Directed-Job Match with Heterogeneity*

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August 8, 2007

Abstract

Matching process involves three stages of selection: application, candidates selection and job acceptance. In traditional matching models all three stages are assumed random, while in the directed-search literature only the first stage is generally assumed 'directed'. This paper develops a job-matching model where all three selection stages are directed, by introducing heterogeneous preferences of firms and workers. Both firm-level and aggregate matching functions are derived, which in a comparison with random-matching models reveals that the coordination failure problem is worse under heterogeneous directed-match when the number of vacancies is small, but is better when it is large. Furthermore directed-search limits friction when the market consists of fewer but larger firms.

JEL Classification: J41, J64

Keywords: matching function, directed-search, heterogeneity

*I am grateful to Sir James Mirrlees, Alan Manning, Bob Evans, Sandeep Kapur and three anonymous referees for their valuable comments. All errors are the author's. Correspondence to: Kenjiro Hori, School of Economics, Mathematics and Statistics, Birkbeck College, Malet Street, London, WC1E 7HX, UK. e-mail: k.hori@bbk.ac.uk.

1 Introduction

In a job market the matching process involves three stages of selection: the application stage, where workers select the firm(s) to apply to; the candidates selection stage, where firms select the worker(s) to offer jobs to; and the job acceptance stage, where workers, if they have more than one job offers, select one job to accept. Random-matching models, such as Pissarides (1979), Blanchard and Diamond (1994) and Julien *et al.* (2000), assume random selection in all three of the stages, which leads to coordination failure resulting in frictional unemployment. The directed-search literature on the other hand assume that there is a criteria, such as the wage level or the probability of a successful job match, upon which workers are attracted to certain firms. However once firms and workers meet the selections there-onwards, i.e. those in the latter two stages, are generally assumed random. There are at least two questions that can be raised with this set-up. One is that of symmetry in heterogeneity: if workers are attracted to certain firms, firms should also be attracted to certain workers. The second is that once job offers are made, workers with a choice of job offers should also select the job to accept, using their original preference orderings with which they chose the firms to make applications to. This paper attempts to resolve these questions by developing a model of heterogeneous directed-search where all three selection stages are directed. This is done by allowing workers and firms to be heterogeneous, either in their skills-set possessed (by workers) or required (by firms), and/or in other non-monetary job characteristics preferred or offered, that give rise to better matches between certain firms and workers. Workers then apply to their most preferred jobs, while firms select their best matched candidates, after which workers each accept their most preferred job. As Pissarides (2000) states a matching function is "a modeling device that captures the implications of the costly trading process without the need to make the *heterogeneities and other features* that give rise to it explicit" (emphasis the author's). The aim of this paper is to model explicitly and comprehensively the heterogeneities.

The set-up here is that of workers and firms being represented by points on a multi-dimensional heterogeneity domain, depending on their individual heterogeneity factors described above. The distances between the points represent the closeness of their match, and the workers apply to $a \geq 1$ 'nearest' jobs. The firms then select the nearest candidates and make job offers of the number which is the smaller of the number of available vacancies and the number of received applications. For the purpose of deriving the matching functions here the domain considered is a simple one-dimensional circular domain (similar to Salop's (1979) monopolistic competition model). The workers who are situated around the circumference apply to $a = 2$ nearest firms, one on each side of them, and the firms compete for the workers in between. The unknowns in the model are the number of workers choosing to apply to any given two neighbouring firms (or in the measure where workers are uniformly distributed, as assumed here, the distances between the firms), and the number of vacancies offered at each firm. If these variables were known, then firms and workers would simply choose their optimal matches and there will be no coordination failure. The number of successful hirings then depends on the relative size of the two unknown variables, given the number of the firm's own vacancies. Taking expectations over these yields the firm-level matching function, which is found to be concave in the number of own-vacancies. The aggregate matching function is derived by evaluating the probability of a worker receiving at least one job offer from the two firms that he applies to, and multiplying it by the total number of applicants. This is found to exhibit increasing returns with respect to the aggregate vacancies and unemployment levels when the number of firms is assumed fixed.

A question that is raised then is whether coordination failure problem under heterogeneous directed-search is more or less severe than under random-matching. An initial intuition may be that, as it allows better-matched workers and firms to attract each other, directed-search may dominate random-matching in aggregate matching success. However this is found not so: a comparison shows that while the heterogeneity model does outperform the random-matching model when the number of aggregate vacancies is large, the order is reversed when the number is small. This is because the heterogeneity effect is more acute for smaller number of vacancies. Further it is also shown that, given the aggregate number of vacancies, the coordination failure under heterogeneity is less severe when the job-matching market consists of fewer but larger firms. This last result agrees with that attained by Burdett *et al.* (2001).[1]

Other models of directed-search in the literature include Moen (1997), Lagos (2000) and Galenianos and Kircher (2005, 2007). In Moen (1997) the matching market is divided into submarkets, each with exogenously assigned wage levels. The firms' and workers' choice of a submarket to join results from a trade-off between the wage level and the matching rate within it. Once in a submarket the matching mechanism is that of the traditional random-matching. Lagos (2000) derives a matching function in a frame-work where homogeneous taxi drivers choose locations to meet the customers, given a distribution of expected profit at each meeting points. Again there is a trade-off between expected profit and the matching rate. Lagos calls the resulting friction the 'equilibrium friction' and distinguishes it from the usual random matching frictions, in that the agents 'choose' the equilibrium level of friction. In Galenianos and Kircher (2005) the application process is modelled as an optimal portfolio selection problem, where the willingness of workers to apply to jobs with different wages (and hence different job-match probabilities) incentivise homogeneous firms to post different wages. Finally in Galenianos and Kircher (2007) firms with heterogeneous productivity post different wages to attract homogeneous workers. In all these cases even though in the application stage workers are attracted to different firms for different reasons, selections in the candidates selection, and where appropriate in the acceptance stages, are assumed random. As already stated in this paper even these two latter stages depend on firms' and workers' heterogeneous preferences.

The rest of the paper is organised as follows. In Section 2 the heterogeneous directed-search is introduced and both firm-level and aggregate matching functions are derived. Section 3 compares this model with the existing random-matching models, to suggest conditions under which coordination failure is more severe in one than the other. Section 4 then gives concluding remarks.

[1] Their model consists of buyers each wanting to buy one unit, and sellers each offering a deterministic number of goods. Viewing the prices posted by different sellers, the buyers choose a mixed strategy of selecting sellers to approach. However once the buying interests are received the sellers select the buyers randomly. Therefore in their model only the first selection stage is directed.

2 From Random Matching to Heterogeneous Directed-Search

2.1 Workers

In this model the heterogeneity of workers can be in their skills-set, and/or in their preferences for job characteristics such as the nature of the job, work environment, geographical location or friendliness of the colleagues. The only requirement is that both workers and firms have heterogeneous preferences for their prospective firms or workers. Workers then apply to their most preferred firms, and the firms choose their best-matched workers. For example in the case that workers differ in their skills-set, where a better match leads to a higher productivity, if the wage levels are determined by a Nash bargaining then workers and firms would select their most profitable matches. Workers and firms can therefore be represented as different points on a multi-dimensional heterogeneity domain, with distances between them indicating how close a match they are with each other. For the purpose of deriving the matching function, I consider here a simple one-dimensional circular domain with circumference Ω. This is similar to Salop's (1979) model of monopolistic competition where competitors are situated on a circular market, the idea which was later employed to a clearing labour market by Hamilton *et al.* (2000).[2] The measure is also chosen in which workers are distributed uniformly around Ω, as depicted in Figure 1. Then if the total number of workers searching for jobs in the job-matching market is u, the distance between two neighbouring workers is $\frac{\Omega}{u}$. It is assumed that no two workers are identical, i.e. each point on the circumference represents a single applicant.

2.2 Random Matching

First consider the case of random matching, where workers each apply to $a \geq 1$ jobs. The case of multiple-applications matching was first analysed by Albrecht, Gautier, Tan and Vroman (2004) (hereafter AGTV), and was later refined by Hori (2007a).

[2] They assume continuously and uniformly distributed workers along a circular skills space, with equally spaced firms.

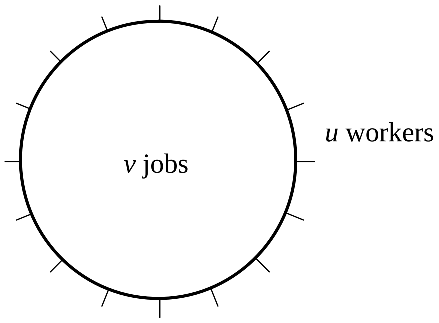

Figure 1: Random Matching

As with AGTV assume for now that each firm consists of one job vacancy, i.e. $F = v$ where F is the number of firms and v is the total number of vacancies in the job market. In this paper's set-up the v jobs can be thought of as being situated within the circle, as shown in Figure 1, where the u workers on the circumference select a jobs randomly. This is analogous to the urn-ball set-up were workers select a balls out of an urn containing v balls. Firms receiving more than one applications then randomly select one candidate and make a job offer. Finally workers with more than one job offers selects one and accepts, and the match is complete. Working in the limit that $u, v \to \infty$ but $\frac{v}{u} = \theta \ll \infty$, the probability that the firm a workers applies to receives $\alpha \geq 0$ other job applications is given by a Poisson distribution with rate $\frac{au}{v} = \frac{a}{\theta}$, and therefore the probability of a job offer by the firm, denoted by Ψ, is[3]

$$\begin{aligned}\Psi &= \sum_{\alpha=0}^{u-1} \frac{1}{\alpha+1} \frac{\left(\frac{a}{\theta}\right)^{\alpha} e^{-\frac{a}{\theta}}}{\alpha!} \\ &= \frac{\theta}{a}\left(1 - e^{-\frac{a}{\theta}}\right). \qquad (1)\end{aligned}$$

As discussed in AGTV (2004) and Hori (2007a), in the limit $u, v \to \infty$ this probability of a job offer at each firm is independent from each other, and therefore the probability that a worker receives at least one job offer from the a applications is $1 - (1 - \Psi)^a$. Hence the expected number of matches M in a random-matching job market with

[3]For the case of finite u and v see AGTV (2004) or Hori (2007a).

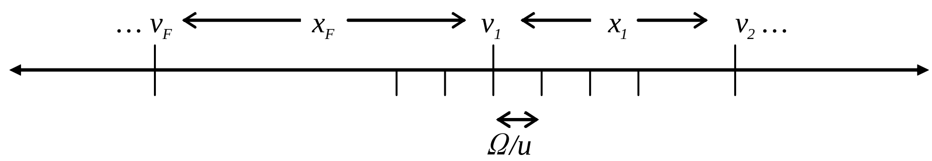

Figure 2: Multiple-Vacancies Firms (above) and Multiple-Applications Workers (below) on a Circular Heterogeneity Domain

single-vacancy firms can be calculated by the following matching function,

$$m(u, v; a) = u \left\{ 1 - \left[1 - \frac{\theta}{a} \left(1 - e^{-\frac{a}{\theta}} \right) \right]^a \right\}. \tag{2}$$

2.3 Directed-Search

Next consider the case of directed-search. The firms are now also represented by points on the heterogeneity circumference, (Figure 2 shows a segment of the circumference), and the workers and jobs now match depending on their 'closeness' in the heterogeneity domain. In contrast to the random-matching model outlined above, it is now also assumed that each job can advertise a multiple number of vacancies $\{v_1, v_2, ..., v_F\}$, with $v = \sum_{i=1}^{F} v_i$. Distances between the firms are given by $\{x_1, x_{2,...,}x_F\}$, with $\sum_{i=1}^{F} x_i = \Omega$. In this analysis I concentrate on the case $a = 2$, and hence the workers apply to their nearest firm on either side of them on the circumference. Having received those applications, firms in turn make the exact number of job offers as the number of their vacancies to their 'nearest' candidates. The workers who receive more than one job offers then accept their nearest job, and the match is complete. This differs from the set-ups in other models of directed-search in that, whilst in those candidate selection by firms and job acceptance by workers are still assumed random, here they are modelled explicitly as being directed, as discussed in the introduction section of this paper. In this simple representation of a heterogeneous job-matching market then a hiring firm has two rivals, one on each side along the circumference, with whom it would compete for the workers situated between them. The workers in turn would receive either zero, one or two job offers.

There are two sources of randomness in the model. One is the number of workers who choose to apply to any two neighbouring firms. In the chosen measure with uniformly distributed workers, this translates to random distances x_i between the firms. The number of firms in any given distances is assumed Poisson distributed with rate $\frac{F}{\Omega}$. Then x_i is the 'arrival distance' of the nearest firm, which is exponentially distributed with support $[0, \Omega]$ and mean $\frac{\Omega}{F}$.[4] The probability density functions ϕ_{x_i} for the distances x_i is then,

$$\phi_{x_i} = \frac{F}{\Omega} e^{-\frac{F}{\Omega} x_i}, \tag{3}$$

with $\int_0^\Omega \phi_{x_i} dx_i = 1$ in the limit of large Ω. The other source of randomness is the number of vacancies v_i at each firm. The aggregate number of vacancies v is deterministically known. In the limit $v \to \infty$, $v_i \in \{0, 1, ..., v\}$ can be assumed to be independently distributed for all $i = 1, ..., F$ with the following Poisson density with rate $\frac{v}{F}$,

$$\phi_{v_i} = \frac{\left(\frac{v}{F}\right)^{v_i} e^{-\frac{v}{F}}}{v_i!}. \tag{4}$$

For example then there is a $e^{-\frac{v}{F}}$ probability that the firm actually offers no vacancy. The incomplete information assumption leads to coordination failure in both stages of application and candidate selection; if these variables were known, then firms and workers will simply choose their optimal matches. Other aggregate numbers u and F of applicants and firms are deterministically known.

Investigate first now the firm-level matching rate. This depends on the relative size of the two sources of randomness above where, if the number of vacancies at the rival firm is small relative to the distance then the success rate is high, while if it is large then the firm will be less successful in filling its vacancies. Consider then a firm i. Its nearest competitors are firms $i-1$ and $i+1$, at distances x_{i-1} and x_i on its left- and right-hand sides respectively. These distances are unknown to firm i. Firm i knows the number of its own vacancies v_i, but not those of its rivals v_{i-1} and v_{i+1}. The number of candidates available for firm i between it and the nearest rivals are $\frac{u}{\Omega} x_{i-1}$ and $\frac{u}{\Omega} x_i$ respectively. Since the firm does not know its success rates on either

[4] See for example Ross (2003) Ch 5.3.3.

side of it along the circumference, it makes $\frac{v_i}{2}$ offers each to workers on its either side. Here, and for the rest of the paper, the additional complication of the number of vacancies being an odd number is ignored. Now denote by M_{iR} the number of vacancies filled for firm i on its right-hand side after a round of job matching. This depends on the different scenarios of the relative sizes of v_i, v_{i+1} and x_i, which I now consider individually.

Firstly if the number of offers on the right-hand side $\frac{v_i}{2}$ is less than half the available workers $\frac{1}{2}\frac{u}{\Omega}x_i$, all $\frac{v_i}{2}$ workers will accept the offer from firm i as firm i is the nearest firm for these. If however $\frac{v_i}{2}$ is greater than $\frac{1}{2}\frac{u}{\Omega}x_i$, then firm i will now be offering jobs to some of the workers for whom firm $i+1$ is the nearest, and therefore it may not get all its offers accepted. This would certainly be the case if additionally $v_{i+1} \geq v_i$, in which case both firms get $\frac{1}{2}\frac{u}{\Omega}x_i$ workers each. If however $\frac{v_i}{2}$ is greater than $\frac{1}{2}\frac{u}{\Omega}x_i$ but $v_{i+1} < v_i$, then there can be three further scenarios. First if the total number of job offers by the two firms $\frac{v_i}{2} + \frac{v_{i+1}}{2}$ for the workers within x_i is less than the available workers $\frac{u}{\Omega}x_i$, then there will be no overlap between the two firms' offers, and both will get their full quota of workers, i.e. $M_{iR} = \frac{v_i}{2}$. If however $\frac{v_i}{2} + \frac{v_{i+1}}{2}$ is greater than the available workers $\frac{u}{\Omega}x_i$, but $\frac{v_{i+1}}{2}$ is less than $\frac{1}{2}\frac{u}{\Omega}x_i$, then firm i will be able to claim all the workers that are closer to firm $i+1$ but do not get offers from the firm, i.e. $\frac{u}{\Omega}x_i - \frac{v_{i+1}}{2}$. Finally if $\frac{v_{i+1}}{2}$ is also greater than $\frac{1}{2}\frac{u}{\Omega}x_i$ both will simply get their closest workers $\frac{1}{2}\frac{u}{\Omega}x_i$. In summary then,

$$
\begin{aligned}
M_{iR} &= \frac{v_i}{2} && \text{if } \frac{v_i}{2} \leq \frac{1}{2}\frac{u}{\Omega}x_i \\
&= \frac{1}{2}\frac{u}{\Omega}x_i && \text{if } v_i \leq v_{i+1} \text{ and } \frac{v_i}{2} > \frac{1}{2}\frac{u}{\Omega}x_i \\
&= \frac{v_i}{2} && \text{if } v_i > v_{i+1} \text{ and } \frac{v_i}{2} + \frac{v_{i+1}}{2} \leq \frac{u}{\Omega}x_i \leq v_i \qquad (5) \\
&= \frac{u}{\Omega}x_i - \frac{v_{i+1}}{2} && \text{if } v_i > v_{i+1} \text{ and } v_{i+1} \leq \frac{u}{\Omega}x_i \leq \frac{v_i}{2} + \frac{v_{i+1}}{2} \\
&= \frac{1}{2}\frac{u}{\Omega}x_i && \text{if } v_i > v_{i+1} \text{ and } \frac{u}{\Omega}x_i \leq v_{i+1}.
\end{aligned}
$$

Note that in theory one can envisage a situation where, if required, firm i may be able to employ workers on the other side of firm $i+1$ once firm $i+1$ has filled its vacancies. However for this to happen it would require a combination of relatively large v_i, small x_i, even smaller v_{i+1}, large x_{i+1} and small v_{i+2}, the joint probability of which I consider to be too small to affect the results obtained here.

Then,

Proposition 1 (Firm-Level Matching Function) *The expected number of vacancies filled M_i for firm i given v_i, v, u and F is given by the following firm-level matching function,*

$$m_i(v_i, v, u, F) = \frac{u}{F}\left(1 - e^{-\frac{F}{u}v_i}\right) + \frac{u}{F}\sum_{v_{i+1}=1}^{v_i-1}\phi_{v_{i+1}}\left(e^{-\frac{F}{u}\frac{v_i}{2}} - e^{-\frac{F}{u}\frac{v_{i+1}}{2}}\right)^2 \tag{6}$$

where $\phi_{v_{i+1}}$ is given by (4).

Proof. The expected number of filled jobs for firm i on its right-hand side is calculated by taking the expectations of (5) over the two random variables x_i and v_{i+1} in the limit of large Ω,

$$\begin{aligned} M_{iR} &= \int_{v_i\frac{\Omega}{u}}^{\infty}\frac{v_i}{2}\phi_{x_i}dx_i + \sum_{v_{i+1}=v_i}^{v}\phi_{v_{i+1}}\int_0^{v_i\frac{\Omega}{u}}\frac{ux_i}{2\Omega}\phi_{x_i}dx_i + \sum_{v_{i+1}=1}^{v_i-1}\phi_{v_{i+1}}\left[\int_{\left(\frac{v_i+v_{i+1}}{2}\right)\frac{\Omega}{u}}^{v_i\frac{\Omega}{u}}\frac{v_i}{2}\phi_{x_i}dx_i\right. \\ &\quad \left. + \int_{v_{i+1}\frac{\Omega}{u}}^{\left(\frac{v_i+v_{i+1}}{2}\right)\frac{\Omega}{u}}\left(\frac{ux_i}{\Omega} - \frac{v_{i+1}}{2}\right)\phi_{x_i}dx_i + \int_0^{v_{i+1}\frac{\Omega}{u}}\frac{ux_i}{2\Omega}\phi_{x_i}dx_i\right], \end{aligned} \tag{7}$$

where ϕ_{x_i} and $\phi_{v_{i+1}}$ are given respectively by (3) and (4). By symmetry the expected number of vacancies filled on the firm's left-hand side, M_{iL}, is also given by (7). Hence the total expected number of filled vacancies M_i is twice (7). By subtracting $\frac{ux_i}{2\Omega}$ from the last three integrands and adding them to the second then,

$$\begin{aligned} M_i &= \int_{v_i\frac{\Omega}{u}}^{\infty} v_i\phi_{x_i}dx_i + \int_0^{v_i\frac{\Omega}{u}}\frac{ux_i}{\Omega}\phi_{x_i}dx_i + \sum_{v_{i+1}=1}^{v_i-1}\phi_{v_{i+1}}\left[\int_{\left(\frac{v_i+v_{i+1}}{2}\right)\frac{\Omega}{u}}^{v_i\frac{\Omega}{u}}\left(v_i - \frac{ux_i}{\Omega}\right)\phi_{x_i}dx_i\right. \\ &\quad \left. + \int_{v_{i+1}\frac{\Omega}{u}}^{\left(\frac{v_i+v_{i+1}}{2}\right)\frac{\Omega}{u}}\left(\frac{ux_i}{\Omega} - v_{i+1}\right)\phi_{x_i}dx_i\right]. \end{aligned} \tag{8}$$

Now in substituting in $\phi_{x_i} = \frac{F}{\Omega} e^{-\frac{F}{\Omega} x_i}$, the first two terms in (8) are,

$$\begin{aligned} \int_{v_i \frac{\Omega}{u}}^{\infty} v_i \phi_{x_i} dx_i &= v_i e^{-\frac{F}{u} v_i}, \\ \int_0^{v_i \frac{\Omega}{u}} \frac{u x_i}{\Omega} \phi_{x_i} dx_i &= \frac{u}{F} - \left(v_i + \frac{u}{F} \right) e^{-\frac{F}{u} v_i}. \end{aligned}$$

The second integration is evaluated by parts. These add up to,

$$\text{First two terms of (8)} = \frac{u}{F} \left(1 - e^{-\frac{F}{u} v_i} \right).$$

Similarly evaluating the latter two terms within [.] in (8),

$$\begin{aligned} \int_{\left(\frac{v_i + v_{i+1}}{2}\right)\frac{\Omega}{u}}^{v_i \frac{\Omega}{u}} \left(v_i - \frac{u x_i}{\Omega} \right) \phi_{x_i} dx_i &= \left[\left(\frac{v_i - v_{i+1}}{2} \right) - \frac{u}{F} \right] e^{-\frac{F}{u}\left(\frac{v_i + v_{i+1}}{2}\right)} + \frac{u}{F} e^{-\frac{F}{u} v_i}, \\ \int_{v_{i+1} \frac{\Omega}{u}}^{\left(\frac{v_i + v_{i+1}}{2}\right)\frac{\Omega}{u}} \left(\frac{u x_i}{\Omega} - v_{i+1} \right) \phi_{x_i} dx_i &= \left[-\left(\frac{v_i - v_{i+1}}{2} \right) - \frac{u}{F} \right] e^{-\frac{F}{u}\left(\frac{v_i + v_{i+1}}{2}\right)} + \frac{u}{F} e^{-\frac{F}{u} v_{i+1}}. \end{aligned}$$

These add up to,

$$\begin{aligned} \text{Terms within [.] of (8)} &= \frac{u}{F} \left[e^{-\frac{F}{u} v_i} - 2 e^{-\frac{F}{u}\left(\frac{v_i + v_{i+1}}{2}\right)} + e^{-\frac{F}{u} v_{i+1}} \right] \\ &= \frac{u}{F} \left(e^{-\frac{F}{u} \frac{v_i}{2}} - e^{-\frac{F}{u} \frac{v_{i+1}}{2}} \right)^2. \end{aligned}$$

Substituting these back into (8) yields (6). ■

The first term in (6) represents the cases of filling, on firm i's either side, $\frac{v_i}{2}$ if $\frac{v_i}{2} < \frac{u x_j}{2\Omega}$, and $\frac{u x_j}{2\Omega}$ if otherwise, where $j \in \{i-1, i\}$. These are the two possible outcomes if firms $i-1$, i and $i+1$ were identical. The second term represents the cases for filling on top of $\frac{u x_j}{2\Omega}$, when $\frac{v_i}{2} > \frac{u x_j}{2\Omega}$, for example on its right-hand side extra $\frac{v_i}{2} - \frac{u x_i}{2\Omega}$ if $\frac{v_i + v_{i+1}}{2} < \frac{u x_i}{\Omega}$, and extra $\frac{u x_i}{2\Omega} - \frac{v_{i+1}}{2}$ if $\frac{v_i + v_{i+1}}{2} > \frac{u x_i}{\Omega}$ but $\frac{v_{i+1}}{2} < \frac{u x_i}{2\Omega}$. These outcomes occur in the asymmetric cases $v_i > v_{i-1}$ and $v_i > v_{i+1}$.

Now the firm-level matching function exhibits the following properties,

Property 1 $m_i(v_i, v, u, F)$ *has the following properties,*

1. $m_i(0, v, u, F) = m_i(v_i, v, 0, F) = 0.$
2. $\frac{\partial m_i}{\partial u} > 0$ *and* $\lim_{u \to \infty} m_i = v_i$.
3. $\frac{\partial m_i}{\partial v_i} > 0$ *and* $\frac{u}{F} < \lim_{v_i \to \infty} m_i = \frac{u}{F}\left[1 + e^{-\frac{v}{F}}\left(e^{\frac{v}{F}e^{-\frac{F}{u}}} - 1\right)\right] < \frac{2u}{F}$.
4. $\frac{\partial^2 m_i}{\partial v_i^2} < 0.$

Proof. See Appendix A. ■

Properties 3 and 4 above imply a diminishing rate of success of filling vacancies on the firm-level as the number of own-vacancies increases, with the number of successful match bounded above. This property is demonstrated in the simulated graphs in Figures 3 and 4. This is caused by the heterogeneity factor: as a firm widens its net for potential candidates, the compatibility of workers on the periphery decreases, reducing the firm's chance of success in hiring the extra workers.

Now there are two ways of calculating the total expected number of filled vacancies M for the economy. One is to average the number of firm-level matches in (6) over all possible values of v_i, and then multiplying it by the number of hiring firms F,

$$m(u, v, F) = F \sum_{v_i=1}^{v} \phi_{v_i} m_i(v_i, v, u, F) \tag{9}$$

where ϕ_{v_i} is again given by (4). The other way is,

Proposition 2 (Aggregate Matching Function) *The total expected number of filled vacancies* $M = m(u, v, F)$ *for the job-matching market* (u, v, F) *is,*

$$m(u, v, F) = u\left[1 - e^{-\frac{2v}{F}\left(1 - e^{-\frac{F}{2u}}\right)}\right]. \tag{10}$$

Proof. An applicant will be matched to a job for certain if he receives one or more job offers. Let the probability that an applicant receives a job offer from a firm

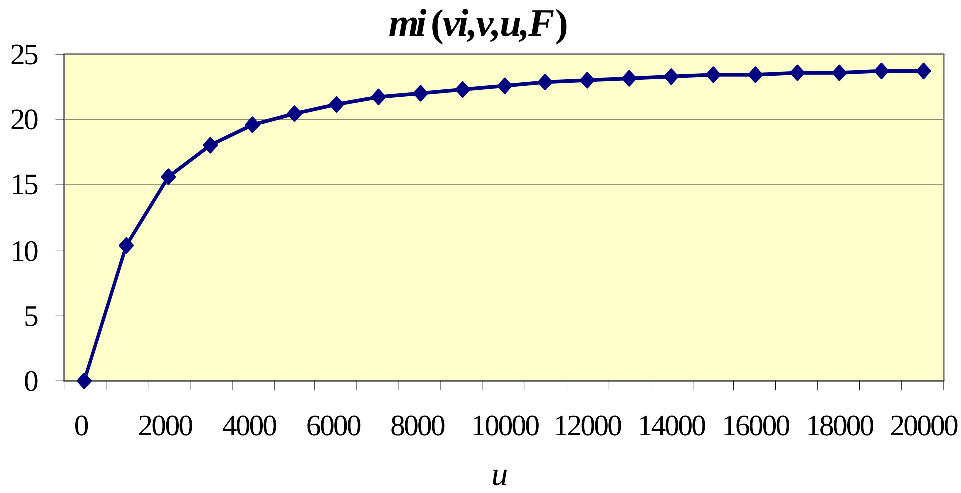

Figure 3: $m_i(v_i, v, u, F)$ vs u: $v_i = 25$, $v = 1000$, $F = 100$; $\lim_{U\to\infty} M_i = 25$

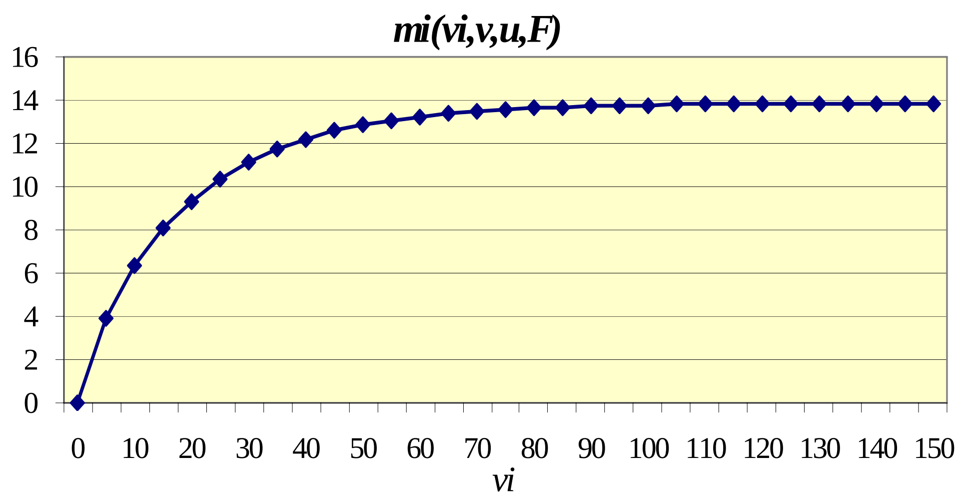

Figure 4: $m_i(v_i, v, u, F)$ vs v_i: $u = 1000$, $v = 1000$, $F = 100$; $\lim_{v_i\to\infty} M_i = 13.86$

that he applies to be Ψ. Then in the set-up here where each applicant applies to the two nearest firms, the probability that he has at least one job offer is given by,

$$\text{Probability of at least one job offer} = 1 - (1-\Psi)^2 . \tag{11}$$

The aggregate number of matches is then this multiplied by u. Now for the nearest firm i on his right-hand side, an applicant will receive a job offer if the number of vacancies v_i, offered by i to workers on its left-hand side, is greater than the number of applicants nearer to the firm than him, i.e. $\frac{v_i}{2} \geq \frac{u}{\Omega} x_i$, where x_i is the distance to the firm. Hence the probability of a job offer from firm i is,

$$\Psi_i = \int_0^\Omega \sum_{v_i=1}^{v} \phi_{v_i} \chi_{\frac{v_i}{2} \geq \frac{u}{\Omega} x_i} \phi_{x_i} dx_i, \tag{12}$$

where ϕ_{v_i} and ϕ_{x_i} are given by (4) and (3) respectively,[5] and $\chi_{\frac{v_i}{2} \geq \frac{u}{\Omega} x_i}$ is an indicator function which takes the value 1 if $\frac{v_i}{2} \geq \frac{u}{\Omega} x_i$, and 0 if otherwise. Evaluating this,

$$\begin{aligned}
\Psi_i &= \sum_{v_i=1}^{v} \phi_{v_i} \int_0^{\frac{v_i \Omega}{2u}} \frac{F}{\Omega} e^{-\frac{F}{\Omega} x_i} dx_i \\
&= \sum_{v_i=1}^{v} \frac{\left(\frac{v}{F}\right)^{v_i} e^{-\frac{v}{F}}}{v_i!} \left(1 - e^{-\frac{v_i F}{2u}}\right) \\
&= \sum_{v_i=1}^{v} \frac{\left(\frac{v}{F}\right)^{v_i} e^{-\frac{v}{F}}}{v_i!} - e^{-\frac{v}{F}\left(1-e^{-\frac{F}{2u}}\right)} \sum_{v=1}^{v} \frac{\left(\frac{v}{F} e^{-\frac{F}{2u}}\right)^{v_i} e^{-\frac{v}{F} e^{-\frac{F}{2u}}}}{v_i!} \\
&= \left(1 - e^{-\frac{v}{F}}\right) - e^{-\frac{v}{F}\left(1-e^{-\frac{F}{2u}}\right)} \left(1 - e^{-\frac{v}{F} e^{-\frac{F}{2u}}}\right) \\
&= 1 - e^{-\frac{v}{F}\left(1-e^{-\frac{F}{2u}}\right)}.
\end{aligned} \tag{13}$$

The fourth equality uses the fact that $\sum_{v_i=0}^{v} \frac{\lambda^{v_i} e^{-\lambda}}{v_i!} = 1$ for $\lambda = \frac{v}{F}$ or $\frac{v}{F} e^{-\frac{F}{2u}}$. Replacing Ψ in (11) with this Ψ_i and multiplying by u yields (10). ∎

Note that if the indicator function in (12) was replaced by 1, i.e. the worker is the

[5] Note in particular that the distribution of the distance to the nearest firm is the same whether your starting point is a firm or an applicant.

preferred candidate by the applied firm with certainty, then the probability of a job offer (13) would equal $1 - e^{-\frac{v}{F}}$. This is the probability that the best matched firm has a strictly positive number of vacancies, when the number of vacancies is given by a Poisson distribution with rate $\frac{v}{F}$. On top of this (13) includes an adjustment factor $1 - e^{-\frac{F}{2u}}$, which is the probability that at least one firm chooses the worker's application as its most preferred application out of a total of $2u$ applications in the market.

The aggregate matching function exhibits the following properties,

Property 2 *$M = m(u, v, F)$ has the following properties,*

1. *$m(0, v, F) = m(u, 0, 0) = 0$.*
2. *$\lim_{u \to \infty} m(u, v, F) = v$ for given v and F.*
3. *$\lim_{v \to \infty} m(u, v, F) = u$ for given u and F.*
4. *$m(u, v, F)$ exhibits constant returns to scale in (u, v, F), and increasing returns to scale in (u, v) for given F.*

Proof. Trivial using (10). ■

An example of a graph of $m(u, v; F)$ is shown in Figure 5. One unresolved issue in the matching function literature is the homogeneity of the aggregate matching function. As noted by Petrongolo and Pissarides (2001), "testing for homogeneity, or constant returns to scale, has been one of the preoccupations of the empirical literature." The conclusion from their extensive survey is that the "stylized fact that emerges from the empirical literature is that there is a stable aggregate matching function of a few variables that satisfies the Cobb-Douglas restrictions with constant returns to scale in vacancies and unemployment." For example Pissarides (1986) in using UK men's data from 1967-83, estimates the elasticities of u and v to be 0.7 and 0.3 respectively. However there are also more recent studies that show evidence of increasing returns, such as Blanchard and Diamond (1990), Warren (1996), and

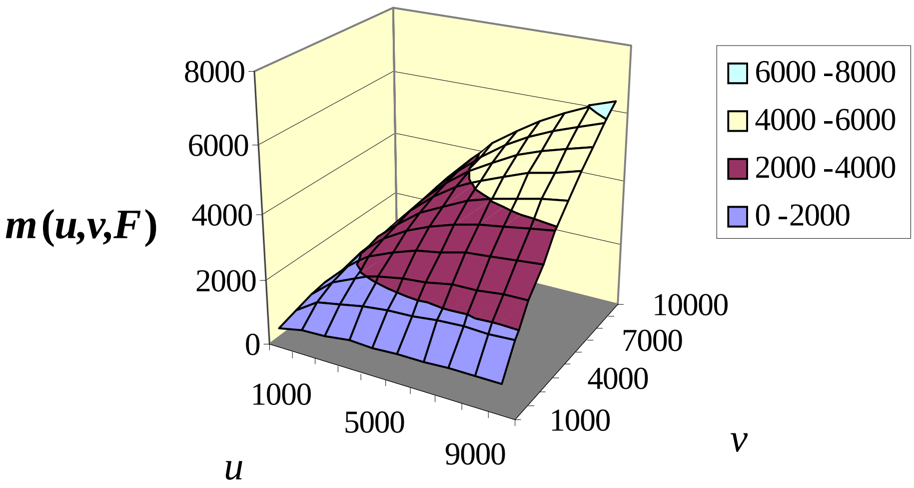

Figure 5: $m(u, v, F)$ vs u and v: $F = 100$

Münich *et al.* (1999).[6] Theoretical studies are also divided in this issue. Whilst Blanchard and Diamond (1994) and Julien *et al.* (2000) (see eqns (15) and (16) below) suggest unit homogeneity in u and v, other derived functions such as Coles and Smith (1998) suggest increasing returns.[7] The derived matching function here supports both claims: if an increase in the size of the labour market involves also a proportional increase in the number of advertising firms (i.e. the average number of vacancies per firm is stable) then the aggregate number of matches is homogeneous of degree 1, while if the number of firms is relatively stable then the successful matching rates $\frac{M}{v}$ and $\frac{M}{u}$ will increase as the market size increases.

[6] The data sets used are as follows: Blanchard and Diamond (1990) the US data, 1968-81; Warren (1996) the US manufacturing data, 1969-73; and Münich *et al.* (1999) the Czech Republic and Slovakia data, 1991-96. Then for example Warren estimates the sum of the elasticities to be 1.33.

[7] Coles and Smith follow the 'stock-flow' approach where there are stocks U and V of unemployed workers and vacancies who attempt to match with new flows of unemployed and vacancies u and v. When there is a probability α of the match being unsuccessful, then the derived matching function is

$$M = v\left(1 - \alpha^U\right) + u\left(1 - \alpha^V\right)$$

Gregg and Petrongolo (1997) however claim that the increasing returns can be ruled out when extra congestion externalities are considered between the newly unemployed.

3 Is Directed-Match Better Than Random-Match?

As noted by AGTV (2004), under the condition $u, v \to \infty$ where probabilities Ψ of a job offer from firms can be considered as being independent, the general aggregate matching function when workers make $a \geq 1$ applications is given by,

$$m(u, v; a) = u \left\{1 - (1 - \Psi)^a\right\}. \tag{14}$$

For random matching with single-vacancy firms, Ψ was given in (1). This function incorporates the two sources of coordination failure as pointed out by Albrecht *et al.* (2006): the urn-ball friction, where some firms receive more job applications than their number of advertised vacancies while others less, and the multiple-applications friction, where analogously some applicants receive more than one job offers while others none. For example letting $a = 1$ yields the matching function when workers each make a single application that captures solely the urn-ball friction,

$$m(u, v; 1) = u\Psi = v(1 - e^{-\frac{u}{v}}). \tag{15}$$

This is the function derived by, amongst others, Blanchard and Diamond (1994)[8] in an urn-ball set-up that models solely the first selection (i.e. application) stage of the matching process.[9] On the other hand if $a = v$,

$$\begin{aligned} m(u, v; v) &= u\left\{1 - \left[1 - \frac{1}{u}\left(1 - e^{-\frac{1}{u}}\right)\right]^v\right\} \\ &\approx u\left[1 - \left(1 - \frac{1}{u}\right)^v\right] \quad \text{for large } u, \end{aligned} \tag{16}$$

which is the function derived by Julien *et al.* (2000), in whose model the matching trade is an auctioning process where firms compete for agents who have reservation

[8] Blanchard and Diamond's derived matching function is in fact $m = v\left(1 - e^{-\frac{bu}{v}}\right)$, where b is the exogenous probability of making an acceptable application, which "reflects the intensity of search by workers and firms, as well as the skill and geographical distributions of workers and jobs" (Blanchard and Diamond).

[9] The second and third (i.e. candidates selection and acceptance) stages, are made deterministic by the assumption of single-application workers.

wages. By cutting out the application stage, this set-up solely captures the multiple-applications friction in the candidates selection stage of job-matching.

In this paper's model of heterogeneous directed-match, the probability of a job offer by a firm applied to was given in (13). Substituting this into (14) for $a = 2$ yields the derived matching function (10). While under heterogeneous directed-search better-matched workers and firms do attract each other, it is still prone to both of the frictions above in that it does not prevent more or less workers being attracted to a firm than its number of vacancies, nor does it stop some workers receiving a multiple number of job offers while others none, under the incomplete information structure assumed here. The question is whether it aggravates or alleviates these sources of coordination failure in comparison to random-match. To investigate this compare the two aggregate matching functions (2) and (10) when $a = 2$. Strictly speaking the two results differ in that the former assumes firms each with a single vacancy, while the latter assumes multiple-vacancies firms. However a comparison would still be a worthwhile exercise for an intuitive result which turns out to be quite intriguing. For example random-matching produces more matches than directed-search when,

$$(2) > (10) \Leftrightarrow 2u\left[1 - e^{-\frac{v}{F}\left(1-e^{-\frac{F}{2u}}\right)}\right] < v\left(1 - e^{-\frac{2u}{v}}\right). \tag{17}$$

But for $F < 2u$, $e^{-\frac{F}{2u}} > 1 - \frac{F}{2u}$ and hence for the left-hand expression,

$$2u\left[1 - e^{-\frac{v}{F}\left(1-e^{-\frac{F}{2u}}\right)}\right] < 2u\left(1 - e^{-\frac{v}{2u}}\right).$$

Therefore the sufficient condition that (17) holds is,

$$2u\left(1 - e^{-\frac{v}{2u}}\right) < v\left(1 - e^{-\frac{2u}{v}}\right) \Leftrightarrow v < 2u. \tag{18}$$

Hence the random-matching assumption (albeit for single-vacancy firms) produces unambiguously larger match than the heterogeneous directed-search when the number of advertised vacancies is less than the total number of applications, irrespective of

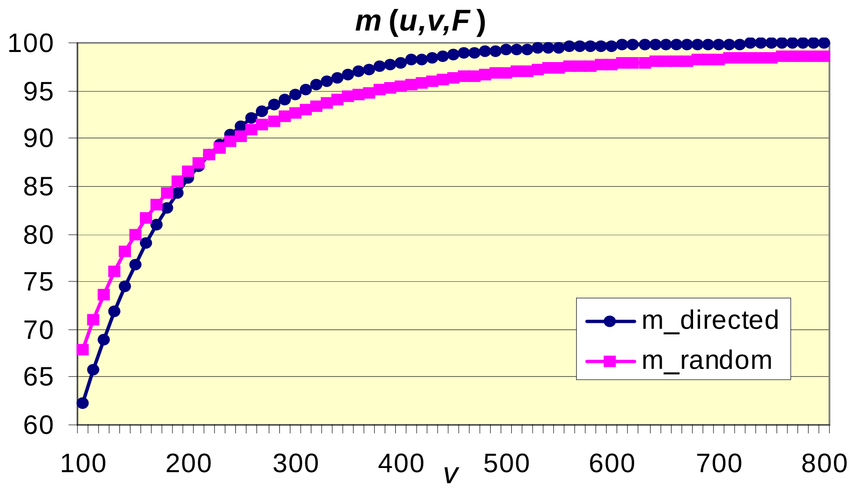

Figure 6: Random-Matching vs Heterogeneous Directed-Search: $u = 100$, $F = 10$

the number of firms F. On the other hand the directed-search produces a better result when v becomes large for given u and F, although in the limit that $v \to \infty$ the aggregate number of matches under both methods approach u. This is demonstrated in Figure 6. Further, for given v, an increase in F reduces the matching success in the directed-search as,

$$\frac{\partial m(u,v,F)}{\partial F} = -\frac{2uv}{F^2}\left[1 - \left(1 + \frac{F}{2u}\right)e^{-\frac{F}{2u}}\right] e^{-\frac{2v}{F}\left(1-e^{-\frac{F}{2u}}\right)} < 0. \tag{19}$$

This is strictly negative as $1 + \frac{F}{2u} < e^{\frac{F}{2u}}$ for $F < 2u$ and hence the term in [] is strictly positive. This implies that the coordination failure problem is alleviated in the heterogeneous directed-search when the labour market consists of a fewer firms but with larger number of advertised vacancies, a result that agrees with the one attained by Burdett *et al.* (2001).[10] This result was in fact already indirectly stated when it was shown that the aggregate matching function (6) exhibits increasing returns when F is held fixed.

[10] In a set-up already described in footnote 1, they conclude that "the frictions are more problematic when there are more locations with limited capacity."

4 Conclusion

In this paper a new model of directed-search was developed by introducing heterogeneous preferences of firms and workers, which affected not only the application stage of matching, but also the candidates selection and the acceptance stages. In this respect the model has a flavour of Cole and Smith (1998) in which friction is also entirely caused by heterogeneity, where buyers and sellers fail to match due to mismatch in the characteristics of the goods sought / offered. However in their model all agents have complete information, and hence there is no coordination problem. Here incomplete information leads to coordination failure, but in a heterogeneous directed-search framework rather than that of random-matching. In this new framework both the firm-level and the aggregate matching functions were here derived, and the performances of the heterogeneous search and the random-matching models were compared. Finally it was predicted that the friction effect is smaller in a market where there are fewer but larger firms.

There are many possible extensions to this model, not least of it being an extension to a $n > 1$ dimensional heterogeneity domain allowing $a > 2$ applications per worker. However the more interesting cases would be the relaxations of the two elements of randomness that still remain in the model. One is the number of workers who choose to apply to each pair of neighbouring firms. For example one may consider a model that incorporates workers' and firms' decision processes for their preferences of counterparts that reflect heterogeneous skills match between them. This would lead to a formal discussion about microeconomic wage-determination processes in a job-matching market. Adding in further other non-monetary characteristics preferred / offered by workers and firms may lead to interesting predictions about market frictions or wage dispersion. A related work is given in Hori (2007b) where the optimal application strategy is derived for workers with a preference for higher wages (i.e. higher productivity skills match), who also have heterogeneous job offer probabilities at each firm. The other remaining random element is the number of vacancies offered by firms. A relaxation of this would require a formal modelling of the firms' labour demand. This may depend on the decreasing returns nature of the firm-level match demonstrated in this paper, as well as factors affecting firms' production, such as product market demand, technology, or even the availability of appropriately skilled workers. This potential extension would push the randomness assumed in the model down to the levels of workers' innate skills, technology shocks and product market demand fluctuations. This will be a step towards developing a microeconomic theory of unemployment, for which this paper provides a useful framework.

A Proof of Property 1

1. Trivial using (6).

2. This is easiest seen by partially differentiating (8) by u. The derivative terms corresponding to the terms in the limits cancel out, leaving,

$$\begin{aligned}\frac{\partial m_i}{\partial u} &= \int_0^{v_i \frac{\Omega}{u}} \frac{x_i}{\Omega} \phi_{x_i} dx_i \\ &+ \sum_{v_{i+1}=1}^{v_i - 1} \phi_{v_{i+1}} \left[- \int_{\left(\frac{v_i + v_{i+1}}{2}\right)\frac{\Omega}{u}}^{v_i \frac{\Omega}{u}} \frac{x_i}{\Omega} \phi_{x_i} dx_i + \int_{v_{i+1}\frac{\Omega}{u}}^{\left(\frac{v_i + v_{i+1}}{2}\right)\frac{\Omega}{u}} \frac{x_i}{\Omega} \phi_{x_i} dx_i \right].\end{aligned}$$

This is positive as

$$\int_0^{v_i \frac{\Omega}{u}} \frac{x_i}{\Omega} \phi_{x_i} dx_i = \sum_{v_{i+1}=0}^{v} \phi_{v_{i+1}} \int_0^{v_i \frac{\Omega}{u}} \frac{x_i}{\Omega} \phi_{x_i} dx_i > \sum_{v_{i+1}=1}^{v_i - 1} \phi_{v_{i+1}} \int_{\left(\frac{v_i + v_{i+1}}{2}\right)\frac{\Omega}{u}}^{v_i \frac{\Omega}{u}} \frac{x_i}{\Omega} \phi_{x_i} dx_i.$$

For the limit $u \to \infty$, letting $e^{-\frac{F}{u} v_i} \approx 1 - \frac{F}{u} v_i$ in (6) yields $\lim_{u\to\infty} m_i = v_i$.

3. In partially differentiating (8) by v_i once again the terms corresponding to the limits cancel out, yielding

$$\frac{\partial m_i}{\partial v_i} = \int_{v_i \frac{\Omega}{u}}^{\infty} \phi_{x_i} dx_i + \sum_{v_{i+1}=1}^{v_i - 1} \phi_{v_{i+1}} \int_{\left(\frac{v_i + v_{i+1}}{2}\right)\frac{\Omega}{u}}^{v_i \frac{\Omega}{u}} \phi_{x_i} dx_i > 0. \tag{20}$$

For the limit let $v_i \to \infty$ in (6),

$$\begin{aligned}\lim_{v_i \to \infty} m_i &= \frac{u}{F} + \frac{u}{F} \sum_{v_{i+1}=1}^{\infty} \frac{\left(\frac{v}{F}\right)^{v_{i+1}} e^{-\frac{v}{F}}}{v_{i+1}!} e^{-\frac{F}{u} v_{i+1}} \\ &= \frac{u}{F} \left[1 + e^{-\frac{v}{u}\left(1 - e^{-\frac{F}{u}}\right)} \sum_{v_{i+1}=1}^{\infty} \frac{\left(\frac{v}{F} e^{-\frac{F}{u}}\right)^{v_{i+1}} e^{-\frac{v}{F} e^{-\frac{F}{u}}}}{v_{i+1}!} \right] \\ &= \frac{u}{F} \left[1 + e^{-\frac{v}{u}\left(1 - e^{-\frac{F}{u}}\right)} \left(1 - e^{-\frac{v}{F} e^{-\frac{F}{u}}} \right) \right] \\ &= \frac{u}{F} \left[1 + e^{-\frac{v}{u}} \left(e^{\frac{v}{F} e^{-\frac{F}{u}}} - 1 \right) \right].\end{aligned}$$

This is greater than $\frac{u}{F}$ and smaller than $\frac{2u}{F}$.

4. Differentiate (20) once again by v_i,

$$\begin{aligned}
\frac{\partial^2 m_i}{\partial v_i^2} &= -\frac{F}{u}e^{-\frac{F}{u}v_i} + \sum_{v_{i+1}=1}^{v_i-1} \phi_{v_{i+1}} \left[\frac{F}{u}e^{-\frac{F}{u}v_i} - \frac{F}{2u}e^{-\frac{F}{u}\left(\frac{v_i+v_{i+1}}{2}\right)}\right] \\
&= -\frac{F}{u}e^{-\frac{F}{u}v_i}\left(1 - \sum_{v_{i+1}=1}^{v_i-1} \phi_{v_{i+1}}\right) - \frac{F}{2u}\sum_{v_{i+1}=1}^{v_i-1} \phi_{v_{i+1}} e^{-\frac{F}{u}\left(\frac{v_i+v_{i+1}}{2}\right)}.
\end{aligned}$$

which is negative unambiguously.■

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