THE EXCLUDED 3-MINORS FOR VF-SAFE DELTA-MATROIDS

JOSEPH E. BONIN, CAROLYN CHUN, AND STEVEN D. NOBLE

ABSTRACT. Vf-safe delta-matroids have the desirable property of behaving well under certain duality operations. Several important classes of delta-matroids are known to be vf-safe, including the class of ribbon-graphic delta-matroids, which is related to the class of ribbon graphs or embedded graphs in the same way that graphic matroids correspond to graphs. In this paper, we characterize vf-safe delta-matroids and ribbon-graphic delta-matroids by finding the minimal obstructions, called excluded 3-minors, to membership in the class. We find the unique (up to twisted duality) excluded 3-minor within the class of set systems for the class of vf-safe delta-matroids. In the literature, binary delta-matroids appear in many different guises, with appropriate notions of minor operations equivalent to that of 3-minors, perhaps most notably as graphs with vertex minors. We give a direct explanation of this equivalence and show that some well-known results may be expressed in terms of 3-minors.

1. INTRODUCTION

A *set system* is a pair $S = (E, \mathcal{F})$, where E, or $E(S)$, is a set, called the *ground set*, and $\mathcal{F}$, or $\mathcal{F}(S)$, is a collection of subsets of E. (All set systems in this paper have finite ground sets.) The members of $\mathcal{F}$ are the *feasible sets*. We say that S is *proper* if $\mathcal{F} \neq \emptyset$.

A matroid M has many associated set systems with $E = E(M)$. The most important of these from the perspective of this paper has $\mathcal{F} = \mathcal{B}(M)$, the set of bases of M. Recall that the bases of a matroid satisfy the following exchange property: for any $B_1, B_2 \in \mathcal{B}(M)$ and for each element $x \in B_1 - B_2$, there is a $y \in B_2 - B_1$ for which $B_1 \triangle \{x, y\} \in \mathcal{B}(M)$. To get the definition of a delta-matroid, replace set differences by symmetric differences. Thus, as introduced by Bouchet in [2], a *delta-matroid* is a proper set system $D = (E, \mathcal{F})$ for which $\mathcal{F}$ satisfies the *delta-matroid symmetric exchange axiom*:

(SE) for all triples (X, Y, u) with X and Y in $\mathcal{F}$ and $u \in X \triangle Y$, there is a $v \in X \triangle Y$ (perhaps u itself) such that $X \triangle \{u, v\}$ is in $\mathcal{F}$.

Clearly every matroid $(E(M), \mathcal{B}(M))$ is a delta-matroid.

Just as there is a mutually-enriching interplay between matroid theory and graph theory, the theory of delta-matroids has substantial connections with the theory of embedded graphs or equivalently ribbon graphs; see [17, 18]. Brijder and Hoogeboom [12, 13, 14] introduced the operation of loop complementation, which we define in the next paragraph. This operation is natural for the class of binary delta-matroids and its subclass of ribbon-graphic delta-matroids. These classes are closed under loop complementation, but that is not true for the class of all delta-matroids. We investigate when loop complementation of a delta-matroid yields a delta-matroid.

For a set system $S = (E, \mathcal{F})$ and $e \in E$, we define $S + e$ to be the set system

$$S + e = (E, \mathcal{F} \triangle \{F \cup e : e \notin F \in \mathcal{F}\}). \tag{1.1}$$

Date: February 13, 2019.

(As in matroid theory, we often omit set braces from singletons.) Note that $(S+e)+e = S$ and that $S + e$ is proper if and only if S is proper. It is straightforward to check that if $e_1, e_2 \in E$ then $(S + e_1) + e_2 = (S + e_2) + e_1$. Thus if $X = \{e_1, \dots, e_n\}$ is a subset of E, then the set system $S + X$ is unambiguously defined by

$$S + X = ((S + e_1) + \cdots) + e_n. \tag{1.2}$$

This operation is called the *loop complementation of S on X*. There is a natural operation of embedded graphs, namely *partial Petriality*, to which loop complementation corresponds. More precisely if two embedded graphs are partial Petrials of each other then their ribbon graphic delta-matroids are related by a loop complementation [18, Section 4].

For a delta-matroid D and element $e \in E(D)$, the set system $D + e$ need not be a delta-matroid. Consider, for example, the delta-matroid $D_3 = (\{a, b, c\}, 2^{\{a,b,c\}} - \{\{a, b, c\}\})$. Then $D_3 + \{a, b, c\}$ is the set system $(\{a, b, c\}, \{\emptyset, \{a, b, c\}\})$. This is not a delta-matroid. In fact, it is an excluded minor for the class of delta-matroids [1].

Another operation on delta-matroids is the twist. For $A \subseteq E$, the *twist of S on A*, which is also called the *partial dual of S with respect to A*, denoted $S * A$, is given by

$$S * A = (E, \{F \triangle A \, : \, F \in \mathcal{F}\}).$$

Note that $(S * A) * A = S$. The *dual* S^* of S is $S * E$. In contrast with loop complementation, each twist of a delta-matroid is a delta-matroid. Apart from the dual, the twists of a matroid $(E(M), \mathcal{B}(M))$ are generally not matroids, as discussed in [18, Theorem 3.4].

Two set systems are said to be *twisted duals* of one another if one may be obtained from the other by a sequence of twists and loop complementations. Following [14], a delta-matroid is said to be *vf-safe* if all of its twisted duals are delta-matroids. (The term vf-safe is short for 'vertex-flip safe'. Both of the terms vf-safe and loop complementation are named for operations on graphs representing binary delta-matroids [12], which we discuss in Section 5.)

Delta-matroids belonging to certain important classes are known to be vf-safe. In fact, every twisted dual of a ribbon-graphic delta-matroid is a ribbon-graphic delta-matroid [18, Theorem 2.1,Theorem 4.1], and every twisted dual of a binary delta-matroid is a binary delta-matroid [14, Theorem 8.2]. Brijder and Hoogeboom showed that quaternary matroids are vf-safe [15], although, as mentioned earlier, matroids are not closed under twists.

In the main result of this paper, Theorem 4.4, we identify D_3 as essentially the unique obstacle for a delta-matroid to be vf-safe. More precisely, we show that the excluded 3-minors for the class of vf-safe delta-matroids within the class of set systems comprise the 28 set systems that are the twisted duals of D_3. These set systems are given in Tables 2–7. In the final section of the paper, we relate 3-minors to other minor operations that have appeared in the literature, and we apply Theorem 4.4 to recast some known results using short lists of excluded 3-minors.

2. BACKGROUND

Let $S = (E, \mathcal{F})$ be a proper set system. An element $e \in E$ is a *loop* of S if no set in $\mathcal{F}$ contains e. If e is in every set in $\mathcal{F}$, then e is a *coloop*. If e is not a loop, then the *contraction of e from S*, written S/e, is given by

$$S/e = (E - e, \{F - e : e \in F \in \mathcal{F}\}).$$

If e is not a coloop, then the *deletion of e from S*, written $S \backslash e$, is given by

$$S \backslash e = (E - e, \{F \subseteq E - e : F \in \mathcal{F}\}).$$

If e is a loop or a coloop, then one of S/e and $S\backslash e$ has already been defined, so we can set $S/e = S\backslash e$. Any sequence of deletions and contractions, starting from S, gives another set system S', called a *minor* of S. Each minor of S is a proper set system.

The order in which elements are deleted or contracted can matter. See [1] for an example. However, for disjoint subsets X and Y of E, if some set in $\mathcal{F}$ is disjoint from X and contains Y, then the deletions and contractions in $S\backslash X/Y$ can be done in any order, and

$$S\backslash X/Y = (E-(X\cup Y), \{F-Y \,:\, F\in\mathcal{F} \text{ and } Y\subseteq F\subseteq E-X\}).$$

The following lemma, which is [1, Lemma 2.1], shows that all minors of a proper set system are of this type.

Lemma 2.1. *For any minor S' of a proper set system $S=(E,\mathcal{F})$, there are disjoint subsets X and Y of E such that*

$$S' = S\backslash X/Y = (E-(X\cup Y), \{F-Y \,:\, F\in\mathcal{F} \textit{ and } Y\subseteq F\subseteq E-X\}).$$

Bouchet and Duchamp [5] showed that, if S is a delta-matroid, X and Y are any disjoint subsets of E, and $S' = S\backslash X/Y$, then S' is a delta-matroid and S' is independent of the order of the deletions and contractions.

The following definition from [12] is equivalent to that given in equations (1.1)–(1.2). Equivalence can be shown by a routine induction on $|A|$.

Definition 2.2. *If $S=(E,\mathcal{F})$ is a set system and A is a subset of E, then the* loop complementation of S by A*, denoted by $S+A$, is the set system on E such that F is feasible in $S+A$ if and only if S has an odd number of feasible sets F' with $F-A\subseteq F'\subseteq F$.*

Note that if $A=\{e\}$, then the feasible sets of $S+e$ that do not contain e are the same as those of S, and a set F containing e is feasible in $S+e$ if and only if one but not both of F and $F-e$ is feasible in S. That is, so long as e is not a loop or coloop,

$$\mathcal{F}(S+e) = \mathcal{F}(S\backslash e)\cup\{F\cup e : F\in\mathcal{F}(S\backslash e)\bigtriangleup\mathcal{F}(S/e)\}.$$

If e is a loop, then $\mathcal{F}(S+e)=\mathcal{F}\cup\{F\cup e : F\in\mathcal{F}\}$. If e is a coloop, then $S+e=S$.

The twist and loop complementation operations interact in the following way. If A and B are disjoint subsets of E then $(S+A)*B = (S*B)+A$ (a two-element case check and routine induction suffice to verify this), but in general $(S*A)+A \neq (S+A)*A$. However $((S+A)*A)+A = ((S*A)+A)*A$ (see [12]). It follows that there are at most six twisted duals of S with respect to a fixed set A. These relations ensure that any twisted dual of S is of the form $((S*X)+Y)*Z$ for suitably chosen subsets X, Y and Z of E with $X\subseteq Z$. The first relation is used in the proof of the following result.

Lemma 2.3. *Let $S=(E,\mathcal{F})$ be a proper set system, and let a and b be distinct elements of E. Then*

(i) $S+a\backslash a = S\backslash a$*,*
(ii) $S+a\backslash b = S\backslash b + a$*, and*
(iii) $S+a/b = S/b+a$*.*

Proof. If a is a coloop of S, then $S+a=S$, so statement (i) follows. Also, a is not a coloop of S if and only if it is not a coloop of $S+a$, in which case the feasible sets avoiding a are the same in S and $S+a$ by the definition.

For statement (ii), observe that b is a coloop of $S+a$ if and only if it is a coloop of S. When b is not a coloop of S, statement (ii) holds since for each side, the feasible sets are the sets F with $b\notin F$ for which an odd number of the sets X with $F-a\subseteq X\subseteq F$ are in $\mathcal{F}$. When b is a coloop of S, we need to show that $S+a/b=S/b+a$. This holds since for each side, the feasible sets are the sets F with $b \notin F$ for which an odd number of the sets X with $(F-a)\cup b \subseteq X \subseteq F \cup b$ are in $\mathcal{F}$.

It is easy to check that $S'/e = S' * e\backslash e$, so, using statement (ii), we get statement (iii):

$$S + a/b = ((S+a)*b)\backslash b = ((S*b)+a)\backslash b = ((S*b)\backslash b) + a = S/b + a. \qquad \square$$

The counterpart, for contractions, of statement (i) is false, as taking $S = D_3$ shows.

3. 3-MINORS

We introduce a third minor operation on set systems. For a proper set system S, we define $S \ddagger e$ to be the set system $(S+e)/e$. This minor operation was first defined by Ellis–Monaghan and Moffatt [19] for ribbon graphs and in an equivalent way by Brijder and Hoogeboom [13] for delta-matroids. The notation $\ddagger$ is new, but it seems appropriate to shorten the unwieldy $+e/e$ notation. Motivation for this definition comes from two directions. First, one way to define the Penrose polynomial of a ribbon graph is by specifying a recursive relation analogous to the deletion-contraction recurrence of the chromatic polynomial with this minor operation replacing contraction. For this reason, following a suggestion of Iain Moffatt [22], we propose calling the operation *Penrose contraction*. Second, there is a class of combinatorial objects called multimatroids [8, 9, 10], of which tight 3-matroids are a particular subclass. Brijder and Hoogeboom [13] showed that tight 3-matroids are equivalent (in a sense that we do not make precise here) to vf-safe delta-matroids. Tight 3-matroids have three minor operations corresponding to deletion, contraction, and Penrose contraction in vf-safe delta-matroids.

Any sequence of the three minor operations, starting from S, gives another proper set system S', called a 3-*minor* of S. A collection $\mathcal{C}$ of proper set systems is 3-*minor closed* if every 3-minor of every member of $\mathcal{C}$ is in $\mathcal{C}$. Given such a collection $\mathcal{C}$, a proper set system S is an *excluded* 3-*minor* for $\mathcal{C}$ if $S \notin \mathcal{C}$ and all other 3-minors of S are in $\mathcal{C}$. A proper set system belongs to $\mathcal{C}$ if and only if none of its 3-minors is an excluded 3-minor for $\mathcal{C}$. Thus, the excluded 3-minors determine $\mathcal{C}$; they are the 3-minor-minimal obstructions to membership in $\mathcal{C}$.

For a given class $\mathcal{C}$, there may be substantially fewer excluded 3-minors than excluded minors. For instance, in [21], Geelen and Oum found 166 delta-matroids that, up to twists, are the excluded minors for ribbon-graphic delta-matroids within the class of binary delta-matroids. In contrast, in Theorem 5.8, we show that every binary matroid that does not have a twisted dual of one of three delta-matroids as a 3-minor is ribbon-graphic.

An element e is called a *pseudo-loop* of S if e is a loop of $S+e$. The next lemma characterizes these elements.

Lemma 3.1. *For an element e in a proper set system S, the following statements are equivalent:*

(i) *e is a loop of $S+e$, that is, a pseudo-loop of S,*
(ii) *$F \cup e \in \mathcal{F}(S)$ if and only if $F \in \mathcal{F}(S)$, and*
(iii) *$S * e = S$.*

Pseudo-loops of S are neither loops nor coloops of S. Furthermore, $S \ddagger e = S \setminus e = S/e$ if and only if e is a loop, a coloop, or a pseudo-loop of S.

Proof. The equivalence of statements (i)–(iii) is immediate from the definitions. Statement (ii) implies that pseudo-loops are neither loops nor coloops. If e is a loop of S, then $S \ddagger e = S \setminus e$ since $\mathcal{F}(S+e) = \mathcal{F}(S) \cup \{F \cup e : F \in \mathcal{F}(S)\}$; also, $S \setminus e = S/e$ by definition. If e is a coloop of S, then $S \ddagger e = S/e$ since $S + e = S$; also, $S \setminus e = S/e$ by definition. If e is a pseudo-loop of S, then statements (i) and (ii) gives the equality. If e is not a loop, a coloop, or a pseudo-loop of S, then $S \setminus e \neq S/e$ by the failure of statement (ii) and the fact that some, but not all, sets in $\mathcal{F}(S)$ contain e. $\square$

The following two results show that one may choose the operations used to form a 3-minor in such a way that they commute.

Lemma 3.2. *Let $S = (E, \mathcal{F})$ be a proper set system, and let X, Y, and Z be pairwise disjoint subsets of E. If there is a set F with*

$$F \subseteq E - (X \cup Y \cup Z) \quad \textit{and} \quad |\mathcal{F} \cap \{F' : F \cup Y \subseteq F' \subseteq F \cup Y \cup Z\}| \textit{ is odd,} \tag{3.1}$$

then the minor operations in $S \setminus X/Y \ddagger Z$ can be done in any order and a set F is feasible in $S \setminus X/Y \ddagger Z$ if and only if it satisfies Condition (3.1).

Proof. A set F meets Condition (3.1) if and only if $F \subseteq E - (X \cup Y \cup Z)$ and $F \cup Y \cup Z$ is in $\mathcal{F}(S + Z)$. If there is at least one set satisfying Condition (3.1), the remarks preceding Lemma 2.1 imply that the deletions and contractions in forming $(S + Z) \setminus X/(Y \cup Z)$ from $S + Z$ may be done in any order and a set F is feasible in $(S + Z) \setminus X/(Y \cup Z)$ if and only if it satisfies Condition (3.1). Lemma 2.3 implies that we may defer taking a loop complementation of an element in Z until just before it is contracted. The result follows. $\square$

We next show that for every 3-minor of a proper set system, there are pairwise disjoint sets X, Y and Z satisfying Condition (3.1).

Lemma 3.3. *Let S' be a 3-minor of a proper set system $S = (E, \mathcal{F})$. Then there are pairwise disjoint subsets X, Y, and Z of E such that $S' = S \setminus X/Y \ddagger Z$ and there is a set F satisfying Condition (3.1).*

Proof. Suppose we get S' from S by, for each of $e_1, e_2, \ldots, e_k$ in turn, performing one the three minor operations, giving the sequence of minors $S_0 = S, S_1, \ldots, S_k = S'$. Let X be the set of elements e_i in $\{e_1, \ldots, e_k\}$ that satisfy at least one of the following conditions:

(1) e_i is a loop or a pseudo-loop of S_{i-1}, so $S_i = S_{i-1} \setminus e_i$, or
(2) e_i is not a coloop of S_{i-1} and $S_i = S_{i-1} \setminus e_i$.

Let Y be the set of elements e_i in $\{e_1, \ldots, e_k\} - X$ such that e_i is either a coloop of S_{i-1} or $S_i = S_{i-1}/e_i$. Note that if $e_i \in Y$ then it is not a loop in S_{i-1}. Finally let $Z = \{e_1, \ldots, e_k\} - (X \cup Y)$, so that Z comprises the elements e_i in $\{e_1, \ldots, e_k\}$ for which $S_i = S_{i-1} \ddagger e_i$ but e_i is not a loop, pseudo-loop or coloop. Then there is always at least one set F satisfying Condition (3.1). $\square$

It follows from the definition of minors in a multimatroid [9] and the equivalence of tight 3-matroids and vf-safe delta-matroids [13], that if S is a vf-safe delta-matroid, then changing the order of a sequence of 3-minor operations on S never affects the result.

Table 1 shows the result of applying one of the three minor operations that remove e after taking one of the six twisted duals, with respect to e, of a proper set system. If instead the minor operation removes a different element from that used for the twisted dual, then these operations commute.

We next show that any 3-minor of a twisted dual of a proper set system S is a twisted dual of some 3-minor of S. It is easy to see that the converse is also true.

Lemma 3.4. *Suppose S is a proper set system and S' is a twisted dual of S. If S'' is a 3-minor of S', then S has a 3-minor that is a twisted dual of S''.*

	$/e$	$\backslash e$	$\ddagger e$
S	S/e	$S\backslash e$	$S\ddagger e$
$S*e$	$S\backslash e$	S/e	$S\ddagger e$
$S+e$	$S\ddagger e$	$S\backslash e$	S/e
$(S+e)*e$	$S\backslash e$	$S\ddagger e$	S/e
$(S*e)+e$	$S\ddagger e$	S/e	$S\backslash e$
$((S*e)+e)*e$	S/e	$S\ddagger e$	$S\backslash e$

TABLE 1. Interaction of minor operations and twisted duality.

Proof. There are subsets A and B of $E(S)$ such that we obtain S'' from S by first forming a twisted dual for each element of A and then performing one of the three minor operations for each element of B. The remarks before this lemma imply that one may reorder these operations to first deal with the elements of $A\cap B$, one by one, forming a twisted dual for an element and then a 3-minor before moving on to the next element. According to Table 1 each of these pairs of operations may be replaced by a single 3-minor operation. Next a 3-minor is formed for each element of $B-A$. The resulting set system is a twisted dual of S'' with respect to the elements of $A-B$. □

4. CHARACTERIZATIONS BY EXCLUDED 3-MINORS

Brijder and Hoogeboom [14] showed that the class of vf-safe delta-matroids is minor-closed. A computer search for excluded minors for this class turns up many examples with apparently little structure. The class of vf-safe delta-matroids is defined using both the twist and loop complementation operations, so it is natural to try to characterize this class using 3-minors. By Lemma 4.1 below, the class of vf-safe delta-matroids is closed under Penrose contraction, so, with the result in [14], it is closed under 3-minors. The main result of this section, Theorem 4.4, gives the excluded 3-minors for the class of vf-safe delta-matroids within the class of set systems.

Lemma 4.1. *If S is vf-safe and $e\in E(S)$, then $S\ddagger e$ is vf-safe.*

Proof. If S is vf-safe, then all of its twisted duals are vf-safe by definition, so $S+e$ is vf-safe. Theorem 8.3 in [14] states that every minor of a vf-safe delta-matroid is vf-safe. Thus $S\ddagger e = S+e/e$ is vf-safe. □

Let

$$S_i = (\{e_1,e_2,\dots,e_i\},\{\emptyset,\{e_1,e_2,\dots,e_i\}\}).$$

Let $\mathcal{S}$ be the set of all twists of the set systems in $\{S_3,S_4,\dots\}$. Let

- $T_1 = (\{a,b,c\},\{\emptyset,\{a,b\},\{a,b,c\}\})$;
- $T_2 = (\{a,b,c\},\{\emptyset,\{a,b\},\{a,c\},\{a,b,c\}\})$;
- $T_3 = (\{a,b,c\},\{\emptyset,\{a\},\{a,b\},\{a,b,c\}\})$;
- $T_4 = (\{a,b,c\},\{\emptyset,\{a\},\{a,b\},\{a,c\},\{a,b,c\}\})$;
- $T_5 = (\{a,b,c,d\},\{\emptyset,\{a,b\},\{a,b,c,d\}\})$;
- $T_6 = (\{a,b,c,d\},\{\emptyset,\{a,b\},\{a,c\},\{a,b,c,d\}\})$;
- $T_7 = (\{a,b,c,d\},\{\emptyset,\{a,b\},\{a,c\},\{a,d\},\{a,b,c,d\}\})$;
- $T_8 = (\{a,b,c,d\},\{\emptyset,\{a\},\{a,b\},\{a,c\},\{a,d\},\{a,b,c,d\}\})$.

Let $\mathcal{T}$ be the set of all twists of the set systems in $\{T_1,T_2,\dots,T_8\}$. By the following result from [1, Theorem 5.1], these are all of the excluded minors for delta-matroids within the class of set systems.

Theorem 4.2. *A proper set system S is a delta-matroid if and only if S has no minor isomorphic to a set system in $\mathcal{S} \cup \mathcal{T}$.*

The following lemma is key for finding the excluded 3-minors for vf-safe delta-matroids within the class of set systems.

Lemma 4.3. *Let S be an excluded 3-minor for the class of vf-safe delta-matroids. Then S has a twisted dual that is isomorphic to a set system in $\mathcal{S} \cup \mathcal{T}$.*

Proof. Such an excluded 3-minor S either is not a delta-matroid and all of its other minors are delta-matroids, or it is a delta-matroid and has a twisted dual S' that is not a delta-matroid. In the former case S is isomorphic to a set system in $\mathcal{S} \cup \mathcal{T}$ and the lemma holds. In the latter case S' has a minor S'' isomorphic to a member of $\mathcal{S} \cup \mathcal{T}$. By Lemma 3.4, S has a 3-minor S''' that is a twisted dual of S''. Therefore S''' is not a vf-safe delta-matroid. The only 3-minor of S that is not a vf-safe delta-matroid is S itself. Hence $S = S'''$ and the lemma holds. □

To connect the next result with the remarks in Section 1, note that $D_3 + \{a, b, c\} = S_3$.

Theorem 4.4. *A proper set system is a vf-safe delta-matroid if and only if it has no 3-minor that is isomorphic to a twisted dual of S_3.*

Proof. All proper set systems with two elements are delta-matroids, and therefore each one is vf-safe, so the twisted duals of S_3 are excluded 3-minors for the class of vf-safe delta-matroids. By Lemma 4.3 every excluded 3-minor for the class of vf-safe delta-matroids must be a twisted dual of a set system in $\mathcal{S} \cup \mathcal{T}$. We first consider the set systems with three-element ground sets. We have $T_1^* + c = S_3$ and $T_2^* + \{b, c\} \simeq T_3 + a = T_1$ and $T_4 + a = T_2$, so every excluded 3-minor of size three is a twisted dual of S_3.

Lastly, we show that no other set system in $\mathcal{S} \cup \mathcal{T}$ is an excluded 3-minor. Lemma 3.4 implies that each twisted dual of an excluded 3-minor is an excluded 3-minor, so it suffices to show that each of T_5, T_6, T_7, T_8, and S_n, for $n \geq 4$, has a smaller member of $\mathcal{S} \cup \mathcal{T}$ as a 3-minor. Indeed, $S_n \ddagger e_n = S_{n-1}$, for $n \geq 4$, $T_5 \ddagger d = T_1$, $T_6 \ddagger d = T_8 \ddagger d = T_2$, and $T_7 \ddagger d = T_4$. □

As stated in the introduction, there are 28 twisted duals of S_3, up to isomorphism. These excluded 3-minors are listed in Tables 2–7.

5. 3-MINORS AND VERTEX MINORS

In this section we explain the link between 3-minors of binary delta-matroids and vertex minors of graphs. As we shall see later, up to twisted duality, each binary delta-matroid may be represented by a loopless simple graph, and this equivalence is preserved under appropriate 3-minor operations in the binary delta-matroid and vertex minors in the graph.

Vertex minors are well-studied, but are defined only for graphs. In contrast, the concept of a 3-minor is relatively unstudied, but is important due to its direct correlation with ribbon-graph operations and its applicability beyond binary delta-matroids. By combining several existing results from the literature, one may show that the notions of 3-minors and vertex minors are equivalent in a sense to be made precise later, but to see this equivalence one must piece together results involving several combinatorial structures, which are not obviously related. For this reason, and for completeness, we give a full explanation of the link here. Although the key ideas presented here are not new, as far as we know this direct link has not previously been fully described.

A delta-matroid is *normal* if the empty set is feasible. A delta-matroid is *even* if for every pair F_1 and F_2 of its feasible sets $|F_1 \triangle F_2|$ is even. Equivalently, the sizes of all its feasible sets are of the same parity. Let M denote a symmetric binary matrix with rows and columns indexed by $[n] = \{1, \dots, n\}$. Take $E = [n]$ and $\mathcal{F}$ to be the collection of subsets S of $[n]$ for which the principal submatrix of M comprising the rows and columns indexed by elements of S is non-singular. Bouchet [5] showed that $D(M) = (E, \mathcal{F})$ is a delta-matroid. We call such delta-matroids *basic binary*. (In the literature, what we have called basic binary delta-matroids are often called graphic, but we prefer to avoid this term to prevent confusion with ribbon-graphic delta-matroids.) A delta-matroid is *binary* [5] if it is a twist of a basic binary delta-matroid.

It follows immediately from the definition that every basic binary delta-matroid is normal and that a basic binary delta-matroid is uniquely determined by its feasible sets of size at most two. A well-known result of linear algebra states that a symmetric matrix with an odd number of rows (and columns) and zero diagonal is singular. Consequently a basic binary delta-matroid is even if and only if it has no feasible sets of size one.

Let A be a matrix over an arbitrary field with rows and columns indexed by $[n]$, and let X be a subset of $[n]$ such that the principal submatrix $P = A[X]$ is non-singular. Suppose without loss of generality that $A = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$. Then the matrix $A * X$ is defined by

$$A * X = \begin{pmatrix} P^{-1} & -P^{-1}Q \\ RP^{-1} & S - RP^{-1}Q \end{pmatrix}.$$

Note that if A is a symmetric binary matrix then $A * X$ is symmetric. The following result is due to Tucker [24].

Theorem 5.1. *Let A be a matrix over an arbitrary field with rows and columns indexed by $[n]$, and let X be a subset of $[n]$ such that the principal submatrix $P = A[X]$ is non-singular. Then for every subset Y of $[n]$, we have*

$$\det\big((A * X)[Y]\big) = \frac{\det(A[X \triangle Y])}{\det(A[X])}.$$

In particular for any subset Y of $[n]$, the principal submatrix $(A * X)[Y]$ is non-singular if and only if the principal submatrix $A[X \triangle Y]$ is non-singular.

The following corollary is immediate.

Corollary 5.2. *Suppose that A is a binary matrix, and X is a feasible set of $D(A)$. Then $D(A) * X = D(A * X)$.*

See [5] for an alternative proof of this result that holds for arbitrary fields. A consequence of this corollary is that every normal twist of a basic binary delta-matroid is basic binary.

A *looped simple graph* is a graph without parallel edges but in which each vertex is allowed to have up to one loop. Much of the time we will forbid loops; we call such graphs *loopless simple graphs*. Recall that basic binary delta-matroids are completely determined by their feasible sets with size two or fewer. Clearly basic binary delta-matroids on the set $[n]$ are in one-to-one correspondence with looped simple graphs with vertex set $[n]$; likewise, even basic binary delta-matroids on $[n]$ are in one-to-one correspondence with loopless simple graphs with vertex set $[n]$.

We take adjacency matrices to always be binary. Given a looped simple graph G and its adjacency matrix A, we let $D(G)$ denote the basic binary delta-matroid $D(A)$. If X is a subset of the edges of G, then X labels a subset of the rows and columns of A, and we define $G * X$ to be the looped simple graph with adjacency matrix $A * X$.

We now consider various transformations that may be applied to G and their effect on $D(G)$.

The loop complementation operation of Brijder and Hoogeboom was first defined in terms of basic binary delta-matroids. If G is a looped simple graph and v is a vertex of G, then the loop complementation $G+v$ is formed by toggling the loop at v, that is, removing a loop if there is one at v and adding one at v if there is no loop there.

The following lemma from [12] is straightforward.

Lemma 5.3. *Let G be a looped simple graph with vertex v. Then $D(G+v) = D(G)+v$.*

Our next operation is local complementation. There are several variations in the definition of local complementation: see, for instance, [23]. We will only require certain cases of what is defined there. For a looped simple graph G with vertex v, let $N_G(v)$ denote the *open neighbourhood* of v, that is, the set of vertices, excluding v, that are adjacent to v in G. We will generally write N instead of N_G when there is no possibility of confusion. The *local complementation* of G at v, denoted by G^v, is formed by toggling the adjacencies between pairs of neighbours of v, that is, for every distinct pair x, y from the neighbourhood of v, delete edge xy if x and y are adjacent in G and add edge xy if x and y are not adjacent in G. Additionally, if there is a loop at v, then the loop status of every vertex in the open neighbourhood of v is toggled. In both cases, adjacencies involving one or more non-neighbours of v or v itself are unchanged and the presence or not of a loop at v is unaffected. To distinguish the two cases, it will be helpful to refer to local complementation at v as *simple local complementation* when v is loopless, and *non-simple local complementation* when there is a loop at v.

For delta-matroid D and subset $A \subseteq E(D)$, let $D \bar{*} A$ denote the *dual pivot on* A, which is equal to $D + A * A + A$. The following result is implicit in the results of [23], but is not expressed in this form.

Proposition 5.4. *Let G be a loopless simple graph with vertex v. Then $D(G^v) = (D(G)\bar{*}v) + N(v)$.*

Proof. Let A be the adjacency matrix of G. Then A is symmetric and all of its diagonal entries are zero. Notice that the simple local complementation G^v can be formed by adding a loop at v, performing the non-simple local complementation at v and then removing the loops added at v and all of its neighbours.

We have $D(G+v) = D(G)+v$. Assume without loss of generality that $v = 1$ and let $Z = [n]-1$. Then the adjacency matrix of $G+v$ is $\begin{pmatrix} 1 & c \\ c^t & A[Z] \end{pmatrix}$ for some vector c. Then it follows from Corollary 5.2 that $(D(G)+v)*v = D((G+v)*v) = D(A')$ where $A' = \begin{pmatrix} 1 & c \\ c^t & A[Z]+c^tc \end{pmatrix}$.

A diagonal entry of c^tc is non-zero if it corresponds to a neighbour of v and an off-diagonal entry of c^tc is non-zero if both the row and column correspond to neighbours of v. Thus $(D(G)+v)*v = D(G')$ where G' is formed from G by adding a loop at v and performing the non-simple local complementation at v. Now G' has loops at v and at all neighbours of v, so

$$D(G^v) = D(G'+v+N(v)) = D(G')+v+N(v) = (D(G)\bar{*}v)+N(v). \qquad \square$$

The corollary below is well-known and follows from the previous result.

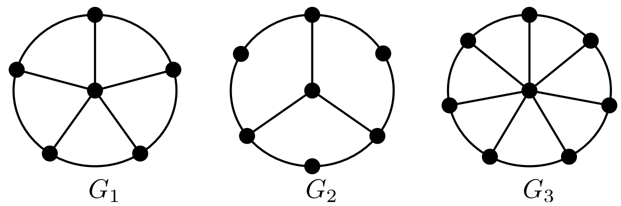

FIGURE 1. A complete set of circle graph obstructions.

Corollary 5.5. *Let G be a loopless simple graph with adjacent vertices v and w. Then $D(((G^v)^w)^v) = D(G) * \{v, w\}$.*

Proof. We have

$$D(((G^v)^w)^v) = ((D(G)\bar{*}v + N(v))\bar{*}w + N_{G^v}(w))\bar{*}v + N_{(G^v)^w}(v).$$

It follows from the discussion before Lemma 2.3 that one may reorder the loop complement and twist operations so that those involving a particular vertex of G are done consecutively. The result follows by considering the effect of the operations involving each vertex of G separately and noting that

(1) a common neighbour of v and w in G is a neighbour of v but not w in both G^v and $(G^v)^w$,
(2) a vertex other than w that is a neighbour of v but not w in G is a neighbour of both v and w in G^v and of w but not v in $(G^v)^w$, and
(3) a vertex other than v that is a neighbour of w but not v in G is a neighbour of both v and w in $(G^v)^w$ and of w but not v in G^v. □

A *vertex minor* of a looped simple graph G is formed from G by a sequence of local complementations and deletions of vertices. It is easy to check that if v and w are different vertices of an unlooped simple graph, then $(G^v) \setminus w = (G \setminus w)^v$ and thus one may assume that all the local complementations are done first.

Perhaps the most important use of vertex minors is Bouchet's characterization of circle graphs. A *chord diagram* is a collection of chords of a circle. Chord diagrams are in one-to-one correspondence with orientable ribbon graphs with one vertex. (For more information on ribbon graphs, see [20] or [18].) To see this think of the circle and its interior as the vertex of a ribbon graph and for each chord attach a ribbon to the vertex at the points corresponding to the endpoints of the chord. Clearly two chords intersect if and only if the corresponding ribbons e_1 and e_2 are interlaced, that is, as one traverses the vertex one meets an end of e_1, then an end of e_2, then the other end of e_1, and finally the other end of e_2. An unlooped simple graph is a *circle graph* if it is the intersection graph of the chords in a chord diagram, that is, there is a vertex corresponding to each chord and they are adjacent if and only if the chords cross. Equivalently a circle graph is the interlacement graph of an orientable ribbon graph with one vertex: it has a vertex for each ribbon and two vertices are adjacent if the corresponding ribbons are interlaced. Bouchet established the following result [7].

Theorem 5.6. *An unlooped simple graph is a circle graph if and only if it has no vertex minor isomorphic to the graphs G_1, G_2 or G_3 depicted in Figure 1.*

We are now ready to state the link between 3-minors and vertex minors.

Theorem 5.7. (1) *Let G be a unlooped simple graph and let H be a vertex minor of G. Then $D(H)$ is a 3-minor of $D(G)$.*
(2) *Let D be a twisted dual of a basic binary delta-matroid and let D' be a 3-minor of D. Then there are graphs G and G' such that $D(G)$ and $D(G')$ are twisted duals of D and D' respectively, and G' is a vertex minor of G.*

Proof. For part (1), note that a vertex minor of an unlooped simple graph is obtained by a sequence of local complementations and vertex deletions. The result follows from Proposition 5.4 and the fact that if v is a vertex of G then $D(G \setminus v) = D(G) \setminus v$.

For part (2), let F be a feasible set of D and let

$$B = \{e \in E(D) \,:\, \{e\} \in \mathcal{F}(D * F)\}.$$

The remark following Corollary 5.2 implies that $D * F$ is basic binary, so $(D*F)+B$ is an even basic binary delta-matroid, so $(D*F)+B = D(G)$ for some unlooped simple graph G. It follows from Lemma 3.4 that there is a delta-matroid D'' that is a 3-minor of $D(G)$ and a twisted dual of D'. We shall prove by induction on k that if G is an unlooped simple graph and D'' is a 3-minor of $D(G)$ with k fewer elements, then there is an unlooped simple graph G' that is a vertex minor of G and such that $D(G')$ is a twisted dual of D''. The result then follows.

If $k = 0$, then take $G' = G$. Otherwise D'' is obtained from $D(G)$ by a sequence of k deletions, contractions and Penrose contractions. Suppose that the first operation is the deletion of v. Then take $G'' = G \setminus v$, which is a vertex minor of G. Furthermore $D(G) \setminus v = D(G'')$ and D'' is a 3-minor of $D(G'')$ with $k-1$ fewer edges. Hence, by induction, there is an unlooped simple graph G' that is a vertex minor of G'' and hence of G, and such that $D(G')$ is a twisted dual of D''. Suppose next that the first operation is the Penrose contraction of v. Then take $G'' = (G^v) \setminus v$. We have

$$\begin{aligned}
D(G'') &= D(G^v \setminus v) \\
&= ((((D(G)+v)*v)+v)+N(v)) \setminus v \\
&= ((((D(G)*v)+v)*v) \setminus v) + N(v) \\
&= (((D(G)*v)+v)/v) + N(v) \\
&= (D(G)\ddagger v) + N(v).
\end{aligned}$$

(The last equality uses Table 1.) Now $D(G'')$ is a twisted dual of $D(G)\ddagger v$, so it has a 3-minor D''' with $k-1$ fewer elements that is a twisted dual of D''. Hence, by induction, there is an unlooped simple graph G' that is a vertex minor of G'' such that $D(G')$ is a twisted dual of D''' and consequently of D''. In the final case the first operation is the contraction of v. If v is an isolated vertex of G then v appears in no feasible set of $D(G)$ of size at most two and consequently in no feasible set of $D(G)$ of any size. Thus v is a loop of $D(G)$ and $D(G)/v = D(G) \setminus v = D(G \setminus v)$. If v is not an isolated vertex of v then let w be a neighbour of v. We have

$$\begin{aligned}
D(((G^v)^w)^v \setminus v) &= D(((G^v)^w)^v) \setminus v \\
&= (D(G) * \{v,w\}) \setminus v \\
&= (D(G)/v) * w.
\end{aligned}$$

The proof of this case is completed in a similar way to the case of Penrose contraction. □

Having described the equivalence of 3-minors and vertex minors, we note how the equivalence may be seen from existing results involving binary delta-matroids and other combinatorial structures, namely, isotropic systems, binary tight 3-matroids (both defined by Bouchet in [3] and [8], respectively) and isotropic matroids (defined by Traldi in [23]). The equivalence of 3-minors of binary delta-matroids and minors of binary tight 3-matroids is explained in [13]. The equivalence of minors of binary tight 3-matroids, isotropic matroids and isotropic systems is explained in [16]. Finally vertex minors of graphs are equivalent to minors of isotropic matroids by [23] or to minors of isotropic systems by [4].

From the preceding result we obtain the following translation of Bouchet's result, determining the three binary delta-matroids that are the excluded 3-minors for ribbon-graphic delta-matroids. In [7], Bouchet points out that his result may be stated in terms of minors of isotropic systems, which informally are much closer to 3-minors of binary delta-matroids than to vertex-minors of graphs.

Theorem 5.8. *A binary delta-matroid is ribbon-graphic if and only if it has no* 3*-minor that is a twisted dual of* $D(G_1)$*,* $D(G_2)$ *or* $D(G_3)$*.*

Proof. If D is a binary delta-matroid and v is an element of D then D is ribbon-graphic if and only if $D + v$ is ribbon graphic, because it follows from Theorem 4.1 of [18] that if R is a ribbon graph with $D = D(R)$ then $D + v$ is the delta-matroid corresponding to the ribbon graph formed from R by applying a half-twist to v. Let

$$B = \{e \in E(D) : \{e\} \in \mathcal{F}(D)\}.$$

Then $D + B$ is even and, furthermore, $D + B$ is ribbon-graphic if and only if D is ribbon-graphic. Now $D + B = D(G)$ where G is an unlooped simple graph. Bouchet's Theorem 5.6 states that G is a circle graph if and only if G has no vertex minor isomorphic to G_1, G_2 or G_3. Equivalently $D + B$ is ribbon-graphic if and only if it has no 3-minor that is a twisted dual of $D(G_1)$, $D(G_2)$ or $D(G_3)$. As $D + B$ is a twisted dual of D, the result follows. □

We close by presenting excluded 3-minor results for the classes of binary delta-matroids and ribbon graphic delta-matroids that follow from Theorem 4.4. Bouchet [6] proved that every minor of a binary delta-matroid is binary and gave the following excluded-minor characterization of binary delta-matroids.

Theorem 5.9. *A delta-matroid is binary if and only if it does not have a minor isomorphic to any of the following five delta-matroids or their twists.*

(1) $B_1 = (\{a, b, c\}, \{\emptyset, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\})$;
(2) $B_2 = S_3 + \{a, b, c\}$;
(3) $B_3 = (\{a, b, c\}, \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}\})$;
(4) $B_4 = (\{a, b, c, d\}, \{\emptyset, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\})$;
(5) $B_5 = (\{a, b, c, d\}, \{\emptyset, \{a, b\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, c, d\}\})$.

We obtain corollaries of this result characterizing binary delta-matroids in terms excluded 3-minors. The first result is equivalent to a recent result of Brijder, stated in terms of binary tight 3-matroids [11].

Corollary 5.10. *A vf-safe delta-matroid is binary if and only if it has no* 3*-minor that is a twisted dual of* B_1*.*

Proof. Theorem 8.2 of [14] states that every twisted dual of a binary delta-matroid is a binary delta-matroid. In particular every binary delta-matroid is vf-safe. Moreover, every 3-minor of a binary delta-matroid is binary. The delta-matroid B_1 is vf-safe and all of its 3-minors are binary. Thus all of its twisted duals are excluded 3-minors for the class of binary delta-matroids.

Suppose that D is a vf-safe delta-matroid that is not binary. Then Theorem 5.9 implies that D has a minor isomorphic to a twist of B_i for $1 \leq i \leq 5$. The delta-matroid B_2 is not vf-safe and $B_4 \ddagger d = B_2$, so D has no minor isomorphic to a twist of B_2 or of B_4. Furthermore $(B_3 + a)^* = B_1$, and $B_5 \ddagger d \simeq B_3$. Thus D has a 3-minor that is isomorphic to a twisted dual of B_1. □

By combining this result with Theorem 4.4, we obtain the following.

Corollary 5.11. *A proper set system is a binary delta-matroid if and only if it has no 3-minor that is a twisted dual of B_1 or S_3.*

Finally we combine the last two results with Theorem 5.8.

Corollary 5.12. *A proper set system is a ribbon graphic delta-matroid if and only if it has no 3-minor that is a twisted dual of B_1, S_3, $D(G_1)$, $D(G_2)$ or $D(G_3)$.*

6. Appendix: The twisted duals of S_3

As proved in Theorem 4.4, these twisted duals of S_3 are the excluded 3-minors for vf-safe delta-matroids.

S_3	$\emptyset$ $\{a,b,c\}$
$S_3 * \{a\}$	$\{a\}$ $\{b,c\}$

Table 2. All twists of S_3 up to isomorphism.

$\emptyset$ $\{a\}$ $\{a,b,c\}$	$\emptyset$ $\{b,c\}$ $\{a,b,c\}$
$S_3 + \{a\}$	$(S_3 + \{a\})^*$
$\emptyset$ $\{a\}$ $\{b,c\}$	$\{a\}$ $\{b,c\}$ $\{a,b,c\}$
$(S_3 + \{a\}) * \{a\}$	$(S_3 + \{a\}) * \{b,c\}$
$\{b\}$ $\{a,b\}$ $\{a,c\}$	$\{b\}$ $\{c\}$ $\{a,c\}$
$(S_3 + \{a\}) * \{b\}$	$(S_3 + \{a\}) * \{a,c\}$

Table 3. All twists of $S_3 + \{a\}$ up to isomorphism. Dual pairs are side by side.

$\emptyset$	$\{a\}$ $\{b\}$	$\{a,b\}$	$\{a,b,c\}$	$\emptyset$	$\{c\}$	$\{a,c\}$ $\{b,c\}$	$\{a,b,c\}$
$S_3+\{a,b\}$				$(S_3+\{a,b\})^*$			
$\emptyset$	$\{a\}$ $\{b\}$	$\{a,b\}$ $\{b,c\}$			$\{a\}$ $\{c\}$	$\{a,c\}$ $\{b,c\}$	$\{a,b,c\}$
$(S_3+\{a,b\})*\{a\}$				$(S_3+\{a,b\})*\{b,c\}$			
	$\{c\}$	$\{a,b\}$ $\{a,c\}$ $\{b,c\}$	$\{a,b,c\}$	$\emptyset$	$\{a\}$ $\{b\}$ $\{c\}$	$\{a,b\}$	
$(S_3+\{a,b\})*\{c\}$				$(S_3+\{a,b\})*\{a,b\}$			

TABLE 4. All twists of $S_3+\{a,b\}$ up to isomorphism. Dual pairs are side by side.

$\emptyset$	$\{a\}$ $\{b\}$ $\{c\}$	$\{a,b\}$ $\{a,c\}$ $\{b,c\}$			$\{a\}$ $\{b\}$ $\{c\}$	$\{a,b\}$ $\{a,c\}$ $\{b,c\}$	$\{a,b,c\}$
$S_3+\{a,b,c\}$				$(S_3+\{a,b,c\})^*$			
$\emptyset$	$\{a\}$ $\{b\}$ $\{c\}$	$\{a,b\}$ $\{a,c\}$	$\{a,b,c\}$	$\emptyset$	$\{b\}$ $\{c\}$	$\{a,b\}$ $\{a,c\}$ $\{b,c\}$	$\{a,b,c\}$
$S_3+\{a,b,c\}*\{a\}$				$S_3+\{a,b,c\}*\{b,c\}$			

TABLE 5. All twists of $S_3+\{a,b,c\}$ up to isomorphism. Dual pairs are side by side.

	$\{a\}$	$\{a,b\}$ $\{b,c\}$	$\{a,b,c\}$	$\emptyset$	$\{a\}$ $\{c\}$	$\{b,c\}$	
$(S_3*\{a\})+\{a,b\}$				$((S_3*\{a\})+\{a,b\})^*$			

$\emptyset$	$\{b\}$	$\{b,c\}$	$\{a,b,c\}$
$((S_3*\{a\})+\{a,b\})*\{a\}$			

	$\{a\}$ $\{c\}$	$\{a,b\}$ $\{a,c\}$	
$((S_3*\{a\})+\{a,b\})*\{b\}$			

TABLE 6. All twists of $(S_3*\{a\})+\{a,b\}$ up to isomorphism. Dual pairs are side by side.

$\{a\}$ $\{a,b\}$ $\{a,c\}$ $\{b,c\}$	$\{a\}$ $\{b\}$ $\{c\}$ $\{b,c\}$
$(S_3 * \{a\}) + \{a,b,c\}$	$((S_3 * \{a\}) + \{a,b,c\})^*$
$\emptyset$ $\{b\}$ $\{c\}$ $\{a,b,c\}$	$\emptyset$ $\{a,b\}$ $\{a,c\}$ $\{a,b,c\}$
$((S_3 * \{a\}) + \{a,b,c\}) * \{a\}$	$((S_3 * \{a\}) + \{a,b,c\}) * \{b,c\}$
$\{a\}$ $\{c\}$ $\{a,b\}$ $\{a,b,c\}$	$\emptyset$ $\{c\}$ $\{a,b\}$ $\{b,c\}$
$((S_3 * \{a\}) + \{a,b,c\}) * \{b\}$	$((S_3 * \{a\}) + \{a,b,c\}) * \{a,c\}$

TABLE 7. All twists of $(S_3 * \{a\}) + \{a,b,c\}$ up to isomorphism. Dual pairs are side by side.

REFERENCES

[1] J. Bonin, C. Chun, and S. Noble, Delta-matroids as subsystems of sequences of Higgs lifts. Preprint.
[2] A. Bouchet, Greedy algorithm and symmetric matroids, *Math. Program.* **38** (1987) 147–159.
[3] A. Bouchet, Isotropic systems, *European J. Combin.* **8** (1987) 231–244.
[4] A. Bouchet, Graphic presentations of isotropic systems, *J. Combin. Theory Ser. B* **45** (1988) 58–76.
[5] A. Bouchet. Representability of Δ-matroids. Combinatorics (Eger, 1987), Colloq. Math. Soc. János Bolyai, **52**, North-Holland, Amsterdam, (1988) 167–182.
[6] A. Bouchet. Representability of delta-matroids over $GF(2)$, *Linear Algebra Appl.*, **78**, (1991) 67–78.
[7] A. Bouchet. Circle graph obstructions, *J. Combin. Theory Ser. B*, **60** (1994) 107–144.
[8] A. Bouchet, Multimatroids I. Coverings by independent sets, *SIAM J. Discrete Math.* **10** (1997) 626–646.
[9] A. Bouchet, Multimatroids II. Orthogonality, minors and connectivity, *Electron. J. Combin.* **8** (1998) R8.
[10] A. Bouchet, Multimatroids III. Tightness and fundamental graphs, *European J. Combin.* **22** (2001) 657–677.
[11] R. Brijder. Orienting transversals and transition polynomials of multimatroids, *Adv. in Appl. Math.* **94** (2018) 120–155.
[12] R. Brijder and H. Hoogeboom. The group structure of pivot and loop complementation on graphs and set systems, *European J. Combin.* **32** (2011) 1353–1367.
[13] R. Brijder and H. Hoogeboom. Interlace polynomials for multimatroids and delta-matroids, *European J. Combin.* **40** (2014) 142–167.
[14] R. Brijder and H. Hoogeboom. Nullity and loop complementation for delta-matroids, *SIAM J. Discrete Math.* **27** (2013) 492–506.
[15] R. Brijder and H. Hoogeboom. Quaternary matroids are vf-safe. Preprint, `arXiv:1302.4415v2`.
[16] R. Brijder and L. Traldi. Isotropic matroids I: Multimatroids and neighbourhoods, *Electron. J. Combin.* **23** (2016) P4.1 R8.
[17] C. Chun, I. Moffatt, S. D. Noble, and R. Rueckriemen, Matroids, delta-matroids and embedded graphs. Preprint, `arXiv:1403.0920v2`.
[18] C. Chun, I. Moffatt, S. D. Noble and R. Rueckriemen. On the interplay between embedded graphs and delta-matroids. To appear in *Proc. Lond. Math. Soc.* `arXiv:1602.01306`.
[19] J. A. Ellis-Monaghan and I. Moffatt. A Penrose polynomial for embedded graphs, *European J. Combin.* **34** (2013) 424–445.
[20] J. Ellis-Monaghan and I. Moffatt, *Graphs on surfaces: Dualities, Polynomials, and Knots*, Springer, (2013).
[21] J. Geelen, S. Oum, Circle graph obstructions under pivoting, *J. Graph Theory* **61** (2009) 1–11.
[22] I. Moffatt. Private communication, (2017).
[23] L. Traldi. Binary matroids and local complementation, *European J. Combin.* **45** (2015) 21–40.
[24] A. W. Tucker. A combinatorial equivalence of matrices. *Combinatorial Analysis, Proc. Symposia Appl. Math., vol. X*, American Mathematical Society, Providence (1960) 129–140.

(J. Bonin) DEPARTMENT OF MATHEMATICS, THE GEORGE WASHINGTON UNIVERSITY, WASHINGTON, D.C. 20052, USA
Email address, J. Bonin: `jbonin@gwu.edu`

(C. Chun) DEPARTMENT OF MATHEMATICS, UNITED STATES NAVAL ACADEMY, ANNAPOLIS, MD, 21402, USA
Email address, C. Chun: `chun@usna.edu`

(S. Noble) DEPARTMENT OF ECONOMICS, MATHEMATICS AND STATISTICS, BIRKBECK, UNIVERSITY OF LONDON, LONDON WC1E 7HX, UNITED KINGDOM
Email address, S. Noble: `s.noble@bbk.ac.uk`