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Walter Beckert* and Richard Blundell†

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Abstract

This paper considers structural nonparametric random utility models for continuous choice variables. It provides sufficient conditions on the structural model to yield reduced-form systems of nonparametric stochastic demand functions that constitute a global homeomorphism between demands and random utility components. Such homeomorphic relationships are essential for global identification of the structural model, the existence of well-specified probability models for choice variables and for the analysis of revealed stochastic preference.

Keywords: nonparametric random utility model, stochastic demand, global homeomorphism, coherency

JEL classification: C14, C31, C51, D1

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1 Introduction

Structural micro-econometric demand analysis is concerned with modelling consumer demands in terms of a behavioral choice model, taking account of randomness in consumer choice data. One interpretation of randomness in consumer choice data, given prices and total expenditure, is unobserved heterogeneity in consumer preferences. Unobserved preference heterogeneity can be modelled in terms of random utility $U(\mathbf{x}, \epsilon)$ where $x \in \mathbb{R}^J_+$ is a vector of continuous consumption amounts of J goods and $\epsilon \in \mathbb{R}^{J-1}$ is a $J-1$ dimensional vector representing unobserved heterogeneity in preferences. Then, given prices $\mathbf{p} \in \mathbb{R}^{J-1}_{++}, p_J \equiv 1$, and total expenditure $m > 0$, stochastic demand functions $h(\mathbf{p}, m, \epsilon)$ for the $J-1$ inside goods $\mathbf{x}_{-J} = (x_1, \dots x_{J-1})'$ solve

$$\begin{aligned} \mathbf{p} &= \mathbf{MRS}(\mathbf{x}_{-J}, m - \mathbf{p}'\mathbf{x}_{-J}, \epsilon) \\ \mathbf{x}_{-J} &= h(\mathbf{p}, m, \epsilon), \end{aligned}$$

where $\mathbf{MRS}(\mathbf{x}, \epsilon) = \left[\frac{\partial}{\partial x_j} U(\mathbf{x}, \epsilon) / \frac{\partial}{\partial x_J} U(\mathbf{x}, \epsilon)\right]_{j=1,\dots,J-1}$ is the $J-1$ dimensional vector of stochastic marginal rates of substitution. This paper addresses the question under which conditions on the structural model $U(x, \epsilon)$ or $\mathbf{MRS}(\mathbf{x}, \epsilon)$ the mapping between demands $\mathbf{x}_{-J}$ and unobserved preference heterogeneity ϵ is a homeomorphism, given $\mathbf{p}$ and m. This property is necessary for global nonparametric identification of $U(\mathbf{x}, \epsilon)$ (Brown and Matzkin (1995)), the existence of well-specified probability models for choice variables $\mathbf{x}_{-J}$, given $\mathbf{p}$ and m, and, hence, for the analysis of revealed stochastic preference (McFadden and Richter (1971, 1990) and McFadden (2004)). Section 2 lays out the formal framework and notation for this analysis. Section 3 presents a result for local invertibility of demand functions, primarily as a reference point for the main parts of the paper. Section 4 presents results on global invertibility when unobserved preference heterogeneity enters the model in a separable fashion. Section 5 presents invertibility results for models with nonseparable heterogeneity. Section 6 concludes.

2 Framework for Analysis

Conditions under which the relationship between demands and heterogeneity terms is a global homeomorphism are of interest for two main reasons. They guarantee global nonparametric identification of the structural random utility model (Brown and Matzkin (1995), following the approach taken by Roehrig (1988)) as well as the existence of a well-specified probability model for the induced choice variables. They thereby eliminate potential problems in the analysis of revealed stochastic preference which relies on proper probability models for observable choices. In the Appendix we present an example of a deficient probability model in which there are continuous choice variables but they do not have a joint density. In the absence of a proper probability model the postulates of revealed stochastic preference cannot be verified.

The analysis in this paper proceeds within the following setup. Let $(\mathbf{X}, \mathcal{X})$ be a metric space of choice variables, where $\mathbf{X} \subset \mathbb{R}^J$ and $\mathcal{X}$ is the Borel σ-algebra of subsets of $\mathbf{X}$.

Denote by $(\mathbf{U}, \mathcal{U}, P_\epsilon)$ the probability space defined over all random (direct) utility functions $U : \mathbb{R}^J_+ \times \mathbb{R}^{J-1} \to \mathbb{R}$, i.e. $U(\mathbf{x}, \epsilon)$, where $\mathbf{x} \in \mathbb{R}^J_+$ is a vector of continuous consumption amounts, $\epsilon \in \mathbb{R}^{J-1}$ is a $J-1$ dimensional random component representing unobserved preference heterogeneity, distributed according to probability measure P_ϵ. $\mathcal{U}$ is the Borel σ-algebra of subsets of $\mathbf{U}$. Elements $U \in \mathbf{U}$ in this probability space satisfy the following assumptions:

Assumption A1: For each ϵ, $U \in \mathcal{U}$ is continuous in its arguments, continuously differentiable in $\epsilon, \mathbf{x}$, strongly monotone, concave and strictly quasi-concave in $\mathbf{x}$.

Assumption A2: Let $\mathbf{MRS}(\mathbf{x}, \epsilon) = \left[\frac{\partial}{\partial x_j} U(\mathbf{x}, \epsilon) / \frac{\partial}{\partial x_J} U(\mathbf{x}, \epsilon)\right]_{j=1,\dots,J-1}$. Suppose that the $(J-1) \times (J-1)$ matrix $\nabla_\epsilon \mathbf{MRS}(\mathbf{x}, \epsilon)$ has full rank $J-1$ for all ϵ.

Assumption A3: (*smoothness in the sense of Debreu*) The bordered Hessian satisfies

$$\begin{vmatrix} \nabla_{\mathbf{w}\mathbf{w}'} U(\mathbf{x}, \epsilon) & \nabla_{\mathbf{w}} U(\mathbf{x}, \epsilon) \\ \nabla_{\mathbf{w}'} U(\mathbf{x}, \epsilon) & 0 \end{vmatrix} \neq 0$$

for all $\mathbf{w}' = (\mathbf{x}', \epsilon')$.

Assumptions A1 - A3 guarantee that the reduced form system of stochastic demands $h(\mathbf{p}, m, \epsilon)$ is a system of continuously differentiable demand functions. In other words, under these assumptions, the system

$$g(\mathbf{x}_{-J}, m, \mathbf{p}, \epsilon) = \mathbf{MRS}(\mathbf{x}_{-J}, m - \mathbf{p}'\mathbf{x}_{-J}, \epsilon) - \mathbf{p}$$

associates a unique value of $\mathbf{x}_{-J}$ with any $\mathbf{p}$, m and ϵ, i.e. it has a well-defined reduced form $\mathbf{x}_{-J} = h(\mathbf{p}, m, \epsilon)$. Assumptions A1-A3, thus, amount to coherency conditions, as introduced by Gourieroux et al. (1980) for parametric simultaneous linear equation systems.

Let $(\mathbf{h}, \mathcal{H}, P_h)$ denote the probability space of demands, where P_h is the probability measure induced by P_ϵ through the nonlinear transformation $h(\mathbf{p}, m, \epsilon)$, given $\mathbf{p}$ and m; and let $\mathcal{H}$ be the Borel σ-field of subsets of h. In then terminology of revealed stochastic preference (McFadden (2004)), the probability spaces $(\mathbf{U}, \mathcal{U}, P_\epsilon)$ and $(\mathbf{h}, \mathcal{H}, P_h)$ are consistent (or $\mathbf{h}$ is $\mathbf{U}$-rational), if i.a. for any $\mathbf{x}_{-J}$ satisfying $\mathbf{p}'\mathbf{x} \leq m$, $x_J \geq 0$, the inverse image of $\mathbf{x}_{-J} = h(\mathbf{p}, m, \epsilon)$ with respect to ϵ, given $\mathbf{p}$ and m, is in $\mathcal{U}$, i.e. $P_h(h(\mathbf{p}, m, \epsilon)) = P_\epsilon(\tilde{U}(\mathbf{p}, m, \mathbf{x}_{-J}))$, where $\tilde{U}(\mathbf{p}, m, \mathbf{x}_{-J})) = \{U \in \mathbf{U} : (\mathbf{x}_{-J}, m - \mathbf{p}'\mathbf{x}_{-J})' = (h(\mathbf{p}, m, \epsilon), x_J)' = \arg\max_{\mathbf{p}'\mathbf{x} \leq m} U(\mathbf{x}, \epsilon)\} \in \mathcal{U}$. In order for unambiguous revelation of stochastic preferences from stochastic demands, this inverse should be unique. This paper provides conditions on the structural model $U \in \mathbf{U}$ that are sufficient for $\mathbf{x}_{-J} = h(\mathbf{p}, m, \epsilon)$ and ϵ to be globally one-to-one - or a global homeomorphism -, given any $\mathbf{p}$ and m.

It is worth noting that arguments establishing global homeomorphisms rest on applications of the theorems by Gale and Nikaido (1965) or Mas-Colell (1979). These theorems provide sufficient conditions for the existence of global homeomorphisms. Hence, within the constraints of these theorems, there is no scope to determine necessary conditions for global homeomorphisms. To distinguish the conditions for global invertibility of $h(\mathbf{p}, m, \epsilon)$ from the substantially weaker requirements for local invertibility, the following section presents a local invertibility result, while subsequent sections are devoted to conditions for global invertibility. Local invertibility is necessary for global invertibility, and hence the analysis of local invertibility sheds some light on necessary conditions for global invertibility.

3 Local Invertibility

Definition: The random variable $\mathbf{x} \in \mathbb{R}^J$ has dimension J, denoted by $\dim(\mathbf{x}) = J$, if it has a non-degenerate distribution on $\mathbb{R}^J$.

Assumption A4: $\dim(\epsilon) = J - 1$.

Assumption 5: $\mathrm{Im}\left(\mathbf{MRS}(\mathbf{x}, \epsilon)\right) = \left\{\mathbf{MRS}(\mathbf{x}, \epsilon) : \epsilon \in \mathbb{R}^{J-1}\right\} = \mathbb{R}^{J-1}_{++}$.

Lemma 3.1: (necessary conditions for $\dim(\mathbf{x}_{-J}) = J - 1$; omitted; see Beckert (2004))

Lemma 3.2: (*Local Invertibility*) *Suppose A1-A5 hold. Consider the system of demand* *functions for the $J-1$ inside goods $\mathbf{x}_{-J} = h(\mathbf{p}, m, \epsilon)$. Fix $\epsilon_0 \in \mathbf{R}^{J-1}$. Then, there exists $\delta > 0$ such that, on the sets*

$$\begin{aligned} \mathcal{E}(\epsilon_0; \delta) &:= \{\epsilon \in \mathbf{R}^{J-1} : ||\epsilon - \epsilon_0|| < \delta\} \\ \mathcal{X}(\epsilon_0; \delta) &:= \{\mathbf{z} \in \mathbb{R}_+^{J-1} : \mathbf{z} = h(\mathbf{p}, m, \epsilon) \text{ for } \epsilon \in \mathcal{E}(\epsilon_0; \delta)\}. \end{aligned}$$

$\mathbf{x}_{-J} = h(\mathbf{p}, m, \epsilon)$ and ϵ are one-to-one, given any $(\mathbf{p}', m)' > \mathbf{0}$, and hence the distribution of $x_{-J} \in \mathcal{X}(\epsilon_0; \delta)$, conditional on $\mathbf{p}$ and m, is non-degenerate.

The proof is an application of the Implicit Function Theorem and is omitted. Local invertibility is not enough for global identification of unobserved preference heterogeneity and hence of random utility. The result does, however, point to a necessary condition for global invertibility. Suppose that $h(\mathbf{p}, m, \epsilon)$ is invertible with respect to $\mathbf{p}$. Denote this inverse by $q(\mathbf{x}_{-J}, m, \epsilon)$. The mapping between $\mathbf{x}_{-J}$ and ϵ being homeomorphic, given $\mathbf{p}$ and m, is equivalent to $\mathbf{p} - q(\mathbf{x}_{-J}, m, \epsilon) = \mathbf{0}$ being an implicit homeomorphism between $\mathbf{x}_{-J}$ and ϵ, given $\mathbf{p}$ and m. Let $B_{-J}(\mathbf{p}, m) = \{\mathbf{x}_{-J} \in \mathbb{R}_+^{J-1} : \mathbf{p}'\mathbf{x}_{-J} + x_J = m, x_J \geq 0\}$. Since

$$\mathbf{p} = q(\mathbf{x}_{-J}, m, \epsilon) = \mathbf{MRS}(\mathbf{x}_{-J}, m - \mathbf{p}'\mathbf{x}_{-J}, \epsilon),$$

this implies that $\mathbf{x}_{-J}$ and ϵ are one-to-one, given $\mathbf{p}$ and m, if, and only if, for any $\mathbf{x}_{-J} \in B_{-J}(\mathbf{p}, m)$, $\mathbf{MRS}(\mathbf{x}_{-J}, m - \mathbf{p}'\mathbf{x}_{-J}, \epsilon)$ is an implicit homeomorphism between $\mathbf{x}_{-J}$ and ϵ. Under conventional smoothness assumptions, the rank condition on the matrix $\nabla_\epsilon \mathbf{MRS}(\mathbf{x}, \epsilon)$ in A2, $\text{rk}(\nabla_\epsilon \mathbf{MRS}(\mathbf{x}, \epsilon)) = J - 1$ on $\mathbf{x}_{-J} \in B_{-J}(\mathbf{p}, m)$, is a necessary, though not sufficient condition for this.

4 Global Invertibility with Separable Heterogeneity

This section examines structural model specifications in which unobserved preference heterogeneity ϵ enters in a separable form. Specifically, it considers models for marginal rates of substitution in which unobserved preference heterogeneity enters in a multiplicative fashion. Such specifications permit higher order derivatives of random utility to depend on unobserved heterogeneity as well, allowing i.a. for heterogeneous curvature of utility and heterogenous substitution elasticities. They include the model of Brown and Matzkin (1995) as a special case.

The following additional assumptions are maintained:

Assumption A4': In addition to A4, assume that supp(ϵ) is a rectangle.

Assumption A6: $\mathbf{MRS}(\mathbf{x},\epsilon)$ is multiplicatively separable with respect to ϵ:

$$\mathbf{MRS}(\mathbf{x},\epsilon) = K(\mathbf{x})\psi(\epsilon),$$

where $K(\mathbf{x})$ is a $(J-1)\times(J-1)$ matrix with full rank and span equal to R_{++}^{J-1}, and $\psi : \mathbf{R}^{J-1} \to \mathbf{R}^{J-1}$ satisfies the Gale and Nikaido or Mas-Colell conditions.[1]

Lemma 4.1: *Suppose A1, A2, A3, A4' and A6 hold. Then, for any* $\mathbf{p}$ *and* m, $h(\mathbf{p},m,\epsilon)$ *is globally invertible for all* $\mathbf{x}_{-J} \in B_{-J}(\mathbf{p},m)$*, and, hence,* $\mathbf{x}_{-J}$ *has a non-degenerate distribution on* $B_{-J}(\mathbf{p},m)$*, given any* $\mathbf{p}$ *and* m.

Proof: From the first-order conditions and A8,

$$\phi(\epsilon) = K(\mathbf{x})^{-1}\mathbf{p},$$

and the result follows from an application of the Gale Nikaido or the Mas-Colell Theorem. □

This result can be slightly generalized, using the following

Assumption A6': $\mathbf{MRS}(\mathbf{x},\epsilon)$ is multiplicatively separable with respect to ϵ:

$$\mathbf{MRS}(\mathbf{x},\epsilon) = v(\mathbf{x}) + K(\mathbf{x})\psi(\epsilon),$$

where $v(\mathbf{x})$ is a $(J-1)\times 1$ vector of nonnegative functions, $K(\mathbf{x})$ is a $(J-1)\times(J-1)$ matrix with full rank and span equal to R_{++}^{J-1}, and $\psi : \mathbf{R}^{J-1} \to \mathbf{R}^{J-1}$ satisfies the Gale and Nikaido or Mas-Colell conditions.

Lemma 4.2: *Suppose A1, A2, A3, A4' and A6' hold. Then, for any* $\mathbf{p}$ *and* m, $h(\mathbf{p},m,\epsilon)$ *is globally invertible for all* $\mathbf{x}_{-J} \in B_{-J}(\mathbf{p},m)$*, and, hence,* $\mathbf{x}_{-J}$ *has a non-degenerate distribution on* $B_{-J}(\mathbf{p},m)$*, given any* $\mathbf{p}$ *and* m.

The proof proceeds as in for Lemma 4.1. As an illustration of the these results, consider the random utility model

$$U(\mathbf{x},\epsilon) = u(\mathbf{x}_{-J})'\psi(\epsilon) + \nu(\mathbf{x}),$$

[1]Gale and Nikaido: support of ϵ is compact and convex, and the Jacobian of $\phi(\epsilon)$ is a P matrix for every ϵ, i.e. every principal minor has positive sign; Mas-Colell's conditions are slightly weaker.

where $u(\cdot)$ is defined on $\mathbb{R}_+^{J-1}$, monotonically increasing and weakly concave, $\nu(\cdot)$ is defined on $\mathbb{R}_+^J$ and satisfies A1 and A3, and $\psi(\epsilon)$ as in A4'. In this model, the J goods are nonseparable, and marginal utilities may involve any subset of the components of ϵ. Then,

$$\begin{aligned}\mathbf{MRS}(\mathbf{x},\epsilon) &= \left[\frac{\frac{\partial}{\partial x_j}\nu(\mathbf{x})}{\frac{\partial}{\partial x_J}\nu(\mathbf{x})}\right]_{j=1,\ldots,J-1} + \left[\frac{\partial}{\partial x_J}\nu(\mathbf{x})\right]^{-1}\left[\frac{\partial}{\partial x_j}u(\mathbf{x}_{-J})'\right]_{j=1,\ldots,J-1}\psi(\epsilon) \\ &= v(\mathbf{x}) + K(x)\psi(\epsilon),\end{aligned}$$

where $v(\mathbf{x}) = \left[\frac{\frac{\partial}{\partial x_j}\nu(\mathbf{x})}{\frac{\partial}{\partial x_J}\nu(\mathbf{x})}\right]_{j=1,\ldots,J-1} \in \mathbb{R}_+^{J-1}$ and $K(\mathbf{x}) = \left[\frac{\partial}{\partial x_J}\nu(\mathbf{x})\right]^{-1}\left[\frac{\partial}{\partial x_j}u(\mathbf{x}_{-J})'\right]_{j=1,\ldots,J-1}$. A1 and A3 imply that $K(\mathbf{x})$ has full rank and that its span is $\mathbb{R}_{++}^{J-1}$. The model due to Brown and Matzkin (1995) can be obtained by choosing $u(\cdot)$ and $\psi(\epsilon)$ the respective identity functions, i.e. $u(\mathbf{x}_{-J}) = \mathbf{x}_{-J}$ for any $\mathbf{x}_{-J} \in \mathbb{R}_+^{J-1}$, and $\psi(\epsilon) = \epsilon$ for any ϵ, and $\nu(\mathbf{x}) = \phi(\mathbf{x}) + x_J$, so that $U(\mathbf{x},\epsilon) = \phi(\mathbf{x}) + \mathbf{x}_{-J}'\epsilon + x_J$. Brown and Matzkin's model implies that marginal rates of substitution are additive in ϵ, hence invertibility follows directly from the first-order conditions and no recourse to the Gale Nikaido or Mas-Colell results is necessary, so that ϵ need not have rectangular support. Another illustration is provided by a random coefficient Cobb-Douglas utility model, where the random coefficients are functions of ϵ satisfying Gale Nikaido or Mas-Colell conditions.

5 Global Invertibility with Non-separable Heterogeneity

Non-separable cases require stronger assumptions.

Assumption A7: $U(\mathbf{x},\epsilon)$ satisfies $\nabla_\epsilon \mathbf{MRS}(\mathbf{x},\epsilon)\nabla_{\mathbf{x}_{-J}}\mathbf{MRS}(\mathbf{x},\epsilon) = A(\mathbf{x},\epsilon)$ negative definite almost surely.

Lemma 5.1: *Suppose that A1, A3, A4', and A7 hold. Then, for any* $\mathbf{p}$ *and* m, $h(\mathbf{p},m,\epsilon)$ *is globally invertible for all* $\mathbf{x}_{-J} \in B_{-J}(\mathbf{p},m)$, *and, hence,* $\mathbf{x}_{-J}$ *has a non-degenerate distribution on* $B_{-J}(\mathbf{p},m)$, *given any* $\mathbf{p}$ *and* m.

Proof: By A3, $\nabla_{\mathbf{x}_{-J}}\mathbf{MRS}(\mathbf{x}_{-J}, m-\mathbf{p}'\mathbf{x}_{-J},\epsilon)$ has full rank, so that its inverse exists. From the first-order conditions,

$$\nabla_\epsilon h(\mathbf{p},m,\epsilon) = -\left[\nabla_{\mathbf{x}_{-J}}\mathbf{MRS}(\mathbf{x}_{-J}, m-\mathbf{p}'\mathbf{x}_{-J},\epsilon)\right]^{-1}\nabla_\epsilon\mathbf{MRS}(\mathbf{x}_{-J}, m-\mathbf{p}'\mathbf{x}_{-J},\epsilon).$$

The Gale Nikaido Theorem requires that this $(J-1)\times(J-1)$ Jacobian matrix has all principal minors positive. Magnus and Neudecker, Chapt.1 Theorem 29, establishes that for symmetric matrices this is equivalent to it being positive definite. Therefore, A7 implies

$$\nabla_\epsilon \mathbf{MRS}(\mathbf{x}_{-J}, m-\mathbf{p}'\mathbf{x}_{-J}, \epsilon) = A(\mathbf{x}_{-J}, m-\mathbf{p}'\mathbf{x}_{-J}, \epsilon)\left[\nabla_{\mathbf{x}_{-J}} \mathbf{MRS}(\mathbf{x}_{-J}, m-\mathbf{p}'\mathbf{x}_{-J}, \epsilon)\right]^{-1},$$

so that $\nabla_\epsilon h(\mathbf{p}, m, \epsilon)$ is seen to be positive definite. □

The following corollary follows immediately.

Corollary: *Suppose that A1, A3, A4', and A7 hold. Then,* $\nabla_\epsilon h(\mathbf{p}, m, \epsilon)$ *is positive definite symmetric for all* $\mathbf{p}, m, \epsilon$.

(A constructed) Example: Suppose that

$$U(x_1, x_2, x_3) = \left(\alpha \exp(\frac{x_1}{\epsilon_1}) + \exp(x_3)\right)^{\epsilon_1} + \left(\beta \exp(\frac{x_2}{\epsilon_2}) + \exp(x_3)\right)^{\epsilon_2},$$

where α, β are positive parameters and ϵ_1, ϵ_2 are random components; $\epsilon_i > 0$ is necessary and sufficient for strict monotonicity and strict quasi-concavity, and $\epsilon_i < 1$ is required for concavity, $i = 1, 2$. Here, (x_1, x_3) and (x_2, x_3) are nonseparable. Moreover,

$$\mathbf{MRS}(\mathbf{x}, \epsilon) = \begin{bmatrix} \alpha \exp(\frac{x_1}{\epsilon_1} - x_3) \\ \beta \exp(\frac{x_2}{\epsilon_2} - x_3) \end{bmatrix}$$

shows that the model is nonseparable in the stochastic components. Since

$$\begin{aligned} \nabla_{\mathbf{x}_{-3}} \mathbf{MRS}(\mathbf{x}, \epsilon) &= \begin{bmatrix} \frac{\alpha}{\epsilon_1} \exp(\frac{x_1}{\epsilon_1} - x_3) & 0 \\ 0 & \frac{\beta}{\epsilon_2} \exp(\frac{x_2}{\epsilon_2} - x_3) \end{bmatrix} \\ \nabla_\epsilon \mathbf{MRS}(\mathbf{x}, \epsilon) &= \begin{bmatrix} -\frac{\alpha}{\epsilon_1^2} \exp(\frac{x_1}{\epsilon_1} - x_3) & 0 \\ 0 & -\frac{\beta}{\epsilon_2^2} \exp(\frac{x_2}{\epsilon_2} - x_3) \end{bmatrix} \\ &= \begin{bmatrix} -\frac{x_1}{\epsilon_1} & 0 \\ 0 & -\frac{x_2}{\epsilon_2} \end{bmatrix} \nabla_{\mathbf{x}_{-3}} \mathbf{MRS}(\mathbf{x}, \epsilon), \end{aligned}$$

it follows that

$$\nabla_\epsilon \mathbf{MRS}(\mathbf{x}, \epsilon) \nabla_{\mathbf{x}_{-3}} \mathbf{MRS}(\mathbf{x}, \epsilon) = \begin{bmatrix} -\frac{x_1}{\epsilon_1} & 0 \\ 0 & -\frac{x_2}{\epsilon_2} \end{bmatrix} \left[\nabla_{\mathbf{x}_{-3}} \mathbf{MRS}(\mathbf{x}, \epsilon)\right]^2,$$

a diagonal matrix with negative elements on the diagonal, almost surely. Hence A7 is met.

An assumption slightly weaker than A7 is[2]

Assumption A8: $U(\mathbf{x}, \epsilon)$ is strictly concave in $\mathbf{x}_{-J}$ and linear in the outside good x_J, and $\nabla_\epsilon \mathbf{MRS}(\mathbf{x}, \epsilon)$ is positive definite almost surely.

Lemma 5.2: *Suppose that A1, A3, A4', and A8 hold. Then, for any* $\mathbf{p}$ *and* m, $h(\mathbf{p}, m, \epsilon)$ *is globally invertible for all* $\mathbf{x}_{-J} \in B_{-J}(\mathbf{p}, m)$*, and, hence,* $\mathbf{x}_{-J}$ *has a non-degenerate distribution on* $B_{-J}(\mathbf{p}, m)$*, given any* $\mathbf{p}$ *and* m.

Proof: A8 implies that $-\nabla_{\mathbf{x}_{-J}} \mathbf{MRS}(\mathbf{x}, \epsilon)$ is positive definite for all $\mathbf{x}$ and ϵ, and symmetric. Its inverse inherits these properties. Horn and Johnson, Theorem 7.6.3, then implies that its product with a positive definite matrix $\nabla_\epsilon \mathbf{MRS}(\mathbf{x}, \epsilon)$ is diagonalizable, i.e. similar[3] to a diagonal matrix, whose eigenvalues are positive. Similarity means that there exists a nonsingular transformation S of $x_{-J} = h(\mathbf{p}, m, \epsilon)$, possibly dependent on $\mathbf{p}, m, \epsilon$, such that the transformed vector of demands has a distribution, conditional on $\mathbf{p}$ and m, that can be deduced from the distribution of ϵ by evaluation at the inverse function and multiplication by a Jacobian which is diagonal. Then, one diagonalization is

$$\nabla_\epsilon h(\mathbf{p}, m, \epsilon) = S(\mathbf{p}, m, \epsilon) D(\mathbf{p}, m, \epsilon) S(\mathbf{p}, m, \epsilon)^{-1},$$

where $S(\mathbf{p}, m, \epsilon)$ is a nonsingular matrix consisting of the $J-1$ eigenvectors of $\nabla_\epsilon h(\mathbf{p}, m, \epsilon)$ and $D(\mathbf{p}, m, \epsilon)$ is a diagonal matrix with the positive eigenvalues of $\nabla_\epsilon h(\mathbf{p}, m, \epsilon)$ on its diagonal. This is necessary and sufficient for $\nabla_\epsilon h(\mathbf{p}, m, \epsilon)$ to be positive definite almost surely. Note that under A10 $\nabla_\epsilon h(\mathbf{p}, m, \epsilon)$ is not necessarily symmetric, so that the Magnus and Neudecker result cannot be applied and the Gale Nikaido conditions need to be verified. For $k = 1, \ldots, J-1$, define $k \times (J-1)$ matrices $E_k = [\mathbf{I}_k, \mathbf{0}]$, where $\mathbf{0}$ is a $(J-1-k) \times (J-1)$ matrix of zeros. Then, the kth principal minor of the Jacobian matrix $\nabla_\epsilon h(\mathbf{p}, m, \epsilon)$ is

$$|\nabla_\epsilon h(\mathbf{p}, m, \epsilon)_k| = |E_k \nabla_\epsilon h(\mathbf{p}, m, \epsilon) E_k'| = |E_k S(\mathbf{p}, m, \epsilon) D(\mathbf{p}, m, \epsilon) S(\mathbf{p}, m, \epsilon)^{-1} E_k|.$$

Therefore, for any $\mathbf{y} \in \mathbf{R}^k, \mathbf{y} \neq \mathbf{0}$, and any $k = 1, \ldots, J-1$,

$$\begin{aligned} \mathbf{y}' \nabla_\epsilon h(\mathbf{p}, m, \epsilon)_k \mathbf{y} &= \mathbf{y}' E_k \nabla_\epsilon h(\mathbf{p}, m, \epsilon) E_k' \mathbf{y} \\ &= (E_k' \mathbf{y})' \nabla_\epsilon h(\mathbf{p}, m, \epsilon)(E_k' \mathbf{y}) > 0, \end{aligned}$$

[2] A8 appears to be weaker than A9 because it does not imply symmetry of the Jacobian $\nabla_\epsilon h(\mathbf{p}, m, \epsilon)$.

[3] An $n \times n$ matrix A is similar to an $n \times n$ matrix B if there exists a nonsingular $n \times n$ matrix S such that $B = S^{-1} A S$. Similarity is an equivalence relation. See Horn and Johnson, section 1.3, for further details.

where $E_k' \mathbf{y} \neq 0$ and the last inequality follows because $\nabla_\epsilon h(\mathbf{p}, m, \epsilon)$ is positive definite almost surely. Hence, the Jacobian has all principal submatrices positive definite almost surely. Therefore, for any $k = 1, \ldots, J-1$, there exists a full-rank $k \times k$ matrix $P_k(\mathbf{p}, m, \epsilon)$ such that

$$\begin{aligned} \nabla_\epsilon h(\mathbf{p}, m, \epsilon)_k &= P_k(\mathbf{p}, m, \epsilon) P_k(\mathbf{p}, m, \epsilon)' \\ \Rightarrow |\nabla_\epsilon h(\mathbf{p}, m, \epsilon)_k| &= |P_k(\mathbf{p}, m, \epsilon) P_k(\mathbf{p}, m, \epsilon)'| \\ &= |P_k(\mathbf{p}, m, \epsilon)|^2 > 0. \end{aligned}$$

Therefore, the Gale Nikaido conditions are satisfied. □

Comment: Note that for the Brown and Matzkin (1995) model,

$$\begin{aligned} \nabla_{\mathbf{x}_{-J}} \mathbf{MRS}(\mathbf{x}, \epsilon) &= \left[\left[\frac{\partial}{\partial x_J} u(\mathbf{x}) \right]^{-1} \nabla_{\mathbf{x}_{-J} \mathbf{x}'_{-J}} u(\mathbf{x}) \right] \\ &\quad - \left[\frac{\partial}{\partial x_J} u(\mathbf{x}) \right]^{-2} \left[\nabla_{\mathbf{x}_{-J}} u(\mathbf{x}) + \epsilon \right] \left[\frac{\partial^2}{\partial \mathbf{x}_{-J} \partial x_J} u(\mathbf{x}) \right]' \\ \nabla_\epsilon \mathbf{MRS}(\mathbf{x}, \epsilon) &= \left[\frac{\partial}{\partial x_J} u(\mathbf{x}) \right]^{-1} \mathbf{I}_{J-1}. \end{aligned}$$

The first matrix is nonsingular as a consequence of A3. If U is quasi-linear in the outside good, concavity implies that the first matrix is negative definite.

Quasi-linearity leaves the possibility that the demand for the outside good may be negative. Avoiding quasi-linearity of $U(\mathbf{x}, \epsilon)$ in the outside good x_J may come at a high price, as the following result shows. It uses

Assumption A9: Both $\nabla_\epsilon \mathbf{MRS}(\mathbf{x}, \epsilon)$ and $-\left[\nabla_{\mathbf{x}_{-J}} \mathbf{MRS}(\mathbf{x}, \epsilon)\right]^{-1}$ are strictly totally positive almost surely, and all the minors of $\nabla_\epsilon \mathbf{MRS}(\mathbf{x}, \epsilon)$ are bounded almost surely.

Lemma 5.3: *Suppose that A1, A3, A4', and A8 hold. Then, for any* $\mathbf{p}$ *and* m, $h(\mathbf{p}, m, \epsilon)$ *is globally invertible for all* $\mathbf{x}_{-J} \in B_{-J}(\mathbf{p}, m)$, *and, hence,* $\mathbf{x}_{-J}$ *has a non-degenerate distribution on* $B_{-J}(\mathbf{p}, m)$, *given any* $\mathbf{p}$ *and* m.

Proof: Again, it needs to be shown that the Gale Nikaido conditions are satisfied. Let $A = -\left[\nabla_{\mathbf{x}_{-J}} \mathbf{MRS}(\mathbf{x}, \epsilon)\right]^{-1}$ and $B = \nabla_\epsilon \mathbf{MRS}(\mathbf{x}, \epsilon)$. Furthermore, let α and β form a partition of $\{1, \ldots, J-1\}$, and let $A(\alpha, \beta)$ be the sub-matrix composed of the rows and columns of A

indexed by α and β, and similarly for $B(\alpha, \beta)$. By the Cauchy-Binet formula,

$$|\nabla_\epsilon h(\mathbf{p}, m, \epsilon)_k| = \sum_\gamma |A(\{1, \ldots, k\}, \gamma)||B(\gamma, \{1, \ldots, k\})|,$$

where γ indexes all partitions of $\{1, \ldots, J-1\}$ of cardinality k, $k = 1, \ldots, J-1$. Since both A and B are strictly totally positive, it follows from A9 that $|B(\gamma, \{1, \ldots, k\})|$ as a function of γ is bounded a.s. and has no sign changes. Therefore, for any $k = 1, \ldots, J-1$, $|\nabla_\epsilon h(\mathbf{p}, m, \epsilon)_k|$ is the convolution of a bounded function without sign changes, B, with a strictly totally positive kernel, A. It then follows from Karlin (1968), chapter 5 Theorem 3.1, that $|\nabla_\epsilon h(\mathbf{p}, m, \epsilon)_k|$ as a function of k does not exhibit any sign changes. Hence, the Gale Nikaido conditions are satisfied. □

Example: (*3 goods case*)

Suppose $\mathbf{x} \in \mathbf{R}^3_+$, i.e. there are two inside goods $(x_1, x_2)'$ and the outside good x_3, and let $U(\mathbf{x}, \epsilon) \in \mathcal{U}$ be strictly concave. Denote $MU_i(\mathbf{x}, \epsilon) = \frac{\partial}{\partial x_i} U(\mathbf{x}, \epsilon)$, for $i = 1, 2, 3$. Suppose, furthermore, that

$$\nabla_{\mathbf{x}_{-3}} \mathbf{MRS}(\mathbf{x}, \epsilon) = \begin{bmatrix} \frac{\partial}{\partial x_1} \frac{MU_1(\mathbf{x},\epsilon)}{MU_3(\mathbf{x},\epsilon)} & \frac{\partial}{\partial x_2} \frac{MU_1(\mathbf{x},\epsilon)}{MU_3(\mathbf{x},\epsilon)} \\ \frac{\partial}{\partial x_1} \frac{MU_2(\mathbf{x},\epsilon)}{MU_3(\mathbf{x},\epsilon)} & \frac{\partial}{\partial x_2} \frac{MU_2(\mathbf{x},\epsilon)}{MU_3(\mathbf{x},\epsilon)} \end{bmatrix}, \ \operatorname{sgn}\left(\nabla_{\mathbf{x}_{-3}} \mathbf{MRS}(\mathbf{x}, \epsilon)\right) = \begin{bmatrix} - & + \\ + & - \end{bmatrix};$$

here, the negative diagonal elements follow from the assumed strict concavity of $U(\mathbf{x}, \epsilon)$. Then,

$$\operatorname{sgn}\left(-\left[\nabla_{\mathbf{x}_{-3}} \mathbf{MRS}(\mathbf{x}, \epsilon)\right]^{-1}\right) = \operatorname{sgn}\left(-\det\left(\nabla_{\mathbf{x}_{-3}} \mathbf{MRS}(\mathbf{x}, \epsilon)\right)^{-1} \begin{bmatrix} - & - \\ - & - \end{bmatrix}\right).$$

For this matrix to be strictly totally positive, it suffices that the own effects dominate cross effects, in the sense that $\left|\frac{\partial}{\partial x_i} \frac{MU_i(\mathbf{x},\epsilon)}{MU_3(\mathbf{x},\epsilon)}\right| > \left|\frac{\partial}{\partial x_i} \frac{MU_j(\mathbf{x},\epsilon)}{MU_3(\mathbf{x},\epsilon)}\right|$, for $i, j = 1, 2$ and $i \neq j$. Under these assumptions, provided that $\frac{\partial}{\partial x_3} \mathbf{MRS}(\mathbf{x}, \epsilon) > \mathbf{0}$ a.s.,

$$\nabla_{\mathbf{p}} h(\mathbf{p}, m, \epsilon) = \left[\nabla_{\mathbf{x}_{-3}} \mathbf{MRS}(\mathbf{x}, \epsilon)\right]^{-1} \left[\mathbf{I}_2 + \frac{\partial}{\partial x_3} \mathbf{MRS}(\mathbf{x}, \epsilon) h(\mathbf{p}, m, \epsilon)'\right],$$

where $\mathbf{x}' = [h(\mathbf{p}, m, \epsilon)', m - \mathbf{p}' h(\mathbf{p}, m, \epsilon)]$, has all entries negative a.s., and therefore goods 1 and 2 are (symmetric) gross complements and goods 1 and 3 are (not necessarily symmetric) gross substitutes, as are goods 2 and 3. Under the above assumptions, nothing can be said about net substitutability. The Jacobian of the Hicksian demand system depends, via the Slutsky decomposition of the Jacobian of Marshallian demands, on the vector of income effects, about which the assumptions are tacit.

Analogous assumptions yield that $\nabla_\epsilon MRS(\mathbf{x}, \epsilon)$ is strictly totally positive.

Example: (*4 goods case*)

The previous example may be generalized as follows. Suppose $\mathbf{x} \in \mathbf{R}^4_+$, i.e. there are three inside goods $(x_1, x_2, x_3)'$ and the outside good x_4, and let $U(\mathbf{x}, \epsilon) \in \mathcal{U}$ be strictly concave. For notational simplicity, let $A = \nabla_{\mathbf{x}_{-4}} MRS(\mathbf{x}, \epsilon)$, a 3×3 matrix. Suppose further that

$$\text{sgn}(A) = \begin{bmatrix} - & + & + \\ + & - & + \\ + & + & - \end{bmatrix},$$

and $\det(A(\alpha', \alpha')) > 0$ a.s. for $\alpha' \in \{(1,2), (1,3), (2,3)\}$ and $\det(A) < 0$ a.s., A having full rank a.s.

It then needs to be shown that $-A^{-1}$ is strictly totally positive almost surely. Notice first that

$$\det(-A^{-1}) = -\det(A^{-1}) = -(\det(A))^{-1} > 0.$$

Moreover, the assumption on the determinants of the principal minors of A implies

$$-A^{-1} = -|A|^{-1} \begin{bmatrix} \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} & -\begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} & \begin{vmatrix} A_{21} & A_{21} \\ A_{31} & A_{31} \end{vmatrix} \\ -\begin{vmatrix} A_{12} & A_{13} \\ A_{32} & A_{33} \end{vmatrix} & \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} & -\begin{vmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{vmatrix} \\ \begin{vmatrix} A_{12} & A_{13} \\ A_{22} & A_{23} \end{vmatrix} & -\begin{vmatrix} A_{11} & A_{13} \\ A_{21} & A_{23} \end{vmatrix} & \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} \end{bmatrix},$$

and therefore,

$$\begin{aligned} \text{sgn}(-A^{-1}) &= \text{sgn}\left(-|A|^{-1} \begin{bmatrix} \begin{vmatrix} - & + \\ + & - \end{vmatrix} & -\begin{vmatrix} + & + \\ + & - \end{vmatrix} & \begin{vmatrix} + & - \\ + & + \end{vmatrix} \\ -\begin{vmatrix} + & + \\ + & - \end{vmatrix} & \begin{vmatrix} - & + \\ + & - \end{vmatrix} & -\begin{vmatrix} - & + \\ + & + \end{vmatrix} \\ \begin{vmatrix} + & + \\ - & + \end{vmatrix} & -\begin{vmatrix} - & + \\ + & + \end{vmatrix} & \begin{vmatrix} - & + \\ + & - \end{vmatrix} \end{bmatrix} \right) \\ &= \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & + \end{bmatrix}. \end{aligned}$$

Also, for $\alpha', \beta' \in \{(1,2),(1,3),(2,3)\}$,

$$\det(-A^{-1}(\alpha',\beta')) = (-1)^{\sum_{i\in\alpha} i+\sum_{j\in\beta} j}\det(-A(\beta',\alpha'))/\det(A)^2$$

(cp. Horn and Johnson (1985)) implies

$$\begin{aligned}
\det\left(-A^{-1}((1,2),(1,2))\right) &= \det\left(A^{-1}((1,2),(1,2))\right) = \det(A((1,2),(1,2)))/\det(A)^2 > 0 \\
\det\left(-A^{-1}((1,3),(2,3))\right) &= \det\left(A^{-1}((1,3),(2,3))\right) = -\det(A((2,3),(1,3)))/\det(A)^2 > 0,
\end{aligned}$$

and analogously for the remaining minors of $-A^{-1}$. This completes the argument.

Conjecture: (n goods)

Suppose $\mathbf{x} \in \mathbf{R}^n_+$, i.e. there are $n-1$ inside goods $(x_1,\ldots,x_{n-1})'$ and the outside good x_n, and let $U(\mathbf{x},\epsilon) \in \mathcal{U}$ be strictly concave. For notational simplicity, let $A = \nabla_{\mathbf{x}_{-n}} MRS(\mathbf{x},\epsilon)$, an $(n-1)\times(n-1)$ matrix. Suppose further that

$$\operatorname{sgn}(A) = \begin{bmatrix} - & + & + & \cdots \\ + & - & + & \cdots \\ + & + & - & \cdots \\ \vdots & & & \ddots \end{bmatrix},$$

and, for α any partition of $\{1,\ldots,n-1\}$, suppose that $\det(A(\alpha',\alpha')) > 0$ a.s. for $\dim(\alpha)$ even, and $\det(A(\alpha',\alpha')) < 0$ a.s. for $\dim(\alpha)$ odd.[4] Then, $-A^{-1}$ is strictly totally positive almost surely. If true, the proof would presumably proceed by induction on n.

The requirement of strict total positivity (A9) seems rather strong, especially since A9 imposes it on the inverse of $-\nabla_{\mathbf{x}_{-J}}\mathbf{MRS}(\mathbf{x},\epsilon)$, which is difficult to interpret. Moreover, it is difficult to check in applications. The preceding examples suggest, however, that sign restrictions on the entries of $\nabla_{\mathbf{x}_{-J}}\mathbf{MRS}(\mathbf{x},\epsilon)$ may permit a weakening of A9. The following result provides one avenue to do that. It uses a result due to Fiedler and Pták (1962) about matrices with such sign restrictions.

Assumption A10: (i) $\nabla_{\mathbf{x}_{-J}}\mathbf{MRS}(\mathbf{x},\epsilon)$ has negative diagonal and non-negative off-diagonal entries, a.s.; (ii) $(-1)^J\nabla_\epsilon\mathbf{MRS}(\mathbf{x},\epsilon)$ has positive diagonal and non-positive off-diagonal entries, and all its principal minors are positive, a.s.; and (iii) $(-1)^J\nabla_{\mathbf{x}_{-J}}\mathbf{MRS}(\mathbf{x},\epsilon) - \nabla_\epsilon\mathbf{MRS}(\mathbf{x},\epsilon) \geq \mathbf{0}$, a.s.

[4] I.e., for α any partition of $\{1,\ldots,n-1\}$, suppose that $(-1)^{\dim(\alpha)}\det(A(\alpha',\alpha')) > 0$ a.s.

Matrices, having properties as in (ii), are sometimes referred to as M-matrices. See, e.g., Horn and Johnson (1991). Let $\mathbf{Z}$ be the class of square matrices whose off-diagonal elements are all non-positive, as in Fiedler and Pták's definition (4,1). And let $\mathbf{K}$ be those elements in $\mathbf{Z}$ which have all principal minors positive, as in Fiedler and Pták's definition (4,4). Lemma 5.4 below uses Fiedler and Pták's Theorem (4,6): If $A \in \mathbf{K}$, $B \in \mathbf{Z}$ and $B - A \geq \mathbf{0}$, then, i.a., $B^{-1}A \in \mathbf{K}$.

Lemma 5.4: *Suppose that A1, A3, A4', and A10 hold. Then, for any* $\mathbf{p}$ *and* m, $h(\mathbf{p}, m, \epsilon)$ *is globally invertible for all* $\mathbf{x}_{-J} \in B_{-J}(\mathbf{p}, m)$, *and, hence,* $\mathbf{x}_{-J}$ *has a non-degenerate distribution on* $B_{-J}(\mathbf{p}, m)$, *given any* $\mathbf{p}$ *and* m.

Proof: By A10(i), $-\nabla_{\mathbf{x}_{-J}}\mathbf{MRS}(\mathbf{x}, \epsilon)$ has positive diagonal and non-positive off-diagonal entries. Hence it belongs to the class $\mathbf{Z}$; and by A10(ii), $(-1)^J \nabla_\epsilon \mathbf{MRS}(\mathbf{x}, \epsilon)$ belongs to the class $\mathbf{K}$. Hence, using A10(iii), by Fiedler and Pták, Theorem (4,6),

$$\begin{aligned} h(\mathbf{p}, m, \epsilon) &= -\left[\nabla_{\mathbf{x}_{-J}}\mathbf{MRS}(\mathbf{x}, \epsilon)\right]^{-1} \nabla_\epsilon \mathbf{MRS}(\mathbf{x}, \epsilon) \\ &= (-1)^J \left[-\nabla_{\mathbf{x}_{-J}}\mathbf{MRS}(\mathbf{x}, \epsilon)\right]^{-1} \nabla_\epsilon \mathbf{MRS}(\mathbf{x}, \epsilon) \in \mathbf{K}, \end{aligned}$$

i.e. all its principal minors are positive, so that the Gale Nikaido conditions are satisfied. □

6 Conclusions

This paper provides various conditions on structural preference models for continuous choices under which the induced stochastic demand system for the inside goods is globally invertible. This broadens the class of random utility models suitable for microeconometric analysis. The synopsis of these conditions emphasizes the view that microeconometric modelling of demand aknowledging unobserved preference heterogeneity comes at the price of additional restrictions on preference, beyond those imposed by microeconomic theory.

Appendix

Consider the following random utility model: $U(x_1, x_2, \epsilon) = \min\{x_1 + x_2, \epsilon x_1 + \frac{1}{2}x_2\}$, where $\text{supp}(\epsilon) = (\frac{1}{2}, +\infty)$. Indifference curves associated with this random utility model are kinked, and the location and angle of the kink are determined by ϵ. Depending on relative prices p_{x_1}/p_{x_2}

and given any income m, various types of solutions to the consumer's utility maximization problem can arise. Interior solutions for $p_{x_1}/p_{x_2} \in (1, 2\epsilon)$ are characterized by $\frac{\bar{x}_2^\star}{\bar{x}_1^\star} = 2(\epsilon - 1)$ and yield $\bar{x}_i^\star = (2(\epsilon-1))^{i-1} m/(p_{x_1} + 2(\epsilon-1)p_{x_2})$, $i = 1, 2$; set-valued solutions arise when either $p_{x_1}/p_{x_2} = 1$, in which case $x_1^\star + x_2^\star = u^\star$, where $u^\star = m/(p_{x_1} + p_{x_2})$ and $x_1^\star \in [\bar{x}_1^\star, m/p_{x_1}]$, $x_2^\star \in [0, \bar{x}_2^\star]$; or when $p_{x_1}/p_{x_2} = 2\epsilon$, in which case $\epsilon x_1^\star + \frac{1}{2}x_2^\star = u^\star = m/2p_{x_2}$ and $x_1^\star \in [0, \bar{x}_1^\star]$, $x_2^\star \in [\bar{x}_2^\star, m/p_{x_2}]$; corner solutions arise when $p_{x_1}/p_{x_2} < 1$ or $p_{x_1}/p_{x_2} > 2\epsilon$.

Now suppose that for a consumer with income $m = 27$, consumption choices $x_1 = 3$ and $x_2 = 18$ are observed at prices $p_{x_1} = 3, p_{x_2} = 1$. Assuming the consumer maximizes $U(x_1, x_2, \epsilon)$, this could either be a corner solution, in which case one infers $\epsilon_1 = 4$; or it could be an element of a set-valued solution, in which case one infers $\epsilon_2 = 3/2$. This amounts to a lack of identification of the structural model. If, in the spirit of revealed preference type comparisons, the price of good one changes to $p_{x_1} = 1$, then ϵ_1 induces another solution in the set $x_1 \in [3, 27]$ and $x_2 = 27 - x_1$, while this ϵ_2 induces a solution in the larger set $x_1 \in [27/2, 27]$ and $x_2 = 27 - x1$. Note that, in fact, given $p_{x_1} = p_{x_2} = 1$ any choice pair $\{(x_1, x_2) : x_1 \leq 27/2, x_1 + x_2 = 27\}$ can be induced by a continuum of values of ϵ, namely all $\epsilon \geq \frac{1}{2}\frac{x_2}{x_1} + 1 \geq \frac{3}{2}$. This implies that any such (x_1, x_2) is observed with positive probability induced by ϵ, $\Pr(\epsilon \geq \frac{1}{2}\frac{x_2}{x_1} + 1)$. This is a deficient probability probability model, since x_1 and x_2 are continuous choice variables, but do not have a joint density.

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