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Abstract

We analyse deflationary bubbles in a model where money is the only financial asset. We show that such bubbles are consistent with the household's transversality condition if and only if the nominal money stock is falling. Our results are in sharp contrast to those in several prominent contributions to the literature, where deflationary bubbles are ruled out by appealing to a non-standard transversality condition, originally due to Brock ([4], [5]). This condition, which we dub the *GABOR* condition, states that the consumer must be indifferent between reducing his money holdings by one unit and leaving them unchanged and enjoying the discounted present value of the marginal utility of that unit of money forever. We show that the *GABOR* condition is not part of the necessary and sufficient conditions for household optimality nor is it sufficient to rule out deflationary bubbles. Moreover, it rules out Friedman's optimal quantity of money equilibrium and, when the nominal money stock is falling, it rules out deflationary bubbles that are consistent with household optimality.

We also consider economies with real and nominal government debt and small open economies where the government can lend to and borrow from abroad. In these cases, deflationary bubbles may be possible, even when the nominal money stock is rising. Their existence is shown to depend on the rules governing the issuance of government debt.

1 Introduction

This paper revisits the existence of deflationary bubbles and the conditions that rule them out. We focus on two standard dynamic optimizing models with unbacked government-issued money. In the first model, money is the only store of value. In this framework we obtain new results, demonstrating that the existence of deflationary bubbles depends on whether the money stock is increasing or decreasing. In the second model, money co-exists with non-monetary financial instruments, such as government bonds, that can be in negative net supply. We show that the rules governing the issuance of these instruments determine whether or not deflationary bubbles exist.

The literature we are extending goes back to two seminal papers by Brock [4], [5]. Brock analyzes a closed-economy model with money, but no government bonds. In addition to the standard transversality condition (henceforth, the ST condition) associated with the household's optimisation problem, that the present value of the terminal stock of real money balances is zero, Brock introduces an additional restriction. This restriction (henceforth the $GABOR$ condition) was later adopted by Obstfeld and Rogoff [27], [28] [29], Gray [14] and Azariadis [1]. It states that it must not be possible to increase consumer welfare by increasing (reducing) current consumption by a small amount through a reduction (increase) in the stock of current money balances and holding this lower (higher) stock of money balances forever after.

In this paper we establish the following. First, unlike the ST condition, the $GABOR$ condition is not part of the necessary and sufficient conditions for an optimal consumption-money demand programme.

Second, in a model where money is the only financial instrument, we demonstrate that deflationary bubbles are consistent with the ST condition if and only if money growth is strictly negative. This contrasts with results that have been obtained with the $GABOR$ condition. Even when the money stock is rising, the $GABOR$ condition alone is not sufficient to rule out deflationary bubbles. We show that when the nominal money stock is falling, the *GABOR* condition rules out deflationary bubbles that are consistent with household optimality and the other equilibrium conditions of the model.

Third, we show that the *GABOR* condition rules out Bailey's [2] and Friedman's [13] stationary optimal quantity of money equilibrium, where the nominal interest rate is zero, the (negative) inflation rate equals the rate of decline of the stock of nominal money balances and there is satiation in real money balances. This is true even when satiation is achieved at a finite level of real money balances.

Fourth, without the *GABOR* condition, deflationary bubbles can exist in the closed economy model with both money and bonds, even when the nominal money stock is rising. An example of a fiscal rule that supports deflationary bubbles was suggested by Woodford ([38], pp. 131-135), who discusses and provides examples of fiscal rules that rule out what he calls "deflationary traps".

Fifth, suppose that in the money-and-bonds model the solvency constraints of the household and the government are asymmetric in that households view money as a redeemable asset while the government views money as irredeemable or inconvertible (see Buiter [6] and [7]). Then if the government satisfies its solvency constraint with equality, we show that deflationary bubbles do not exist unless the money stock is falling.

Sixth, in a small open economy, we show that deflationary bubbles can exist even when the money supply is growing and even when the government views money as irredeemable and satisfies its solvency constraint with equality.

2 Deflationary Bubbles when Money is the Only Financial Instrument

2.1 The households

The economy is inhabited by a representative household and its government. Each period, the household receives an exogenous endowment of the single perishable consumption good and pays a lump-sum tax. It consumes the good and saves in the form of non-interest-bearing unbacked money issued by the state. The household receives liquidity services from its money holdings and has preferences defined over paths of consumption and holdings of real balances represented by

$$U = \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t u(c_t, M_t^d/P_t),\ 0 < \beta < 1, \tag{1}$$

where $c_t \geq 0$ is time-t consumption, $M_t^d \geq 0$ is the household's end-of-period-t demand for nominal money balances to be carried over into period $t+1$, P_t is the period-t money price of the good and u is the extended real-valued utility function on $\mathbb{R}_+^2$. We assume that u is strictly increasing in its first argument, concave and continuously differentiable on $\mathbb{R}_{++}^2$, with $u_c(c,m) \to \infty$ as $c \searrow 0$ and $u_m(c,m) \to \infty$ as $m \searrow 0$. We only consider equilibria where the infinite sum in equation (1) converges.

We temporarily assume that money is valued and thus, $1/P_t > 0$ for every $t \geq 0$. We then only consider outcomes where this is true. There is, however, always a non-monetary equilibrium where $1/P_t = 0$ for every $t \geq 0$. In this outcome, money is not held and the household consumes its after-tax endowment each period.

Assumption 1. For every $c \in \mathbb{R}_{++}$, there exists $[\underline{u}(c), \bar{u}(c)] \subset \mathbb{R}_{++}$ such that $\underline{u}(c) \leq u_c(c,m) \leq \bar{u}(c)$ for every $m \in \mathbb{R}_+$

We make one of the following mutually exclusive assumptions:

Assumption 2a (Satiation in real balances) For every $c \in \mathbb{R}_{++}$ there exists $\hat{m}(c) \in \mathbb{R}_{++}$ such that $u_m(c,m) > (=)\ 0$ if $m < (\geq)\ \hat{m}(c)$ and u is strictly concave and twice differentiable on $\{(c,m) : 0 < m < \hat{m}(c), c \in \mathbb{R}_{++}\}$.

Assumption 2b (Bounded utility in real balances; no satiation for finite real balances) The function u is strictly concave and twice differentiable on $\mathbb{R}_{++}^2$, $u_m(c,m) > 0$ and $u(c,m)$ is bounded from above in m.

Assumption 2c (Unbounded utility). The function u is strictly concave and twice differentiable on $\mathbb{R}_{++}^2$, $u_m(c,m) > 0$ and $u_m(c,m) \to 0$ as $m \longrightarrow \infty$.

The household's within-period budget constraint is

$$M_t^d/P_t \leq y - \tau_t - c_t + M_{t-1}^d/P_t,\ t \geq 0, \tag{2}$$

where $y > 0$ is the constant per-period endowment and τ_t is the period-t real lump-sum tax. In this section, we only consider outcomes where

$$\tau_t \leq y + M_{t-1}^d / P_t, \ t \geq 0 \tag{3}$$

The household maximises utility (equation (1)) subject to (2), taking as given initial money holdings $M_{-1}^d > 0$.

Sufficient conditions for optimality are given by the household period budget constraints (2) (with equality), the Euler equation

$$u_c(c_{t,} M_t^d / P_t) = u_m(c_t, M_t^d / P_t) + (\beta P_t / P_{t+1})\, u_c(c_{t+1}, M_{t+1}^d / P_{t+1}), \quad t \geq 0, \tag{4}$$

and the standard transversality (ST) condition[1]

$$\lim_{t \to \infty} \beta^t u_c(c_t, M_t^d / P_t) M_t^d / P_t = 0. \tag{5}$$

The Euler equation and the budget constraint are also necessary for optimality. (See Lucas and Stokey [25], p. 97.) The necessity of the transversality condition has been a more difficult issue. Weitzman [34] shows that it is necessary when within-period utility is bounded. Kamihigashi [19] generalises this result to the case where, at an optimum, the sequence of discounted within-period utilities are summable.

Equation (4) is typical of the Euler equations that characterise investment in a consumer durable and has the following interpretation. The household is indifferent between a marginal increase in period-t consumption, yielding utility of $u_c(c_t, M_t^d / P_t)$, and foregoing this consumption and acquiring money, receiving utility of $u_m(c_t, M_t^d / P_t)$ from its liquidity services and using it to purchase consumption next period, with associated discounted utility of $(\beta P_t / P_{t+1})\, u_c(c_{t+1}, M_{t+1}^d / P_{t+1})$.

Equation (5) implies that either the optimal value of the state variable, M_t^d / P_t, goes

[1] See Lucas and Stokey [25], p. 98.

to zero as time goes to infinity or that its marginal contribution to the maximized value of the objective function, $\beta^t u_c(c_t, M_t^d/P_t)$, goes to zero.

Solving (4) forward yields

$$u_c\left(c_t, M_t/P_t\right) = \sum_{s=0}^{\infty} \frac{\beta^s P_t u_m(c_{t+s}, M_{t+s}^d/P_{t+s})}{P_{t+s}} + \lim_{T\to\infty} \frac{\beta^T P_t u_c(c_{t+T}, M_{t+T}^d/P_{t+T})}{P_{t+T}}. \quad (6)$$

2.2 Brock's restriction on optimal programmes

Brock [4], [5] proposes a further restriction on permissible optimal programmes:

$$u_c(c_t, M_t^d/P_t) = \sum_{s=0}^{\infty} \left(\beta^s P_t/P_{t+s}\right) u_m(c_{t+s}, M_{t+s}^d/P_{t+s}). \quad (7)$$

By equation (6) this is equivalent to

$$\lim_{T\to\infty} \left(\beta^T P_t/P_{t+T}\right) u_c(c_{t+T}, M_{t+T}^d/P_{t+T}) = 0. \quad (8)$$

His justification is that the consumer must be indifferent between reducing his money holdings by one unit and enjoying an increase in marginal utility of consumption (the left-hand side of equation (7)) and leaving his money holdings unchanged and enjoying the discounted present value of the marginal utility of that unit of money forever (the right-hand side of equation (7)).[2] Obstfeld and Rogoff [27], [28] and [29] make the same argument, as do Gray [14] and Azariadis [1][3].

The Gray-Azariadis-Brock-Obsfeld-Rogoff (*GABOR*) condition given in (8) (or, equiv-

[2] Brock assumes $u(c, m) = u(c) + v(m)$. There is no public spending; hence, in equilibrium $c = y$. Denoting time-t real balances by m_t, he says, "At some point in time, T, the act of taking one dollar out of cash balances will yield him $u'(y)/P_T$ utils at the margin. His cash balances are depleted by one dollar for all $s \geq T$. This loss of money services generates a utility loss $\sum_{t=T}^{\infty} \beta^{t-T} v'(m_t)/P_t$..."(Brock [5], p. 140). The same argument is also made in Brock [4], p. 762.

[3] Gray's intuitive motivation of the *GABOR* condition (in [14], Section 4.1, and especially the argument starting on p. 107 leading up to equation (29) on p. 110), is essentially the same as Brock's. The *GABOR* condition can be written as $\lim_{T\to\infty} \beta^T u_c(c_{t+T}, M_{t+T}^d/P_{t+T})/P_{t+T} = 0$. The ST condition is $\lim_{T\to\infty} \beta^T u_c(c_{t+T}, M_{t+T}^d/P_{t+T}) M_{t+T}/P_{t+T} = 0$. Gray notes correctly ([14], footnote (21)), that when the nominal money stock becomes a positive constant after some date, the ST condition and the *GABOR* condition are equivalent.

alently, (7)) cannot in general be a requirement for optimality. The reason is that if a consumer converts a dollar into consumption (or vice versa) in period t and never undoes that shift, he loses (gains) not just the reduction (increase) in money holdings in period t and forever after. He also loses (gains) the terminal consumption that is associated with a permanent reduction in money holdings. Without this eventual reduction in consumption, the proposed perturbation of the optimal consumption and money demand programme is in general not feasible because it violates the sequence of household within-period budget constraints.

To see this, suppose that a consumer lowers his holdings of real balances by one unit and increases his consumption in period t and then lowers his consumption $T \geq 1$ periods later to restore his original money holdings. This leads to the period-t utility gain $u_c(c_t, M_t^d/P_t)$. The loss of liquidity services before he undoes the shift equals $\sum_{s=0}^{T-1} (\beta^s P_t/P_{t+s})\, u_m(c_{t+s}, M_{t+s}^d/P_{t+s})$. In period $t+T$ the reduction in real balances is reversed. There is no utility loss from lower money balances in period $t+T$ or later, but the restoration of money balances has been affected through a reduction in period $t+T$ consumption by an amount P_t/P_{t+T}. The associated discounted utility loss is $(\beta^T P_t/P_{t+T})u_c(c_{t+T}, M_{t+T}^d/P_{t+T})$. When a household reduces its real balances in period t by one unit and never reverses this, it is still required to reduce its terminal consumption by an amount $\lim_{T\to\infty} P_t/P_{t+T}$. The associated discounted utility loss is $\lim_{T\to\infty}(\beta^T P_t/P_{t+T})u_c(c_{t+T}, M_{t+T}^d/P_{t+T})$.

Ignoring the last term in (6) or requiring it to equal zero may thus violate the household's sequence of within-period budget constraints.[4] Reducing consumption in period t and increasing money holdings in period t and never reversing the shift is feasible, but not rational. The correct characterisation of the perturbations of the infinite horizon optimal programme that should be utility neutral is therefore (5). Household optimisation does not, in general require that (8) hold. Because of the prominence of the *GABOR* condition

[4] Whether it does or not depends on the relationship between inflation and the discount factor. If $\lim_{t\to\infty} P_{t+s}/P_{t+s-1} > \beta$, then the last term in (6) is zero if $c_t \nrightarrow 0$.

in the literature, we state the following result.

Proposition 1 *Given the Euler equation (4), the sequence of within-period budget constraints (2) (with equality), and the initial value of the nominal money stock, the GABOR condition (8) is not part of the necessary and sufficient conditions for an optimum of the household's consumption-money demand programme.*

This is obvious from Kamihigashi's [19]) result that the household within-period budget constraints, (2), the Euler equations (4) and the transversality condition (5) are the set of necessary and sufficient conditions for an optimum. Condition (6) is an implication of the Euler equation (4). Condition (8) is an additional restriction on (5), and therefore on the necessary and sufficient conditions for an optimum.

2.3 The government

The government is the consolidated fiscal and monetary authorities. Its within-period budget constraint is

$$M_t/P_t \geq g - \tau_t + M_{t-1}/P_t,\ t \geq 0, \tag{9}$$

where $g \geq 0$ is the constant per-period real public spending and M_t is the money supply. Assuming $g < y_t$ ensures that the assumed restriction (3) can be satisfied. We assume that (9) holds with equality.

We assume that the authorities adopt a constant proportional growth rate for the money stock so that

$$M_{t+1}/M_t = \mu \geq 0. \tag{10}$$

The sequence of real lump-sum taxes is endogenously determined to make the exogenous public spending programme and the constant proportional growth rate of the nominal money stock consistent with the sequence of within-period government budget constraints.

2.4 Equilibrium

In equilibrium, $M_t^d = M_t,\ t \geq 0$ and

$$c_t = c = y - g,\ t \geq 0 \tag{11}$$

Let $m_t \equiv M_t/P_t$. Substitute (11) into (4) and (5). We define an equilibrium as follows:

Definition 2 *An equilibrium is a strictly positive sequence* $\{m_t\}_{t=0}^{\infty}$ *such that*

$$\beta u_c(c, m_{t+1})m_{t+1} = \mu[u_c(c, m_t) - u_m(c, m_t)]m_t,\ t \geq 0 \tag{12}$$

$$\lim_{t \to \infty} \beta^t u_c(c, m_t)m_t = 0. \tag{13}$$

By equation (9), the taxes associated with the equilibrium satisfy

$$\mu\tau_t = (1 - \mu)\, m_t + \mu g,\ t \geq 0. \tag{14}$$

The *GABOR* condition (8) can be rewritten as

$$\lim_{t \to \infty} (\beta/\mu)^t\, u_c(c, m_t)m_t = 0. \tag{15}$$

The *ST* condition (13) implies the *GABOR* condition (15) when $\mu \geq 1$ and the *GABOR* condition implies the *ST* condition when $\mu \leq 1$. The two conditions are equivalent when the nominal money stock is constant ($\mu = 1$).

There are two potential types of monetary equilibria. First, given our constant fundamentals (y, g, μ), there is a *fundamental* equilibrium where $m_t = \bar{m} > 0$ for every $t \geq 0$. Constant real balances clearly satisfy (13). By (12) such an equilibrium has

$$L(\bar{m}) \equiv \mu u_m(c, \bar{m}) = (\mu - \beta)u_c(c, \bar{m}) \equiv R(\bar{m}). \tag{16}$$

We make the following assumption, which given our previous assumptions, is necessary and sufficient for a fundamental monetary equilibrium to exist:

$$\mu \left\{ \begin{array}{c} \geq \\ > \end{array} \right\} \beta \text{ if } \left\{ \begin{array}{c} \text{Assumption 2a holds} \\ \text{Assumption 2b or 2c holds} \end{array} \right\}. \tag{17}$$

If $\mu < \beta$, then $L(m) \geq 0 > R(m)$ for every $m > 0$ and no fundamental monetary equilibrium exists. If $\mu = \beta$ and Assumption 2b or 2c hold, then $L(m) > 0 = R(m)$ for every $m > 0$ and no fundamental monetary equilibrium exists. If $\mu = \beta$ and Assumption 2a holds then any $m \geq \hat{m}$ satisfies equation (16).[5] Such an outcome is a Friedman Optimal Quantity of Money (OQM) equilibrium, where the nominal stock of money declines proportionally at the rate of time preference and the household is satiated at a finite stock of real balances.

When $\mu > \beta$ a simple fixed-point argument can be made to establish the existence of a fundamental monetary equilibrium.[6] For this case, the additional restriction that real balances are a normal good at any fixed point (that is, $u_c u_{mm} - u_m u_{cm} < 0$) ensures that the fundamental monetary equilibrium is unique.

In additional to fundamental monetary equilibria, there can be a gamut of non-fundamental (or non-stationary) equilibria. (See Azariadis [1]). A monetary equilibrium can be stable, with monotonic or cyclical convergence; it can be unstable, with either monotonic or cyclical divergence; there can be limit cycles and there can be chaotic behaviour. We are interested in monetary equilibria where nominal real balances go to infinity; such equilibria are called deflationary bubbles.

Definition 3 *A deflationary bubble is an equilibrium where* $m_t \to \infty$.

When the nominal money stock is constant, a deflationary bubble has the price level

[5] As c equals $y-g$, which is constant, we surpress the notational dependence of $\hat{m}$ and $\underline{u}$ and $\bar{u}$ (defined in Assumption 1) on c.

[6] By Assumption 1, $R(m) \in \rho \equiv [(\mu-\beta)\underline{u}, (\mu-\beta)\bar{u}]$. By the continuity of u_c in m and Assumption 2a, 2b or 2c, L^{-1}exists on ρ and $L^{-1}(R(m))$ is a continuous mapping from the compact convex set $[L^{-1}(((\mu-\beta)\bar{u}), L^{-1}((\mu-\beta)\underline{u})]$ into itself and, thus, by Brouwer's theorem, there exists a fixed point $\bar{m}$ that satisfies equation(16).

going to zero - the standard definition of sustained deflation. With positive growth in the nominal money stock, a deflationary bubble can occur even with a rising price level. Along such a path however, inflation will be less than inflation in the associated fundamental equilibrium.

Proposition 4 *Consider the class of preferences satisfying both Assumption 1 and one of Assumptions 2a, 2b or 2c. If* $\mu > 1$*, then the ST condition is sufficient to rule out deflationary bubbles; the GABOR condition is not. If* $1 \geq \mu > \beta$*, then neither the ST condition nor the GABOR condition are sufficient to rule out deflationary bubbles for all preferences. If* $1 > \mu$*, the GABOR condition rules out deflationary bubbles not ruled out by the ST condition.*

Proof. Suppose that $\mu > 1$. By (12), $u_c(c, m_{t+1})m_{t+1}/(u_c(c, m_t)m_t) = (\mu/\beta)\left[1 - u_m(c, m_t)/u_c(c, m_t)\right] \leq \mu/\beta$. If the equilibrium is a bubble, then $m_t \rightarrow \infty$ and $u_m(c, m_t) \rightarrow 0$. Thus, $\forall \epsilon > 0, \exists\, t^* > 0$ and finite such that $u_c(c, m_{t+1})m_{t+1}/u_c(c, m_t)m_t > \mu/\beta - \epsilon \ \forall t \geq t^*$. Let $\epsilon = (\mu - 1)/\beta$. Then we have $u_c(c, m_{t^*+T})m_{t^*+T} > (1/\beta)^T u_c(c, m_{t^*})m_{t^*}, \forall T \geq 1$. This implies $\lim_{T\rightarrow\infty} \beta^T u_c(c, m_{t^*+T})m_{t^*+T} > u_c(c, m_{t^*})m_{t^*} > 0$ which violates the ST condition (13).

If $u_m(c, m) = 1/\ln(m)$ for large values of m then a deflationary bubble equilibrium exists and the *GABOR* condition is satisfied for every $\mu > \beta$ (See Obstfeld and Rogoff [28]) and the ST condition is satisfied for $\mu = 1$.

Suppose $\mu < 1$. Then examples of deflationary bubbles that satisfy the ST condition, but not the *GABOR* condition are easy to find. See the text following. ■

The intuition for why the ST condition rules out deflationary bubbles when the nominal money stock is not falling is as follows. In equilibrium, in each period the household must be indifferent between spending a unit of money on the consumption good and holding the unit of money, enjoying its liquidity services, and spending it on the consumption good the following period. Thus, because money provides liquidity services, the discounted shadow value of money used to purchase the consumption good must be falling over time.[7] Along a bubble path, real balances go to infinity and the liquidity services of

[7] The time-t discounted shadow value to the household of a unit of money used for purchasing the good is the incremental contribution to U in equation (1), $(\beta^t/P_t)u_c(c, M_t/P_t)$.

a unit of money go to zero. Thus, the rate of decrease over time in the discounted shadow value of money used to purchase consumption goes to zero. If the nominal money stock is constant or growing at a constant rate over time, this implies that the value of the entire money stock when priced at the discounted shadow value of money used to purchase the consumption good, becomes constant or rises over time. This is inconsistent with the ST condition, which says that the present discounted shadow value of the economy's money stock must go to zero in the long run. If the nominal money stock is falling at a constant rate over time, then the discounted shadow value of the entire money stock when used to purchase the consumption good must be falling as well.[8]

When $\mu > 1$, the $GABOR$ condition is a weaker condition than the ST condition and it is not sufficient to rule out deflationary bubbles. Obstfeld and Rogoff [28] show that the $GABOR$ condition is sufficient to rule out deflationary bubbles if Assumptions 2a or 2b are satisfied. Brock [4] provides a weaker condition.

Proposition 5 *(Brock's Result) The GABOR condition is sufficient to rule out deflationary bubbles if there exists $\lambda < 0$ and finite $m_0 > 0$ such that $m > m_0$ implies $v'(m) \leq m^{\lambda}$.*

When $\beta < \mu < 1$, imposing the $GABOR$ condition rules out deflationary bubbles that satisfy the necessary and sufficient conditions of the consumer's problem. For example, suppose that $u(c,m) = h(c) + \left(m^{1-\theta} - 1\right)/(1-\theta)$ when $0 < \theta \neq 1$ and $u(c,m) = h(c) + \ln c$ when $\theta \to 1$. Let $\alpha \equiv 1/h'(c)$. Then, there is a steady state at $\bar{m} = [\alpha\mu/(\mu - \beta)]^{1/\theta}$. We have $dm_{t+1}/dm_t = \mu\beta - [(1-\theta)(\mu-\beta)/\beta]\left(m_t/\bar{m}\right)^{-\theta} > 0$ if $m_t > \bar{m}$ and $dm_{t+1}/dm_t = \mu\beta - (1-\theta)(\mu-\beta)/\beta > 1$ if $m_t = \bar{m}$. Thus, the steady state is not stable and if $m_0 > \bar{m}$, the equilibrium is a deflationary bubble. This utility function satisfies Brock's condition; hence, it does not satisfy the $GABOR$ condition. We have $m_{t+1}/m_t = (\mu/\beta)\left(1 - \alpha m_t^{1-\theta}\right)$; hence $\lim_{t\to\infty} \beta^t u_c(c,m_t)m_t = (1/\alpha)\lim_{t\to\infty} \mu^t \prod_{s=0}^{t}\left(1 - \alpha m_t^{1-\theta}\right) = 0$. Thus, the equilibrium is not ruled out by the ST condition. A particularly transparent

[8] This argument suggests that if $\mu < 1$, then any sequence of real balances that goes to infinity and satisfies (12) must satisfy the ST condition and this is true. In this case, by (12), $u_c(c,m_{t+1})m_{t+1}/(u_c(c,m_t)m_t) < \mu/\beta$. This implies $\beta^T u_c(c,m_{t+T})m_{t+T} < \mu^T u_c(c,m_t)m_t \to 0$.

example is the case where $\theta = 1$. Then $m_t = (m_0 - \bar{m})(\mu/\beta)^t + \bar{m}$ satisfies (12). If $m_0 > \bar{m}$ then $m_t \to \infty$ and clearly the ST condition is satisfied, but not the $GABOR$ condition.

Proposition 6 *The GABOR condition rules out Friedman's Optimal Quantity of Money equilibrium; the ST condition does not.*

Proof. Friedman's OQM equilibrium (any fundamental equilibrium supported by $\beta = \mu$) has $(\beta/\mu)^t u_c(c, m_t) m_t = u_c(c, \bar{m})\bar{m} > 0$, with $\bar{m} \geq \hat{m}$; hence $GABOR$ is not satisfied. It has $\beta^t u_c(c, m_t) m_t = \beta^t u_c(c, \bar{m})\bar{m} \to 0$; hence, the ST condition is satisfied. ■

The $GABOR$ condition rules out Friedman's where $\mu = \beta$ and households are satiated in real balances.[9] It is easily verified that in the model of this section and in the model with money and bonds in Section 3 that if Assumption 2a holds, Friedman's OQM equilibrium is the household's most preferred outcome.

The intuition for why the $GABOR$ condition cannot be part of the household's optimal programme is particularly stark in this case. The $GABOR$ condition is a statement that the household must be indifferent between reducing his money holdings by one unit and using this extra unit of money to purchase consumption and holding the unit of money *forever* and enjoying the discounted present value of the resulting liquidity services. However, as the household is satiated in real balances in the OQM equilibrium, it would hold an extra unit of money as a pure store of value only in order to increase its consumption at some time in the future. At some point in the future, the incremental unit of money must be exchanged for the consumption good if the earlier sacrifice of consumption is to be rational.

We now show that when money growth equals the discount factor and there is satiation in real balances, deflationary bubbles cannot exist. This is a consequence of the Euler equation, rather than the ST condition.

Proposition 7 *If $\mu = \beta < 1$ and Assumption 2a holds, then deflationary bubbles cannot exist.*

[9] The neoclassical or intertemporal public finance theory of the optimal quantity of money is a vast subject. See, for example, Bailey [2] , Friedman [13], Wilson [37] and Cohen [9],

Proof. When $\mu = \beta$, equation (12) becomes $u_c(c, m_{t+1})m_{t+1} = [u_c(c, m_t) - u_m(c, m_t)]m_t,\ t \geq 0$. If there exists a solution that has money balances rising without bound, then at some point, $m_t > \hat{m}(c)$. By Assumption 2a, if $m > \hat{m}(c)$ then $u_m(c, m) = 0$, and we have $u_c(c, m_{t+1})m_{t+1} = u_c(c, m_t)m_t$. For this to support m_t rising without bound, we require $u_c(c, m_t)/u_c(c, m_{t+1}) > 1$. For this to be be possible, $u_{cm}(c, m) < 0$. This is a contradiction, however, as $u_{cm}(c, m_t) = 0$ for $m_t > \hat{m}(c)$. ■

We have assumed that $\mu \geq \beta$. If this is not the case, it is easy to find examples of non-fundamental equilibria, but deflationary bubbles cannot exist. This result is also a consequence of the Euler equation, rather than the ST condition.

Proposition 8 *If $\mu < \beta$ then deflationary bubbles cannot exist.*

Proof. Rewrite equation (12) as $m_{t+1} = (\mu/\beta)[u_c(c, m_t)/u_c(c, m_{t+1}) - u_c(c, m_t)/u_c(c, m_{t+1})]m_t$. For $\{m_t\} \to \infty$, there must be a subsequence $\{m_{t_k}\}$ such that $u_c(c, m_{t_k})/u_c(c, m_{t_k+1}) > \beta/\mu > 1$. This implies, $u_c(c, m_{t_k}) \to 0$, which is not possible as it violates Assumption 1. ■

3 Deflationary Bubbles with Money and Government Bonds

In the money-only model of Section 2, the household transversality condition requires that the present discounted value of the household's terminal stock of real money balances is zero. When there are financial and/or real assets besides money, and asset markets are efficient, then the corresponding household transversality condition (plus a solvency constraint) implies that the present discounted value of the household's terminal *aggregate* net non-human wealth is zero. In the money-only model, if the nominal money stock is not falling, then any path of real balances that satisfies the household's Euler equation and goes to infinity produces a path of present discounted values of real money balances that eventually rises or remains constant at some positive value, violating the household's transversality condition.

Woodford [38] suggests that if an additional financial asset were available that could have a strictly negative supply, then the household transversality condition might be satisfied even if the discounted value of the household's real balances does not go to zero. In this section we analyse the case were the government issues real and nominal debt as well as unbacked money and we show that Woodford's conjecture is correct when the government follows a tax rule which implies that the government's solvency constraint is satisfied with strict inequality along a deflationary bubble path. We also show that the conjecture is not correct if the government follows a tax rule which implies that the government satisfies its solvency constraint with equality. This is a consequence of the asymmetry of the government and household budget constraints. The household views its terminal money stock as an asset. The government views money as irredeemable; hence, it does not view the terminal money stock as a liability.

3.1 Households

We now assume that the government issues nominal and real bonds, in addition to money. Nominal bonds pay a nominal interest rate of i_t in period t on debt acquired in period $t-1$; real bonds pay a real interest rate of r_t in period t on debt acquired at $t-1$. Since the nominal interest rate on money is assumed to be zero, an equilibrium with valued nominal bonds requires that the nominal interest rate be non-negative. We only consider monetary rules that support such equilibria. Equilibria where both nominal and real bonds are held require that the returns on these assets are equalised. Thus

$$(1+r_{t+1})P_{t+1}/P_t = 1+i_{t+1},\ t \geq 0. \tag{18}$$

Denote the period-t household demand for nominal bonds by B_t^d and the period-t demand for real bonds by d_t^d. Let $a_t^d \equiv m_t^d + b_t^d + d_t^d$ be the real value of the household's time-t demand for financial wealth, where $b_t^d \equiv B_t^d/P_t$. The household's within-period budget constraint is

$$a_t^d = (1+r_t)a_{t-1}^d + y - \tau_t - c_t - \left(i_t P_{t-1}/P_t\right) m_{t-1}^d. \tag{19}$$

The household faces a solvency constraint: it cannot run a Ponzi scheme where it borrows ever-increasing amounts to service its previously accumulated debt. Thus, the present discounted value of the household's terminal financial wealth has to be non-negative:

$$\lim_{t\to\infty} a_t^d / \textstyle\prod_{s=0}^{t}(1+r_s) \geq 0. \tag{20}$$

We assume that the household's initial holdings of money, nominal bonds and real bonds, $M_{-1} > 0$, B_{-1} and d_{-1}, respectively, are given.[10]

The (interior) optimality conditions for the household are, for $t \geq 0$, are (19) and

$$\frac{u_m(c_t, m_t^d)}{u_c(c_t, m_t^d)} = \frac{i_{t+1}}{1+i_{t+1}} \tag{21}$$

$$\beta(1+r_{t+1})u_c\left(c_{t+1}, m_{t+1}^d\right) = u_c\left(c_t, m_t^d\right) \tag{22}$$

$$\lim_{t\to\infty} \beta^t u_c(c_t, m_t^d) a_t^d = 0. \tag{23}$$

Equation (21) is the familiar efficiency condition relating period-t money demand to period-t consumption. Equation (22) is the household's consumption Euler equation. Equation (23) is the the ST condition for the money-and-bonds model.

It is common in the economics literature to replace the single transversality condition (23) with multiple transversality conditions, one for each component of financial wealth. Turnovsky [33], p. 389 and McCallum [26], for example, model households that hold money and bonds and impose two transversality conditions, one requiring that the discounted terminal shadow value of debt be zero and one requiring that the discounted

[10] In the money-and-bonds model too, we do not consider the non-monetary equilibrium with $P_t^{-1} = 0,\ t \geq 0$.

terminal shadow value of money be zero. Other recent papers that take this approach are Jha, Wang and Yip [18], Chuang and Huo [8] and Schabert [31]. While multiple transversality conditions may help rule out inflationary bubbles and explosive debt, only one transversality condition is part of the necessary and sufficient conditions for a household optimum. When the consumer's optimisation problem is written as a discrete-time Hamiltonian control problem, there is a single *state variable*, real aggregate financial wealth, a^d, and two *control variables,* consumption, c, and real money balances, m^d. There is a single transversality condition involving the state variable a^d. There is no separate transversality condition requiring the present discounted shadow value of terminal real money balances to equal zero. The economic reason for this is that financial markets are assumed to be frictionless: households can costlessly and instantaneously change the composition of their portfolios between money and bonds.

It follows from equations (22), (23) and Assumption 1, that the household solvency constraint (20) holds with equality:

$$\lim_{t\to\infty} a_t^d / \prod_{s=0}^{t}(1+r_s) = 0. \tag{24}$$

3.2 The government

The government's outstanding stocks of nominal and real bonds at the beginning of period t are denoted by B_{t-1} and d_{t-1}, respectively. Let $b_t \equiv B_t/P_t$ and $a_t \equiv m_t + b_t + d_t$. The government's period-t budget constraint is

$$a_t = (1+r_t)a_{t-1} + g - \tau_t - (i_t P_{t-1}/P_t)\, m_{t-1},\ t \geq 1. \tag{25}$$

The government's solvency constraint, given in (26), is that the present discounted value of the government's terminal *non-monetary* liabilities is non-positive. This in contrast to the household solvency constraint, given in (20), which requires the present discounted value of the household's terminal total net financial liabilities, monetary and non-monetary, to be non-positive. The rationale for the *asymmetric* specification of the private and public sectors' solvency constraints is that, while the household views money as an asset that can be realised at any time, the government recognises that, unlike bonds, unbacked base money is *irredeemable* or *inconvertible.* (see Buiter [6], [7]).[11] Unbacked base money is perceived to be an asset by the private sector, even in the long run, but is not treated as an effective liability in the long run by the government. Reflecting this, the government's solvency constraint is

$$\lim_{t\to\infty} f_t / \prod_{s=0}^{t}(1+r_s) \leq 0, \tag{26}$$

where $f_t \equiv b_t + d_t$.

To demonstrate the possible existence of deflationary bubbles when the government issues non-monetary as well as monetary financial instruments, we specify a simple government tax rule: we suppose that the government keeps the real value of its net stock of debt (monetary and non-monetary) constant at the initial level $a_{-1} \geq 0$. From the government's within-period budget constraint (25) it follows that taxes are given by

$$\tau_t = g + r_t a_{-1} - (i_t P_{t-1}/P_t)\, m_{t-1}. \tag{27}$$

Under this tax rule, the discounted terminal value of the government's aggregate debt, both monetary and non-monetary, is clearly zero. Thus, the government satisfies its

[11] This explains why there is no government solvency constraint in the money-only model of Section 2. If we were to impose, in the money-only model, a government solvency constraint analogous to (**??**), we would have to constraint the terminal behaviour of the stock of money balances, in addition to the sequence of within-period budget constraints given in (9). Precisely what this constraint would be is not clear. It cannot be $\lim_{t\to\infty} m/\prod_{s=0}^{t}(1+r_s) \leq 0$, because there are no market interest rates in the money-only model. Perhaps $\lim_{t\to\infty} m_t\beta^t \leq 0$ could be a candidate.

solvency constraint (26).[12] By (18) it follows that taxes are given by:

$$\tau_t = g + \left(\frac{(1+i_t)\, m_t}{\mu m_{t-1}} - 1 \right) a_{-1} - \frac{i_t m_t}{\mu}. \tag{28}$$

3.3 Market clearing

Market clearing requires that $m_t^d = m_t$ and $a_t^d = a_t,\ t \geq 0$. As before, the resource constraint implies that $c_t = c \equiv y - g,\ t \geq 0$. Then by equations (18), (21) and (22) we have the following definition:

Definition 9 *A monetary equilibrium is a sequence of pairs* $\{(m_t, i_t)\}_{t=0}^{\infty}$ *such that* $m_t > 0$ *and* $i_t \geq 0$ *for every* $t \geq 0$ *and*

$$\beta u_c(c, m_{t+1}) m_{t+1} = \mu [u_c(c, m_t) - u_m(c, m_t)] m_t,\ t \geq 0 \tag{29}$$

$$i_{t+1} = \frac{u_m(c, m_t)}{u_c(c, m_t) - u_m(c, m_t)},\ t \geq 0 \tag{30}$$

$$\lim_{t \to \infty} \beta^t u_c(c, m_t) a_t = a_{-1} \lim_{t \to \infty} \beta^t u_c(c, m_t) = 0. \tag{31}$$

As before, a unique monetary steady state exists. It has the associated nominal interest rate $\bar{\imath} = (\mu - \beta)/\beta$ and real interest rate $\bar{r} = (1-\beta)/\beta$.

Deflationary bubble paths result in the nominal interest rate going to zero and, by (28) - (30),

$$\tau_t \to g + \frac{1-\beta}{\beta} a_{-1} - \lim_{t \to \infty} \frac{u_m(c, m_t) m_t}{\mu u_c(c, m_t)}. \tag{32}$$

With constant real aggregate financial liabilities ($a_t = a_{-1},\ t \geq 0$), the household's transversality condition and the household's and government's solvency constraints are satisfied even when real money balances rise without bound and the present discounted value of the terminal money stock is strictly positive. As the stock of real money balances

[12] This requires that long-run real interest rate is positive. With the restrictions we imposed to ensure a unique monetary steady state, the long-run real interest rate will always be positive. Were this not the case, setting $a_t = 0,\ t \geq 0$ would guarantee solvency. The tax sequence would be given by $\tau_0 = g + (1 + r_0) a_{-1} - (i_0 P_{-1}/P_0)\, m_{-1}$ and $\tau_t = g - (i_t P_{t-1}/P_t)\, m_{t-1},\ t \geq 0$.

rises without bound along a deflationary bubble path, the real value of government bonds falls without bound, keeping the real value of the sum of these two portfolio components constant.

Consider the example from the previous section where $u(c,m) = h(c) + \ln m$. Taxes are constant, since $\tau_t = g + (1-\beta)a_{-1}/\beta - (\mu-\beta)\bar{m}/(\mu\beta)$, and the real interest rate equals $(1-\beta)/\beta$. Suppose $\mu > \beta$.[13] Then $m_t = (m_0 - \bar{m})\left(\mu/\beta\right)^t + \bar{m}$ and $i_t = (\mu-\beta)\bar{m}/[\mu m_{t-1} - (\mu-\beta)\bar{m}]$. If $m_0 > \bar{m}$, then real balances go to infinity and the nominal interest rate goes to zero. If the nominal money stock is rising ($\mu > 1$), then $\beta^t u_c(c, m_t)m_t \to \infty$. It follows that $\beta^t u_c(c, m_t)(b_t + d_t) \to -\infty$ to keep $\lim_{t\to\infty} \beta^t u_c(c, m_t)a_t = 0$. If the nominal money stock is constant ($\mu = 1$), then $\beta^t u_c(c, m_t)m_t \to (m_0 - \bar{m})h'(c)$ and the present discounted value of government bonds goes to $-(m_0 - \bar{m})h'(c)$.

Many other tax rules would support deflationary bubbles in the money-and-bonds model. They all share the property that if the present discounted value of real balances goes to infinity, the present discounted real value of government bonds goes to minus infinity, thus ensuring that the present discounted value of the aggregate financial liabilities of the government goes to zero in the long run.

Proposition 10 *With a non-decreasing nominal money stock, deflationary bubbles cannot be ruled out when the government issues both money and bonds if the present discounted value of government"s terminal non-monetary liabilities can be negative (if $\mu = 1$) or if they can go to minus infinity (if $\mu > 1$).*

The government's solvency constraint (26) requires the discounted value of its terminal *non-monetary* debt to be non-positive. The fiscal rule $a_t \equiv f_t + m_t = a_{-1}, t \geq 0$ that was shown to support deflationary bubbles ensures that the government solvency constraint (26) is satisfied, since $m_t \geq 0$. However, unless the present value of terminal real balances is zero, (27) satisfies (26) with strict inequality. We now show that when the government adopts instead of (27) a fiscal rule that ensures its solvency constraint is satisfied with

[13] The Friedman rule, $\mu = \beta$, implies an an infinite stock of real money balances, so there is no *OQM* equilibrium that can be implemented in this case.

equality, deflationary bubbles cannot exist unless the growth rate of the nominal money stock is negative.

An example of a tax rule that causes (26) to hold with equality is one that keeps constant the real value of the government's bonds, that is, $f_t = f_{-1},\ t \geq 0$. This rule is among the most common *ad hoc* rules in the macroeconomics literature. Associated taxes are given by

$$\tau_t = g + r_t f_{-1} - \left(\frac{\mu - 1}{\mu}\right) m_t \tag{33}$$

Proposition 11 *If the government satisfies its solvency constraint (26) with equality, then all of the Propositions for the money-only model of Section 2 hold for the model with money and government debt.*

Proof. Market clearing and equations (22) and (26) (with equality) imply

$$\lim_{t\to\infty} \beta^t u_c(c, m_t) f_t = 0. \tag{34}$$

For the household sector, the combined solvency constraint and transversality condition from the previous section, condition (24), continues to hold. This, market clearing and (22) imply

$$\lim_{t\to\infty} \beta^t u_c(c, m_t) a_t = 0. \tag{35}$$

Together, equations (34) and (35) imply

$$\lim_{t\to\infty} \beta^t u_c(c, m_t) m_t = 0. \tag{36}$$

■

The proof demonstrates that if the government solvency constraint holds with equality, then there is an equilibrium requirement that the present discounted value of the terminal money stock be zero. This does not come solely from the household's transversality and solvency conditions, as was the case with the money-only model. Instead it is an implication of these conditions, the assumption of irredeemable money (and the associated asymmetry between the household and government solvency constraints), the requirement that the government solvency constraint hold with equality and the market-clearing conditions. With the requirement restored that the present value of the terminal money stock be zero, all propositions concerning the existence and non-existence of deflationary bubbles and about the *GABOR* condition, derived for the money-only model in Section 2, now also apply unchanged in the money-and-bonds model.

Neither the specific rule in (33) nor the requirement that the government's solvency constraint hold with equality are derived from optimising government behaviour. The assumption that the government's solvency constraint holds with equality sounds sensible, in that optimising private economic agents typically satisfy their budget constraints with equality - satiation in commodities is not a common feature of standard models of household behaviour. Without the possibility of bubbles and with distortionary taxes or real tax administration and compliance costs, a benevolent optimising government would choose to satisfy its budget constraint with equality. However because of the possibility of bubbles, this assumption is not innocuous here. The rule $a_t = a_{-1},\ t \geq 0$ satisfies the government's constraint (26) of this section with strict inequality unless $\lim_{t\to\infty} m_t / \prod_{s=0}^{t}(1 + r_s) = 0$. If nominal money growth is strictly positive, it can produce a deflationary bubble. As welfare is higher in the deflationary bubble equilibrium than it would be in the associated fundamental equilibrium, household welfare is higher in a bubble equilibrium under the rule $a_t = a_{-1},\ t \geq 0$ than it is under the rule $f_t = f_{-1}, t \geq 0$ which satisfies the budget constraint of this section with equality and which supports only the fundamental equilibrium.[14]

When households can hold physical capital, but not government bonds, the logic of the money-only model prevails, and deflationary bubbles are not possible when the nominal money stock is non-decreasing. The household transversality condition (23) would still

[14] We assume that the growth rate of the nominal money stock and the level of real public spending are the same under both fiscal rules.

hold, but with $a_t^d = m_t^d + k_t^d$, where k_t^d is the household's demand for real capital. Even if an individual household can hold negative quantities of capital (equity), in equilibrium the aggregate real capital stock is constrained to be non-negative and cannot play the role of government bonds in permitting deflationary bubbles.

In both the money-only and the money-and-bonds models we introduce a motive for holding money and generate money demand functions that depend on the nominal interest rate by including real money balances as an argument in the direct utility function, along the lines pioneered by Sidrauski [32]. This way of introducing a motive for holding money is not crucial for any of our results, however. Other ways of making the demand for money sensitive to the nominal interest rate, such as a flexible cash-in-advance model with cash goods and credit goods (see, for example, Lucas and Stokey [24]) or a shopping, time-savings model (see, for example, Feenstra [10]), would produce identical results to those obtained here. Indeed, all our key results hold also for the constant-velocity cash-in-advance models of Lucas [23], Helpman [17] and Sargent [30], Chapter 5.[15]

It is clear that none of the results depend on the interest-sensitivity of the demand for real money balances. The key assumption that makes the household transversality condition bite is that the marginal utility of *consumption* is positive. With the unitary velocity of circulation of the simple cash-in-advance models (say $M_t \geq P_t(c_t + g)$, with $M_t = P_t(c_t + g)$ if $i_t > 0$), and a period utility function $u(c)$, with u increasing, twice continuously differentiable, strictly concave and satisfying the Inada conditions) deflationary bubbles exist or fail to exist under the same conditions that they do with the money in the utility function under Assumptions 1, 2A (satiation) and normality of real money balances. The simple cash-in-advance model has the equivalent of 'satiation' at $\hat{m} = c + g$.

[15] Deflationary bubbles cannot exist in an overlapping-generations model where money is the only store of value and is only held as a store of value. Consider the case where households live for two periods, only the young receive a positive endowment of the good and the nominal money stock is constant. Consumption by the old then equals the value of the stock of real money balances. A deflationary bubble would mean that old consumers' demand for the good is growing without bound. As the amount of the good supplied by the young is bounded above by the sum of their endowments, this cannot be an equilibrium. See, for example, Hahn [15], p. 10.

4 Deflationary Bubbles in a Small Open Economy

This approach to the existence of deflationary bubbles when the nominal money stock is non-decreasing can be applied to the small open economy model of Obstfeld and Rogoff [29], pp. 536-543. This model has a single tradable commodity, a freely floating exchange rate and perfect international capital mobility. The domestic household holds home and foreign real bonds. Home money is held by the home household, but not by foreign households. The world rate real interest rate of interest, r^*, is exogenous and assumed to equal the domestic consumer's time preference rate: $r_t = r^* = (1-\beta)/\beta,\ t \geq 0$. The government balances its budget each period and has no initial debt; hence $f_t = 0$, $t \geq -1$ and its within-period budget constraint is

$$M_t - M_{t-1} \equiv P_t(g - \tau_t),\ t \geq 0. \tag{37a}$$

While the government does not borrow or lend, households can freely buy and sell bonds on the international capital market. Let b^*_t denote the time-t real value of net household claims on the rest of the world. The household's period budget constraint is still (19), but now $a_t \equiv m_t + b^*_t$. The household solvency constraint (20), and the household optimality conditions (21), (22) and (23) are the same as in the closed economy model of Sections 3 and 4. The closed economy resource constraint $c_t = y - g$ no longer constrains private consumption. Instead, the private and public sector budget constraints imply the nation's within-period resource constraint is:

$$b^*_t \equiv (1 + r^*)\, b^*_{t-1} + y - g - c_t. \tag{38}$$

The household transversality and solvency conditions require that $\lim_{t\to\infty} \beta^t u_c(c, m_t)$ $(m_t + b^*_t) = \lim_{t\to\infty}(m_t + b^*_t)/(1+r^*)^t = 0$. This can be satisfied if the present discounted value of the terminal money stock is strictly positive, provided the present discounted value of the household's terminal holdings of net external assets takes on a matching negative value.[16] Because the government views money as non-redeemable, with no home government non-monetary debt debt ($f_t = 0$), the government's solvency condition 26 is trivially satisfied as well.[17]

5 Conclusion

The paper ties up a number of loose ends in the deflationary bubbles literature; we have detailed our main results in the introduction. What remains to be done in future research is a reconciliation of the neoclassical public finance approach to monetary policy, both positive and normative, of which this paper is an example, and the more diverse and eclectic literature on deflation, debt deflation, monetary policy ineffectiveness, financial fragility, recession and depression. This deflationary crisis literature goes back at least to Fisher [11], [12], Wicksell [35, 36], Keynes [20], and Hayek [16] and other Austrian School economists and figures prominently in the more recent policy-oriented contributions of authors like Bernanke [3], King [21] and Krugman [22]

In the neoclassical public finance approach, deflation and zero nominal interest rates are not a serious cause for concern. The deflationary bubbles analysed in this paper all support the maximum feasible level of private consumption as an equilibrium in every period, and the real money stock converges to its satiation value or higher. The stationary Optimal Quantity of Money equilibrium, characterised by steady deflation and a zero nominal interest rate, represents the social optimum in the models considered in our paper. In the deflationary crisis literature, deflation and the zero bound are to be avoided and are a source of concern for monetary policy makers. The difference between the two approaches goes well beyond the fact that the deflationary equilibria considered in this paper concern fully anticipated deflations, while the deflationary crisis literature

[16] As the home country is small, this does not violate any foreign transversality condition. One might argue, however, that if the home country is borrowing ever increasing amounts from the rest of the world that the small economy assumption would eventually cease to be sensible, unless the rest of the world is growing at a rate at least equal to the rate of interest.

[17] Obstfeld and Rogoff appeal to a *GABOR* argument to rule out deflationary bubbles in this open economy example also (see [29], pp. 542-543).

is concerned with the response of the economic system to unexpected shocks - mainly unexpected contractionary demand shocks (leading to 'bad' deflations) but also, more recently, unexpected expansionary supply shocks (leading to 'good' deflations). The deflationary crisis literature uses models that emphasize nominal and real rigidities (both *ad-hoc* nominal wage or price rigidities, and real rigidities based on asymmetric information considerations), incomplete markets and non-competitive behaviour by enterprises and financial intermediaries. The gap between these two approaches is so wide that it may not be possible to come up with a tractable analytical model that encompasses them. However, both approaches offer important insights, and the hope that a synthesis can be achieved that is more than the sum of the parts should encourage a renewed focus on a research agenda that aims to join these two perspectives.

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