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On the Nash Equilibria for the FCFS Queueing System with Load-Increasing Service Rate

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On the Nash equilibria for the FCFS queueing system with load-increasing service rate

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Abstract

We consider a service system (Q_S) that operates according to the FCFS discipline, and in which the service rate is an increasing function of the queue length. Customers arrive sequentially to the system and decide whether or not to join, using decision rules based upon the queue length on arrival to Q_S. Each customer is interested in selecting a rule that meets a certain optimality criterion with regards to their expected sojourn time in the system; as a consequence, the decision rules of other customers need to be taken into account. Within a particular class of decision rules for an associated infinite player game, the structure of the Nash equilibrium routing policies is characterized. We prove that within this class, there exist a finite number of Nash equilibria, and that at least one of these is non-randomized. Finally, we explore the extent to which the Nash equilibria are characteristic of customer joining behaviour under a learning rule based on system-wide data with the aid of simulation experiments.

KEYWORDS: QUEUES, STATE DEPENDENT SERVICE RATE; NON-COOPERATIVE GAME; NASH EQUILIBRIUM; SIMULATION

1 Introduction

This paper looks at customer joining behaviour into a *first come first served* single server queueing system where the service rate responds to changes in the queue size.

Customers are prepared to join the system only if their expected time there is projected to be not too high. However, due to the nature of the service rate function, the routing decisions of customers that arrive on the scene in the future could affect the sojourn times of customers that are already present in the system.

In order to make an assessment as to whether or not quality of service requirements will be met upon joining such a system, assumptions regarding the form of the routing decisions (as to whether to join or balk) taken by other customers will have to be made. The routing decision

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to be taken by each customer is whether or not to join the system on the basis of quantities which may depend on the number of customers observed on arrival to the system.

This leads us to analyze the problem as an infinite-player non-cooperative (stationary) game, where the expected sojourn times at particular entry states are considered. Here, the aim is to characterize the conditions under which Nash equilibrium routing policies exist, and to explore the structure of such policies.

We also examine a scenario in which customers base their joining decisions on sample mean sojourn times of customers that have previously passed through the system, again, at particular entry states. Thus, routing decisions are subject to dynamic learning. Using simulation methods, we explore the extent to which the long-term (non-transient) behaviour of the system under the learning rule adheres to that under the Nash equilibria.

The seminal and most relevant work on the game-theoretic analysis of this class of queueing system was carried out by Altman and Shimkin [2], in which a processor sharing system was investigated. They established the existence and uniqueness of a symmetric Nash equilibrium joining policy for the stationary game; it was also demonstrated via simulation methods (and, in [1], using the theory of the Stochastic Approximations(SA) algorithm) that it can be used to characterize the convergent behaviour of the system (in an *almost sure* sense) when customers base their joining decisions on a certain class of dynamic learning rule.[1]. Buche and Kushner [6, 7] analyzed a modified learning rule for the processor sharing system in which a discount factor was incorporated: this allowed the most recent system data to be weighted more heavily than that from the distant past. Again, using theory related to Stochastic Approximations, they show that their learning rule converges to that of the symmetric Nash equilibrium in a *weak sense*.

The general theory developed in [2] was applied to a multiple server retrial system in Brooms [4], and to a FIFO system where the service rate is non-increasing in the system load in Brooms [5]. The analysis of the processor sharing system was extended in Ben-Shahar et. al. [3] to the case where customers arrive to the system with differing quality of service requirements (although with the same exponential service distributions). Existence of a Nash equilibrium for this class-heterogeneous scenario was established; uniqueness was also asserted albeit under the proviso that the inter-arrival times were also i.i.d. exponential (in order to facilitate a coupling argument).

The notion of individual optimality pertains neither to the processor sharing system of [2], nor to the one considered in this paper, unless the routing decisions of future arrivals are taken into account; this is because the sojourn time of a customer who enters such a system will depend on the joining rules adopted by future arriving customers. Systems in which characterization of an individually optimal policy is possible, without having to condition on the decision rules of others, have been considered by Naor [10], and Yechiali [11], for example. Lippman and Stidham [8] also considered an exponential service system, consisting of a FCFS queue with a concave increasing, and bounded, service rate. However, they make the key assumption that a '...customer's holding time is not affected by future arrivals': thus individual optimality can be characterized there.

The rest of the paper is organized as follows. In the next section, we specify the model in

[1]We consider this learning rule for our system later in this paper.

detail, which includes a more thorough description of the decision rules used by the arriving customers. In Section 3 the generic random variables, and various processes defined with respect to these, will be introduced. In Section 4, we use coupling arguments to establish stochastic order results for the sojourn time in Q_S with respect to the entry state. This is followed up, in Section 5, by a discussion on monotonicity and continuity of the sojourn time with respect to *symmetric threshold policies*. The properties established in these latter two sections are then brought together in Section 6 to characterize the existence, and structure, of symmetric Nash equilibrium joining policies for the stationary game. An algorithm for finding the symmetric Nash equilibrium policies is outlined in Section 7. A simulation of the system under a similar learning rule to the one proposed in [2] is presented in Section 8. Plots of the empirical average sojourn times and of the entrance probabilities, against time, for various arrival states are presented, and we argue that these show a close correspondence with the behaviour under the stationary game when the Nash equilibrium is unique. We conclude our discussion in Section 9.

2 The Model

We take $\mathbb{Z}^+ = \{1, 2, \ldots\}$, $\mathbb{N} = \mathbb{Z}^+ \cup \{0\}$, and $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ throughout the paper. An arriving customer has to choose between either joining a shared service system, which is a FCFS queue (denoted by Q_S), or balking.

It is assumed that Q_S has a buffer size B, which may be finite or infinite. Any customer which arrives when the buffer is full is not permitted to enter the system.

The departure process in Q_S at queue length x forms a Poisson process at rate $\mu(x)$, where $\mu(x)$ is a *strictly increasing* and *bounded* function on $x \in \{1, 2 \ldots, B\}$, with $\mu(0) = 0$. Set $\overline{\mu} = \sup\{\mu(x) : x = 1, 2, \ldots\}$.

Let θ be the quality of service requirement. If an arriving customer perceives that the (expected/empirical) sojourn time in Q_S is greater than this value, then it will be reluctant to enter the system.

It is assumed that $\mu(1)^{-1} < \theta$. This condition ensures that it is always worthwhile for a customer to enter Q_S if the system is empty upon arrival.

Let the number of customers in Q_S at time t be denoted by $X(t)$ with initial state $X(0) = x_0$. Let A_k be the arrival time of the k-th customer to the system, where $0 = A_0 < A_1 < A_2 < \ldots$; denote this k-th customer by the label C_k, $k \in \mathbb{N}$, where it is assumed that C_0 arrives at time A_0 (i.e. at time 0). Call the sequence of customers $C_0, C_1, \ldots, C_k, \ldots$ the (overall) *arrival stream*.

The decision as to whether C_k enters Q_S or not, is taken on the basis of $X(A_k)$, the queue length in Q_S just *prior* to its arrival.

A customer within the arrival stream can either be *controlled*, or *uncontrolled*, with probabilities $1 - p$, and p, respectively, independently of all other customers, and irrespective of the state of the system upon arrival. If the customer that arrives at time A_k, say, is uncontrolled, then it will enter Q_S if and only if $X(A_k) < B$;

Label the r-th *controlled* customer who could potentially arrive at (albeit not necessarily enter) the system by $C^{(r)}$, $r \in \mathbb{Z}^+$.

Further define $T(r)$ to be the 'arrival index' corresponding to the r-th *controlled* customer; the r-th *controlled* customer receives the label $C_{T(r)}$, and arrives at time $A_{T(r)}$, within the overall arrival stream. The precise construction of the function $T(\cdot) : \mathbb{Z}^+ \longmapsto \mathbb{N}$ will be given in Section 3.

A decision rule, $u(\cdot) : \{0, 1, \ldots, B-1\} \longmapsto [0, 1]$, is defined to be a function that specifies the probability that a customer adhering to it enters Q_S, which is equal to $u(x)$ if the number of customers in Q_S is equal to x just prior to its arrival. Let $\mathbb{U}$ denote the set of all such decision rules. Define $u_k(\cdot)$ to be the decision rule associated with customer C_k, $k \in \mathbb{N}$, and $u^{(r)}(\cdot)$, $r \in \mathbb{Z}^+$, to be the decision rule associated with the r-th *controlled* customer $C^{(r)}$. A policy, $\pi = (u^{(1)}, u^{(2)}, \ldots, u^{(r)}, \ldots) \in \mathbb{U}^\infty$, is a collection of decision rules, whose r-th member, $u^{(r)}(\cdot)$, $r \in \mathbb{Z}^+$, represents the decision rule associated with $C^{(r)}$.

Let $v_k(x, \pi)$, $x \in \{0, 1, \ldots, B-1\}$, be the sojourn time of C_k in Q_S, given that x customers were present in Q_S just prior to its arrival, and that any *controlled* customer arriving in the future adheres to its decision rule inferred by π. Further define $V_k(x, \pi)$ to be the expected value of $v_k(x, \pi)$.

Define $k' = \max\{r \in \mathbb{N} : T(r) \leq k\}$. We draw attention to the slight abuse of terminology insofar that π need only represent the collection $(u^{(k'+1)}, u^{(k'+2)}, \ldots)$: for given x, and by the assumption that C_k joins Q_S, $(u^{(0)}, u^{(1)}, \ldots, u^{(k'-1)}, u^{(k')})$ do not provide any additional information about $v_k(x, \pi)$.

Also, let $v^{(k)}(x, \pi)$, $x \in \{0, 1, \ldots, B–1\}$, be the sojourn time of the k-th *controlled* customer to enter Q_S, given that x customers were present in Q_S just prior to its arrival, and any *controlled* customer arriving in the future adheres to its decision rule inferred by π. Further define $V^{(k)}(x, \pi)$ to be the expected value of $v^{(k)}(x, \pi)$. Here, π need only represent $(u^{(k+1)}, u^{(k+2)}, \ldots)$.

There is no collaboration between customers, and each *controlled* customer seeks to choose an optimal joining rule with regard to some measure of their projected sojourn time in Q_S and the quality of service requirement. Bearing these points in mind, we are lead to analyze this system within the paradigm of the infinite player non-cooperative game.

A decision rule $u_k(\cdot)$ for the k-th customer in the overall arrival stream is said to be *optimal* against the policy π if

$$u_k(x) = \begin{cases} 1 & \text{if } V_k(x, \pi) < \theta \\ 0 & \text{if } V_k(x, \pi) > \theta \\ q \in [0, 1] & \text{if } V_k(x, \pi) = \theta \end{cases} \tag{1}$$

for $x \in \{0, 1, \ldots, B-1\}$.

The collection of all possible decision rules of C_k which are optimal against π is denoted by $\mathbb{U}_k(\pi)$.

A policy $\pi = (u^{(1)}, u^{(2)}, \ldots)$ is said to be a Nash equilibrium policy if, for every $r \in \mathbb{Z}^+$, the decision rule of the r-th *controlled* customer, $u^{(r)}(\cdot)$, is optimal against π.

Under this regime, there is no guarantee that *uncontrolled* customers will ever be exhibiting optimal behaviour.

3 Random Variables and Processes

Let $\{M_i : i \in \mathbb{Z}^+\}$, $\{N_j : j \in \mathbb{Z}^+\}$, $\{U_k : k \in \mathbb{Z}^+\}$, $\{U_k^\phi : k \in \mathbb{N}\}$, and $\{U_l^{'} : l \in \mathbb{Z}^+\}$, be mutually independent sequences of random variables, where:
• $\{M_i : i \in \mathbb{Z}^+\}$ is a sequence of independent and identically distributed (i.i.d.) random variables with mean $0 < \lambda^{-1} < \infty$;
• $\{A_k : k \in \mathbb{N}\}$ is the sequence of arrival times to the system, where $A_0 := 0$, and $A_k := \sum_{i=1}^k M_i$, $k \in \mathbb{Z}^+$;
• $\{U_k : k \in \mathbb{N}\}$ is a sequence of i.i.d. random variables which are uniformly distributed on the interval $(0, 1]$. The random variable U_k will be used to decide whether or not customer C_k enters Q_S;
• $\{U_k^\phi : k \in \mathbb{N}\}$ is a sequence of i.i.d. uniform random variables on $(0, 1]$ used for determining whether an arrival to the system is *controlled*, or *uncontrolled*;
• $\{N_j : j \in \mathbb{Z}^+\}$ is a sequence of i.i.d. exponential random variables, with mean $\overline{\mu}^{-1} < \infty$;
• $\{S_l : l \in \mathbb{Z}^+\}$ is the sequence of *potential* service completion times for customers in Q_S, where $S_l := \sum_{j=1}^l N_j$;
• $\{U_l^{'} : l \in \mathbb{Z}^+\}$ is a sequence of i.i.d. uniform random variables on $(0, 1]$ used for determining whether a potential departure time corresponds to an *actual departure*, or a *dummy event*.

Further define $\{t_n : n \in \mathbb{Z}^+\}$ to be the *order statistics* for the set $\{A_k\} \cup \{S_l\}$, where $t_i < t_j$ for $i < j$.

The specifications of the arrival decisions, and departures from Q_S, are presented at the end of the section. These will provide further motivation for the formal definitions of the stochastic processes which are given next.

Queue length process:
For a given initial state $X(0) = x_0$, and policy π, let $\{X(t) : t \geq 0\}$ be the queue length process, where $X(t)$ represents the number of customers in the system at time t.
This process is defined to be left-continuous, piecewise constant, with its potential jumps described by the following relations:

$$\begin{array}{ll} X(A_k^+) = X(A_k) + \mathbf{1}\{U_k < u_k(X(A_k))\} & k \in \mathbb{N} \\ X(S_l^+) = x_l - \mathbf{1}\{U_l^{'} < \mu(X(S_l))/\overline{\mu}\} & l \in \mathbb{Z}^+ \end{array} \tag{2}$$

where $\mathbf{1}\{\cdot\}$ is the indicator function .
Note that if the $\{U_l\}$ were chosen to be uniform on the interval $[0, 1]$ rather than $(0, 1]$, then we would need to additionally include $X(S_l) > 0$ inside the indicator function in the second of the two relations.

The 'remaining service transitions' (RST)-process
Let $\{Z(t) : t \geq 0\}$ be the RST-process. This process is defined to be left continuous, piecewise-constant, and non-increasing, where:
$Z(0) = X(0) = x_0$, and with its potential jumps (coinciding with times of departure) satisfying

the following relation:

$$Z(S_l^+) = Z(S_l) - \mathbf{1}\{Z(S_l) > 0, U_l' < \mu(X(S_l))/\overline{\mu}\} \quad l \in \mathbb{Z}^+ \quad . \tag{3}$$

When C_0 is in the queue, then $Z(t)$ represents the number of customers present less those that are residing behind C_0 (or the number of actual service transitions that still need to occur before C_0 exits) at time t, and $Z(t) = 0$ if customer C_0 is not present at time t;

Service Time
It is clear that if C_0 actually enters Q_S, then its sojourn time would be equal to

$$v_0 = \min\{t : Z(t) = 0\}.$$

Arrivals to Q_S:
At time A_k, customer C_k arrives at Q_S and enters the system with probability γ, say, which depends on its decision rule and the value of $X(A_k)$. The actual decision is based on the value of the random variable U_k and γ in the following way:
C_k enters Q_S if and only if $U_k \le \gamma$.

The mapping $T(\cdot) : \mathbb{Z}^+ \mapsto \mathbb{N}$ is defined more precisely as follows:

$$T(r) = \min\{n \in \mathbb{N} : \sum_{l=0}^{n} \mathbf{1}\left\{U_l^{\phi} > p\right\} = r\}$$

for $r \in \mathbb{Z}^+$.

On each realization of the $\{U_l^{\phi}\}$, the set of indices corresponding to *controlled* customers in the overall arrival stream is

$$\mathcal{I} = \{m \in \mathbb{N} : \exists\, r \in \mathbb{Z}^+ \text{ s.t. } m = T(r)\}.$$

Thus, for $k \in \mathcal{I}$, $u_k(x) = u^{(T^{-1}(k))}(x)$, and for $k \notin \mathcal{I}$, $u_k(x) \equiv 1$ for $x \in \{0, 1, \ldots, B-1\}$.

Service at Q_S:
For ease of exposition, define x_l to be equal to $X(S_l)$, the queue length in Q_S just prior to a potential departure at time S_l.
If $U_l' \in (\mu(x_l)/\overline{\mu}, 1]$, then S_l is considered to be a dummy service completion instant; otherwise any customer at the server completes service and departs from the system.

The above procedure invokes a uniformization technique (see [9]). The fact that such a procedure generates actual departure times with the correct distribution can be seen as follows. As long as the queue length remains at $x \in \mathbb{Z}^+$, the next potential departure is generated from a Poisson process with rate $\mu(x)$. Now consider a Poisson process in which events occur at the *uniform* rate of $\overline{\mu}$, the fastest rate at which departures could possibly occur. Whenever the queue length is x, and an event from the Poisson process with rate $\overline{\mu}$ occurs, then it corresponds to an actual departure with probability $\mu(x)/\overline{\mu}$, independently of all other events. But since this corresponds to a Bernoulli sampling of a Poisson process, then departures at queue length x are Poisson with rate $\overline{\mu} \times \mu(x)/\overline{\mu} = \mu(x)$, as anticipated.

4 Monotonicity with respect to Entry Queue Size

We show, in the sense of stochastic dominance, and in the sense of expectation, that $v_0(x, \pi)$ is an increasing function of x for any π which is member of a certain class of policies: this class is defined below.

Definition
Let $\mathbb{T}$ be the class of decision rules which are non-increasing functions of the queue length $x \in \{0, 1, \ldots, B-1\}$. Also, let $\mathbb{T}^\infty$ be the class of policies in which the decision rule for each *controlled* customer is a member of $\mathbb{T}$. We also note that the decision rule corresponding to an *uncontrolled* customer trivially belongs to $\mathbb{T}$: thus, with regard to the proofs in this section, no special distinction needs to be made between *controlled* and *uncontrolled* customers.

Evaluating the distribution of $v_0(x, \pi)$ appears to be less than straightforward for anything but the simplest cases. This difficulty will be circumvented by utilizing stochastic coupling and forward induction techniques. The collections of random variables and stochastic processes, which we shall parenthetically refer to as 'systems', upon which these stochastic comparisons will be based, are introduced next.

'System' $\mathcal{X}$
This is characterized by the sets of random variables, decision rules, and stochastic processes listed below.
(I): $\mathcal{M} = \{M_i\}$, $\mathcal{N} = \{N_j\}$, $\mathcal{U} = \{U_k\}$, $\mathcal{U}^\phi = \{U_k^\phi\}$, $\mathcal{U}' = \{U_l'\}$;
(II): the arrival time sequence $\mathcal{A} = \{A_k\}$, and the potential departure time sequence $\mathcal{S} = \{S_l\}$;
(III): customer C_0 enters Q_S at time $A_0 = 0$, with all other *controlled* customers adhering to policy $\pi \in \mathbb{T}^\infty$;
(IV): the queue length process $\{X(t) : t \geq 0\}$ with $X(0) = x$;
(V): the RST-process $\{Z(t) : t \geq 0\}$ with $Z(0) = x$;
(VI): v_0, the sojourn time of C_0 in Q_S.

'System' $\widetilde{\mathcal{X}}$
This is characterized in a similar way to $\mathcal{X}$, except that the quantities of $(\widetilde{\text{II}})$-$(\widetilde{\text{VI}})$ are defined in terms of those of $(\widetilde{\text{I}})$ in the obvious way.
$(\widetilde{\text{I}})$: $\widetilde{\mathcal{M}} = \{\widetilde{M}_i\}$, $\widetilde{\mathcal{N}} = \{\widetilde{N}_j\}$, $\widetilde{\mathcal{U}} = \{\widetilde{U}_k\}$, $\widetilde{\mathcal{U}}^\phi = \{\widetilde{U}_k^\phi\}$, $\widetilde{\mathcal{U}}' = \{\widetilde{U}_l'\}$, which have the same distributions as $\mathcal{M}$, $\mathcal{N}$, $\mathcal{U}$, $\mathcal{U}^\phi$, and $\mathcal{U}'$, respectively;
$(\widetilde{\text{II}})$: the arrival time sequence $\widetilde{\mathcal{A}} = \{\widetilde{A}_k\}$, and the potential departure time sequence $\widetilde{\mathcal{S}} = \{\widetilde{S}_l\}$;
$(\widetilde{\text{III}})$: as in (III) for $\mathcal{X}$ with $\widetilde{A}_0 = 0$;
$(\widetilde{\text{IV}})$: the queue length process $\{\widetilde{X}(t) : t \geq 0\}$ with $\widetilde{X}(0) = x + 1$;
$(\widetilde{\text{V}})$: the RST-process $\{\widetilde{Z}(t) : t \geq 0\}$ with $\widetilde{Z}(0) = x + 1$;
$(\widetilde{\text{VI}})$: $\widetilde{v}_0$, the sojourn time of C_0 in Q_S.

We intend to relate $\mathcal{X}$ with $\widetilde{\mathcal{X}}$ to each other using the following device:

Coupling $\mathcal{C}$

Set

$$\begin{array}{rl} M_i = \widetilde{M_i} & i \in \mathbb{Z}^+ \\ N_j = \widetilde{N}_j & j \in \mathbb{Z}^+ \\ U_k = \widetilde{U}_k & k \in \mathbb{N} \\ U_k^{\phi} = \widetilde{U}_k^{\phi} & k \in \mathbb{N} \\ U_l^{'} = \widetilde{U}_l^{'} & l \in \mathbb{Z}^+ \end{array}.$$

The effect of this procedure is that arrival instants $\{A_k\}$, potential departure times $\{S_l\}$, the $\{U_k\}$, the positions of *controlled* customers in the overall arrival sequence, and the $\{U_l^{'}\}$, under $\mathcal{X}$, take the same values as their counterparts under $\widetilde{\mathcal{X}}$, on each realization.

The next result allows us to infer that under the above coupling, $v_0 \leq \widetilde{v}_0$.

Lemma 1 *For systems $(\mathcal{X}, \widetilde{\mathcal{X}})$ under coupling $\mathcal{C}$, in which $\pi \in \mathbb{T}^{\infty}$, one of the following sets of relations will hold at each time $t \in \mathbb{R}^+$:*

$$\begin{array}{c} X(t) + 1 = \widetilde{X}(t) \\ Z(t) + 1 = \widetilde{Z}(t) \end{array} \tag{4}$$

$$\begin{array}{c} X(t) = \widetilde{X}(t) \\ Z(t) = \widetilde{Z}(t) \end{array} \tag{5}$$

$$\begin{array}{c} X(t) = \widetilde{X}(t) \\ Z(t) + 1 = \widetilde{Z}(t). \end{array} \tag{6}$$

Proof By definition of $\mathcal{X}$ and $\widetilde{\mathcal{X}}$,

$$Z(0^+) = X(0^+) = x + 1 < x + 2 = \widetilde{X}(0^+) = \widetilde{Z}(0^+). \tag{7}$$

Assume that t_{n+1} corresponds to an arrival, with $t_{n+1} = A_r$, or a departure, with $t_{n+1} = S_m$, such that C_0 is still present in Q_S under $\mathcal{X}$.

Before proceeding, we observe that if t_{n+1} does indeed correspond to an arrival time, then due to the class of policies and the coupling being considered, one of the following three scenarios must arise:
(i) C_r enters Q_S under both $\mathcal{X}$ and $\widetilde{\mathcal{X}}$,
(ii) C_r enters Q_S under $\mathcal{X}$ only,
(iii) C_r balks under both $\mathcal{X}$ and $\widetilde{\mathcal{X}}$.

Case 1: Suppose that (4) holds at time t_n^+.

$t_{n+1} \in \{A_k\}$

Under (i),

$$X(t_{n+1}^+) = X(t_n^+) + 1 = \widetilde{X}(t_n^+) = \widetilde{X}(t_{n+1}^+) - 1. \tag{8}$$

Under (ii),

$$X(t_{n+1}^+) = X(t_n^+) + 1 = \widetilde{X}(t_n^+) = \widetilde{X}(t_{n+1}^+). \tag{9}$$

Under (iii), the states of the queue-length processes for $\mathcal{X}$ and $\widetilde{\mathcal{X}}$ at time t_{n+1}^+ are the same as they were at time t_n^+.

Also, since C_r would reside behind C_0 if it were to enter Q_S, then there would be no change in the RST-processes in each of the above scenarios, i.e.

$$Z(t_{n+1}^+) = Z(t_n^+) = \widetilde{Z}(t_n^+) - 1 = \widetilde{Z}(t_{n+1}^+) - 1. \tag{10}$$

Thus, at time t_{n+1}^+, (4) holds under scenarios (i) and (iii), whereas (6) holds under scenario (ii).

<u>$t_{n+1} \in \{S_l\}$</u>
Since $x_m < \widetilde{x}_m$, by assumption, then $\mu(x_m) < \mu(\widetilde{x}_m)$.
If $U'_m \le \mu(x_m)/\overline{\mu}$, then an actual departure occurs under both $\mathcal{X}$ and $\widetilde{\mathcal{X}}$. Therefore

$$X(t_{n+1}^+) = X(t_n^+) - 1 = (\widetilde{X}(t_n^+) - 1) - 1 = \widetilde{X}(t_{n+1}^+) - 1 \tag{11}$$
$$Z(t_{n+1}^+) = Z(t_n^+) - 1 = (\widetilde{Z}(t_n^+) - 1) - 1 = \widetilde{Z}(t_{n+1}^+) - 1. \tag{12}$$

Thus (4) holds at time t_{n+1}^+.
If $U'_m \in (\mu(x_m)/\overline{\mu}, \mu(\widetilde{x}_m)/\overline{\mu}]$, then an actual departure occurs under $\widetilde{\mathcal{X}}$ but not under $\mathcal{X}$. Therefore,

$$X(t_{n+1}^+) = X(t_n^+) = \widetilde{X}(t_n^+) - 1 = \widetilde{X}(t_{n+1}^+) \tag{13}$$
$$Z(t_{n+1}^+) = Z(t_n^+) = \widetilde{Z}(t_n^+) - 1 = \widetilde{Z}(t_{n+1}^+). \tag{14}$$

Thus, (5) holds at time t_{n+1}^+.

If $U'_m > \mu(\widetilde{x}_m)/\overline{\mu}$, then the states of the processes remain unchanged, i.e.

$$X(t_{n+1}^+) = X(t_n^+) = \widetilde{X}(t_n^+) - 1 = \widetilde{X}(t_{n+1}^+) - 1 \tag{15}$$
$$Z(t_{n+1}^+) = Z(t_n^+) = \widetilde{Z}(t_n^+) - 1 = \widetilde{Z}(t_{n+1}^+) - 1 \tag{16}$$

i.e. (4) holds at time t_{n+1}^+.

Case 2: Suppose that (5) holds at time t_n^+.
It follows that (5) holds at time t_{n+1}^+ also, as the following argument shows.

<u>$t_{n+1} \in \{A_k\}$</u>
Since the queue lengths are identical, as are the decision rules for C_r, under both $\mathcal{X}$ and $\widetilde{\mathcal{X}}$, then the decision as to whether or not to enter Q_S will be the same under both systems. Hence

$$X(t_{n+1}^+) = \widetilde{X}(t_{n+1}^+). \tag{17}$$

Again, since C_r would reside behind C_0 should it actually enter Q_S, then the states of the RST-processes remain unchanged.

$\underline{t_{n+1} \in \{S_l\}}$
Since $x_m = \widetilde{x}_m$, then $\mu(x_m) = \mu(\widetilde{x}_m)$.
If $U'_m \leq \mu(x_m)/\overline{\mu}$, then there is an actual departure under both $\mathcal{X}$ and $\widetilde{\mathcal{X}}$, and so

$$X(t_{n+1}^+) = X(t_n^+) - 1 = \widetilde{X}(t_n^+) - 1 = \widetilde{X}(t_{n+1}^+) \tag{18}$$
$$Z(t_{n+1}^+) = Z(t_n^+) - 1 = \widetilde{Z}(t_n^+) - 1 = \widetilde{Z}(t_{n+1}^+). \tag{19}$$

If $U'_m > \mu(x_m)/\overline{\mu}$ then there are no actual departures under both $\mathcal{X}$ and $\widetilde{\mathcal{X}}$, and so there is no change in either the queue-length, or RST, processes, i.e.

$$X(t_{n+1}^+) = X(t_n^+) = \widetilde{X}(t_n^+) = \widetilde{X}(t_{n+1}^+) \tag{20}$$
$$Z(t_{n+1}^+) = Z(t_n^+) = \widetilde{Z}(t_n^+) = \widetilde{Z}(t_{n+1}^+). \tag{21}$$

Case 3: Suppose that (6) holds at time t_n^+.
It follows that (6) holds at time t_{n+1}^+ also, as the argument below shows.

$\underline{t_{n+1} \in \{A_k\}}$
For the same reasons as in the previous case, the queue lengths remain equal, i.e.

$$X(t_{n+1}^+) = \widetilde{X}(t_{n+1}^+), \tag{22}$$

and there are no changes in the RST-processes.

$\underline{t_{n+1} \in \{S_l\}}$
Since $x_m = \widetilde{x}_m$, then $\mu(x_m) = \mu(\widetilde{x}_m)$. Again, for the same reasons as in the previous case, if $U'_m \leq \mu(x_m)/\overline{\mu}$ then

$$X(t_{n+1}^+) = X(t_n^+) - 1 = \widetilde{X}(t_n^+) - 1 = \widetilde{X}(t_{n+1}^+) \tag{23}$$
$$Z(t_{n+1}^+) = Z(t_n^+) - 1 = (\widetilde{Z}(t_n^+) - 1) - 1 = \widetilde{Z}(t_{n+1}^+) - 1. \tag{24}$$

On the other hand, if $U'_m > \mu(x_m)/\overline{\mu}$ then there is no change in either the queue-length, or the RST, processes. □

Recalling the definition for the sojourn time of C_0 in Q_S, the following Lemma now holds.

Lemma 2 *For all $\pi \in \mathbb{T}^\infty$ and $k \in \mathbb{N}$, $V_k(x,\pi)$ is strictly increasing in x, in the sense that for $x \in \{0, 1, \ldots, B-2\}$,*

$$V_k(x+1,\pi) - V_k(x,\pi) \geq \delta_x > 0$$

uniformly in π.

Proof Without loss of generality, and for concreteness, consider customer C_0. Consider systems $(\mathcal{X}, \widetilde{\mathcal{X}})$ under coupling $\mathcal{C}$. From the definitions of v_0 and $\widetilde{v}_0$, and by Lemma 1,

$$v_0 \leq \widetilde{v}_0$$

which implies that $E[v_0] \leq E[\widetilde{v}_0]$. To establish the sharp inequality, define the event D_x:

$$D_x = \{S_{x+1} < A_1, U'_m \leq \mu(x_m)/\overline{\mu},\ m=1,\ldots,x+1\}.$$

This is the event that customer C_1 arrives no earlier than the $(x+1)$-st departure under both systems, where C_0 leaves under $\mathcal{X}$ at time S_{x+1}, but becomes the only customer left in Q_S under $\widetilde{\mathcal{X}}$ at that time. By conditioning on this event, noting that $v_0 < \widetilde{v}_0$ on D_x and that $\mathbb{P}(D_x) > 0$, then the result follows. □

5 Monotonicity and Continuity with respect to Threshold Policies

This section examines the behaviour of the sojourn time in Q_S with respect to a certain type of threshold rule, which is introduced below.

Definition
For $L \in \mathbb{N}$ and $q \in [0,1]$, an $[L,q]$-threshold decision rule $u(\cdot)$ is defined as follows:

$$u(x) = \begin{cases} 1 & \text{if} \quad x < L \\ q & \text{if} \quad x = L \\ 0 & \text{if} \quad x > L \end{cases}.$$

This may be represented more compactly by $[L,q]$, or indeed $[g]$, where $g = L+q$. Of course, for $B < \infty$, $[B]$ is equivalent to $[g]$ whenever $g > B$.

We are ultimately interested in the characterization of *symmetric* policies: these are policies in which each and every *controlled* customer adopts the same decision rule.

A policy π in which the decision rule for each *controlled* customer is given by $[g]$, is denoted by $[g]^\infty$: we call this a *symmetric threshold policy.*

Next we introduce another two 'systems' which will facilitate the proofs of the results for this section.

'System' $\mathcal{G}$
(I),(II),(IV), (V) and (VI) are exactly as in $\mathcal{X}$. However (III) becomes
(III): customer C_0 enters Q_S at time $A_0 = 0$, with all other *controlled* customers adhering to policy $[g]^\infty$, where $g \in [0,B)$.

'System' $\widetilde{\mathcal{G}}$

$(\widetilde{\text{I}})$, $(\widetilde{\text{II}})$, and $(\widetilde{\text{VI}})$ are precisely the same as for $\widetilde{\mathcal{X}}$;
$(\widetilde{\text{III}})$: customer C_0 enters Q_S at time $A_0 = 0$, with all other *controlled* customers adhering to the policy $[\widetilde{g}]^\infty$, with $g < \widetilde{g} \leq B$ (where the latter inequality is strict if $B = \infty$);
$(\widetilde{\text{IV}})$: the queue length process $\{\widetilde{X}(t) : t \geq 0\}$ with $\widetilde{X}(0) = x$;
$(\widetilde{\text{V}})$: the RST-process $\{\widetilde{Z}(t) : t \geq 0\}$ with $\widetilde{Z}(0) = x$.

It is implicit from the definition of g and $\widetilde{g}$ that $L < B$.

From now on, $g \in [0, B]$ will be taken to mean $0 \leq g \leq B$ when B is finite, and $0 \leq g < B$ when B is non-finite, unless specified to the contrary.

The next two results will be used to infer results about $V_k(\cdot, [g]^\infty)$ on the intervals $[0, 1]$ and $[1, B]$.

Lemma 3 *For systems $(\mathcal{G}, \widetilde{\mathcal{G}})$ under coupling $\mathcal{C}$, the following set of relations hold at each time $t \in \mathbb{R}^+$:*

$$X(t) \leq \widetilde{X}(t) \tag{25}$$

$$Z(t) \geq \widetilde{Z}(t). \tag{26}$$

Proof Assume that t_{n+1} corresponds to an arrival time, where $t_{n+1} = A_r$, or a potential departure time, where $t_{n+1} = S_m$, such that C_0 is still present in Q_S under $\mathcal{G}$.

First note that

$$Z(0^+) = X(0^+) = x + 1 = \widetilde{X}(0^+) = \widetilde{Z}(0^+).$$

Now suppose that

$$X(t_n^+) \leq \widetilde{X}(t_n^+) \tag{27}$$

$$Z(t_n^+) \geq \widetilde{Z}(t_n^+) \quad . \tag{28}$$

Case 1: Relation (27) is strict.

$t_{n+1} \in \{A_k\}$

If $\widetilde{X}(t_n^+) < B$ then either C_r enters Q_S under neither, one, or both of $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ (noting that the last of these scenarios certainly occurs if C_r is *uncontrolled*); it follows that

$$X(t_{n+1}^+) \leq \widetilde{X}(t_{n+1}^+)$$

(which holds with equality when C_r enters Q_S under $\mathcal{G}$ only and $X(t_n^+) = \widetilde{X}(t_n^+) - 1$).

If $\widetilde{X}(t_n^+) = B$ then C_r can only enter Q_S under $\mathcal{G}$ (and will certainly enter under $\mathcal{G}$ if C_r is *uncontrolled*). Hence

$$X(t_{n+1}^+) \leq \widetilde{X}(t_{n+1}^+) = B. \tag{29}$$

Since C_r can never reside ahead of C_0 in any of these scenarios, then there can be no change in the RST-processes, so that

$$Z(t_{n+1}^+) = Z(t_n^+) \geq \widetilde{Z}(t_n^+) = \widetilde{Z}(t_{n+1}^+). \tag{30}$$

$\underline{t_{n+1} \in \{S_l\}}$
Since $x_m < \widetilde{x}_m$, then $\mu(x_m) < \mu(\widetilde{x}_m)$.
If $U'_m \leq \mu(x_m)/\overline{\mu}$, then an actual departure occurs under both $\mathcal{G}$ and $\widetilde{\mathcal{G}}$, and so

$$X(t_{n+1}^+) = X(t_n^+) - 1 < \widetilde{X}(t_n^+) - 1 = \widetilde{X}(t_{n+1}^+) \tag{31}$$
$$Z(t_{n+1}^+) = Z(t_n^+) - 1 \geq \widetilde{Z}(t_n^+) - 1 = \widetilde{Z}(t_{n+1}^+). \tag{32}$$

If $U'_m \in (\mu(x_m)/\overline{\mu}, \mu(\widetilde{x}_m)/\overline{\mu}]$, then an actual departure occurs under $\widetilde{\mathcal{G}}$, but not under $\mathcal{G}$, and so

$$X(t_{n+1}^+) = X(t_n^+) \leq \widetilde{X}(t_n^+) - 1 = \widetilde{X}(t_{n+1}^+) \tag{33}$$
$$Z(t_{n+1}^+) = Z(t_n^+) \geq \widetilde{Z}(t_n^+) > \widetilde{Z}(t_n^+) - 1 = \widetilde{Z}(t_{n+1}^+). \tag{34}$$

If $U'_m > \mu(x_m)/\overline{\mu}$ then there is no change, i.e. (25) holds strictly, and (26) also holds, at time t_{n+1}^+.

Case 2: Relation (27) holds with equality.
$\underline{t_{n+1} \in \{A_k\}}$
Here, either C_r enters Q_S under none, or both, of $\mathcal{G}$ and $\widetilde{\mathcal{G}}$, or indeed just $\widetilde{\mathcal{G}}$ alone (and the second of these scenarios will certainly hold if $X(t_n^+) = \widetilde{X}(t_n^+) < B$ and C_r is *uncontrolled*). Therefore

$$X(t_{n+1}^+) \leq \widetilde{X}(t_{n+1}^+). \tag{35}$$

As in the previous case, C_r is not able to reside ahead of C_0, and so, again, (26) holds at time t_{n+1}^+.

$\underline{t_{n+1} \in \{S_l\}}$
Since $x_m = \widetilde{x}_m$, then $\mu(x_m) = \mu(\widetilde{x}_m)$.
If $U'_m \leq \mu(x_m)/\overline{\mu}$, then an actual departure occurs under both $\mathcal{G}$ and $\widetilde{\mathcal{G}}$, and so

$$X(t_{n+1}^+) = X(t_n^+) - 1 = \widetilde{X}(t_n^+) - 1 = \widetilde{X}(t_{n+1}^+) \tag{36}$$
$$Z(t_{n+1}^+) = Z(t_n^+) - 1 \geq \widetilde{Z}(t_n^+) - 1 = \widetilde{Z}(t_{n+1}^+). \tag{37}$$

If $U'_m > \mu(x_m)/\overline{\mu}$, then there is no change, i.e. (25) holds with equality, and (26) holds at time t_{n+1}^+. □

Lemma 4 *For systems* $(\mathcal{G},\widetilde{\mathcal{G}})$ *under coupling* $\mathcal{C}$*, with* $0 < g < \widetilde{g} \leq 1$*, the following set of relations hold at each time* $t\in\mathbb{R}^+$ *throughout the sojourn of* C_0 *(in either system):*

$$X(t) = \widetilde{X}(t) \tag{38}$$
$$Z(t) = \widetilde{Z}(t). \tag{39}$$

Proof Since $X(0) = x = \widetilde{X}(0)$, arrivals into Q_S, actual departures, and dummy events will coincide in the two systems during the sojourn of C_0 at least until such time that a disparity in the arrival decisions occurs. However, since $0 < g < \widetilde{g} \leq 1$, then the first opportunity for a customer to enter Q_S under $\widetilde{\mathcal{G}}$, but not under $\mathcal{G}$, are when the queues are completely empty: but, C_0 will obviously have left by this time. Therefore, as a result, the relations (38) and (39) will hold also. □

Next, we define the following quantity for the ensuing discussion.

$$\widehat{q} := 1 - (1 - \widetilde{q})\mathbf{1}\{L = \widetilde{L}\}. \tag{40}$$

Lemma 5 *For each* $k\in\mathbb{N}$*, and* $x \in \{0, 1, \ldots, B-1\}$
(i) $V_k(x, [g]^\infty)$ *is constant in* g *on* $[0, 1]$*;*
(ii) $V_k(x, [g]^\infty)$ *is strictly decreasing in* g *on* $[1, B]$*.*

Proof Without loss of generality, and for concreteness, consider customer C_0. Consider the systems $(\mathcal{G},\widetilde{\mathcal{G}})$ under coupling $\mathcal{C}$. From the definitions of v_0 and $\widetilde{v}_0$, and by invoking Lemma 3, it is easy to see that

$$E[v_0] \geq E[\widetilde{v}_0] \tag{41}$$

which holds with equality whenever $0 \leq g < \widetilde{g} \leq 1$ by Lemma 4, thereby establishing (i). Thus assume that $1 \leq g < \widetilde{g}$ and define the following events.
For $x < L$,

$$\begin{aligned} F_\alpha \;\; = \;\; & \{A_{L-x} < S_1; A_{L-x+1} > S_{x+1}; U^\phi_{L-x} > p; U_{L-x} \in (q, \widehat{q}]; \\ & U'_1 \in (\mu(X(S_1))/\overline{\mu}, \mu(\widetilde{X}(S_1))/\overline{\mu}]; \\ & U'_m \leq \mu(X(S_m))/\overline{\mu} : m=2, \ldots, x+1\}. \end{aligned}$$

Any realization on F_α under coupling $\mathcal{C}$ results in the following occurrences, in the order presented below:
• $L - x - 1$ customers enter Q_S under both $\mathcal{G}$ and $\widetilde{\mathcal{G}}$, resulting in the queue size moving up to L and C_0 remaining at position $x + 1$ in both cases;
• a *controlled* customer enters Q_S under $\widetilde{\mathcal{G}}$, but not under $\mathcal{G}$: this results in the queue size under $\widetilde{\mathcal{G}}$ moving from L to $L + 1$, but remaining at L under $\mathcal{G}$, with C_0 still at position $x + 1$ in both cases;
• a departure occurs under $\widetilde{\mathcal{G}}$, but not under $\mathcal{G}$: this results in the queue size being equal to L in both cases, C_0 remaining at position $x + 1$ under $\mathcal{G}$, and C_0 moving to position x (or indeed exiting the system if $x = 0$) under $\widetilde{\mathcal{G}}$;
• a further x departures occur under both processes (before the next arrival), resulting in C_0

leaving under $\widetilde{\mathcal{G}}$ (if it is still present), but residing at the head of the queue under $\mathcal{G}$, at time S_{x+1}.

For $L \leq x < B$,

$$\begin{aligned} F_\beta &= \{S_{x-L+1} < A_1 < S_{x-L+2}; S_{x+1} < A_2; U_1^\phi > p; U_1 \in (q, \widehat{q}]; \\ &\quad U'_m \leq \mu(X(S_m))/\overline{\mu} : m=1, \ldots, x+1, m \neq x-L+2; \\ &\quad U'_m \in \Big(\mu(X(S_m))/\overline{\mu}, \mu(\widetilde{X}(S_m))/\overline{\mu}\Big] : m=x-L+2\} \end{aligned}$$

which, in the order presented below, under coupling $\mathcal{C}$, results in the following:
• the queue size under each of $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ moves from $x+1$ down to L; C_0 ends up in position L in both cases;
• a *controlled* customer enters Q_S under $\widetilde{\mathcal{G}}$, but not under $\mathcal{G}$; the resulting queue sizes are L and $L+1$ under $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ respectively; C_0 remains at position L in both cases;
• a departure occurs under $\widetilde{\mathcal{G}}$, but not under $\mathcal{G}$: this results in the queue size being equal to L in both cases, C_0 remaining at position L under $\mathcal{G}$, and C_0 moving to position $L-1$ (or even exiting the system if $L=1$) under $\widetilde{\mathcal{G}}$;
• a further $L-1$ departures occur under both $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ before the second arrival, resulting in C_0 leaving under $\widetilde{\mathcal{G}}$ (if still present), but residing at the head of the queue under $\mathcal{G}$, at time S_{x+1}.
Define

$$F_\zeta = \mathbf{1}\{x < L\}F_\alpha + \mathbf{1}\{x \geq L\}F_\beta.$$

By conditioning on F_ζ, it is easy to establish that $E[v_0] > E[\widetilde{v}_0]$ and (ii) follows.

Note
For clarity of exposition, $\mu(x_m)$ and $\mu(\widetilde{x}_m)$ have been written more explicitly as $\mu(X(S_m))$ and $\mu(\widetilde{X}(S_m))$ in order to avoid confusion with x that appears in the indexing of the random variables.

Next we show that the expected sojourn time of a customer in Q_S, when other controlled customers adhere to the decision rule $[g]$, for particular entry states x, is a continuous function of g. This result is established under the proviso that a certain, albeit not very restrictive, condition holds, which is presented below.

Stability Condition (SC):
Under 'system' $\mathcal{G}$, there exists a bound D_x such that

$$E\left[\sum_{k=1}^{\infty} \mathbf{1}\{A_k \leq v_0\}\right] \leq D_x$$

uniformly in $g \in [0, B]$.

This condition says that the expected number of arrival instants that occur during the sojourn of C_0 in Q_S when *controlled* customers adopt the threshold decision-rule $[g]$, is bounded above by D_x, over all $g \in [0, B]$. This condition will certainly be satisfied in the case where the inter-arrival times, which are given by the $\{M_i\}$, are exponential. To see this, consider a 'system' similar to $\mathcal{G}$ except that the service rate is always $\mu(1)$. Flag quantities which are associated with this 'system' by $'*'$. Under a coupling similar to $\mathcal{C}$ between this 'system' and $\mathcal{G}$, it can be shown that $v_0 \leq v_0^*$ for all $g \in [0, B]$. But is also clear that v_0^* only depends on the $\{N_j\}$ and the $\{U_l'\}$, and, therefore, is independent of the arrival instants $\{A_k\}$.
Hence

$$E\left[\sum_{k=1}^{\infty} \mathbf{1}\{A_k \leq v_0\}\right] \leq E\left[\sum_{k=1}^{\infty} \mathbf{1}\{A_k \leq v_0^*\}\right]$$
$$= \int_0^{\infty} E\left[\sum_{k=1}^{\infty} \mathbf{1}\left\{A_k \leq t\right\}\right] f_{v_0^*}(t)dt = \lambda \int_{t=0}^{\infty} t f_{v_0^*}(t)dt = \lambda E[v_0^*].$$

However $E[v_0^*]$ is just the expected value of the sum of $x+1$ i.i.d. exponential random variables, each with mean $\mu(1)^{-1}$. Thus

$$E\left[\sum_{k=1}^{\infty} \mathbf{1}\{A_k \leq v_0\}\right] \leq \lambda \tfrac{(x+1)}{\mu(1)}.$$

Lemma 6 *Suppose that the stability condition (SC) holds.*
Then for every $k \in \mathbb{N}$, $x \in \{0, 1, \ldots, B-1\}$, $V_k(x, [g]^{\infty})$ is continuous in $g \in [0, B]$.

Proof
Consider customer C_0 for concreteness, and without loss of generality. Consider $(\mathcal{G}, \widetilde{\mathcal{G}})$ under coupling $\mathcal{C}$, with the additional restriction that $g = L+q$ and $\widetilde{g} = L+\widetilde{q}$ such that $0 \leq q < \widetilde{q} \leq 1$.
Define

$$k_0 = \inf\{k \in \mathbb{Z}^+ : A_k < v_0,\ X(A_k) = L, U_k \in (q, \widetilde{q}], U_k^{\phi} > p\}$$

where $\inf\emptyset := \infty$. Indeed if $k_0 = \infty$, then $v_0 = \widetilde{v}_0$.
For $k_0 = k < \infty$, then $\widetilde{v}_0 > A_k$, and

$$E[\widetilde{v}_0 - v_0 | k_0 = k]$$
$$= E[\widetilde{v}_0 - A_k | k_0 = k] - E[v_0 - A_k | k_0 = k]$$
$$\leq (L+1)/\mu(1).$$

This follows from the fact that the first term is bounded above by the expected time to serve $L+1$ customers at the slowest possible rate of $\mu(1)$, and that the second term is bounded below by zero.

It is easy to deduce that $\{A_k < v_0\}$ is independent of $\{U_k \in (q, \widetilde{q}]\}$ (noting that the former is equivalent to $\{Z(A_k) > 0\}$). Therefore

$$\mathbb{P}(k_0 = k) \leq \mathbb{P}(A_k < v_0,\ U_k \in (q, \widetilde{q}])$$
$$= \mathbb{P}(A_k < v_0)\mathbb{P}(U_k \in (q, \widetilde{q}]) = (\widetilde{q} - q)\mathbb{P}(A_k < v_0).$$

Hence

$$E[\widetilde{v}_0 - v_0] \leq \frac{(L+1)}{\mu(1)} \sum_{k=1}^{\infty} \mathbb{P}(k_0 = k)$$

$$\leq (\widetilde{q} - q)\frac{(L+1)}{\mu(1)}\sum_{k=1}^{\infty}\mathbb{P}(A_k < v_0).$$

By the Monotone Convergence Theorem, $\sum_{k=1}^{\infty}\mathbb{P}(A_k < v_0)$ can be expressed as $E[\sum_{k=1}^{\infty}\mathbf{1}\{A_k < v_0\}]$. Further observe that $\widetilde{q} - q = \widetilde{g} - g$. Thus

$$E[\widetilde{v}_0 - v_0] \leq (\widetilde{g} - g)\frac{(L+1)}{\mu(1)}D_x$$

as required. □

6 Structure and Existence of the Nash Equilibrium

In this section, we first explore the required structure of any candidate *symmetric Nash equilibrium policy* (SNEP) within the $\mathbb{T}^{\infty}$ class. We then go on to prove that there exists a finite number of SNEPs within this class, and that at least one of these is characterized by a non-randomized threshold.

Lemma 7 *Suppose that* $\pi \in \mathbb{T}^{\infty}$. *Then*
(a) any optimal decision rule of C_k *against* π *must be a threshold decision rule;*
(b) the set of optimal decision rules of C_k *against* π*, i.e.* $\mathbb{U}_k(\pi)$*, can be found in the following way:*
set $\widehat{L} := \min\{L \in \mathbb{Z}^+ : V_k(L,\pi) \geq \theta\}$;
(i) if $V_k(\widehat{L},\pi) = \theta$ *then* $\mathbb{U}_k(\pi) = \{[\widehat{L}, q] : 0 \leq q \leq 1\}$;
(ii) otherwise, $\mathbb{U}_k(\pi) = \{[\widehat{L}, 0]\}$.

Proof Follows from the definition of an optimal decision rule and the monotonicity result of Lemma 2. Since $V_k(L,\pi) \geq (L+1)/\overline{\mu} \longrightarrow \infty$ as $L \longrightarrow \infty$, then $\widehat{L}$ is well-defined. □

In the following discussion, the best response map of an arbitrary *controlled* customer, $C^{(k)}$ say, is constructed, against the background of other *controlled* customers adhering to the policy $[g]^{\infty}$. For simplicity, and without loss of generality, we construct this map for customer C_0: this is as if to say that $C^{(k)}$ corresponds to C_0 in the overall arrival stream.

Next, we define the mapping $l(\cdot)$ (as introduced in [2] but with a slight modification) as follows.
For $g \in [0, B]$, let

$$l(g) = \min\{n \in \mathbb{N} : n < B, V_0(n, [g]^{\infty}) \geq \theta\} \tag{42}$$

with $\min\emptyset := B$. Since $V_0(x,\cdot) \geq (x+1)/\overline{\mu} \longrightarrow \infty$ as $x \longrightarrow \infty$, then $l(\cdot)$ is well defined.

Further, let $\{g_1, g_2, \ldots, g_J\}$ be the points of discontinuity of $l(g)$ for $g \in [0, B]$, where

$$0 =: g_0 \leq g_1 < g_2 < \ldots < g_{J-1} < g_J \leq g_{J+1} := B.$$

Notice that $g_1 = 0$ if there is a point of discontinuity at the origin, and $g_J = B$ if there is one at B whenever B is finite.

Following a similar methodology to [2], we define the point-to-set mapping $G^*(g) : [0, B] \mapsto 2^{[0,B]}$ as

$$G^*(g) = \{g' \in [0, B] : [g'] \text{ is optimal for } \mathrm{C}_0 \text{ against } [g]^\infty\}. \tag{43}$$

Since $[g]^\infty$ is a member of $\mathbb{T}^\infty$, then we can invoke Lemma 7 to deduce that $G^*(\cdot)$ is given by

$$G^*(g) = \begin{cases} \{l(g) + q : 0 \le q \le 1\} & \text{if } V_0(l(g), [g]^\infty) = \theta,\ l(g) < B \\ l(g) & \text{if } V_0(l(g), [g]^\infty) > \theta,\ l(g) < B \\ B & \text{if } l(g) = B \end{cases} \tag{44}$$

By the aforementioned properties of $V_0(\cdot, [g]^\infty)$, and provided that (SC) holds, it can be easily deduced that

$$l(g) = l(0), \quad 0 \le g \le 1; \tag{45}$$

$$l(g) = l(0) + j, \quad g > 1,\ g_j < g \le g_{j+1}. \tag{46}$$

Thus $G^*(g)$ can be re-expressed as

$$G^*(g) = \begin{cases} l(0) & \text{if } g < g_1 \\ [l(0) + j - 1, l(0) + j] & \text{if } g = g_j,\ j{=}1,\ldots, J \\ [l(0), l(0) + 1] & \text{if } g_0 = g_1 \text{ and } g \le 1 \\ l(0) + j & \text{if } g_j < g < g_{j+1},\ j{=}2,\ldots, J \\ l(0) + 1 & \text{if } g_0 < g_1 < g < g_2, \text{ or } g_0 = g_1 \text{ and } 1 < g < g_2 \end{cases} \tag{47}$$

Observe that the graph of $G^*(\cdot)$ is staircase arc-connected and non-decreasing whenever $g_0 < g_1$; a similar structure will hold when $g_0 = g_1$, except that a rectangular region occurs with bottom left and top right co-ordinates given by $(0, l(0))$ and $(1, l(0) + 1)$, respectively.

Define the map $H(g) := G^*(g) - g$ with the same domain as $G^*(\cdot)$: here $G^*(g) - g$ is taken to mean $[\min\{G^*(g)\} - g, \max\{G^*(g)\} - g]$. The graph of the map $H(\cdot)$ has a 'saw-tooth'-like structure, except that in the case where $g_0 = g_1$, this is modified to include a rhombus on the interval $[0, 1]$ with corners $(0, l(0))$, $(0, l(0) + 1)$, $(1, l(0) - 1)$, and $(1, l(0))$.
We shall employ this construction in the proof of the following theorem.

Theorem 1 *Suppose that (SC) holds. Then in the class of policies $\mathbb{T}^\infty$,*
(i) there exist a finite number of SNEPs;
(ii) at least one of the SNEPs is non-randomized.

Proof The thresholds associated with the SNEPs correspond to the 'zeros' of the map $H(\cdot)$.

By Lemma 5 (i), and using the fact that $l(0) \ge 1$, then $\min\{H(g)\} > 0$ for all $g \in [0, 1)$. Thus, we restrict our discussion to $g \in [1, B]$.

Suppose that B is finite. Since $\min\{G^*(1)\} \ge 1$ and $\max\{G^*(B)\} \le B$, then $\min\{H(1)\} \ge 0$ and $\max\{H(B)\} \le 0$. Thus, by the structure of the graph of $H(\cdot)$, and the *intermediate value theorem*, there must exist a $g^* \in [1, B]$ for which $0 \in H(g^*)$.

Now suppose that B is non-finite. Notice that $V_0(x,[g]^\infty)$ is bounded below by $(x+1)/\overline{\mu}$, which is independent of g, and that $(x+1)/\overline{\mu} \longrightarrow \infty$ as $x \longrightarrow \infty$. Thus, there exists some $n \in \mathbb{Z}^+$ for which $V_0(n,[g]^\infty) > \theta$ for all $g \geq 0$. This implies that $l(g) \leq n$ for all $g \geq 0$, and so the number of (vertical) jump points of $G^*(\cdot)$ (and $H(\cdot)$), which is given by J, must be finite. Therefore, $G^*(g) = l(g_J) + 1 < \infty$ for all $g > g_J$, which implies that $H(g) < 0$ for all g sufficiently large. Together with the fact that $\min\{H(1)\} \geq 0$, the existence of a $g^{**} \in [1,\infty)$ such that $0 \in H(g^{**})$ follows.

As remarked earlier, since the number of vertical jumps, J, is finite, then the number of zeros of $H(\cdot)$ is bounded (for B finite and non-finite), and so finiteness of the number of SNEPs follows. Part (i) is established.

Suppose, for contradiction, that there are no non-randomized SNEPs. Then, by part (i), at least one randomized SNEP exists. Such a point must correspond to a jump point of $H(\cdot)$. Indeed, an SNEP occurs at $g = g_m$ if and only if a zero occurs which is strictly interior to the vertical part of the graph at g_m, i.e. $\min\{H(g_m)\} < 0 < \max\{H(g_m)\}$. However, we know that $\min\{H(1)\} \geq 0$. Thus, by the *intermediate value theorem*, there exists at least one point, $g' \in [1, g_m)$ on a diagonal section of the graph (which is taken to include corners), for which $0 \in H(g')$. But any such point must correspond to a non-randomized threshold, providing the required contradiction for part (ii). □

7 Computation of the SNEPs

The computation of the SNEPs is considered in this section. It will be assumed that condition (SC) holds throughout. As per usual, attention will be restricted to customer C_0, although the argument extends easily to any C_k, with only minor changes in the subscript indexing. The procedures outlined require the calculation of $V_0(x,[g]^\infty)$ for various x and g, the details of which have been deferred to the Appendix.

7.1 Evaluating the jump points of $G^*(\cdot)$

The lowest point of the graph of $G^*(\cdot)$ at the origin is given by $l(0)$. Defining

$$\mathbb{J} = \{1, 2, \ldots, B - l(0)\}$$

then the number of jump points is given by

$$J = \max\{j \in \mathbb{J} : V_0(l(0)+j-1,[0]^\infty) \geq \theta, V_0(l(0)+j-1,[B]^\infty) \leq \theta\}$$

with $\max\emptyset := 0$.

The jump points, $\{g_j\}$, satisfy

$$V_0(l(0)+j-1,[g_j]^\infty) = \theta$$

for $j = 1, \ldots, J$.

From the monotonicity and continuity of $V_0(\cdot,[g]^\infty)$, the j-th equation will have a solution if and only if $V_0(l(0)+j-1,[0]^\infty) \geq \theta$ and $V_0(l(0)+j-1,[B]^\infty) \leq \theta$ (and in the case where $B = \infty$, we might more precisely replace this latter condition by $\lim_{B\to\infty} V_0(l(0)+j-1,[B]^\infty) \leq \theta$). These observations provide the basis for a systematic procedure for evaluating the $\{g_j\}$.

7.2 Finding the SNEPs

If an SNEP exists at $g^* \in (g_j, g_{j+1})$ for $j = 0, 1, \ldots, J$, then the line of unit slope intersects the graph of $G^*(\cdot)$ on the horizontal section interior to g_j and g_{j+1}. Indeed, the height of that section is $l(0) + j$, and so we deduce that g^* is an SNEP in this location if and only if $g_j < l(0) + j < g_{j+1}$: indeed, $g^* = l(0) + j$.

On the other hand, $g^* = g_j$ for some $j = 1, 2, \ldots, J$, if and only if the line of unit slope intersects the vertical section of the graph of $G^*(\cdot)$ at g_j, i.e.

$$g_j \in [l(0) + j - 1, l(0) + j]$$

So with knowledge of the $\{g_j\}$, then, again, these observations provide the basis for a systematic procedure for finding the SNEPs within the $\mathbb{T}^\infty$ class.

8 Behaviour of a dynamic learning scheme

Consider a dynamic learning scheme in which each customer bases their joining decision on data collected by a central entity prior to its arrival to the system. Here it is assumed that the buffer size, B, is finite, and we consider a service rate function of the form

$$\mu(x) = \overline{\mu} \left\{ 1 - \frac{a}{(x+b)^c} \right\}, \qquad x = 1, \ldots, B$$

where $0 < a \leq 1$, $b > 0$, and $c > 0$.
We insist upon the presence of uncontrolled arrivals, in the sense that p is strictly positive.
Under this scheme each *controlled* customer follows the decision rule

$$\mathrm{join\,Q_S\,with\,probability\,S_\epsilon(\theta - \hat{V}_t(X_t))}, \tag{48}$$

where ϵ is a small positive parameter, S_ϵ is an increasing function, with $S_\epsilon(x) = 0$ for $x \leq -\epsilon$ and $S_\epsilon(x) = 1$ for $x \geq \epsilon$, and $\hat{V}_t(x)$ is the empirical average (sample mean) sojourn time of *all* customers who have exited Q_S by time t, but who entered it when the queue length was x.

The capacity for this scheme to 'learn' the Nash equilibria of the system under the stationary game is investigated using simulation. The examples are chosen to explore, in relative terms, three different regimes: (a) small state space, slowly varying service rate, (b) large state space, service rate quickly approaching upper bound, (c) large state space, slowly varying service rate.

<u>Simulation 1</u>
In this example, the arrivals are simulated to form a Poisson process with rate $\lambda = 9$. The buffer size, B, is set equal to 10, with the service rate function, $\mu(x)$, for $x = 1, \ldots, 10$, being specified by the parameter settings $\overline{\mu} = 10$, $a = 0.7$, $b = 0.05$, and $c = 1$. The parameter p, controlling the entrance of uncontrolled arrivals into the system, is set equal to 0.25. Also, the quality of service parameter, θ, is set equal to 0.85 and $S_\epsilon(\cdot)$ chosen to correspond to the Uniform cumulative distribution function on the interval $(\theta - \epsilon, \theta + \epsilon)$. Under these parameter settings, it can be deduced that

$$l(0) = 6, \quad g_1 = 1.8799$$

with the number of jumps, J, of the graph of $G^*(\cdot)$, being equal to 1. There is a unique SNEP in the class $\mathbb{T}^\infty$ located at $g^* = 7$. A simulation of this system is performed over a horizon length of $5,000$ time units, with ϵ equal to 0.03. Figure 1 shows a plot of $\widehat{V}_t(x)$ against time for entry states 6 and 7, along with horizontal lines at $\theta - \epsilon$ and $\theta + \epsilon$. From a very early stage in the simulation, $\widehat{V}_t(x)$ stays well below $\theta - \epsilon$ for entry states $x = 0, \ldots, 5$, and stays well above $\theta + \epsilon$ for states $x = 8$ and $x = 9$. For comparison, we remark that $V_0(6, [7]^\infty)$ and $V_0(7, [7]^\infty)$ are found to be 0.7959 and 0.9028, respectively. The entrance probabilities against the natural logarithm of time, under the learning rule, for entry states 6 and 7 are depicted in Figure 2. The results of this experiment appear to support the hypothesis that the system-wide statistics converge to those corresponding to the stationary threshold at $g^* = 7$.

Simulation 2

For this second example, the parameter settings are exactly the same as those in Simulation 1, except that the service rate function is specified by the parameters $a = 0.9$, $b = 0.05$, and $c = 3$, and the size of the state space is somewhat larger with $B = 25$. The graph of $G^*(\cdot)$ has, again, one jump, with

$$l(0) = 7, \quad g_1 = 1.0666$$

and so it is deduced that a single SNEP resides within the class $\mathbb{T}^\infty$ at $g^* = 8$.

Again, the simulation is performed over 5,000 time units: empirical averages for $x = 0, \ldots, 6$ stay well below $\theta - \epsilon$, and for $x = 9, 10, \ldots, 24$ well above $\theta + \epsilon$. Empirical averages for states $x = 7$, $x = 8$, are depicted in Figure 3 and entrance probabilities in Figure 4. The values of $V_0(7, [8]^\infty)$ and $V_0(8, [8]^\infty)$ are 0.8049 and 0.9046, respectively.

Simulation 3

In this final example, $\lambda = 9$, with the service rate specified by the parameters $\overline{\mu} = 10$, $a = 1$, $b = 1$, and $c = 0.8$, and with B set equal to 25. This time the parameters θ and ϵ are equal to 1.5 and 0.015, respectively.

The graph of $G^*(\cdot)$ displays one jump, with

$$l(0) = 12, \quad g_1 = 13.1683$$

and so it is deduced that a single SNEP resides within the class $\mathbb{T}^\infty$ at $g^* = 12$.

Again, the simulation is performed over 5,000 time units: it is observed that empirical averages for $x = 0, \ldots, 10$ stay well below $\theta - \epsilon$, and for $x = 13, \ldots, 24$ well above $\theta + \epsilon$. Empirical averages for states $x = 11$, and $x = 12$, are depicted in Figure 5 and entrance probabilities in Figure 6. The values of $V_0(11, [12]^\infty)$ and $V_0(12, [12]^\infty)$ are 1.3998 and 1.5116, respectively.

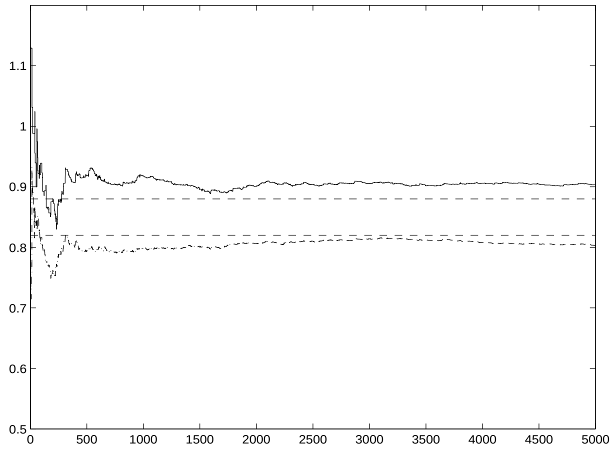

Figure 1: [Simulation 1] Plot of empirical averages against time for entry states $x = 6$ (dash-dot line) and $x = 7$ (solid line) with the bands $\theta \pm \epsilon$ (dashed lines).

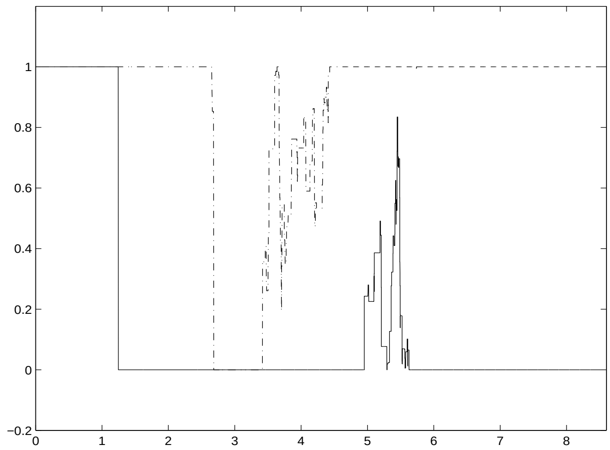

Figure 2: [Simulation 1] Plot of entrance probabilities for controlled customers against the log of time for entry states $x = 6$ (dash-dot line) and $x = 7$ (solid line).

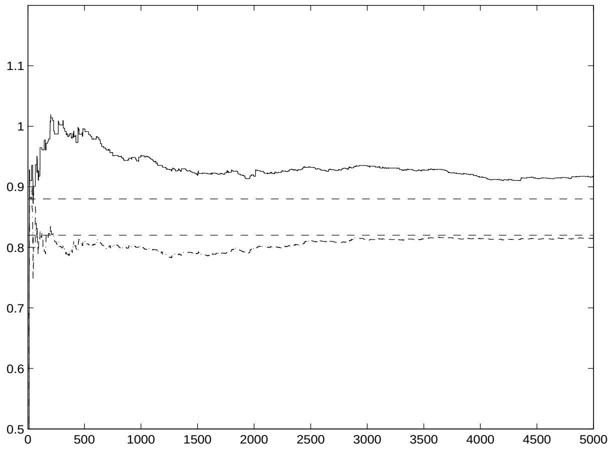

Figure 3: [Simulation 2] Plot of empirical averages against time for entry states $x = 7$ (dash-dot line), and $x = 8$ (solid line), with the bands $\theta \pm \epsilon$ (dashed lines).

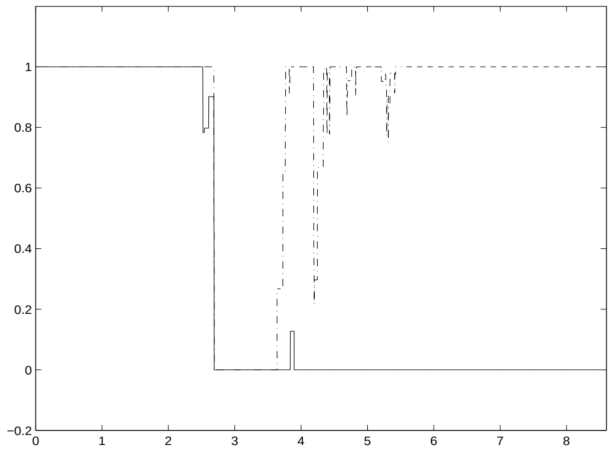

Figure 4: [Simulation 2] Plot of entrance probabilities for controlled customers against the log of time for entry states $x = 7$ (dash-dot line) and $x = 8$ (solid line).

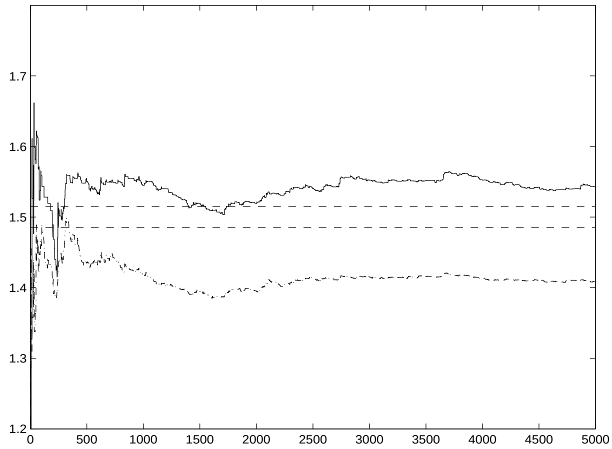

Figure 5: [Simulation 3] Plot of empirical averages against time for entry states $x = 11$ (dash-dot line), and $x = 12$ (solid line), with the bands $\theta \pm \epsilon$ (dashed lines).

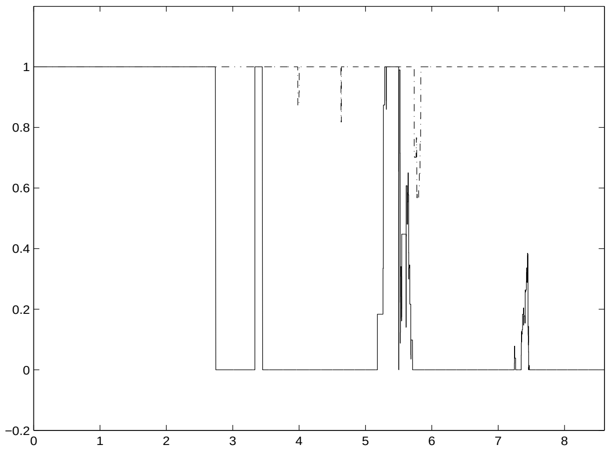

Figure 6: [Simulation 3] Plot of entrance probabilities for controlled customers against the log of time for entry states $x = 11$ (dash-dot line) and $x = 12$ (solid line).

9 Conclusions

In this paper, we established the existence of a certain symmetric Nash equilibrium policy for customers joining a queue with load-increasing service rate, based on the observed queue size on arrival to the system. We proved that at least one of the Nash equilibria is non-randomized: this is a somewhat different phenomenon to that exhibited in [2], [4], and [5]. Constancy of the expected sojourn time in g on the region $[0, 1]$, and strict monotonicity on $[1, B]$, was established as an essential stepping stone for the game theoretic results[2].

Simulation experiments suggest that when a unique (non-randomized) SNEP of the stationary game exists, in the class $\mathbb{T}^{\infty}$, then quantities such as the empirical averages and simulated entrance probabilities, under the learning rule, show a close correspondence to expected sojourn times and entrance probabilities under the SNEP in the associated stationary game. Convergence and stability properties in the case of multiple Nash equilibria are not well understood at this stage; however, simulation experiments appear to suggest that the SNEPs are viable poles of attraction and provide a rough guide to the operating points of the system.

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Appendix: Evaluating $V_0(\cdot, [g]^\infty)$

A procedure for calculating the expected sojourn time of a customer entering Q_S under the symmetric equilibrium policy $[g]^\infty$, with $g = L + q$, and with buffer size B, is presented via a set of linear equations.

We shall further assume that the arrival process consists of the superposition of two independent Poisson processes, which are: (a) the controlled arrival process, at rate λ_c, in which customers are governed by the decision rule $[g]$; (b) the uncontrolled arrival process, at rate λ_u, in which customers always join the system, provided the buffer size is not exceeded.

Set $R(x,0) := 0$ and let $R(x,y)$ be the *expected remaining sojourn time* of a customer who has precisely $y-1$ customers ahead of it in the queue, when the queue length is x for $0 < y \leq x$. Defining $\alpha := \lambda + \overline{\mu}$, then the $\{R(x,y)\}$ satisfy the following set of linear equations.

$$R(x,y) = \frac{1}{\alpha}\left[1 + \mu(x)R(x-1,y-1) + (\overline{\mu} - \mu(x))R(x,y) + \lambda R(x+1,y)\right] \tag{49}$$

where $0 < y \leq x < L \leq B$;

$$R(L,y) = \frac{1}{\alpha}\left[1 + \mu(L)R(L-1,y-1) + \{(\overline{\mu} - \mu(L)) + \lambda_c(1-q)\}R(L,y) + (\lambda_c q + \lambda_u)R(L+1,y)\right] \tag{50}$$

where $0 < y \leq L < B$;

$$R(x,y) = \frac{1}{\alpha}\left[1 + \mu(x)R(x-1,y-1) + \{\overline{\mu} - \mu(x) + \lambda_c\}R(x,y) + \lambda_u R(x+1,y)\right] \tag{51}$$

where $0 < y \leq x$, $L+1 \leq x < B$;

$$R(B,y) = \frac{1}{\alpha}\left[1 + \mu(B)R(B-1,y-1) + \lambda R(B,y)\right] \tag{52}$$

where $0 < y \leq B$.
It is easily seen that $V_0(x, [g]^\infty) = R(x+1, x+1)$.