 1$ then implies that expected revenue is higher in the one-shot game, by continuity of expected revenue in δ, there exists $\check{\delta} \in (0,1)$ such that expected revenues for the sequential game are higher for all $\delta \in [\check{\delta}, 1]$. Part (ii) follows from $y^\star > 0$ in any PBE: In a one shot game, the seller faces all buyers and hence the price $p^\star$ targets all buyers; in the first period of the dynamic game, the seller only faces high valuation buyers with valuations above $y^\star > 0$, and hence the first period price $p_1(y^\star, n, \delta) > p^\star$, targeting only these buyers. $\Box$

6.3 Proof of Theorem 3

The game is solve by backward induction. Bidder i's bidding function in the second auction is

$$b(X_i, R_2, n) = X_i - \int_{R_2}^{X_i} F(u)^{n-1} du / F(X_i)^{n-1}.$$

Bidder i's expected surplus in this auction, conditional on it taking place, is $s_2(X_i, R_2, n) = \int_{R_2}^{X_i} F(u)^{n-1} du / F(y)^{n-1}$, while the seller's expected revenue is

$$\pi_2(y, R_2, n) = E\left[X - \int_{R_2}^{X} F(u)^{n-1} du / F(X)^{n-1}; X \in [R_2, y]\right],$$

where the expectation is taken with respect to the distribution of the maximum of the X_i, i.e. $nF(x)^{n-1} f(x) dx$. The seller chooses R_2 so as to maximizes this expected revenue, so $R_2^\star = R_2(y, n)$ solves

$$\begin{aligned} n\left[F(R_2^\star)^{n-1}(F(y) - F(R_2^\star)) - R_2^\star f(R_2^\star) F(R_2^\star)^{n-1}\right] &= 0 \\ \Leftrightarrow F(y) - F(R_2^\star) - R_2^\star f(R_2^\star) &= 0. \end{aligned}$$

This implies the well-known result that $R_2^\star = R_2(y, n)$ does not (directly) depend on n. As pointed out earlier, it does depend on n indirectly, through the dependence of the equilibrium y on n. By A3, the solution exists and is unique. The last equality implies that $y > R_2(y, n)$, as a consequence of A2. Also, since raising y shifts up the left-hand side, to return to equality $R_2^\star$ has to rise, so that $R_2(y, n)$ is monotonically increasing in y.

Since the marginal bidder's discounted expected surplus in the second auction is

$$\delta s_2(y, R_2(y, n), n) F(y)^{n-1} = \delta \int_{R_2(y,n)}^{y} F(u)^{n-1} du > 0,$$

the bidding function β for the first auction is shaded taking this amount into consideration, beyond conventional shading, so that

$$\beta(X, y, n, \delta) = X - \frac{\int_y^X F(u)^{n-1} du}{F(X)^{n-1}} - \delta \frac{\int_{R_2(y,n)}^{y} F(u)^{n-1} du}{F(X)^{n-1}}.$$

The reserve price of the first auction equals the bid of the marginal bidder. Hence,

$$R_1(y, n, \delta) = \beta(y, y, n, \delta) = y - \delta \frac{\int_{R_2(y,n)}^{y} F(u)^{n-1} du}{F(y)^{n-1}}.$$

Also,

$$\frac{d}{dy} R_1(y, n, \delta) = 1 - \delta + \delta \left(\frac{F(R_2(y, n))}{F(y)}\right)^{n-1} \frac{d}{dy} R_2(y, n) + (n-1) F(y) \frac{\int_{R_2(y,n)}^{y} F(u)^{n-1} du}{F(y)^{n-1}} > 0,$$

i.e. $R_1(y, n, \delta)$ is monotonically increasing in y, since $R_2(y, n)$ is.

It remains to be shown that $y = 1$ can be part of a PBE. To demonstrate this, consider $\delta = 1$ and compare expected revenue for $y = 1$ and $y = 1 - \epsilon$, for some $\epsilon > 0$. In terms of expected revenue, lowering y from 1 to $1 - \epsilon$ entails a gain $g(\epsilon)$ due to a lower reserve price in the second auction, as well as a loss $l(\epsilon)$ due to additional bid shading in the first auction. It will be shown that the net gain at $\epsilon = 0$ is negative - or more precisely, that $\lim_{\epsilon \to 0} \frac{d}{d\epsilon}(g(\epsilon) + l(\epsilon)) < 0$ - so that the PBE involves $y = 1$. First, notice that

$$
\begin{aligned}
\frac{d}{d\epsilon} g(\epsilon)|_{\epsilon=0} &= nR_2(1,n)F(R_2(i,n))^{n-1} f(R_2(1,n)) \frac{d}{dy} R_2(y,n)|_{y=1} > 0 \\
&= nF(R_2(1,n))^{n-1}(1 - F(R_2(1,n)) \frac{d}{dy} R_2(y,n)|_{y=1} \\
&= nF(R_2(1,n))^{n-1}(1 - F(R_2(1,n)) \frac{f(1)}{2f(R_2(1,n)) + R_2(1,n)f'(R_2(1,n))} \\
\frac{d}{d\epsilon} l(\epsilon)|_{\epsilon=0} &= -nf(1) \int_{R_2(1,n)}^{1} F(u)^{n-1} du \\
&= -nf(1)E\left[\max\{X_i, i = 1, \ldots, n-1\}; X_i \in [R_2(1,n), y]\right] < 0.
\end{aligned}
$$

where the second equality uses the first order condition of the auction seller's period two optimization problem, the third the Implicit Function Theorem, applied to the same equation, and the last follows from integration by parts. Also from the same first order condition,

$$(F(y) - F(R_2(y,n)))/f(R_2(y,n)) = R_2(y,n) < y.$$

Raising y must increase expected auction revenue in the second period, as higher valuation bidders now participate. Hence, in the notation used in the proof of Theorem 1,

$$E\left[\max\{\mathrm{MR}(X_i), i = 1, \ldots, n\}; X_i \in [R_2, y]\right] = \int_{R_2}^{y} (1 - F(x)(1 + \epsilon(x)))n(1 - F(x))^{n-1} f(x) dx$$

must rise. Therefore, for all y,

$$
\begin{aligned}
0 \leq\ & (1 - F(y)(1_\epsilon(y)))n(1 - F(y))^{n-1} f(y) - \\
& (1 - F(R_2)(1 + \epsilon(R_2)))n(1 - F(R_2))^{n-1} f(R_2) \frac{d}{dy} R_2.
\end{aligned}
$$

This, together with $y > R_2(y,n)$ implies $f(y) > f(R_2(y,n)) \frac{d}{dy} R_2(y,n)$. Therefore,

$$
\begin{aligned}
\frac{d}{d\epsilon} g(\epsilon)|_{\epsilon=0} &\leq nR_2(1,n)F(R_2(1,n))^{n-1} f(1) \\
\frac{d}{d\epsilon} l(\epsilon)|_{\epsilon=0} &\leq -nf(1)R_2(1,n)F(R_2(1,n))^{n-1}.
\end{aligned}
$$

Thus, $\frac{d}{d\epsilon}(g(\epsilon) + l(\epsilon))|_{\epsilon=0} \leq 0$. By continuity of expected revenue in δ, there exists a δ neighborhood with upper bound 1, $[\bar{\delta}, 1]$, such that the preceding argument holds for $\delta \in [\bar{\delta}, 1]$. This completes the proof of the theorem. □

6.4 Proof of Theorem 4

For $m = 2$, the result is implied by Theorem 2 (which does not require A). Proceed by induction on m. Consider $m > 2$, and suppose the result if true for $m - 1$. The result holds then for $\bar{x}$ as upper bound on X_i, $i = 1, \ldots, n$. Hence, there exists a PBE for any $\bar{y} < \bar{x}$, with

$$\begin{aligned} p_k(\mathbf{y}^\star(\bar{y})) &= \tilde{p}_k(\bar{y}), \quad k = 0, \ldots, m-1, \\ y_k^\star(\bar{y}) &= \tilde{y}(\bar{y}), \quad k, 0, \ldots, m-1, \end{aligned}$$

where the dependence on n and δ is suppressed for notational convenience.

Given $\bar{y}$, choose $p_m(\bar{y})$ such that

$$p_m(\bar{y}) = \bar{y} - \delta(\bar{y} - \tilde{p}_{m-1}(\bar{y})) \left[\frac{F(\bar{y})^n - F(\tilde{p}_{m-1}(\bar{y}))^n}{F(\bar{y}) - F(\tilde{p}_{m-1}(\bar{y}))} \right] \left[\frac{1 - F(\bar{y})}{1 - f(\bar{y})^n} \right].$$

On the basis of this price for period m, expected revenue $\pi(\bar{y})$ can be computed as before, given $\bar{y}$, where the period sales probabilities are based analogously on $\tilde{p}_k(\bar{y})$ and $\tilde{y}(\bar{y})$, $k = 0, \ldots, m-1$. Maximization over $\bar{y}$ yields the PBE values, based on the maximizer $\bar{y}^\star$. Now argue as before that $\bar{y} = 0$ and $\bar{y} = 1$ are not part of a PBE and that $\bar{y}^\star$ is unique. This completes the proof. $\square$

6.5 Proof of Theorem 5

The proof uses the following

Lemma A.1: Suppose $A1 - A5$ hold in the model for m sequential take-it-or-leave-it offers. Let $\xi^m = \max\{p_k^m(\mathbf{y}^{m\star}, n, \delta) - p_{k-1}^m(\mathbf{y}^{m\star}, n, \delta), k = 0, \ldots, m\}$. Then, $\xi^m \to 0$ as $m \to \infty$.

Proof of the Lemma: It follows from the equilibrium condition that, for any $k = 1, \ldots, m-1$,

$$y_k^{m\star} - p_k^m(\mathbf{y}^\star) = (y_k^{m\star} - p_{k+1}^m(\mathbf{y}^\star)\alpha_k(\mathbf{y}^{m\star}),$$

where $\alpha_k(\mathbf{y}^{m\star}) < 1$ denotes the relative odds of acquiring the item in periods k and $k+1$. This equilibrium condition implies that $\max\{p_k^m(\mathbf{y}^{m\star}) - p_{k-1}^m(\mathbf{y}^{m\star}), k = 0, \ldots, m+1\} \to 0$ if, and only if, $\max\{y_k^{m\star} - y_{k-1}^{m\star}, k = 1, \ldots, m-1\} \to 0$. Suppose there exist $q, \epsilon > 0$: $p_k^m \notin [q, q+\epsilon]$ $\forall k = 0, \ldots, m$ and $\forall m$, while in a neighborhood outside of the interval the result holds. Assume w.l.o.g. $q + \epsilon < \bar{x}$.[11] Let $\bar{p}^m = \inf_m \min\{p_k^m : p_k^m \geq q + \epsilon\}$, $\underline{p}^m = \sup_m \max\{p_k^m : p_k^m \leq q\}$, and let $\bar{y}^m$ be the marginal buyer, indifferent between buying at $\bar{p}^m$ and $\underline{p}^m$, and $\bar{y} = \inf_m\{\bar{y}^m\}$. Buyers with valuations in $[\bar{y}^m, \bar{y}^m + a^m]$ buy at $\bar{p}^m \to q + \epsilon$, while buyers with valuations in $[\bar{y}^m - b^m, \bar{y}^m]$ buy at price $\underline{p}^m \to q$, for some $a^m, b^m > 0$. The gap in the price path can only be sustained in equilibrium if there exists a corresponding gap in the limit of the path of the $y_k^{m\star}$s, i.e. if $\lim_m a^m > 0$ or $\lim_m b^m > 0$. The probability of acquiring the item at the price following $\underline{p}^m$,

[11] Boundary cases can be handled analogously.

$\underline{p}^m$, in the limit is simply $\lim_m F(\bar{y}^m - b)^{n-1}$, since, by hypothesis, there is zero probability of ties in the limit, while the probability of aquiring the item at $\bar{p}^m$ in the limit is $\lim_m F(\bar{y}^m)^{n-1}$. By hypothesis, $\underline{p}^m - \underline{p}^m \to 0$ as $m \to \infty$. Hence, the relative odds of acquisition must converge to one and therefore it must be that $\lim_m b^m = 0$. By an analogous argument, $\lim_m a^m = 0$. Hence, in the limit, $\max\{y_k^{m\star} - y_{k-1}^{m\star}, k = 1, \ldots, m-1\} \to 0$. This proves the Lemma. □

Denote PBE prices by p_k^m, $k = 0, \ldots, m+1$. By Theorem 4 and Lemma A1, the probability of an efficient allocation (EA) is

$$\begin{aligned}
\Pr(EA) &= n\left((1 - F(p_{m+1}^m))F(p_{m+1}^m)^{n-1} + \sum_{k=1}^{m+1}(F(p_k^m) - F(p_{k-1}^m))F(p_k^m)^{n-1}\right) \\
&\to n(1 - F(\bar{x}))F(\bar{x})^{n-1} + \int_0^{\bar{x}} \frac{d}{du}F(u)^n du \quad \text{as } m \to \infty \\
&= 1.
\end{aligned}$$

Hence, the highest valuation bidder acquires the item. Lower valuation bidders expect zero surplus. By an appeal to the revenue equivalence theorem, $\lim_{m\to\infty} \pi_S^m = \pi_A$. □

6.6 Details on Examples with $n = 2$

6.6.1 Sequential Take-it-or-leave-it Sales

Consider two buyers with valuations $v_i \sim u[0,1]$, $i = 1, 2$, and two consecutive take-it-or-leave-it sales S1 and S2, with prices p_1 and p_2. Suppose that, given p_1, buyer i's strategy is to accept p_1 if $v_i \geq y$ and to reject it otherwise. Conditional on both buyers rejecting p_1, buyers and seller infer $v_i \sim u[0, y]$. The seller then chooses $p_2 = \arg\max_p p(1 - (p/y)^2)$, which yields $p_2 = y/\sqrt{3}$. Then, given discount factor $\delta \in [0, 1]$, the marginal buyer with valuation $v_i = y$ obtains expected discounted surplus $\delta(y - y/\sqrt{3})y\left(\frac{y/\sqrt{3}}{y} + \frac{y - y/\sqrt{3}}{2y}\right) = \delta\frac{1}{3}y^2$ in the second round. Given p_1 in the first round, the marginal buyer obtains expected surplus $(y - p_1)(y + (1 - y)/2)$ in the first round. Equating yields $p_1 = y - \delta\frac{2y^2}{3(1+y)}$. Hence, the seller's total expected profit is $\pi(y) = \left[y - \delta\frac{2y^2}{3(1+y)}\right](1 - y^2) + \delta(y/\sqrt{3})(1 - ((y/\sqrt{3})/y)^2)y^2$.

6.6.2 Sequential Auctions

Consider two bidder with valuations $v_i \sim u[0,1]$, $i = 1, 2$, and two consecutive auctions A1 and A2. Suppose bidders submit bids in A1 if $v_i \geq y = y(R_1)$ and abstain otherwise, where R_1 (R_2) is the seller's reserve price in auction A1 (A2). Bidder i's bid in A2 is $b(v_i, y, R_2) = v_i - \int_{R_2}^{v_i} udu/v_i$, for $v_i \in [R_2, y]$. Conditional on A2 being held, i receives v_i with probability v_i/y, $v_i \leq y$. Bidder i's expected surplus in A2, conditional on it being held, is $S_{A2}(v_i, y, R_2) = \int_{R_2}^{v_i} udu\frac{1}{v_i}\frac{v_i}{y} = \frac{1}{2}\frac{v_i^2 - R_2^2}{y}$.

A2 is held with probability y^2. Hence, the expected seller revenue in A2 is

$$\begin{aligned}
\pi_{A2}(y, R_2) &= E[\max\{b(v_i, y, R_2), i = 1, 2\}; v_1, v_2 \in [R_2, y]] \\
&= \int_{R_2}^{y} [v - \frac{1}{2}(v^2 - R_2^2)/v] 2v dv \\
&= y^3/3 + R_2^2 y/2 - 4R_2^3/3.
\end{aligned}$$

Maximizing with respect to R_2 yields $R_2 = y/2$. Hence, $S_{A2}(v_i, y, y/2) = \frac{v_i^2}{2y} - \frac{y}{8}$, and $\pi_{A2}(y, y/2) = y^3 \frac{5}{12}$. Expected gains in A2 accrue discounted by $\delta \in [0, 1]$. The marginal bidder with $v_i = y$ obtains discounted expected surplus in A2 of $\delta y S_{A2}(y, y, y/2) = \delta y^2 \frac{3}{8}$. If $\beta(v_i)$ is i's bid in A1, for $v_i \geq y$, then $\beta(y) = R_1$. Then, for $v_i \geq y$, $\beta(v_i) = v_i - \int_y^{v_i} u du/v_i - \delta \frac{3}{8} y^2/v_i = v_i - \frac{v_i^2 - y^2}{2v_i} - \delta \frac{3}{8} y^2/v_i$, and $\beta(y) = R_1 = y(1 - 3\delta/8) > R_2 = y/2$ for any $\delta \in [0, 1]$. Bidder i's expected payment in A1 is $\beta(v_i)v_i$, and expected surplus is $S_{A1}(v_i, y) = v_i^2 - \beta(v_i)v_i = \frac{1}{2}(v_i^2 - y^2) + \delta \frac{3}{8} y^2$. Note that $S_{A1}(y, y) = \delta \frac{3}{8} y^2 = \delta y S_{A2}(y, y, y/2)$, the expected discounted surplus in A2. The expected payment of the winning bidder in A1 is

$$\begin{aligned}
\pi_{A1}(y) &= E[\max\{\beta(v_i), i = 1, 2\}; v_1, v_2 \in [y, 1]] \\
&= 2 \int_{y}^{1} [-\delta 3y^2/8 + v^2 - (v^2 - y^2)/2] dv \\
&= y^2(1 - 3\delta/4)(1 - y) + (1 - y^3)/3.
\end{aligned}$$

Total expected revenue is $\pi(y) = \pi_{A1}(y) + \delta \pi_{A2}(y, y/2)$.

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